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Gauge field theory approach to construct the Navier-Stokes equation

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Abstract

We construct the Navier-Stokes equation from first principle using relativistic bosonic lagrangian which is invariant under local gauge transformations. We show that by defining the bosonic field to represent the dynamic of fluid in a particular form, a general Navier-Stokes equation with conservative forces can be reproduced exactly. It also induces two new forces, one is relevant for rotational fluid, and the other is due to the fluid's current or density. This approach provides an underlying theory to apply the tools in field theory to the problems in fluid dynamics.

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1 Introduction

The Navier-Stokes (NS) equation represents a non-linier system with flow's velocity $\vec{v} \equiv \vec{v}(x_\mu)$, where x_μ is a 4-dimensional space consists of time and spatial spaces, $x_\mu \equiv (x_0, x_i) = (t, \vec{r}) = (t, x, y, z)$. Note that throughout the paper we use natural unit, *i.e.* the light velocity $c = 1$ such that $ct = t$ and then time and spatial spaces have a same dimension. Also we use the relativistic (Minkowski) space, with the metric $g_{\mu\nu} = (1, -\vec{1}) = (1, -1, -1, -1)$ that leads to $x^2 = x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu = x_0^2 - \mathbf{x}^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Since the NS equation is derived from the second Newton's law, in principle it should be derived from analytical mechanics using the principle of at-least action on the hamiltonian as already done in several papers [1]. Some papers also relate it with the Maxwell equation [2]. The relation between NS and Maxwell equations is, however, not clear and intuitively understandable claim since both equations represent different systems. Moreover, some authors have also formulated the fluid dynamics in lagrangian with gauge symmetries [3]. However, in those previous works the lagrangian has been constructed from continuity equation.

Inspired by those pioneering works, we have tried to construct the NS equation from first principle of analytical mechanics, *i.e.* starting from lagrangian density. Also concerning that the NS equation is a system with 4-dimensional space as mentioned above, it is natural to borrow the methods in the relativistic field theory which treats time and space equally. Then we start with developing a lagrangian for bosonic field and put a contraint such that it is gauge invariant. Taking the bosonic field to have a particular form representing the dynamics of fluid, we derive the equation of motion which reproduces the NS equation.

In this paper, we first introduce the bosonic lagrangian and then review briefly the abelian and non-abelian gauge symmetries. After constructing the NS equation through Euler-Lagrange equation, we give some conclusions.

2 Gauge invariant bosonic lagrangian

In the relativistic field theory, the lagrangian (density) for a bosonic field A is written as [5],

$$\mathcal{L}_A = (\partial^\mu A)(\partial_\mu A) + m_A^2 A^2 \ , \tag{1}$$

where m_A is a coupling constant with mass dimension, and $\partial_\mu \equiv \partial/\partial x^\mu$. The bosonic field has the dimension of $[A] = 1$ in the unit of mass dimension $[m] = 1$ ($[x_\mu] = -1$). The bosonic particles are, in particle physics, interpreted as the particles which are responsible to mediate the forces between interacting fermions, ψ's. Then, one has to first start from the fermionic lagrangian,

$$\mathcal{L}_\psi = i\bar{\psi}\gamma^\mu(\partial_\mu \psi) - m_\psi \bar{\psi}\psi \ , \tag{2}$$

where ψ and $\bar{\psi}$ are the fermion and anti-fermion fields with the dimension $[\psi] = [\bar{\psi}] = 3/2$ (then $[m_\psi] = 1$ as above), while γ^μ is the Dirac gamma matrices. In order to expand the theory and incorporate some particular interactions, one should impose some symmetries.

2.1 Abelian gauge theory

For simplicity, one might introduce the simplest symmetry called $U(1)$ (abelian) gauge symmetry. The $U(1)$ local transformation[4] is just a phase transformation $U \equiv \exp\left[-i\theta(x)\right]$ of the fermions, that is $\psi \xrightarrow{U} \psi' \equiv U\,\psi$. If one requires that the lagrangian in Eq. (2) is invariant under this local transformation, *i.e.* $\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L}$, a new term coming from replacing the partial derivative with the covariant one $\partial_\mu \rightarrow D_\mu \equiv \partial_\mu + ieA_\mu$, should be added as,

$$\mathcal{L} = \mathcal{L}_\psi - e(\bar{\psi}\gamma^\mu\psi)A_\mu \; . \tag{3}$$

Here the additional field A_μ should be a vector boson since $[A_\mu] = 1$ as shown in Eq. (1). This field is known as gauge boson and should be transformed under $U(1)$ as,

$$A_\mu \xrightarrow{U} A'_\mu \equiv A_\mu + \frac{1}{e}(\partial_\mu\theta) \; , \tag{4}$$

to keep the invariance of Eq. (3). Here e is a dimensionless coupling constant interpreted as electric charge later on.

The existence of a particle requires that there must be a kinetic term of that particle in the lagrangian. In the case of newly introduced A_μ above, it is fulfilled by adding the kinetic term using the standard boson lagrangian in Eq. (1). However, it is easy to verify that the kinetic term (*i.e.* the first term) in Eq. (1) is not invariant under the transformation of Eq. (4). Then one must modify the kinetic term to keep the gauge invariance. This can be done by writing down the kinetic term in the form of anti-symmetric strength tensor $F_{\mu\nu}$ [4],

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \; , \tag{5}$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, and the factor of 1/4 is just a normalization factor.

On the other hand, the mass term (the second term) in Eq. (1) is automatically discarded in this theory since the quadratic term of A_μ is not invariant (and then not allowed) under transformation in Eq. (4). In particle physics this result justifies the interpretation of gauge boson A_μ as photon which is a massless particle.

Finally, imposing the $U(1)$ gauge symmetry, one ends up with the relativistic version of electromagnetic theory, known as the quantum electrodynamics (QED),

$$\mathcal{L}_{\rm QED} = i\bar{\psi}\gamma^\mu(\partial_\mu\psi) - m_\psi\bar{\psi}\psi - eJ^\mu A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \; , \tag{6}$$

where $J^\mu \equiv \bar{\psi}\gamma^\mu\psi = (\rho, \vec{J}) = (J_0, \vec{J})$ is the 4-vector current of fermion which satisfies the continuity equation, $\partial_\mu J^\mu = 0$, using the Dirac equation governs the fermionic field [5].

2.2 Non-abelian gauge theory

One can furthermore generalize this method by introducing a larger symmetry. This (so-called) non-abelian transformation can be written as $U \equiv \exp\left[-iT_a\theta^a(x)\right]$, where T_a's are matrices called generators belong to a particular Lie group and satisfy certain

[4]The terminology "local" here means that the parameter θ is space dependent, *i.e.* $\theta \equiv \theta(x)$. One needs also to put a preassumption that the transformation is infinitesimal, *i.e.* $\theta \ll 1$.

commutation relation like $[T_a, T_b] = i f_{abc} T_c$, where the anti-symmetric constant f_{abc} is called the structure function of the group [6]. For an example, a special-unitary Lie group $SU(n)$ has $n^2 - 1$ generators, and the subscripts a, b, c run over $1, \cdots, n^2 - 1$.

Following exactly the same procedure as Sec. 2.1, one can construct an invariant lagrangian under this transformation. The differences come only from the non-commutativeness of the generators. This induces $\partial_\mu \to D_\mu \equiv \partial_\mu + i g T_a A_\mu^a$, and the non-zero f_{abc} modifies Eq. (4) and the strength tensor $F_{\mu\nu}$ to,

$$A_\mu^a \xrightarrow{U} A_\mu^{a\prime} \equiv A_\mu^a + \frac{1}{g}(\partial_\mu \theta^a) + f^{abc} \theta^b A_\mu^c \, , \tag{7}$$

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c \, , \tag{8}$$

where g is a particular coupling constant as before. One then has the non-abelian (NA) gauge invariant lagrangian that is analoguous to Eq. (6),

$$\mathcal{L}_{\mathrm{NA}} = i\bar{\psi}\gamma^\mu(\partial_\mu\psi) - m_\psi \bar{\psi}\psi - g J_a^\mu A_\mu^a - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} \, , \tag{9}$$

while $J_a^\mu \equiv \bar{\psi}\gamma^\mu T_a \psi$, and this again satisfies the continuity equation $\partial_\mu J_a^\mu = 0$ as before. For instance, in the case of $SU(3)$ one knows the quantum chromodynamics (QCD) to explain the strong interaction by introducing eight gauge bosons called gluons induced by its eight generators.

We have so far reviewed shortly the basic of the gauge field theory. Now we are ready to jump into the main part of this paper to construct the NS equation from the gauge invariant lagrangian, Eqs. (6) and (9).

3 The NS equation from the gauge field theory

In the fluid dynamics which is governed by the NS equation we are mostly interested only in how the forces are mediated, and not in the transition of an initial state to another final state as concerned in particle physics. Within this interest, we need to consider only the bosonic terms in the total lagrangian. Assuming that the lagrangian is invariant under certain gauge symmetry explained in the preceding section, we have,

$$\mathcal{L}_{\mathrm{NS}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - g J_a^\mu A_\mu^a \, . \tag{10}$$

We put an attention on the current in second term. It should not be considered as the fermionic current as its original version, since we do not introduce any fermion in our system. For time being we must consider J_a^μ as just a 4-vector current, and it is induced by different mechanism than the internal interaction in the fluid represented by field A_μ^a. Actually it is not a big deal to even put $J_a^\mu = 0$ (free field lagrangian), or any arbitrary forms as long as the continuity equation $\partial_\mu J_a^\mu = 0$ is kept.

According to the principle of at-least action for the action $S = \int \mathrm{d}^4x \, \mathcal{L}_{\mathrm{NS}}$, *i.e.* $\delta S = 0$, one obtains the Euler-Lagrange equation,

$$\partial_\mu \frac{\partial \mathcal{L}_{\mathrm{NS}}}{\partial(\partial^\mu A_\nu^a)} - \frac{\partial \mathcal{L}_{\mathrm{NS}}}{\partial A_\nu^a} = 0 \, . \tag{11}$$

Substituting Eq. (10) into Eq. (11), this leads to the equation of motion (EOM) in term of field A_μ^a,

$$\partial_\mu(\partial^\nu A_\nu^a) - \partial^2 A_\mu^a + g J_\mu^a = 0 \; . \tag{12}$$

If A_μ is considered as a field representing a fluid system for each a, then we have multi fluids system governed by a single form of EOM. Inversely, the current can be derived from Eq. (12) to get,

$$J_\mu^a = -\frac{1}{g}\partial^\nu \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) \; , \tag{13}$$

and one can easily verify that the continuity equation is kept. We note that this equation holds for both abelian and non-abelian cases, since the last term in Eq. (8) contributes nothing due to its anti-symmetry. Also, this reproduces the relativistic version of the classical electromagnetic density and current of Maxwell.

The next task is to rewrite the above EOM to the familiar NS equation. Let us first consider a single field A_μ. Then the task can be accomplished by defining the field A_μ in term of scalar and vector potentials,

$$\begin{aligned} A_\mu &= (A_0, A_i) = (\phi, \vec{A}) \\ &\equiv \left(\frac{d}{2}\,|\vec{v}|^2 - V, -d\vec{v}\right) \; , \end{aligned} \tag{14}$$

where d is an arbitrary parameter with the dimension $[d] = 1$ to keep correct dimension for each element of A_μ. $V = V(\vec{r})$ is any potential induced by conservative forces. The condition for a conservative force $\vec{F}$ is $\oint \mathrm{d}\vec{r}\cdot\vec{F} = 0$ with the solution $\vec{F} = \vec{\nabla}\phi$. This means that the potential V must not contain a derivative of spatial space. We are now going to prove that this choice is correct.

From Eq. (12) it is straightforward to obtain,

$$\partial_\mu A_\nu^a - \partial_\nu A_\mu^a = -g \oint \mathrm{d}x_\nu J_\mu^a \; . \tag{15}$$

First we can perform the calculation for $\mu = \nu$ where we obtain trivial relation, that is $J_\mu^a = 0$. Non-trivial relation is obtained for $\mu \neq \nu$,

$$\partial_0 A_i - \partial_i A_0 = g \oint \mathrm{d}x_0 J_i = -g \oint \mathrm{d}x_i J_0 \; . \tag{16}$$

Different sign in the right hand side merely reflects the Minkowski metric we use. Now we are ready to derive the NS equation. Substituting the 4-vector potensial in Eq. (14) into Eq. (16), we obtain $d\,\partial_0 v_i + \partial_i \phi = g\tilde{J}_i$ or,

$$d\,\partial_0 \vec{v} + \vec{\nabla}\phi = g\vec{\tilde{J}} \; , \tag{17}$$

where $\tilde{J}_i \equiv -\oint \mathrm{d}x_0 J_i = \oint \mathrm{d}x_i J_0$. Using the scalar potential given in Eq. (14), we obtain,

$$d\,\frac{\partial \vec{v}}{\partial t} + \frac{d}{2}\vec{\nabla}\,|\vec{v}|^2 - \vec{\nabla}V = g\vec{\tilde{J}} \; . \tag{18}$$

By utilizing the identity $\frac{1}{2}\vec{\nabla}\,|\vec{v}|^2 = (\vec{v}\cdot\vec{\nabla})\vec{v} + \vec{v}\times(\vec{\nabla}\times\vec{v})$, we arrive at,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{1}{d}\vec{\nabla}V - \vec{v}\times\vec{\omega} + \frac{g}{d}\vec{\tilde{J}} \; , \tag{19}$$

where $\vec{\omega} \equiv \vec{\nabla} \times \vec{v}$ is the vorticity. This result reproduces a general NS equation with arbitrary conservative forces ($\vec{\nabla} V$) and some additional forces. This result justifies our choice for the bosonic field in Eq. (14).

Just to mentioned, the potential could represent the already known ones such as,

$$V(\vec{r}) = \begin{cases} P(\vec{r})/\rho(\vec{r}) & : \quad \text{pressure} \\ Gm/|\vec{r}| & : \quad \text{gravitation} \\ (\nu + \eta)(\vec{\nabla} \cdot \vec{v}) & : \quad \text{viscosity} \end{cases} . \tag{20}$$

Here, $P, \rho, G, \nu + \eta$ denote pressure, density, gravitational constant and viscosity as well. We are able to extract a general force of viscosity, $\vec{\nabla} V_{\text{viscosity}} = \eta \vec{\nabla} \left(\vec{\nabla} \cdot \vec{v} \right) + \nu \left(\vec{\nabla}^2 \vec{v} \right) + \nu \left(\vec{\nabla} \times \vec{\omega} \right)$ using the identity $\vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = \vec{\nabla} \times \vec{\omega} + \vec{\nabla}^2 \vec{v}$. This reproduces two terms relevant for both compressible and incompressible fluids, while the last term contributes to the rotational fluid for non-zero $\vec{\omega}$. This provides a natural reason for causality relation between viscosity and turbulence as stated in the definition of Reynold number, $R \propto \nu^{-1}$.

A general NS equation of multi fluids system can finally be obtained by putting the superscript a back to the equation,

$$\frac{\partial \vec{v}^a}{\partial t} + (\vec{v}^a \cdot \vec{\nabla}) \vec{v}^a = \frac{1}{d} \vec{\nabla} V^a - \vec{v}^a \times \vec{\omega}^a + \frac{g}{d} \vec{J}^a , \tag{21}$$

Here the second term in the right hand side is a new force relevant for rotational fluid, while the last term is due to the current or density of fluid.

We would like to note an important issue here. One can take arbitrary current forces in the NS equation (Eq. (21)), as long as the continuity equation is kept, but should set a small number for g. This is very crucial since we will use the perturbation method of field theory to perform any calculation in fluid dynamics starting from the lagrangian in Eq. (10) later on. Taking arbitrary and small enough coupling constant ($g \ll 1$) is needed to ensure that our perturbation works well.

4 Summary and discussion

We have shown that the NS equation can be derived directly from first principle of mechanics using relativistic bosonic lagrangian. Using this method, we can treat any conservative forces in term of the potential V in a general manner. At the same time it predicts an additional current force ($g\vec{J}$) induced by fluid's current or density, and another one ($\vec{v} \times \vec{\omega}$) relevant for rotational fluid.

In this approach we have more freedom on defining the forms of fluid's current and density, and are able to further expand the theory by introducing new forces (interactions) in the lagrangian as long as the continuity equation and symmetry are kept. Also it is straightforward to relate the NS equation with the Maxwell equation, because both of them have been constructed from the same lagrangian and (abelian gauge) symmetry as well.

Another interesting result is, in principle, we are able to discuss multi fluids system consistently, for example, one can consider a system with a lagrangian,

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{NS}}^A + \mathcal{L}_{\text{NS}}^B + \mathcal{L}_{\text{int}}^{AB} , \tag{22}$$

to represent mixing of two fuids A and B. Here the interaction term $\mathcal{L}_{\text{int}}^{AB}$ is choosen such that the symmetry is kept. This opens a possibility to calculate interactions among multi fluids with different conditions characterized by different velocities $\vec{v}^a$ and current forces $g\vec{J}^a$ with only one lagrangian as Eq. (22). These points and applications of using this lagrangian will be discussed in detail in the subsequent paper [8].

We would like to note an important issue in this approach. That is, a more comprehensive explanation to bring the (non-relativistic-like) form of bosonic field A_μ as Eq. (14) is needed. Actually, one should take a (relativistic-like) form of A_μ which can reproduce the relativistic Navier-Stokes equation using the same approach, and it coincides at non-relativistic limit with the form of Eq. (14). We are now working on this issue.

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