TSK Fuzzy Systems Types II and III Stability Analysis: Continuous Case

Assem Sonbol and M. Sami Fadali
Electrical Engineering Department
University of Nevada
Reno, NV 89512

Abstract--- **We propose a new approach for the stability analysis of continuous Sugeno Types II and III fuzzy systems. We introduce the concept of fuzzy positive definite and fuzzy negative definite functions and use them in arguments similar to those of traditional Lyapunov stability theory to derive new conditions for stability and asymptotic stability for continuous Type II/III fuzzy systems. To demonstrate the new approach, we apply it to a numerical example.**

I. Introduction

Although the stability analysis of fuzzy systems has been the subject of extensive research, no general approach is available [1]. Furthermore, the nonlinear structure and many types of fuzzy systems make the development of a general approach highly unlikely. We must therefore start with a classification of fuzzy systems and then proceed to the development of stability analysis tools for each class.

Sugeno [1] classified fuzzy systems into three types. Type I, which was first introduced by Mamdani, uses fuzzy rules of the form:

$$R_{i_1 \ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1 \ldots i_n}\ THEN\ \mathbf{y}_{i_1 \ldots i_n}\ is\ \mathbf{H}_{i_1 \ldots i_n} \tag{1}$$

Type II, or Takagi-Sugeno-Kang fuzzy systems use the simplified fuzzy rules:

$$R_{i_1 \ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1 \ldots i_n}\ THEN\ \mathbf{y}_{i_1 \ldots i_n} = \mathbf{h}_{i_1 \ldots i_n} \tag{2}$$

Type II is a special case of Type III Takagi-Sugeno-Kang (TSK) systems whose rules are in the form:

$$R_{i_1 \ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1 \ldots i_n}\ THEN\ \mathbf{y}_{i_1 \ldots i_n} = \mathbf{f}_{i_1 \ldots i_n}(\mathbf{x}) \tag{3}$$

where $\mathbf{x}=[x_1 \cdots x_n]^T$, $i_j=1,\ldots,N_j$, $j=1,\ldots,n$, $\mathbf{A}_{i_1\ldots i_n}=\left[A_1^{i_1} \cdots A_n^{i_n}\right]^T$, $\mathbf{H}_{i_1\ldots i_n}=\left[H_1^{i_1\ldots i_n} \cdots H_m^{i_1\ldots i_n}\right]^T$, $A_j^{i_j}$ and $H_{i_1\cdots i_n}$ are fuzzy sets, $\mathbf{h}_{i_1\ldots i_n}=\left[h_1^{i_1\ldots i_n} \cdots h_m^{i_1\ldots i_n}\right]^T$, $h_j^{i_1\ldots i_n}$ is a singleton, and $\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})=\left[f_1^{i_1\ldots i_n}(\mathbf{x}) \cdots f_m^{i_1\ldots i_n}(\mathbf{x})\right]^T$. Most authors use the affine form where $f_j^{i_1\ldots i_n}(\mathbf{x})$, $j = 1,\ldots, m$ are of the form:

$$f_j^{i_1\ldots i_n}(\mathbf{x})=\sum_{i=1}^{n} a_{j,i}^{i_1\ldots i_n} x_i + a_{j,0}^{i_1\ldots i_n} = \mathbf{a}_j^{i_1\ldots i_n T}\mathbf{x} + a_{j,0}^{i_1\ldots i_n} \quad \textbf{(4)}$$

where $\mathbf{a}_j^{i_1\ldots i_n T}=\left[a_{j,1}^{i_1\ldots i_n} \cdots a_{j,n}^{i_1\ldots i_n}\right]^T$.

Recently, the stability analysis of Type III systems has attracted considerable interest in the fuzzy system literature [2]-[10]. For a comprehensive review of these results, the reader is referred to [1]. Most of these results require the existence of a common quadratic Lyapunov function [2]-[5]. Unfortunately, conditions for the existence of such functions are restrictive and difficult to establish [11], [12]. For example, the search for a common Lyapunov function can be posed as a convex optimization problem in terms of linear matrix inequalities (LMIs) [9]. However, the standard LMI conditions for quadratic stability are often conservative when applied to fuzzy systems. Moreover, the convex optimization problem typically involves a large number of LMIs and its computational load increases dramatically with the number of inputs [10].

Several authors were able to analyze the stability of fuzzy systems without the need for a common Lyapunov function [6],[9], [10]. Lo and Chen [6] used Kharitonov theory to derive a sufficient condition for fuzzy controller stability. However, Johansen and Slupphaug [7] showed, by a counterexample, that the conditions proposed in [6] are not sufficient for stability. Dvorakova and Husek [8] also analyzed the results in [6] and showed that Lo and Chen’s computational procedure is only valid for fuzzy systems three or fewer rules. Johansson and Rantzer [9] presented a stability analysis approach for fuzzy systems based on piecewise-continuous quadratic Lyapunov functions. The approach gives stability conditions that can be verified via convex optimization over LMIs. Feng and Harris [10] also used a piecewise-continuous quadratic Lyapunov functions. Their approach exploited the properties of the input membership functions to reduce the number of candidate Lyapunov functions and the associated LMIs.

To date, there are no stability tests available for Type I systems [1]. Sugeno [1] and Sonbol and Fadali [13], [14], [15], addressed the issue of stability of fuzzy systems described by fuzzy rules with singleton consequents (Type II). To our knowledge, these are the only published results on Type II system stability. Sugeno gave stability conditions for both discrete-time and continuous time system. Sonbol and Fadali derived general stability results for Type III and then specialized these results to Type II.

In this paper, we introduce the concept of fuzzy positive definite and fuzzy negative definite functions. We then use them to derive Lyapunov conditions for the stability analysis of continuous Type II and the affine case of Type III systems. We provide conditions for stability in the sense of Lyapunov, asymptotic stability, and exponential stability.

The paper is organized as follows. Section II introduces basic definitions and concepts, explains some properties of TSK fuzzy systems. In Section III, we derive conditions for Lyapunov stability, asymptotic stability, and exponential stability of continuous Type II/III fuzzy systems. All proofs are given in the Appendix. Finally, we provide an illustrative example in Section IV.

II. Preliminary Definitions and Concepts

In this section, we introduce concepts and definitions that we need for the stability analysis of Type II/III TSK fuzzy systems. In the sequel, we first provide definitions and derive results for continuous Type III fuzzy systems and then specialize to obtain the corresponding expressions for continuous Type II systems.

We begin with the definition of the class of TSK fuzzy systems to be analyzed throughout this paper.

Definition 1: TSK Fuzzy Systems

TSK fuzzy systems comprise four principal components [11], [16]:

1. A singleton fuzzifier that maps to triangular, complete, and consistent fuzzy sets.
2. A continuous and normal fuzzy rule base of the form (3).
3. A product inference engine.
4. A weighted-average defuzzifier.

Lemma 1: If $\mu_{A_j^{i_j}}(x_j)$ are normal, complete, and consistent triangular membership functions, of the fuzzy sets $A_j^{i_j}$, $i_j = 1,\ldots,N_j, j = 1,\ldots,n$, then

$$\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\prod_{j=1}^{n}\mu_{A_j^{i_j}}(x_j)=1 \tag{5}$$

Using Lemma 1, the TSK output formula can be rewritten as:

$$y=\sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}y^{i_1\ldots i_n}\prod_{j=1}^{n}\mu_{A_j^{l_j}}(x_j) \tag{6}$$

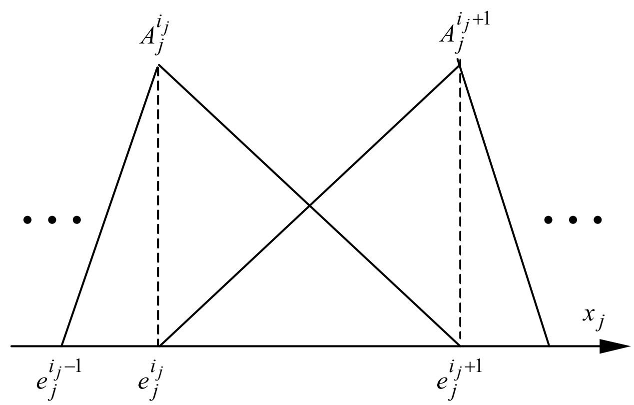

Figure 1 Triangular membership functions.

Definition 2: Continuous dynamic TSK Type II/III Fuzzy Systems

Continuous Type III dynamic fuzzy systems have fuzzy rules of the form:

$$R_{i_1\ldots i_n}: IF\ \mathbf{x}(k)\ is\ \mathbf{A}_{i_1\ldots i_n}\ THEN\ \dot{\mathbf{x}}_{i_1\ldots i_n}=\mathbf{f}_{i_1\ldots i_n}(\mathbf{x}) \tag{7}$$

where $\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})=\left[f_1^{i_1\ldots i_n}(\mathbf{x})\ \cdots\ f_m^{i_1\ldots i_n}(\mathbf{x})\right]^T$ and $f_j^{i_1\ldots i_n}(\mathbf{x})$ are in the affine form of (4).

We can calculate $\dot{\mathbf{x}}$, for normal, complete, and consistent membership functions, by taking the weighted average of consequents as follows:

$$\dot{\mathbf{x}}=\sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}\mathbf{f}_{l_1\ldots l_n}(\mathbf{x})\prod_{j=1}^{n}\mu_{A_j^{l_j}}(x_j) \tag{8}$$

Type II systems are a special case of Type III systems, where the functions $\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})$ reduce to constant vectors $\mathbf{h}_{i_1\ldots i_n}$, and where (7) and (8), respectively, reduce to

$$R_{i_1 \ldots i_n} : IF\ \mathbf{x}(k)\ is\ \mathbf{A}_{i_1 \ldots i_n}\ THEN\ \dot{\mathbf{x}}_{i_1 \ldots i_n} = \mathbf{h}_{i_1 \ldots i_n} \tag{9}$$

$$\dot{\mathbf{x}} = \sum_{l_1 = i_1}^{i_1+1} \cdots \sum_{l_n = i_n}^{i_n+1} \mathbf{h}_{l_1 \ldots l_n} \prod_{j=1}^{n} \mu_{A_j^{l_i}}(x_j) \tag{10}$$

In [1], Proposition 1, Sugeno stated that any 2×1 interval vector **x**, with entries in the range $[a_i, b_i]$, $i = 1,2$, can be represented by a Type II fuzzy system, with triangular membership functions, whose input and output are both equal to the vector **x**. The fuzzy sets in this Type II system can be chosen arbitrarily and then the consequents of the fuzzy rules are chosen to be equal to the value of the vector **x** at the centers of the chosen fuzzy sets. Here, We generalize Sugeno's result to the case of an n×1 vector **x**.

Lemma 2: Any interval vector **x**, with entries $x_i \in [a_i, b_i]$, $i = 1,\ldots,n$, can be represented by a Type II fuzzy system with input and output equal to **x**.

We now define fuzzy definite functions that play the same role in the stability analysis of fuzzy systems that crisp definite functions play in traditional Laypunov stability theory. Positive definite functions serve as " energy" quantifiers while negative definite functions characterize the rate of change of the "energy" along the trajectories of a stable system.

Definition 3: Positive definite fuzzy function

A fuzzy function that comprises the four principal components of Definition 1 and has a scalar output y is positive definite if and only if $y > 0$ for all $x_i \neq 0$, and $y = 0$ for all $x_i = 0$, $i = 1, \ldots, n$.

Definition 4: Positive semi-definite fuzzy function

A fuzzy function that comprises the four principal components of Definition 1 and has a scalar output y is positive semi-definite if and only if $y \geq 0$ for all $x_i \neq 0$, and $y = 0$ for all $x_i = 0$, $i = 1, \ldots, n$.

Definition 5: Negative definite fuzzy function

A fuzzy function that comprises the four principal components of Definition 1 and has a scalar output y is negative definite if and only if $y < 0$ for all $x_i \neq 0$, and $y = 0$ for all $x_i = 0$, $i = 1, \ldots, n$.

Definition 6: Negative semi-definite fuzzy function

A fuzzy function that comprises the four principal components of Definition 1 and has a scalar output y is negative definite if and only if $y \le 0$ for all $x_i \neq 0$, and $y = 0$ for all $x_i = 0$, $i = 1, \ldots, n$.

The following lemma provides sufficient conditions for a fuzzy definite function in terms of its consequents.

Lemma 3: Consider the Type III fuzzy function z comprising:

(i) The four principal components of Definition 1,

(ii) Input sets $\mathbf{A}_{i_1^* \ldots i_n^*} = \left[A_1^{i_1^*} \ \cdots \ A_n^{i_n^*} \right]^T$ such that the rule $R_{i_1^* \ldots i_n^*}$: *IF* $\mathbf{x}$ *is* $\mathbf{A}_{i_1^* \ldots i_n^*}$ *THEN* $y_{i_1^* \ldots i_n^*} = 0$,

(iii) A scalar output y.

1. The fuzzy function z is positive definite if

$$\begin{aligned} & f_1^{i_1 \ldots i_n}\left(\mathbf{e}_{i_1 \ldots i_n}\right) > 0 \\ & f_1^{i_1 \ldots i_n}\left(\mathbf{e}_{(i_1+l_1)\ldots(i_n+l_n)}\right) \ge 0 \end{aligned} \quad \textbf{(11)}$$

2. The fuzzy function z is positive semi-definite if

$$\begin{aligned} & f_1^{i_1 \ldots i_n}\left(\mathbf{e}_{i_1 \ldots i_n}\right) \ge 0 \\ & f_1^{i_1 \ldots i_n}\left(\mathbf{e}_{(i_1+l_1)\ldots(i_n+l_n)}\right) \ge 0 \end{aligned} \quad \textbf{(12)}$$

3. The fuzzy function z is negative definite if

$$\begin{aligned} & f_1^{i_1 \ldots i_n}\left(\mathbf{e}_{i_1 \ldots i_n}\right) < 0 \\ & f_1^{i_1 \ldots i_n}\left(\mathbf{e}_{(i_1+l_1)\ldots(i_n+l_n)}\right) \le 0 \end{aligned} \quad \textbf{(13)}$$

4. The fuzzy function z is negative semi-definite if

$$\begin{aligned} & f_1^{i_1 \ldots i_n}\left(\mathbf{e}_{i_1 \ldots i_n}\right) \le 0 \\ & f_1^{i_1 \ldots i_n}\left(\mathbf{e}_{(i_1+l_1)\ldots(i_n+l_n)}\right) \le 0 \end{aligned} \quad \textbf{(14)}$$

where $i_j = 1,\ldots,N_j - 1$, $i_j \neq i_j^*$, $l_j = 0, 1$, $j = 1,\ldots,n$, $\mathbf{e}_{i_1 \ldots i_n} = \left[e_1^{i_1} \ \cdots \ e_n^{i_n} \right]^T$.

Remark 1: For Type II, conditions (11)-(14) reduce, respectively, to

1. The fuzzy function z is positive definite if and only if

$$h_1^{i_1 \ldots i_n} > 0 \quad \textbf{(15)}$$

2. The fuzzy function z is positive semi-definite system if and only if

$h_1^{i_1 \ldots i_n} \geq 0$

(16)

3. The fuzzy function z is negative definite system if and only if

$h_1^{i_1 \ldots i_n} < 0$

(17)

4. The fuzzy function z is negative semi-definite system if and only if

$h_1^{i_1 \ldots i_n} \leq 0$

(18)

where $i_j = 1,\ldots,N_j,\ i_j \neq i_j^*,\ j = 1,\ldots,n$.

The proofs of (15)-(18) are use almost identical arguments so we only discuss (15) and omit the discussion of (16)-(18). The sufficiency of (15) follows easily from the fact that the weighted sum of positive numbers is positive. To prove the necessity of (15) by contradiction, assume that the fuzzy function z is positive definite but does not satisfy condition (15). Then $y > 0$ for all $\mathbf{x} \neq \mathbf{0}$. Since $y = h_1^{i_1 \ldots i_n}$ when $\mathbf{x}$ *is* $\mathbf{A}_{i_1 \ldots i_n}$, $i_j = 1,\ldots,N_j,\ i_j \neq i_j^*$, $j = 1,\ldots,n$, then $h_1^{i_1 \ldots i_n} > 0$, which contradicts the above assumption.

Lemma 3 allows us to easily check the definiteness of a function using its consequents.

III. Stability Analysis

In this section we give new sufficient conditions for the stability, asymptotic stability, exponential stability of continuous Type II/III systems using arguments similar to those of Lyapunov stability theory [17]. We start by stating the following stability theorem due to, Hahn [18], without proof.

Theorem 1: If there exists a function $V(\mathbf{x})$ for the system $\dot{\mathbf{x}} = f(\mathbf{x})$ with the equilibrium $\mathbf{x} = \mathbf{0}$, which satisfies, $\forall \mathbf{x} \in R^n$, $\mathbf{x} \neq \mathbf{0}$, the conditions

1. $V(\mathbf{x})$ is a positive-definite function.
2. $DV(\mathbf{x}) = \lim_{\Delta t \to 0} \sup \dfrac{V(\mathbf{x}(t+\Delta t)) - V(\mathbf{x}(t))}{\Delta t} < 0$

then the equilibrium $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

•

Remark: $DV(\mathbf{x})$ is the analytic expression of the derivative of $V(\mathbf{x})$ along the trajectories [18]. If $V(\mathbf{x})$ has continuous first order partial derivatives with respect to $\mathbf{x}$, the $DV(\mathbf{x})$ reduces to $\dot{V}(\mathbf{x}) = \lim_{\Delta t \to 0} \dfrac{V(\mathbf{x}(t+\Delta t)) - V(\mathbf{x}(t))}{\Delta t}$.

Theorem 1 shows that stability conditions can be established without requiring continuity of the derivative of $V(\mathbf{x})$ along the trajectories. This give us some freedom in choosing a function $V(\mathbf{x})$ that suits Type II/III dynamic fuzzy systems.

Definition 7: The derivative of a fuzzy function

Consider the continuous Type III dynamic fuzzy system ***z*** and the fuzzy Lyapunov function candidate $V(\mathbf{x})$, the derivative of $V(\mathbf{x})$ in the direction of the trajectories of ***f*** is the fuzzy system $DV(\mathbf{x})$ defined by:

$$DV(\mathbf{x}) = \limsup_{\Delta t \to 0} \frac{V(\mathbf{x}(t+\Delta t)) - V(\mathbf{x}(t))}{\Delta t}$$

The next theorem establishes stability and asymptotic stability for continuous Type III dynamic fuzzy systems.

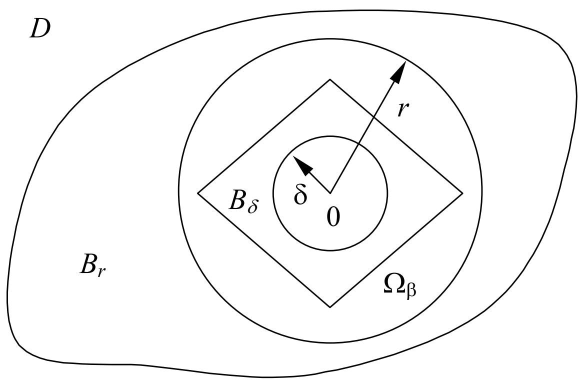

Figure 2 Geometric representation of the sets in the proof of Theorem 1.

Theorem 2: Let ***z*** be a continuous Type III dynamic fuzzy system. Consider the fuzzy Lyapunov function candidate $V(\mathbf{x})$ defined by

$$R_{i_1 \ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1 \ldots i_n}\ THEN\ V_{i_1 \ldots i_n} = C_{i_1 \ldots i_n} \tag{19}$$

Consider the derivative $DV(\mathbf{x})$ defined by:

$$R_{i_1 \ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1 \ldots i_n}\ THEN\ DV_{i_1 \ldots i_n} = \left[\frac{C_{(j_1+1)\ldots j_n} - C_{j_1 \ldots j_n}}{e_1^{j_1+1} - e_1^{j_1}} \;\vdots\; \cdots \;\vdots\; \frac{C_{j_1 \ldots (j_n+1)} - C_{j_1 \ldots j_n}}{e_n^{j_n+1} - e_n^{j_n}} \right] \mathbf{f}_{i_1 \ldots i_n}(\mathbf{x}) \tag{20}$$

where $C_{i_1 \ldots i_n}$ are positive constants that satisfy the condition

$$C_{i_1 \ldots i_j i_{j+1} \ldots j_n} - C_{i_1 \ldots (i_j+1) i_{j+1} \ldots j_n} - C_{i_1 \ldots i_j (i_{j+1}+1) \ldots j_n} + C_{i_1 \ldots (i_j+1)(i_{j+1}+1) \ldots j_n} = 0 \tag{21}$$

1. If ∃ $C_{i_1 \ldots i_n} > 0$ such that

$$\left[\frac{C_{(j_1+1)\ldots j_n} - C_{j_1 \ldots j_n}}{e_1^{j_1+1} - e_1^{j_1}} \,\vdots\, \cdots \,\vdots\, \frac{C_{j_1 \ldots (j_n+1)} - C_{j_1 \ldots j_n}}{e_n^{j_n+1} - e_n^{j_n}} \right] \mathbf{f}_{i_1 \ldots i_n}(\mathbf{e}_{(i_1+1)\cdots(i_n+1)}) \leq 0 \tag{22}$$

   where $i_j = 1,\ldots,N_j-1$, $l_j = 0, 1$, and $j = 1,\ldots,n$, then z is stable in the sense of Lyapunov.

2. If ∃ $C_{i_1 \ldots i_n} > 0$ such that

$$\left[\frac{C_{(j_1+1)\ldots j_n} - C_{j_1 \ldots j_n}}{e_1^{j_1+1} - e_1^{j_1}} \,\vdots\, \cdots \,\vdots\, \frac{C_{j_1 \ldots (j_n+1)} - C_{j_1 \ldots j_n}}{e_n^{j_n+1} - e_n^{j_n}} \right] \mathbf{f}_{i_1 \ldots i_n}(\mathbf{e}_{(i_1+1)\cdots(i_n+1)}) < 0 \tag{23}$$

where $i_j = 1,\ldots,N_j-1$, $l_j = 0, 1$, and $j = 1,\ldots,n$, then z is asymptotically stable.

Corollary 1: Let z be a continuous Type II dynamic fuzzy system. Consider the fuzzy Lyapunov function candidate $V(\mathbf{x})$ defined by (19) with derivative $DV(\mathbf{x})$ defined by:

$$R_{i_1 \ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1 \ldots i_n}\ THEN\ DV_{i_1 \ldots i_n} = \left[\frac{C_{(j_1+1)\ldots j_n} - C_{j_1 \ldots j_n}}{e_1^{j_1+1} - e_1^{j_1}} \,\vdots\, \cdots \,\vdots\, \frac{C_{j_1 \ldots (j_n+1)} - C_{j_1 \ldots j_n}}{e_n^{j_n+1} - e_n^{j_n}} \right] \mathbf{h}_{i_1 \ldots i_n} \tag{24}$$

1. If ∃ $C_{i_1 \ldots i_n} > 0$ such that $DV(\mathbf{x}) \leq 0$, then z is stable in the sense of Lyapunov.

2. If ∃ $C_{i_1 \ldots i_n} > 0$ such that $DV(\mathbf{x}) < 0$, then z is asymptotically stable.

The next theorem establishes exponential stability for continuous Type III dynamic fuzzy systems.

Theorem 3: A continuous Type III dynamic fuzzy system z is exponentially stable if there exists $\mathbf{b} = [b_1 \ldots b_n]^T > \mathbf{0}$ such that

$$\mathbf{b}^T \mathbf{S}(\mathbf{x}) \mathbf{f}_{i_1 \ldots i_n}(\mathbf{e}_{(i_1+l_1)\ldots(i_n+l_n)}) \leq -\alpha_{i_1 \ldots i_n} \mathbf{b}^T \left| \mathbf{e}_{i_1 \ldots i_n} \right| \tag{25}$$

where $\mathbf{S}(\mathbf{x}) = diag\{\mathrm{sgn}(x_1), \cdots, \mathrm{sgn}(x_n)\}$, $i_j = 1,\ldots,N_j-1$, $l_j = 0, 1$, and $j = 1,\ldots,n$, and $0 < \alpha_{i_1 \ldots i_n} \leq s_\alpha$.

Corollary 2: A continuous Type II dynamic fuzzy system z is exponentially stable if there exists $\mathbf{b} = [b_1 \ldots b_n]^T > \mathbf{0}$ such that

$$\mathbf{b}^T \mathbf{S}(\mathbf{x}) \mathbf{h}_{i_1 \ldots i_n} \leq -\alpha_{i_1 \ldots i_n} \mathbf{b}^T \left| \mathbf{e}_{i_1 \ldots i_n} \right| \tag{26}$$

where $\mathbf{S}(\mathbf{x}) = diag\{\mathrm{sgn}(x_1), \cdots, \mathrm{sgn}(x_n)\}$, and $0 < \alpha_{i_1 \ldots i_n} \leq s_\alpha$.

In the next section, we provide an example that demonstrates stability testing using fuzzy Lyapunov analysis. In the example, we present the rule base of a fuzzy system in the form of a table of consequents.

IV. Example

Determine the stability of the discrete Type II fuzzy system with the consequents of Table 1.

Table 1 Consequents for the fuzzy system.

x_2 \ x_1		e_1^1	e_1^2	e_1^3
		-2	0	2
e_2^1	-2	$\begin{bmatrix} 8 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} -5 \\ 3 \end{bmatrix}$
e_2^2	0	$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
e_2^3	2	$\begin{bmatrix} 6 \\ -3 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -2 \end{bmatrix}$	$\begin{bmatrix} -4 \\ -5 \end{bmatrix}$

By Corollary 1, the consequents of $DV(\mathbf{x})$ are as shown in Table 2. We next select a vector $\mathbf{b} = [b_1 \ \ b_2]^T > \mathbf{0}$ that makes $DV(\mathbf{x})$ a fuzzy negative-definite function. We can check for the existence of the vector **b** by solving a linear programming feasibility problem using widely available software, such as MATLAB, Maple, or Lingo.

Table 2 Consequents of DV(x).

x_2 \ x_1		e_1^1	e_1^2	e_1^2	e_1^3
		-2	0	0	2
e_2^1	-2	$\mathbf{b}^T\begin{bmatrix} -8 \\ -4 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} -2 \\ -2 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} 2 \\ -2 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} -5 \\ -3 \end{bmatrix}$
e_2^2	0	$\mathbf{b}^T\begin{bmatrix} -4 \\ 1 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} -2 \\ -1 \end{bmatrix}$
e_2^2	0	$\mathbf{b}^T\begin{bmatrix} -4 \\ -1 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
e_2^3	2	$\mathbf{b}^T\begin{bmatrix} -6 \\ -3 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} 1 \\ -2 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} -1 \\ -2 \end{bmatrix}$	$\mathbf{b}^T\begin{bmatrix} -4 \\ -5 \end{bmatrix}$

Clearly, if we choose $\mathbf{b} = [1.55 \ 3.0]^T$ then $DV(\mathbf{x})$ is negative definite and the system is asymptotically stable. This is confirmed by the simulation results of Figure 3.

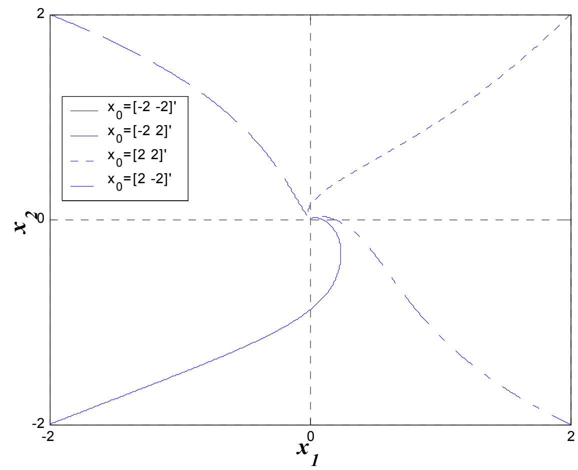

Figure 3 Trajectories of the fuzzy system.

For all initial conditions tested, the trajectories of the system converge asymptotically to the origin. $V(\mathbf{x})$, shown in Figure 4, is indeed a positive definite function because it only assumes positive values except at $\mathbf{x} = [0\ 0]^T$. The contours, Figure 5, show that $V(\mathbf{x})$ is a piecewise linear function. $DV(\mathbf{x})$, shown in Figure 6, is a negative-definite function because it only assumes negative values except at $\mathbf{x} = [0\ 0]^T$.

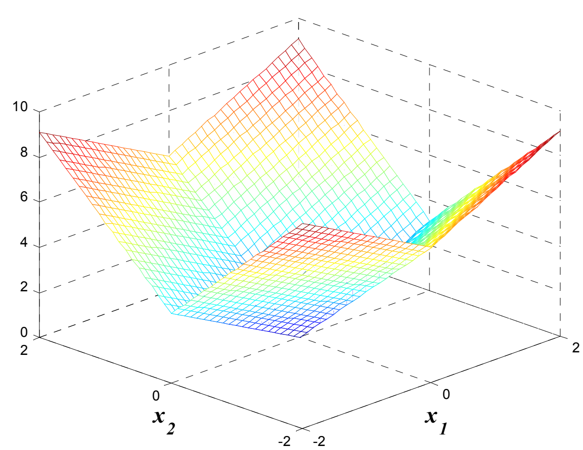

Figure 4 The function $V(\mathbf{x})$.

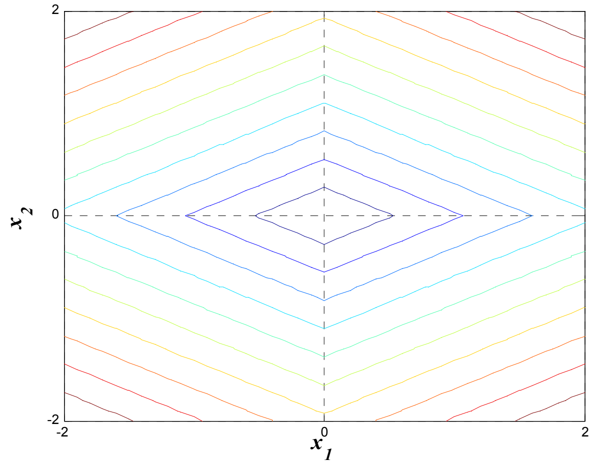

Figure 5 The contours of $V(\mathbf{x})$.

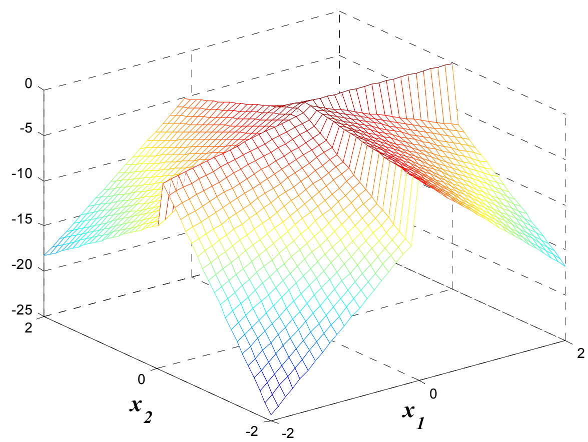

Figure 6 The function $DV(\mathrm{x})$.

V. Conclusion

This paper introduces a new approach for the stability analysis of continuous Sugeno Types II/III fuzzy systems. We use fuzzy positive definite and fuzzy negative definite functions in arguments similar to those of standard Lyapunov stability theory. We show that if a fuzzy positive definite function has fuzzy negative definite changes along the trajectories of a discrete Type II/III dynamic fuzzy system, then the system is asymptotically stable. Similarly, we derive conditions for stability in the sense of Lyapunov and for exponential stability. The main contribution of this work is that it eliminates the difficult condition of a common Lyapunov function. In addition, it simultaneously solves the stability problem for Type II and Type III systems.

Appendix A

In this appendix, we will prove all the lemmas and theorems of this paper.

Proof of Lemma 1:

From Figure 1, we define $\mu_{A_j^{i_j}}(x_j)$ for $x_j \in [e_j^{i_j-1}, e_j^{i_j+1}]$ as follows:

$$\mu_{A_j^{i_j}}(x_j) = \begin{cases} \dfrac{e_j^{i_j+1} - x_j}{e_j^{i_j+1} - e_j^{i_j}}, & x_j \in [e_j^{i_j}, e_j^{i_j+1}] \\ \dfrac{x_j - e_j^{i_j-1}}{e_j^{i_j} - e_j^{i_j-1}}, & x_j \in [e_j^{i_j-1}, e_j^{i_j}] \end{cases} \tag{27}$$

Then for normal and consistent fuzzy sets

$$\mu_{A_j^{i_j}}(x_j) = 1 - \mu_{A_j^{i_j+1}}(x_j) \tag{28}$$

From Figure 1, (5) can be rewritten as:

$$\sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \prod_{j=1}^{n} \mu_{A_j^{l_j}}(x_j) = 1 \tag{29}$$

Expanding the last summation in the RHS of (29) and using (28) gives

$$\begin{aligned} &\sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_{n-1}=i_{n-1}}^{i_{n-1}+1} \prod_{j=1}^{n-1} \mu_{A_j^{l_j}}(x_j) \left\{ \left(1 - \mu_{A_n^{i_n+1}}(x_n)\right) + \mu_{A_n^{i_n+1}}(x_n) \right\} \\ &\quad = \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_{n-1}=i_{n-1}}^{i_{n-1}+1} \prod_{j=1}^{n-1} \mu_{A_j^{l_j}}(x_j) \end{aligned}$$

Repeating the last step for l_i, $i = 1,\ldots,n-1$, completes the proof.

•

Proof of Lemma 2:

Define a Type II fuzzy system with input $\mathbf{x}$ and with the corresponding fuzzy sets and rule base:

$$R_{i_1 \ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1 \ldots i_n}\ THEN\ \mathbf{y}_{i_1 \ldots i_n} = \mathbf{e}_{i_1 \ldots i_n} \tag{30}$$

Let triangular membership functions $\mu_{A_j^{i_j}}(x_j)$ characterize the input fuzzy sets $A_j^{i_j}$. Then the output of the fuzzy system of (30) is calculated as

$$\mathbf{y} = \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \mathbf{e}_{l_1 \ldots l_n} \prod_{j=1}^{n} \mu_{A_j^{l_j}}(x_j) \tag{31}$$

Expanding the last summation in (31)

$$\begin{aligned}
\mathbf{y} &= \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_{n-1}=i_{n-1}}^{i_{n-1}+1} \prod_{j=1}^{n-1} \mu_{A_j^{l_j}}(x_j) \\
&\quad \left\{ \mathbf{e}_{l_1 \ldots i_n} \left(1 - \mu_{A_n^{i_n+1}}(x_n) \right) + \mathbf{e}_{l_1 \ldots i_n+1} \mu_{A_n^{i_n+1}}(x_n) \right\} \\
&= \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_{n-1}=i_{n-1}}^{i_{n-1}+1} \prod_{j=1}^{n-1} \mu_{A_j^{l_j}}(x_j) \\
&\quad \left\{ \mathbf{e}_{l_1 \ldots i_n} + \left[0 \ 0 \ \cdots \ (e_n^{i_n+1} - e_n^{i_n}) \right]^T \mu_{A_n^{i_n+1}}(x_n) \right\}
\end{aligned} \tag{32}$$

We substitute for $\mu_{A_n^{i_n+1}}(x_n)$ from (27) into (32) to obtain

$$\begin{aligned}
\mathbf{y} &= \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_{n-1}=i_{n-1}}^{i_{n-1}+1} \left[\mathbf{e}_{l_1 \ldots i_n} + \left[0 \ 0 \ \cdots \ (x_n - e_n^{i_n}) \right]^T \right] \prod_{j=1}^{n-1} \mu_{A_j^{l_j}}(x_j) \\
&= \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_{n-1}=i_{n-1}}^{i_{n-1}+1} \left[\mathbf{e}_{l_1 \ldots l_{n-1}} \,\vdots\, x_n \right]^T \prod_{j=1}^{n-1} \mu_{A_j^{l_j}}(x_j)
\end{aligned}$$

where $\mathbf{e}_{i_1 \ldots i_{n-1}} = \left[e_1^{i_1} \ \cdots \ e_{n-1}^{i_n} \right]^T$.

Repeating the last step for l_i, $i = 1,\ldots,n-1$, completes the proof.

•

Proof of Lemma 3:

Using the rule base (3) and (4), we write the output of z as

$$y = \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \left(\mathbf{a}_1^{l_1 \ldots l_n\,T} \mathbf{x} + a_{1,0}^{l_1 \ldots l_n} \right) \prod_{j=1}^{n} \mu_{A_j^{l_i}}(x_j)$$

Next, we substitute for $\mathbf{x}$ from (31) and simplify

$$\begin{aligned}
y &= \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \\
&\left[\mathbf{a}_1^{l_1 \ldots l_n\,T} \left(\sum_{k_1=i_1}^{i_1+1} \cdots \sum_{k_n=i_n}^{i_n+1} \mathbf{e}_{k_1 \ldots k_n} \prod_{j=1}^{n} \mu_{A_j^{k_i}}(x_j) \right) + a_{1,0}^{l_1 \ldots l_n} \right] \prod_{j=1}^{n} \mu_{A_j^{l_i}}(x_j) \\
&= \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \\
&\left[\sum_{k_1=i_1}^{i_1+1} \cdots \sum_{k_n=i_n}^{i_n+1} \left(\mathbf{a}_1^{l_1 \ldots l_n\,T} \mathbf{e}_{k_1 \ldots k_n} + a_{1,0}^{l_1 \ldots l_n} \right) \prod_{j=1}^{n} \mu_{A_j^{k_i}}(x_j) \right] \prod_{j=1}^{n} \mu_{A_j^{l_i}}(x_j) \\
&= \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \\
&\left[\sum_{k_1=i_1}^{i_1+1} \cdots \sum_{k_n=i_n}^{i_n+1} f_1^{k_1 \ldots k_n} \left(\mathbf{e}_{k_1 \ldots k_n} \right) \prod_{j=1}^{n} \mu_{A_j^{k_i}}(x_j) \right] \prod_{j=1}^{n} \mu_{A_j^{l_i}}(x_j)
\end{aligned}$$

Substituting conditions (11)-(14) completes the proof for 1-4, respectively.

•

Proof of Theorem 1:

Using Lemma 1, the output of $V(\mathbf{x})$ can be written as

$$V(\mathbf{x}) = \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} C_{l_1 \ldots l_n} \prod_{j=1}^{n} \mu_{A_j^{l_j}}(x_j) \tag{33}$$

Using (28) to expand the last summation in (33), we obtain

$$V(\mathbf{x}) = \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_{n-1}=i_{n-1}}^{i_{n-1}+1} \prod_{j=1}^{n-1} \mu_{A_j^{l_j}}(x_j) \left[C_{l_1 \ldots l_n} + \left(C_{l_1 \ldots (l_n+1)} - C_{l_1 \ldots l_n} \right) \mu_{A_n^{(l_n+1)}}(x_n) \right] \tag{34}$$

Expanding the last summation in (34) and using (28), we obtain

$$V(\mathbf{x}) = \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_{n-2}=i_{n-2}}^{i_{n-2}+1} \prod_{j=1}^{n-2} \mu_{A_j^{l_j}}(x_j) \left[C_{l_1 \ldots l_n} + \left(C_{l_1 \ldots (l_n+1)} - C_{l_1 \ldots l_n} \right) \mu_{A_n^{(l_n+1)}}(x_n) \right. \\ \left. \left(C_{l_1 \ldots (l_{n-1}+1) l_n} - C_{l_1 \ldots l_n} \right) \mu_{A_{n-1}^{(l_{n-1}+1)}}(x_{n-1}) \right] \tag{35}$$

Repeating the last step with the remaining summations in (35), we obtain

$$V(\mathbf{x}) = C_{i_1 \ldots i_n} + \sum_{j=1}^{n} \left(C_{i_1 \ldots (i_j+1) \ldots i_n} - C_{i_1 \ldots i_j \ldots i_n} \right) \mu_{A_j^{(i_j+1)}}(x_j) \\ = C_{i_1 \ldots i_n} + \left[C_{(i_1+1) \ldots i_n} - C_{i_1 \ldots i_n} \vdots \cdots \vdots C_{i_1 \ldots (i_n+1)} - C_{i_1 \ldots i_n} \right] \boldsymbol{\mu}(\mathbf{x}) \tag{36}$$

where $\boldsymbol{\mu}(\mathbf{x}) = \left[\mu_{A_1^{(j_1+1)}}(x_1) \cdots \mu_{A_n^{(j_n+1)}}(x_n) \right]^T$ and $\mu_{A_j^{i_j}}(x_j)$ are defined by (27). Using (27), we have:

$$\boldsymbol{\mu}(\mathbf{x}) = diag\left\{ \frac{1}{e_1^{i_1+1} - e_1^{i_1}}, \cdots, \frac{1}{e_n^{i_n+1} - e_n^{i_n}} \right\} \left(\mathbf{x} - \mathbf{e}_{i_1 \ldots i_n} \right) \tag{37}$$

where $\mathbf{e}_{i_1 \ldots i_n} = \left[e_1^{i_1} \cdots e_n^{i_n} \right]^T$.

Using (37), we can write $V(\mathbf{x}(t+\Delta t))$ as

$$V(\mathbf{x}(t+\Delta t)) = C_{i_1 \ldots i_n} + \\ \left[C_{(i_1+1) \ldots i_n} - C_{i_1 \ldots i_n} \vdots \cdots \vdots C_{i_1 \ldots (i_n+1)} - C_{i_1 \ldots i_n} \right] \boldsymbol{\mu}(\mathbf{x}(t+\Delta t)) \tag{38}$$

Using (36) and (38), we can write $DV(\mathbf{x})$ as

$$DV(\mathbf{x}) = \lim_{\Delta t \to 0} \sup \left[C_{(i_1+1)\ldots i_n} - C_{i_1\ldots i_n} \,\vdots \cdots \vdots\, C_{i_1\ldots(i_n+1)} - C_{i_1\ldots i_n} \right] \frac{\boldsymbol{\mu}(\mathbf{x}(t+\Delta t)) - \boldsymbol{\mu}(\mathbf{x}(t))}{\Delta t} \tag{39}$$

Using (37), we can rewrite (39) as

$$DV(\mathbf{x}) = \sup \left[\frac{C_{(i_1+1)\ldots i_n} - C_{i_1\ldots i_n}}{e_1^{i_1+1} - e_1^{i_1}} \,\vdots \cdots \vdots\, \frac{C_{i_1\ldots(i_n+1)} - C_{i_1\ldots i_n}}{e_n^{i_n+1} - e_n^{i_n}} \right] \dot{\mathbf{x}} \tag{40}$$

Because the entries of $\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})$ are affine functions, conditions (22) and (23) are sufficient for the function in (40) to be negative semi-definite and negative definite, respectively.

Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that

$$B_r = \{\mathbf{x}(k) \in \mathbf{R}^n \mid \|\mathbf{x}(k)\| \le r\} \subset D$$

Let $\alpha = \min_{\|\mathbf{x}\|=r} V(\mathbf{x})$. Then $\alpha > 0$ since it is the minimum of a positive continuous function over a compact set. Take $\beta \in (0, \alpha)$ and let $\Omega_\beta = \{\mathbf{x} \in B_r \mid V(\mathbf{x}) \le \beta\}$

Then Ω_β is entirely inside B_r (see Figure 2). Let $DV(\mathbf{x}) \le 0$, then

$$V(\mathbf{x}(t+\Delta t)) \le V(\mathbf{x}(t)) \le \beta$$

Since $V(\mathbf{x})$ is continuous and $V(\mathbf{0}) = 0$, there exist $\delta > 0$ such that

$$\|\mathbf{x}\| \le \delta \Rightarrow V(\mathbf{x}) \le \beta$$

Hence, we have $B_\delta \subset \Omega_\beta \subset B_r$ and

$$\mathbf{x}(t) \in B_\delta \Rightarrow \mathbf{x}(t) \in \Omega_\beta \Rightarrow \mathbf{x}(t+\Delta t) \in \Omega_\beta \Rightarrow \mathbf{x}(t+\Delta t) \in B_r$$

Therefore, $\|\mathbf{x}(t)\| \le \delta \Rightarrow \mathbf{x}(t+\Delta t) < r \le \varepsilon$ and $\mathbf{x} = \mathbf{0}$ is stable in the sense of Lyapunov.

Similarly, we can show that $\mathbf{x} = \mathbf{0}$ is stable in the sense of Lyapunov for the case when $DV(\mathbf{x}) < 0$. To establish asymptotic stability, we prove convergence to the origin. $V(\mathbf{x})$ decreases continuously along the system trajectories and is lower bounded by zero $V(\mathbf{x}) \to L \ge 0$ as $t \to \infty$. We show that L is zero by contradiction. Let $L >$ 0 and consider the set $\Omega_L = \{\mathbf{x} \mid V(\mathbf{x}) \le c\}$

Select a ball $B_d \subset \Omega_L$, then the trajectories of the system remain outside B_d. Let

$-\gamma(k) = \sup_{d \le \|\mathbf{x}(k)\| \le r} DV(\mathbf{x}) < 0$ and consider the function

$$V(\mathbf{x}(t)) = V(\mathbf{x}(0)) + \int_0^t \dot{V}(\mathbf{x}(\tau))d\tau \le V(\mathbf{x}(0)) - \gamma t$$

which tends to $-\infty$ as $t \to \infty$. This contradicts the lower boundedness of $V(\mathbf{x})$.

•

Proof of Corollary 1:

For continuous Type II systems, conditions (22) and (23) reduce, respectively, to

$$\left[\frac{C_{(j_1+1)\ldots j_n} - C_{j_1\ldots j_n}}{e_1^{j_1+1} - e_1^{j_1}} \middle| \cdots \middle| \frac{C_{j_1\ldots (j_n+1)} - C_{j_1\ldots j_n}}{e_n^{j_n+1} - e_n^{j_n}}\right]\mathbf{h}_{i_1\ldots i_n} \le 0,$$
$$\left[\frac{C_{(j_1+1)\ldots j_n} - C_{j_1\ldots j_n}}{e_1^{j_1+1} - e_1^{j_1}} \middle| \cdots \middle| \frac{C_{j_1\ldots (j_n+1)} - C_{j_1\ldots j_n}}{e_n^{j_n+1} - e_n^{j_n}}\right]\mathbf{h}_{i_1\ldots i_n} < 0$$

(41)

These conditions correspond to a fuzzy negative semi-definite or fuzzy negative definite function $DV(\mathbf{x})$, respectively, (see Definitions 5 and 6).

The proof of sufficiency is identical to the proof of sufficiency of Theorem 1.

•

Proof of Theorem 2:

Assume that $V(\mathbf{x})$ and the derivative $DV(\mathbf{x})$ are defined respectively by

$$R_{i_1\ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1\ldots i_n} THEN\ V_{i_1\ldots i_n} = \mathbf{b}^T \left|\mathbf{e}_{i_1\ldots i_n}\right|$$

(42)

$$R_{i_1\ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1\ldots i_n} THEN\ DV_{i_1\ldots i_n} = \mathbf{b}^T \mathbf{S}(\mathbf{x})\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})$$

(43)

where $\mathbf{b} = [b_1 \ldots b_n]^T$, $\mathbf{S}(\mathbf{x}) = diag\{\mathrm{sgn}(x_1), \cdots, \mathrm{sgn}(x_n)\}$.

$$DV(\mathbf{x}) = \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \mathbf{b}^T \mathbf{S}(\mathbf{x})\mathbf{f}_{l_1\ldots l_n}(\mathbf{x}) \prod_{j=1}^{n} \mu_{A_j^{l_j}}(x_j)$$

(44)

Using (25), we can rewrite (44) as

$$DV(\mathbf{x}) \le -\sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1} \alpha_{l_1\ldots l}\mathbf{b}^T\left|\mathbf{e}_{l_1\ldots l_n}\right|$$
$$\le -s_\alpha \sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1} \mathbf{b}^T\left|\mathbf{e}_{l_1\ldots l_n}\right| = -s_\alpha V(\mathbf{x})$$

(45)

•

Proof of Corollary 2:

Assume that $V(\mathbf{x})$ and the derivative $DV(\mathbf{x})$ are defined by (42) and (43), respectively. In the case of Type II systems, (44) reduces to

$$DV(\mathbf{x}) = \sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}\mathbf{b}^T\mathbf{S}(\mathbf{x})\mathbf{h}_{l_1\ldots l_n}\prod_{j=1}^{n}\mu_{A_j^{l_j}}(x_j)$$

(46)

Using (26), we rewrite (46) in the form (45).

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