Stability Region Analysis using polynomial and composite polynomial Lyapunov functions and Sum of Squares Programming

Weehong Tan and Andrew Packard

Abstract

We propose using (bilinear) sum-of-squares programming for obtaining inner bounds of regions-of-attraction for dynamical systems with polynomial vector fields. We search for polynomial as well as composite Lyapunov functions, comprised of pointwise maximums of polynomial functions. Results for several examples from the literature are presented using the proposed methods and the PENBMI solver.

I. Introduction

Finding the stability region or region-of-attraction (ROA) of a nonlinear system is a topic of significant importance and has been studied extensively, for example in [1-12]. It also has practical applications, such as determining the operating envelope of aircraft and power systems.

Most computational methods aim to compute an inner bound on the region-of-attraction, namely a set that contains the equilibium point, and is contained in the region-of-attraction. The methods above can roughly be split into Lyapunov and non-Lyapunov methods. Lyapunov methods (the focus of this paper) are based on local stability theorems and search for functions satisfying conditions which quantitatively prove local stability. Nonlinear programming is used in [1] to optimize (by choice of positive definite matrix) the volume of an ellipsoid contained in the region-of-attraction. Rational Lyapunov functions that approach ∞ on the boundary of the region-of-attraction are constructed iteratively in [2], motivated from Zubov's work. Computational considerations limit the degree of the rational function, and inner estimates to the ROA are obtained.

Department of Mechanical Engineering, University of California, Berkeley. Email: {weehong, pack}@jagger.me.berkeley.edu

Easy to compute estimates are considered in [3], which restricts the Lyapunov function search to low-dimensional manifold of quadratic Lyapunov functions, obtaining analytical simplifications. Following [1], but employing semidefinite programming techniques, [4] aims to maximize the volume of an ellipsoid whose containment in the region-of-attraction can be ascertained with sum-of-squares (SOS) decompositions. Attention is restricted to odd, polynomial vector fields, and SOS optimization is combined with general nonlinear programming. A sequence of functions, called *nested* Lyapunov functions, are introduced in [5] to derive stability region and rate-of-convergence estimates. Both [6] and [7] solve a sequence of linear semidefinite programs, iteratively searching over Lyapunov function candidates and sum-of-squares multipliers. The "coordinatewise" ascent method is generally effective, though no convergence result holds. By contrast, the formulation here is more direct, but yields a single bilinear (nonconvex) SOS program. Closely related to Lyapunov methods are viability methods, which effectively integrate an invariant set backwards in time, obtaining increasing estimates for the region-of-attraction. Both [8] and [9] use discretization (in time) to flow invariant sets backwards along the flow of the vector field, obtaining larger and larger estimates for the region-of-attraction. In [8], the invariant sets are restricted to be sublevel sets of polynomials, and the discretized backwards flow is approximated with semidefinite programming. The approach of [9] also requires discretization in space and suffers from exponential growth in state dimension. Generally, the method is exact, but computation may require exponential growth in state dimension. Depending on the user's point of view, problems of modest (between 4 and 8) state dimension are intractable. Non-Lyapunov methods, [10] and [11] focus on topological properties of regions of attraction. A survey of results, as well as an extensive set of examples and new ideas, is presented in [12].

In this paper, we present a method of using sum-of-squares (SOS) programming to search for polynomial Lyapunov functions that enlarge an inner estimate of the region-of-attraction of nonlinear systems with polynomial vector fields. SOS programming, coupled with polynomial Lyapunov functions has roots that can be traced back at least to Bose and Li [13] and Brockett, [14] and the power transform of Barkin et.al [15], which was used in [16] to find non-quadratic Lyapunov functions for uncertain linear systems. Recent theoretical work, [17], [7] and [18], continues to further the role of this approach. An impediment to using high degree Lyapunov functions is the extremely rapid increase in the number of optimization decision variables as the state dimension and the degree of the Lyapunov function (and the vector field) increase. Here,

we propose using pointwise maximums of polynomial functions to obtain rich functional forms while keeping the number of optimization decision variables relatively low. Pointwise maximum and other composite Lyapunov functions have been used in many instances, [19], [20], [21], including stability and performance analysis of constrained systems and robustness analysis of uncertain systems, where affine and polynomial parameter-dependent Lyapunov functions are also used, [22], [23]. The notation is generally standard, with $\mathcal{R}_n$ denoting the set of polynomials with real coefficients in n variables and $\Sigma_n \subset \mathcal{R}_n$ denoting the subset of SOS polynomials.

II. Estimating a Region of Attraction

Consider an autonomous dynamical system of the form

$$\dot{x}(t) = f(x(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and f is an n-vector of elements of $\mathcal{R}_n$ with $f(0) = 0$. The following lemma on invariant subsets of the region-of-attraction is a modification of ideas from [24, pg. 167] and [25, pg. 122]:

Lemma 1: If there exist continuously differentiable functions $\{V_i\}_{i=1}^q : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(x) := \max_{1 \le i \le q} V_i(x) \text{ is positive definite}, \tag{2}$$

$$\Omega := \{x \in \mathbb{R}^n \,|\, V(x) \le 1\} \text{ is bounded}, \tag{3}$$

$$L_i := \{x \in \mathbb{R}^n \,|\, \max_{1 \le j \le q} V_j(x) \le V_i(x) \le 1\}, \qquad i = 1, \ldots, q \tag{4}$$

$$L_i \setminus \{0\} \subset \{x \in \mathbb{R}^n \,|\, \tfrac{\partial V_i}{\partial x} f(x) < 0\}, \qquad i = 1, \ldots, q, \tag{5}$$

then for all $x(0) \in \Omega$, the solution of (1) exists, satisfies $x(t) \in \Omega$, and $\lim_{t \to \infty} x(t) = 0$. As such, Ω is invariant, and a subset of the region-of-attraction for (1).

Proof: The proof is written for $q = 2$. The extension to $q = 1$ or $q > 2$ is straightforward. Since $L_1 \cup L_2 = \Omega$, condition (5) insures that if $x(0) \in \Omega$, $V(x(t)) \le V(x(0)) \le 1$ while the solution exists. Solutions starting in Ω remain in Ω while the solution exists. Since Ω is compact, the system (1) has an unique solution defined for all $t \ge 0$ whenever $x(0) \in \Omega$.

Take $\epsilon > 0$. Define $S_\epsilon := \{x \in \mathbb{R}^n \,|\, \frac{\epsilon}{2} \le V(x) \le 1\}$, so $S_\epsilon \subseteq (L_1 \cup L_2) \setminus \{0\}$. Note that for each i, $(S_\epsilon \cap L_i) \subseteq L_i \setminus \{0\} \subset \{x \in \mathbb{R}^n \,|\, \frac{\partial V_i}{\partial x} f(x) < 0\}$, so on the compact set $S_\epsilon \cap L_i$, $\exists\, r_{i,\epsilon}$, such that $\frac{\partial V_i}{\partial x} f(x) \le -r_{i,\epsilon} < 0$. Consequently, if $x(t) \in S_\epsilon \cap L_1$ on $[t_A, t_B]$,

then $V(x(t_B)) \leq -r_{1,\epsilon}(t_B - t_A) + V(x(t_A))$. Similarly, if $x(t) \in S_\epsilon \cap L_2$ on $[t_A, t_B]$, then $V(x(t_B)) \leq -r_{2,\epsilon}(t_B - t_A) + V(x(t_A))$. Therefore, if $x(t) \in S_\epsilon \cap (L_1 \cup L_2)$ on $[t_A, t_B]$, then $V(x(t_B)) \leq -r_\epsilon(t_B - t_A) + V(x(t_A))$, where $r_\epsilon = \min(r_{1,\epsilon}, r_{2,\epsilon})$. Since $r_\epsilon > 0$, this implies that $\exists\, t^*$ such that $V(x(t)) < \epsilon$ for all $t > t^*$, i.e. $x(t) \in T_\epsilon := \{x \in \mathbb{R}^n \,|\, V(x) < \epsilon\}$ for all $t > t^*$. This shows that if $x(0) \in \Omega$, $V(x(t)) \to 0$ as $t \to \infty$.

Finally, let $\epsilon > 0$. Define $\Omega_\epsilon := \{x \in \mathbb{R}^n \,|\, \|x\| \geq \epsilon, V(x) \leq 1\}$. Ω_ϵ is compact, with $0 \notin \Omega_\epsilon$. Since V is continuous and positive definite, $\exists\, \gamma$ such that $V(x) \geq \gamma > 0$ on Ω_ϵ. We have already established that $V(x(t)) \to 0$ as $t \to \infty$, so $\exists\, \hat{t}$ such that for all $t > \hat{t}$, $V(x(t)) < \gamma$ and hence $x(t) \notin \Omega_\epsilon$, which means $\|x(t)\| < \epsilon$. So $x(t) \to 0$ as $t \to \infty$. ■

Remarks: Standard modifications to the hypothesis of Lemma 1 can yield global stability conditions as well. However, neither formulation can yield exact results for systems whose region-of-attraction is unbounded, but not all of $\mathbf{R}^n$ (since in Lemma 1, Ω must be is bounded). See section III-C for further details. The constraints in equations (2)-(5) are not convex constraints on V, as illustrated by a 1-dimensional example, [26]. Take $f(x) = -x$, $q = 1$ and $V_1^a(x) = 16x^2 - 19.95x^3 + 6.4x^4$ and $V_1^b(x) = 0.1x^2$. Then V_1^a and V_1^b satisfy (2)-(5), but $0.58V_1^a + 0.42V_1^b$ does not.

In order to enlarge Ω (by choice of V), we define a variable sized region $P_\beta := \{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\}$, and maximize β while imposing the constraint $P_\beta \subseteq \Omega$. Here, $p(x)$ is a fixed, positive definite polynomial, chosen to reflect the relative importance of the states. Applying Lemma 1, the problem is posed as an optimization:

$$\max_{\beta \in \mathbb{R}, V_i \in \mathcal{R}_n} \beta \qquad \text{s.t.} \qquad V_i(0) = 0$$

$$V(x) := \max_{1 \leq i \leq q} V_i(x) \text{ is positive definite,} \tag{6}$$

$$\Omega := \{x \in \mathbb{R}^n \,|\, V(x) \leq 1\} \text{ is bounded,} \tag{7}$$

$$P_\beta \subseteq \Omega \tag{8}$$

$$\{x \in \mathbb{R}^n \,|\, \max_{1 \leq j \leq q} V_j(x) \leq V_i(x) \leq 1\} \setminus \{0\} \subset \{x \in \mathbb{R}^n \,|\, \tfrac{\partial V_i}{\partial x} f(x) < 0\} \tag{9}$$

where (9) holds for $i = 1, \ldots, q$. Let $l_1(x)$ be a fixed, positive definite polynomial. For each V_i, if we require $V_i - l_1 \in \Sigma_n$ for $i = 1, \ldots, q$, then both (6) and (7) are satisfied. Clearly, (8) holds

if and only if

$$\{x \in \mathbb{R}^n \,|\, p(x) \leq \beta\} \subseteq \bigcap_{i=1}^{q} \{x \in \mathbb{R}^n \,|\, V_i(x) \leq 1\}, \tag{10}$$

Introducing another fixed, positive definite polynomial, $l_2(x)$, we can apply Lemmas 2 and 3 (see appendix) to obtain sufficient conditions which ensure constraints (9) and (10) hold. Written as an optimization, the problem is

$$\max \beta \quad \text{over } \beta \in \mathbb{R}, V_i \in \mathcal{R}_n,\ V_i(0) = 0,\ s_{1i}, s_{2i}, s_{3i}, s_{0ij} \in \Sigma_n, i = 1, \ldots q$$

such that

$$V_i - l_1 \in \Sigma_n, \tag{11}$$

$$-\Big((\beta - p)s_{1i} + (V_i - 1)\Big) \in \Sigma_n, \tag{12}$$

$$-\left[(1 - V_i)s_{2i} + \tfrac{\partial V_i}{\partial x} f s_{3i} + l_2\right] - \sum_{\substack{j=1 \\ j \neq i}}^{q} s_{0ij}(V_i - V_j) \in \Sigma_n. \tag{13}$$

All constraints are sum-of-square constraints, however (even for $q = 1$) products of decision variables are present. Therefore, the optimization cannot be translated into a linear semidefinite program, but is converted to a bilinear semidefinite program. Two of the conditions require positivity (beyond nonnegativity), and the fixed positive-definite polynomials, l_1 and l_2 are introduced as offsets to enforce this. Next we present results from several small problems. We have chosen to rely on the PENBMI solver [27], a local bilinear matrix inequality solver from PENOPT to attack our problems. This uses a penalty method. Alternate approaches to BMIs, such as linearization and homotopy, [28] and interior point methods, [29, Chap 7], may yield improved results and/or superior computational efficiency. Resolving these questions is left for further research.

III. Examples

All of the systems considered are locally exponentially stable. The notation n_V denotes the degree of V, specifically each V_i includes all monomials of degree 2 up through n_V. In all examples, p is quadratic, and the degree of s_{1i} is chosen so that the degree of the polynomial in equation (12) is equal to n_V. The integer n_A denotes degree of the polynomial in (13). Once n_V is chosen, and the vector field f is fixed, n_A limits the degrees of the multipliers s_{2i}, s_{3i} and s_{0ij} through simple degree counting. In each case, the positive definite polynomials l_1 and

l_2 are of the form $l_k(x) = \sum_{i=1}^{n} e_{k,i} x_i^2$. For the purposes of computation, the $e_{k,i}$ are treated as additional decision variables, and constrained to satisfy $e_{k,i} \geq 10^{-7}$.

A. Example 1 - Van der Pol equations

The system is $\dot{x}_1 = -x_2, \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$. It has an unstable limit cycle and a stable equilibrium point at the origin. Finding its region-of-attraction has been studied extensively, for example, in [1], [12], [11]. The region-of-attraction for this system is the region enclosed by its limit cycle, which is easily visualized from the numerical solution of the ODE. However, our goal is to use the bilinear SOS formulation. For this example, p is chosen to be $x^T R x$, for two different $R_i \in \mathbb{R}^{2\times 2}$,

$$R_1 := \begin{bmatrix} 0.38 & -0.14 \\ -0.14 & 0.28 \end{bmatrix}, \quad \text{and} \quad R_2 := \begin{bmatrix} 0.28 & 0.14 \\ 0.14 & 0.38 \end{bmatrix},$$

The results using shape factor defined by R_1 using the pointwise maximum of two fixed degree polynomials are listed in Table I. Fig. 1 shows the limit cycle and the level sets of the certifying Lyapunov functions[1]. The level set of pointwise maximum of two 6th degree polynomial functions includes nearly the entire actual region-of-attraction. The dashed line is the level set of p (for $n_V = 6$), which clearly shows that our p has been preselected to "align" closely with the actual region-of-attraction. Of course, this would be impossible to do in general, and we discuss the implications of this later in this section. Our results compare favorably with [11] as well as the degree 6 solution from [7], and the final (40^{th}) iterate from degree 6 solutions of [8], all of which are shown in Figure 2. Clearly, the solution of [8] is a very high quality estimate of the true ROA. Parametrizing the boundaries using polar coordinates reveals that as a function of angle, the radius of [8] exceeds our $n_V = 6$ radius on 52.6% of $[0\ 2\pi]$; is 0.22% larger, on average, than our $n_V = 6$ radius; exceeds our $n_V = 6$ radius by as much as 1.4% in some directions; is smaller than our $n_V = 6$ radius by as much as 0.8% in other directions. We conclude that the result in [8] is very similar, though slightly superior to our result.

It is interesting to observe how the V_i functions interact in, for example, the 6th degree case. Figure 3 shows the level sets $\{x \,|\, V_i(x) \leq 1\}$. For V_1, there are 3 connected components, one

[1]The certifying Lyapunov functions and SOS multipliers for all examples in this paper are available at `http://jagger.me.berkeley.edu/~pack/certificates`

TABLE I

POINTWISE MAX FOR VAN DER POL

q	degree of V	s_{1i}	s_{2i}	s_{3i}	s_{0ij}	β	total no. of decision variables
2	2	0	2	0	2	0.75	38
2	4	2	2	0	2	0.93	120
2	6	4	2	0	2	1.03	338
3	2	0	2	0	2	0.75	73
3	2	0	4	2	4	0.82	265

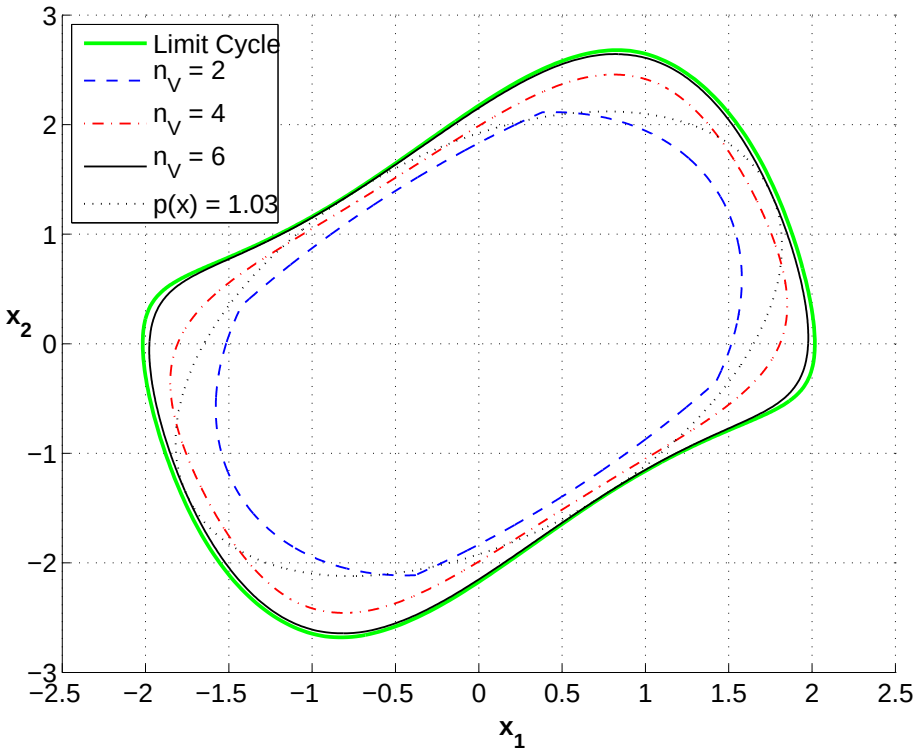

Fig. 1. Provable ROA using pointwise maximum of two polynomial functions, with shape factor $x^T R_1 x$

"large" component centered at the origin (whose boundary is essentially the limit cycle), and 2 "islands" in the 2nd and 4th quadrants. For V_2, the level set is one connected component centered at the origin, visually the same as the large component of V_1. Label the two islands as I_1 and I_2, and the intersection of the two (nearly identical) centered components as Ω.

Inside I_1 and I_2, $\dot{V}_1 \not< 0$, but $V_2 > 1 > V_1$, so I_1 and I_2 are excluded in the set Ω. Moreover, on Ω, $\dot{V}_1 < 0$ where $V_1 \geq V_2$, and $\dot{V}_2 < 0$ where $V_2 \geq V_1$, proving that Ω is a region-of-attraction. Since $\{x \,|\, V_2(x) \leq 1\} \approx \Omega$, it is tempting to assume that V_2 alone can prove the stability claim. However, many points have $\dot{V}_2 \geq 0$ (the shaded region in Ω).

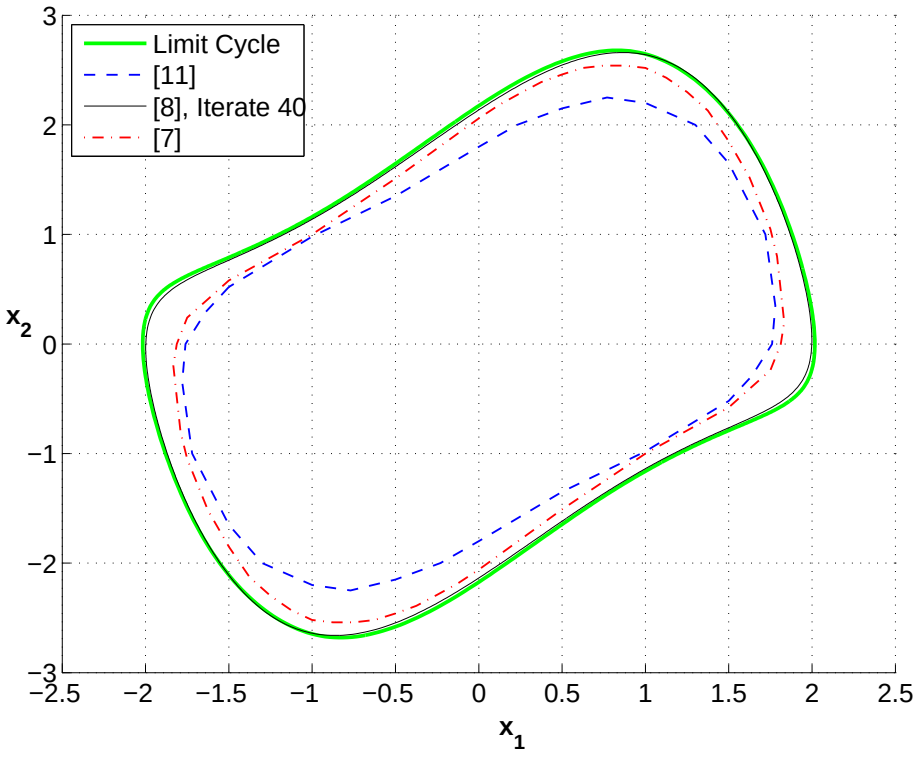

Fig. 2. Provable ROA, from [11], [7] and [8].

In this example, using pointwise maximum of three polynomials does not offer additional benefits (row 1 and 4 of Table I). Better results are obtained (row 5) by increasing the degree of the $\{s_i\}$, but this increases the number of decision variables, so the computational benefit is effectively erased.

Finally, optimizing with the shape factor defined by R_2 yields almost identical results (in terms of Ω). Fig. 4 illustrates the analogous level sets of V, and also shows a level set for this p. Clearly, the level sets for this shape factor are not aligned with the actual region-of-attraction, nevertheless, the optimization performs quite well.

B. 6 examples from [4]

Reference [4] aims to *maximize the volume* of an inner *ellipsoidal* estimate of the region of attraction, presenting results from 6 examples. The volume reported in [4] is normalized: in $\mathbf{R}^2$ it is 2-dimensional area divided by π, while in $\mathbf{R}^3$ it is 3-dimensional volume divided by $\frac{4\pi}{3}$. As an exercise, we solve the same problems here. The results are summarized in Table II. Maximizing volume is not directly compatible with our scalar objective involving the function p (whose level sets may or may not be ellipsoidal). We began with a simple approach: using

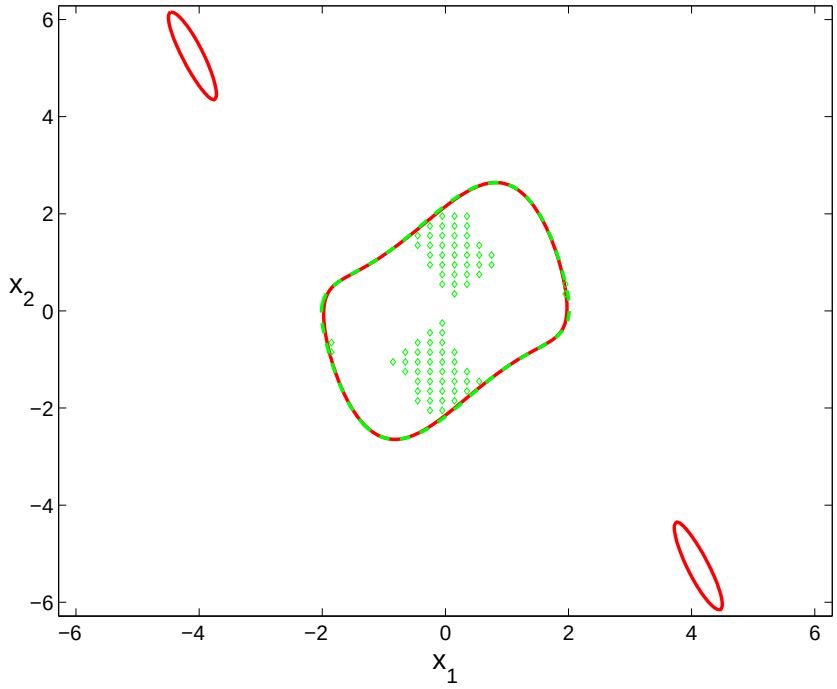

Fig. 3. VDP: Level sets of two 6th degree polynomials at $V_1, V_2 = 1$

a spherical shape factor, $p(x) := x^T x$, solve the optimization problem and then compute the volume of the level set $\{x : V(x) \leq 1\}$ (easily computed for a quadratic V, and estimated with Monte Carlo integration for high degree and pointwise-max V's). Problems S1, S2, S3 and S4 are successfully addressed using this approach. Note the improvement for S2 when the degree of the multipliers is increased (via n_A) even though n_V is held constant. Problem S5 required an alteration, referred to as *bootstrap*, to obtain large volumes. In this calculation, the initial optimization was as above, with a spherical p, using quadratic Lyapunov function candidates. Subsequent optimizations, with richer Lyapunov function candidates used, for p, the obtained quadratic Lyapunov function (as opposed to $x^T x$). Problem S6 is more challenging and the methods we present here do not obtain volumes as large as those reported in [4]. The S6 table entry involving quartic functions is empty, as PENBMI exhibited unreliable behavior on this problem, exposing some genuine deficiences in our overall approach.

C. Unbounded Region-of-Attraction

Consider $\dot{x}_1 = x_2, \dot{x}_2 = -(1 - x_1^2)x_1 - x_2$ from [30]. The region of attraction to the stable equilibrium at $x = 0$ is unbounded, but not all of $\mathbb{R}^2$. Exact methods, such as those in [9], may obtain the correct answer in this problem. By contrast, the formulation in equations (11)-(13)

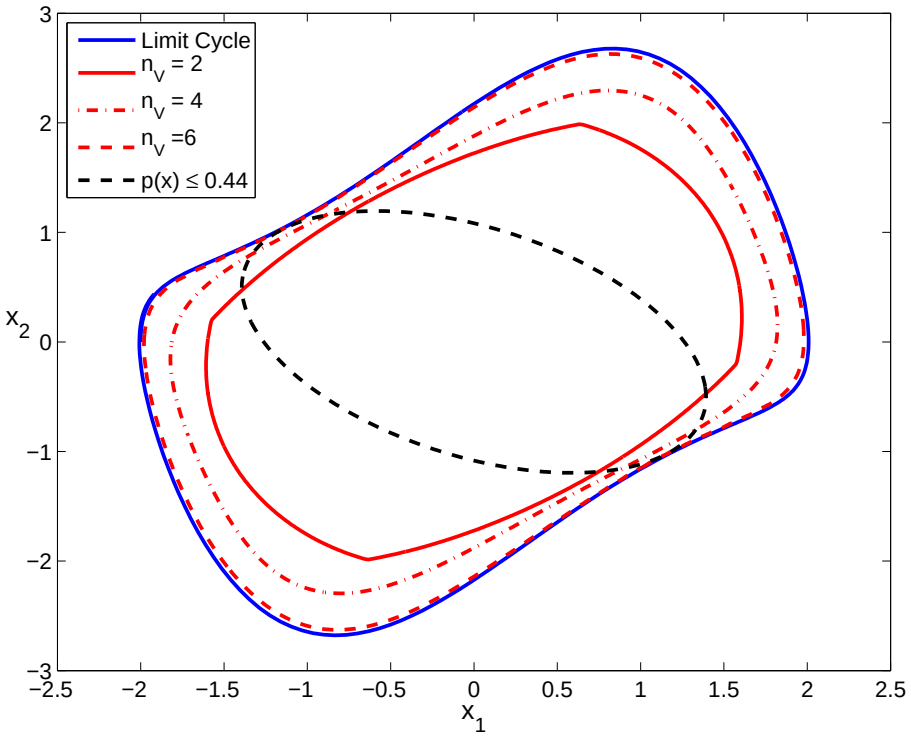

Fig. 4. Provable ROA using pointwise max of two polynomial functions, with shape factor $x^T R_2 x$

TABLE II

CERTIFIED NORMALIZED VOLUME ON EXAMPLES S1-S6 FROM [4]. THE VECTOR FIELDS FOR EXAMPLES S3 AND S4 HAVE DEGREE EQUAL TO 5, WHILE ALL OTHERS HAVE DEGREE EQUAL TO 3.

from [4]	(n_V, n_A)	q	Vol	from [4]	(n_V, n_A)	q	Vol	from [4]	(n_V, n_A)	q	Vol
S1(10.2)	(2,4)	1	7.5	S2(27.1)	(2,4)	1	24.9	S3(9.51)	(2,6)	1	1.68
	(2,4)	2	13.7		(2,4)	2	24.9		(2,6)	2	global
					(2,6)	2	43.0				
S4(0.85)	(2,6)	1	0.83	S5(23.5)	(2,4)	1	21.3	S6(10.9)	(2,4)	1	8.5
	(2,6)	2	0.92		(2,4)	2	21.3		(2,4)	2	9.4
	(4,8)	1	1.12		(4,6)	1	32.9		(4,6)	1	-
	(4,8)	2	1.16						(4,6)	2	-

cannot, since Ω is necessarily compact. Using a simple $p(x) := x_1^2 + x_2^2$ shape factor, we obtain nearly identical results for quadratic and pointwise-max quadratic Lyapunov functions, yielding β such that P_β nearly touches the stability boundary, and the bounded level set $\{x : V(x) \leq 1\}$ is ellipsoidal, roughly aligned with the true region-of-attraction. Using the bootstrap, with $n_V = 6$ yields significant improvement. The two level sets are shown in the left panel of figure 5, along

with some trajectories of the system.

D. An example from reference [2]

Another 2-state example with polynomial vector field comes from example 4 in [2]. The dynamics are $\dot{x}_1 = -0.42x_1 - 1.05x_2 - 2.3x_1^2 - 0.5x_1x_2 - x_1^3; \dot{x}_2 = 1.98x_1 + x_1x_2$. The inner estimate from [2] along with our estimate using quadratic, quartic, pointwise-max quartic, and degree 6 functions are shown in the right panel of Figure 5. Pointwise-max ($q = 2$) degree 6 solutions yielded no appreciable improvement over the $q = 1$ case, and are not shown.

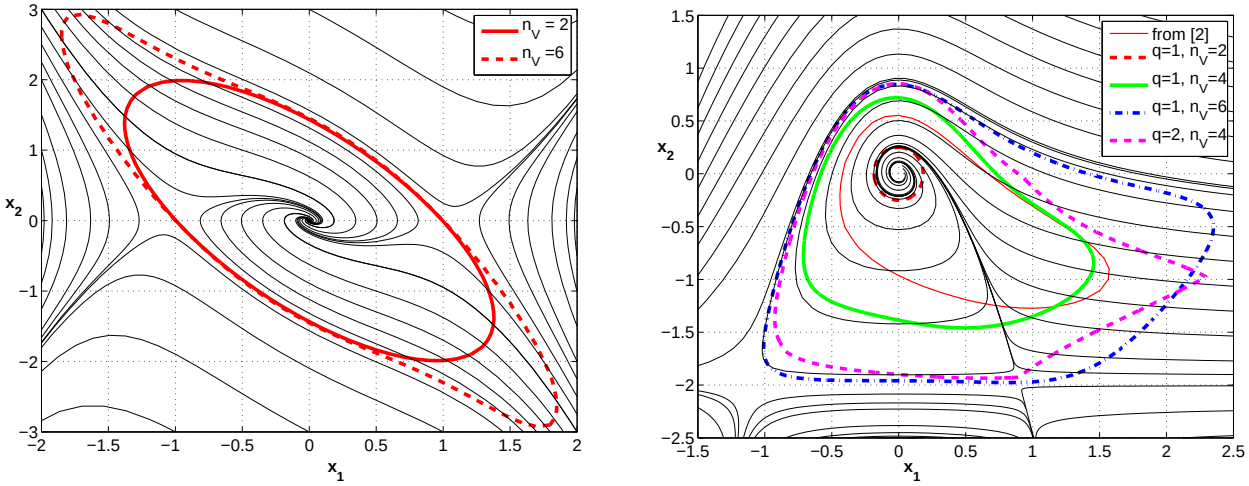

Fig. 5. Trajectories and level sets $V \leq 1$. LEFT panel: example from Section III-C; RIGHT panel: example from Section III-D

IV. Benchmark Study

There are several drawbacks to our approach, most notably searching over the non-convex decision variable space. Given this deficiency, it is useful to investigate how equations (11)-(13), coupled with the PENBMI solver perform on an "easy" nonlinear problem, with respect to "arbitrary" data and increasing problem size. Let $\dot{x} = -Ix + (x^TBx)x$ where $x(t) \in \mathbb{R}^n$, and $B \in \mathbb{R}^{n\times n}$, $B \succ 0$. For this system, inspired by Example 5 of [1], the set $\{x \in \mathbb{R}^n \,|\, x^TBx < 1\}$ is the exact region-of-attraction for the $x = 0$ equilibrium point (use $V(x) := x^TBx$ to prove this). Let $P_\beta := \{x \in \mathbb{R}^n \,|\, x^TRx \leq \beta\}$, $R \in \mathbb{R}^{n\times n}$, $R \succ 0$. The supremum value for β such that $P_\beta \subseteq \{x \in \mathbb{R}^n \,|\, x^TBx < 1\}$ is $\beta = [\lambda_{\max}(R^{-\frac{1}{2}}BR^{-\frac{1}{2}})]^{-1}$. Equations (11)-(13) *can* yield this answer, specifically, take $q = 1$ and for any $\gamma > 1$, choose τ such that $1 < \tau < \gamma$. Then for large enough α (depending on fixed choice of quadratic l_2) the choices $V(x) := \gamma x^TBx$,

$s_2 := 2\alpha\tau x^T Bx$ and $s_3 := \alpha$ satisfy (13), prove that $\{x \,|\, x^T Bx < 1\}$ is in the region-of-attraction. Hence, this class of problems provides a test for any specified BMI solver to actually discover the known-to-exist solution. For each n, 100 trials are performed. Each trial consists of a random choice of positive definite B and R, each with eigenvalues $\exp(2r_i)$ where each r_i is picked from a normal distribution with zero mean and unit variance, and random, orthonormal eigenvectors. For each trial, we run the PENBMI optimizer 3 times (initial point randomly chosen each run). Table III shows the results of the test.

TABLE III

COMPUTATION STATISTICS FOR THE BENCHMARK EXAMPLE

n	variables	successes	worst (in 300)	over 100	time (sec)
2	13	298	0.99995	1.00000	0.70
3	25	296	0.90955	0.99984	1.12
4	48	297	0.07687	0.99999	2.14
6	157	297	0.99997	0.99998	11.2
8	420	300	0.99989	0.99992	99.7

A run is classified *successful* if the solver returns the message "No problems detected", and classified *failure* otherwise. Except for the case of $n = 6$, there are no trials that fail for all 3 runs (for $n = 6$, one trial did fail in all 3 runs, and note that this single instance, 3-trial failure is not taken into account in the table entries described below). Among the successful runs, the quality of the answer is assessed by the nearness of $\beta \times \lambda_{\max}$ to 1. The worst case (smallest) value among the (296-300) successful runs is given. The next column shows the worst case $\beta \times \lambda_{\max}$ over 100 trials, exploiting the 3 repeated attempts and the randomized initial starting point chosen by PENBMI. The entries are ≈ 1, which indicates that repeated runs of the same problem eventually lead to the optimal solution for this example. For this limited benchmark example, although our problem formulation is bilinear in the decision polynomials and the bilinear solver, PENBMI, is a local solver, the results obtained are encouraging.

V. CONCLUSIONS

In this paper, we presented techniques using sum-of-squares programming for finding provable regions-of-attraction for nonlinear systems with polynomial vector fields. Several small examples are presented. For systems with cubic vector fields, analyzing local stability using Lyapunov

functions which are the pointwise-max of quadratic and quartic functions appears to be a useful, and modestly tractable extension to simply using polynomial Lyapunov functions.

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VII. Appendix

A *monomial* m_α in n variables is a function defined as $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}_+^n$. The degree of a monomial is defined, $\deg m_\alpha := \sum_{i=1}^n \alpha_i$. A *polynomial* f in n variables is a finite linear combination of monomials, with $c_\alpha \in \mathbb{R}$:

$$f := \sum_\alpha c_\alpha m_\alpha = \sum_\alpha c_\alpha x^\alpha.$$

Define $\mathcal{R}_n$ to be the set of all polynomials in n variables. The degree of f is defined as $\deg f := \max_\alpha \deg m_\alpha$ (provided the associated c_α is non-zero). Additionally define Σ_n to be the set of sum-of-squares (SOS) polynomials in n variables.

$$\Sigma_n := \left\{ p \in \mathcal{R}_n \,\middle|\, p = \sum_{i=1}^t f_i^2 \quad , \, f_i \in \mathcal{R}_n, \, i = 1, \ldots, t \right\}.$$

Obviously if $p \in \Sigma_n$, then $p(x) \geq 0 \; \forall x \in \mathbb{R}^n$. A polynomial, $p \in \Sigma_n$ iff $\exists \; 0 \preceq Q \in \mathbb{R}^{r \times r}$ such that $p(x) = z^T(x) Q z(x)$, with $z(x)$ a vector of suitable monomials. The set of Q that satisfies $p(x) = z^T(x) Q z(x)$ is an affine subspace, so that semidefinite programming plays the key role in deciding if a given polynomial is SOS. The lemmas below are elementary extensions of the

$\mathcal{S}$-procedure, [32], and very limited special cases of the Positivstellensatz, [33, Theorem 4.2.2]. In both cases, the SOS polynomials $\{s_k\}_{i=1}^m$ are often called the "SOS multipliers."

Lemma 2: Given $p_1, p_2 \in \mathcal{R}_n$, and positive definite $h \in \mathcal{R}_n$, with $h(0) = 0$. If $s_1, s_2 \in \Sigma_n$ satisfy $p_1 s_1 + p_2 s_2 - h \in \Sigma$ then $\{x : p_1(x) \leq 0\} \setminus \{0\} \subset \{x : p_2(x) > 0\}$.

Lemma 3: Given $\{p_i\}_{i=0}^m \in \mathcal{R}_n$. If there exist $\{s_k\}_{i=1}^m \in \Sigma_n$ such that $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, then $\bigcap_{i=1}^m \{x \in \mathbb{R}^n \,|\, p_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \,|\, p_0(x) \geq 0\}$.

SOSTOOLS, [34], [35], GloptiPoly, [36], and YALMIP, [31] automate the translation from SOS programs to semidefinite programs, converting to solver-specific, e.g., SeDuMi [37] or SDPT3 [38], syntax. YALMIP also handles bilinear decision polynomials, using PENBMI [27].

Despite these software tools, and even ignoring the nonconvexity of our formulation, there are significant dimensionality problems as well: [39, Table 6.1] illustrates the unpleasant growth in the number of decision variables with n and the polynomial degree.

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