2008

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Available online at www.sciencedirect.com
ScienceDirect
Chaos, Solitons and Fractals 35 (2008) 688–691
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

# Nonlinear oscillator with discontinuity by parameter-expansion method

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Accepted 19 July 2007

Communicated by Prof. Ji-Huan He

**Abstract**

The parameter-expansion method is applied to a nonlinear oscillator with discontinuity. One iteration is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain. Comparison of the obtained solution with the exact one shows that the method is very effective and convenient.


## 1. Introduction

This paper considers the following nonlinear oscillator with discontinuity [1–3]:

$$u'' + u|u| = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{1}$$

There exists no small parameter in the equation. Therefore, the traditional perturbation methods cannot be applied directly [3].

Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. Many new techniques have appeared in the literature, for example, the homotopy perturbation method [4–10], the variational iteration method [11–14], and the energy balance method [15–17]. A complete review is available in Refs. [3,18]. In this paper, we apply the parameter-expansion method [18–21] to the problem we are discussing.

## 2. Solution procedure

The parameter-expansion method [18–21] entails the bookkeeping parameter method [18,19] and the modified Lindstedt–Poincare method [18,21–23]. Recently, the method has been applied to various nonlinear oscillators, see Refs. [2,24–28]. In order to use the parameter expansion method, we re-write Eq. (1) in the following form [18,21,29]:

$$u'' + 0 \cdot u + 1 \cdot u|u| = 0 \tag{2}$$

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0960-0779/$ - see front matter 
doi:10.1016/j.chaos.2007.07.055

According to the parameter-expansion method, we may expend the solution, $u$, the coefficient of $u$, the zero, and the coefficient of $u|u|$, 1, in series of $p$:

$$u = u_0 + pu_1 + p^2u_2 + \cdots \tag{3}$$

$$0 = \omega^2 + pa_1 + p^2a_2 + \cdots \tag{4}$$

$$1 = pb_1 + p^2b_2 + \cdots \tag{5}$$

Substituting Eqs. (3)–(5) into Eq. (2) and equating the terms with the identical powers of $p$, we have

$$p^0 : u_0'' + \omega^2 u_0 = 0 \tag{6}$$

$$p^1 : u_1'' + \omega^2 u_1 + a_1 u_0 + bu_0|u_0| = 0 \tag{7}$$

$$p^2 : (1+\omega^2)u_2'' + a_1 u_1'' + a_2 u_0'' + b_1\left(|u_0''|u_1'' + u_0''|u_1''|\right) + b_2 u_0''|u_0''| = 0 \tag{8}$$

Considering the initial conditions $u_0(0) = A$ and $u_0'(0) = 0$, the solution of Eq. (6) is $u_0 = A\cos\omega t$. Substituting the result into Eq. (7), we have

$$u_1'' + \omega^2 u_1 + a_1 A \cos\omega t + b_1 A^2 \cos\omega t|\cos\omega t| = 0 \tag{9}$$

It is possible to perform the following Fourier series expansion:

$$\cos\omega t|\cos\omega t| = \sum_{n=0}^{\infty} c_{2n+1}\cos[(2n+1)\omega t] = c_1\cos\omega t + c_3\cos\omega t + \cdots \tag{10}$$

where $c_i$ can be determined by Fourier series, for example

$$c_1 = \frac{2}{\pi}\int_0^{\pi}\cos^2\omega t|\cos\omega t|\,\mathrm{d}(\omega t) = \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\cos^3\tau\,\mathrm{d}\tau - \int_{\frac{\pi}{2}}^{\pi}\cos^3\tau\,\mathrm{d}\tau\right) = \frac{8}{3\pi} \tag{11}$$

Substitution of Eq. (10) into Eq. (9) gives

$$u_1'' + \omega^2 u_1 + \left(a_1 + b_1 A\frac{8}{3\pi}\right)A\cos\omega t + \sum_{n=1}^{\infty} c_{2n+1}\cos[(2n+1)\omega t] = 0 \tag{12}$$

No secular term in $u_1$ requires that

$$a_1 + b_1 A\frac{8}{3\pi} = 0 \tag{13}$$

If the first-order approximation is enough, then, setting $p = 1$ in Eqs. (4) and (5), we have

$$1 = b_1 \tag{14}$$

$$0 = \omega^2 + a_1 \tag{15}$$

From Eqs. (13)–(15), we obtain

$$\omega = \sqrt{\frac{8A}{3\pi}} \approx 2.6667\sqrt{\frac{A}{\pi}} \tag{16}$$

The obtained frequency, Eq. (16), is valid for the whole solution domain, $0 < A < \infty$. The accuracy of frequency can be improved if we continue the solution procedure to a higher order, however, the amplitude obtained by this method is an asymptotic series, not a convergent one. For conservative oscillator

$$u'' + f(u)u = 0, \quad f(u) > 0 \tag{17}$$

where $f(u)$ is a nonlinear function of $u$, we always use the zeroth-order approximate solution. Thus we have

$$u(t) = A\cos\left(t\sqrt{\frac{8A}{3\pi}}\right) \tag{18}$$

Fig. 1 illustrates various cases with different values of $A$.

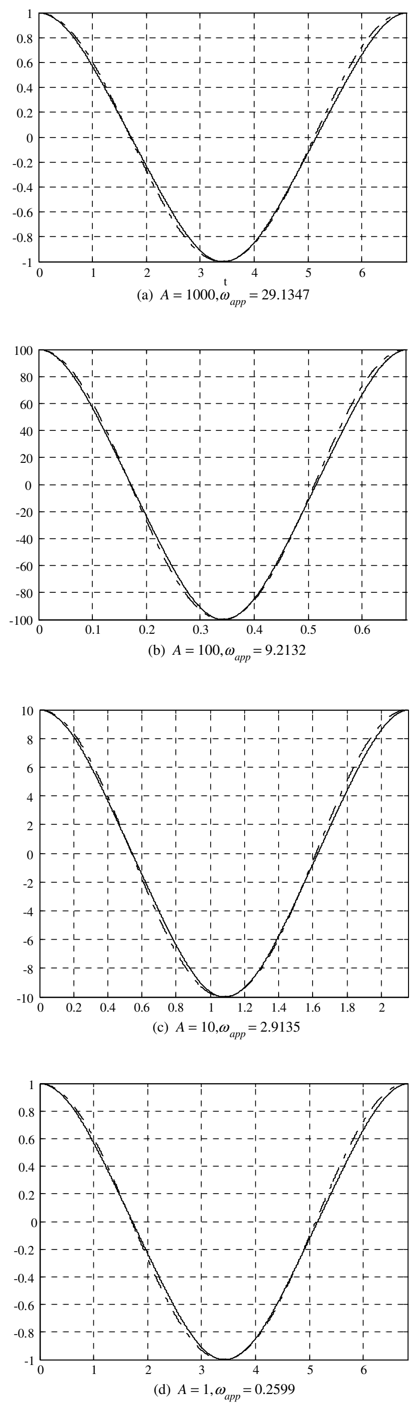


Fig. 1. Comparison of approximate solution, Eq. (18), with exact one. Dashed line: approximate solution; continuous line: exact solution. (a) $A = 1000$, $\omega_{\mathrm{app}} = 29.1347$; (b) $A = 100$, $\omega_{\mathrm{app}} = 9.2132$; (c) $A = 10$, $\omega_{\mathrm{app}} = 2.9135$; and (d) $A = 1$, $\omega_{\mathrm{app}} = 0.2599$.

## 3. Conclusion

The parameter-expansion method is an extremely simple method. One iteration is enough. Furthermore, the obtained frequency is of high accuracy. The method can be applied to many other nonlinear oscillators.

## Acknowledgment

The work was supported by the Program for New Century Excellent Talents in the University.

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