F-rational rings have rational singularities

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Abstract. It is proved that an excellent local ring of prime characteristic in which a single ideal generated by any system of parameters is tightly closed must be pseudorational. A key point in the proof is a characterization of F-rational local rings as those Cohen-Macaulay local rings (R, m) in which the local cohomology module $H^d_m(R)$ (where d is the dimension of R) have no submodules stable under the natural action of the Frobenius map. An analog for finitely generated algebras over a field of characteristic zero is developed, which yields a reasonably checkable tight closure test for rational singularities of an algebraic variety over $\mathbb{C}$, without reference to a desingularization.

With the development of the theory of tight closure by M. Hochster and C. Huneke [HH1], a natural question arose. What information does this powerful new tool provide about the structure of the singularities of an algebraic variety? The main theorem of this paper is the following:

Theorem 3.1. *If an excellent local ring has the property that all ideals generated by a system of paramenters are tightly closed, then the ring is pseudorational.*

Pseudorationality (Definition 1.8) is a desingularization-free analog of the notion of *rational singularities* which makes sense for any scheme. Theorem 3.1 has a characteristic zero version (Theorem 4.3) which can be used to test for rational singularities of a complex algebraic variety. Recently, A. Conca and J. Herzog have used Theorem 3.1 to prove that an interesting class of varieties generalizing Schubert varieties, called the ladder determinantal varieties, have rational singularities [CH].

Theorem 3.1 is not unexpected. Striking similarities had suggested a connection between rings with rational singularities and rings in which all (or certain) ideals are tightly closed. Both are preserved upon passing to direct summands ([Bo], [HH1]). Both are natural settings for the “Briançon-Skoda theorems,” relating powers of ideals to integral closures their powers in a uniform way ([LT], [HH1]). In the graded case, both force strong restrictions on the degrees of nonvanishing elements in local cohomology modules (the so-called a-invariant must be negative; see [FW]). Prior to this paper, several special cases of Theorem 3.1 had been proved before, by Hochster and Huneke, and by R. Fedder and K.-I. Watanabe; see [FW] and its references.

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The converse to Theorem 3.1 is open. Fedder and Watanabe, both jointly and individually, have made contributions to this study (see the references). They jointly coined the term "F-rational" to describe those rings in which ideals generated by a system of parameters are tightly closed; the name indicates the suspicion that such rings may well correspond precisely to those with rational singularities. In particular, the properties are known to be equivalent for graded complete intersections with an isolated singularity [FW] and for all two dimensional graded algebras [F3], [S3].

More recently, several fascinating connections have been discovered between tight closure and other types of singularities. Watanabe has proven under certain additional hypotheses (including the rather strong assumption that the canonical class of R is finite order in the divisor class group) that F-regular rings have log-terminal singularites and the F-pure rings have log canonical singularities [W4]. For two dimensional excellent local rings, the link between quotient singularities F-regular rings is well established [Ha].

Most of the previous progress on the question of rational singularities relied on techniques partial to the graded case. The main point here is a new characterization of F-rationality (Theorem 2.6) which is interesting in its own right: a Cohen-Macaulay local ring (R, m) of dimension d and prime characteristic is F-rational if and only if its highest local cohomology module with support in the maximal ideal, $H^d_m(R)$, has no nontrivial submodules that are stable under the action of Frobenius. Equivalently, (R, m) is F-rational if and only if $H^d_m(R)$ is simple as a left $R[F]$ module, where $R[F]$ denotes the (noncommutative) subring of $End_{\mathbf{Grp}}R$ generated by R (acting by left multiplication) and the Frobenius operator (raising elements to their p^{th} powers). This characterization is sufficiently powerful to imply the pseudorationality of F-rational ring without any assumptions on the singular locus, and without assuming the ring is a complete intersection or even graded.

Prime characteristic techniques have also been used to study rational singularities of Schubert varieties and related cohomological problems by Mehta, Ramanathan, and others (see e.g. [MR], [R]). Indeed, their study of F-split algebraic varieties is closely related to Hochster and Roberts' concept of F-pure rings. The existence of an F-split desingularization for a local scheme V places strong restrictions on the singularities of V and can often be combined with other ideas to prove rational singularities for V; unfortunately, however, the existence of the F-split desingularization is neither necessary nor sufficient for rational singularities. Tight closure appears to narrow this gap, providing a sufficient (and conjectured necessary) condition for the rational singularities of V *without referring* to a desingularization.

The paper is organized as follows. Section 1 contains some preliminary definitions and lemmas about tight closure and about rational singularities. In Section 2, we develop a cohomological characterization of F-rationality for a local ring (Theorem 2.6). Section 3 then uses this characterization to prove the main prime characteristic result: an excellent local F-rational ring is *pseudorational*. The fourth section treats the "geometric case", that is, the case of finitely generated algebras over a field of characteristic zero. Because F-rationality is primarily a prime characteristic concept, we must first define a zero characteristic analog, which we call *F-rational type* (Definition 4.1). We then employ fairly standard "reduction to characteristic p" techniques (similar to, for example, the techniques of [PS]) to prove the main zero-characteristic result, Theorem 4.3. The final section demonstrates how one may use Theorem 4.3 to prove that certain algebraic varieties over $\mathbb{C}$ have rational singularities.

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1. Preliminaries. Throughout this paper R will denote a commutative Noetherian ring and R° the complement of its minimal primes. The characteristic of R will often be a prime number p, and in this case q will always denote p^e, where e is some nonnegative integer. When I is an ideal of R, $I^{[q]}$ will denote the ideal of R generated by the qth powers of the elements of I, or equivalently, of the generators of I. The notation (R, m) will denote a local ring with unique maximal ideal m.

Tight closure. In its primary setting, tight closure is a closure operation performed on ideals in a commutative, Noetherian ring of characteristic $p > 0$. One may define tight closure for modules as well, but we will not treat that issue here. We restrict attention here to the prime characteristic setting, putting off all zero characteristic considerations until Section 4.

Definition 1.1. [HH1] Let I be an ideal of R. The tight closure I^* of I is defined by

$$z \in I^*$$

if and only if there exists $c \in R^\circ$ and a nonnegative integer N such that

$$cz^{p^e} \in I^{[p^e]}$$

for all integers $e \geq N$.

The set of all such z clearly forms an ideal I^* containing I.

If $I^* = I$, we say that I is *tightly closed*. The tight closure operation is a true closure operation in the sense that $(I^*)^* = I^*$. We refer the reader to [HH1], particularly Proposition 4.1, for the elementary properties of tight closure.

Rings in which all ideals are tightly closed (called weakly *F-regular* rings) have distinguished themselves as having exceptionally nice properties. However, nice geometric properties such as normality and Cohen-Macaulayness, and even certain Briançon-Skoda type theorems will hold for rings in which certain key ideals, the *parameter ideals*, are tightly closed.

Definition 1.2. Elements $x_1, x_2, \ldots, x_i$ of a ring R are called parameters if their images form part of a system of paramenters in every local ring R_P of R such that the prime ideal P contains them. Equivalently, $x_1, x_2, \ldots, x_i$ are parameters if they generate an ideal of height at least i (hence equal to i when they do not generate the unit ideal).
An ideal of R is said to be a paramenter ideal if it can be generated by parameters.

When R is a local ring which is both equidimensional and catenary (e.g. excellent), the elements $x_1, x_2, \ldots, x_i$ are parameters if and only if they form part of a system of parameters (s.o.p.) for R.

Definition 1.3. A ring of prime characteristic is F-rational if every parameter ideal is tightly closed.

We recall that an excellent ring R is F-rational if and only if R_m is F-rational for each maximal ideal m of R (theorems 4.2 and 6.27 of [HH2]). Furthermore, for an excellent equidimensional local ring, this is equivalent to some ideal generated by some full s.o.p. being tightly closed (see proposition 6.27 of [HH2], although this was first shown in [FW] with some additional hypotheses). Since the main case of concern to us is the excellent local domain case, we see that the formidable task of checking that all parameter ideals are tightly closed is therefore reduced to verifying that some ideal generated by a full s.o.p. is tightly closed. Moreover, should R happen to be Gorenstein as well, then the fact that some ideal generated by a s.o.p. is tightly closed implies that, in fact, *all* ideals are tightly closed (theorem 4.2g of [HH2]).

Two important properties of F-rational rings are the following: 1) an F-rational ring is always normal (theorem 4.2 of [HH2]); and 2) an excellent local F-rational ring is Cohen-Macaulay (theorem 6.27 of [HH2]).

Of course, the goal of this paper is to add a third property to this list: an excellent local F-rational ring is pseudorational, which is to say, roughly, that it has rational singularities.

We will need the following lemma, whose proof we sketch below although the ideas are implicit throughout [HH2].

LEMMA 1.4. *If (R, m) is an excellent local ring, then R is F-rational if and only if $\hat{R}$ is F-rational.*

Proof. In general, it is not difficult to see that if $R \hookrightarrow S$ is faithfully flat, then any ideal $I \subset R$ such that $IS^* = IS$ must be tightly closed in R. On the other hand, parameters in R will be parameters in S, also by faithful flatness. So if $\hat{R}$ is F-rational, then R is F-rational.

Conversely, if R is F-rational, then it is normal, so $\hat{R}$ is a domain. Since $\hat{R}$ is equidimensional and excellent, it is enough to check that some full system of parameters $x_1, \dots, x_d$ generates a tightly closed ideal I of $\hat{R}$. We may choose these parameters lying in R.

Now if $z \in \hat{R}$ is in $(I\hat{R})^*$ computed in $\hat{R}$, then we may assume that z is in R as well, since I is m-primary. The excellence of R ensures that it has a completely stable test element $c \in R$ (see theorem 6.1 of [HH2]). This means that for $z \in (I\hat{R})^*$ we have

$$cz^q \in I^{[q]}\hat{R}$$

for all q. But then

$$cz^q \in I^{[q]}R$$

for all q as well, by the faithful flatness of $R \hookrightarrow \hat{R}$, whence $z \in I^*$ in R as well. $\square$

Another fact that we will need is the "colon capturing property" of tight closure.

THEOREM 1.5. *Let* (R, m) *be an equidimensional excellent local ring. If* $x_1, \dots, x_d, x_{d+1}$ *are parameters in R, then* $(x_1, \dots, x_d)^* \colon x_{d+1} \subset (x_1, \dots, x_d)^*$ *and* $(x_1, \dots, x_d^{t+s})^* \colon x_d^s \subset (x_1, \dots, x_d^t)^*$.

Proof. This theorem is essentially due to Hochster and Huneke, but does not appear in the literature in this generality. Using the existence of completely stable test elements, however, the proof is easily reduced to the complete case. It is easy to show that Theorem 1.5 follows from [HH1, theorem 7.15a]. This is worked out explicitly in [S1], theorem 1.3.2. $\square$

Rational singularities. We recall some basic definitions and facts about rational singularities.

Definition 1.6. A point x on a normal variety X is said to be a rational singularity if there exists a desingularization $W \xrightarrow{f} X$ such that $(R^i f_* \mathcal{O}_W)_x = 0$ for all $i \geq 1$.

We say that X has *rational singularities* if every point of X is a rational singularity. Obviously, rational singularity is a local property, so it is worth pointing out that when X is affine, $R^i f_* \mathcal{O}_W$ is the sheaf determined by the module $H^i(W, \mathcal{O}_W)$, the usual sheaf cohomology on W.

It is clear that smooth varieties have rational singularities. Moreover, if $x \in X$ is a rational singularity, then the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay. We also have the following characterization of rational singularities (see [KKMS, pp. 50–51]):

THEOREM 1.7. *Let* (R, m) *be a normal local Cohen-Macaulay ring essentially of finite type over a field of characteristic 0. The scheme* $X = \operatorname{Spec} R$ *has rational* *singularities if and only if $f_*\omega_W = \omega_X$, where ω_W is the canonical sheaf for W and ω_X is the canonical sheaf for X.*

Pseudorationality is a property of local rings which is an analog of rational singularities for more general schemes, e.g. rings which may not have a desingularization. When the ring is essentially of finite type over a field of characteristic zero these two notions are the same. We now recall Lipman and Tessier's definition of pseudorationality [LT]:

Definition 1.8. Let (R, m) be a d-dimensional local ring. Then R is pseudorational if it is normal, Cohen-Macaulay, analytically unramified, and if for any proper, birational map $\pi\colon W \to X = \operatorname{Spec} R$ with W normal and closed fiber $E = \pi^{-1}(m)$, the canonical map

$$H^d_m(\pi_*\mathcal{O}_W) = H^d_m(R) \xrightarrow{\delta^d_\pi} H^d_E(\mathcal{O}_W)$$

is injective.

Remark 1.9. When $\operatorname{Spec} R$ admits a desingularization $W \xrightarrow{\pi} \operatorname{Spec} R$, it is enough to check only that the map δ^d_π on local cohomology induced by this π is injective (see [LT, section 2]).

The map δ^d_π appearing in Definition 1.8 is an edge map in the Leray-Serre spectral sequence for the composition of the functors $H^0_E \circ f_* = H^0_{f^{-1}(E)}$. We wish to treat these edge maps in some detail, recording some important facts for future reference.

We are given a map of schemes $f\colon W \to X$, a closed set $E \subset X$, and a quasicoherent sheaf $\mathcal{F}$ on W. The composition of functors

$$H^0_E(f_*(-)) = H^0_{f^{-1}(E)}(-)$$

gives rise to a spectral sequence

$$H^p_E(R^q f_*(\mathcal{F})) \underset{p}{\Longrightarrow} H^{p+q}_{f^{-1}(E)}(\mathcal{F}),$$

for any sheaf $\mathcal{F}$ of $\mathcal{O}_W$ modules. The edge maps in the spectral sequence thus induce a natural map of local cohomology modules for each i and each $\mathcal{F}$:

$$\delta^i_f\colon H^i_E(f_*\mathcal{F}) \to H^i_{f^{-1}(E)}(\mathcal{F}).$$

In particular, this holds when $X = \operatorname{Spec} R$ and $i = d = \dim R$ and we retrieve the map δ^d_f as in Definition 1.8.

An important property of these edge maps is the following "functoriality statement":

LEMMA 1.10. *Given a composition of maps of schemes,*

$$W \xrightarrow{f} Z \xrightarrow{g} X,$$

a closed set $E \subset X$*, and a quasicoherent sheaf* $\mathcal{F}$ *on* W*, we have*

$$H^i_E((gf)_*\mathcal{F}) \xrightarrow{\delta^i_g} H^i_{g^{-1}(E)}(f_*\mathcal{F}) \xrightarrow{\delta^i_f} H^i_{f^{-1}g^{-1}(E)}(\mathcal{F})$$

with $\delta^i_f \circ \delta^i_{(g\circ f)}$.

Proof. The proof is left to the reader. This general statement about the edge map in a spectral sequence is easiest to see from the point of view of the derived category. The reader seeking an explicit proof is referred to [S1], lemma 6.1.7. □

Another essential property of the edge maps is "naturality." Given a map of schemes

$$W \xrightarrow{\pi} X$$

and a closed set $E \subset X$, let $\tilde{E}$ denote the closed set $\pi^{-1}(E) \subset W$. We have two different functors from the category of $\mathcal{O}_W$-modules to the category of abelian groups:

$$\{\mathcal{O}_W - mod\} \xrightarrow{H^i_{\tilde{E}}} \{\mathcal{AB}\}$$
$$\{\mathcal{O}_W - mod\} \xrightarrow{H^i_E \circ \pi_*} \{\mathcal{AB}\}.$$

Fortunately, we have the following naturality relation between them:

LEMMA 1.11. *The transformation* δ^i_π *is a natural transformation from the functor* $H^i_E \circ \pi_*$ *to the functor* $H^i_{\tilde{E}}$.

Explicitly, this means that for each $\mathcal{F} \in \mathbf{Ob}\{\mathcal{O}_W - mod\}$, there is a morphism of abelian groups

$$\delta^i_\pi(\mathcal{F})\colon\ H^i_{\tilde{E}}(\pi_*\mathcal{F}) \to H^i_E(\mathcal{F})$$

such that given any $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \in \mathbf{Mor}\{\mathcal{O}_W - mod\}$, the diagram

$$\begin{array}{ccc} H^i_{\tilde{E}}(\pi_*\mathcal{F}) & \xrightarrow{\delta^i_\pi(\mathcal{F})} & H^i_E(\mathcal{F}) \\ {\scriptstyle H^i_{\tilde{E}}(\pi_*\phi)}\downarrow & & {\scriptstyle H^i_E(\phi)}\downarrow \\ H^i_{\tilde{E}}(\pi_*\mathcal{G}) & \xrightarrow{\delta^i_\pi(\mathcal{G})} & H^i_E(\mathcal{G}) \end{array}$$

commutes.

Proof. We omit the proof of this general statement about edge homomorphisms in the Leray spectral sequence for the composition of two additive functors on abelian categories with enough injectives. The reader seeking an explicit proof is refered to [S1]. □

Assume that the following diagram of schemes

$$\begin{array}{ccc} W' & \xrightarrow{\tau} & W \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ X' & \xrightarrow{\sigma} & X \end{array}$$

commutes, and let E be a closed subscheme of X with inverse image E' in X', $\tilde{E}$ in W, and $\tilde{E}'$ in W'. Suppose that $\mathcal{G}$ is a sheaf of $\mathcal{O}_W$ modules, $\mathcal{F}$ is a sheaf of $\mathcal{O}_{W'}$ modules, and ψ is a map between them, meaning we have a map

$$\mathcal{G} \xrightarrow{\psi} \tau_*(\mathcal{F})$$

of $\mathcal{O}_W$ modules. There is an obvious induced map

$$\pi_*\mathcal{G} \xrightarrow{\phi} \sigma_*(\pi'_*\mathcal{F}).$$

The edge maps involved will all behave nicely, as the next proposition guarantees.

PROPOSITION 1.12. *With notation as in the preceding paragraph, the diagram*

$$\begin{array}{ccccc} H^i_{\tilde{E}'}(\mathcal{F}) & \xleftarrow[\delta_\tau]{} & H^i_{\tilde{E}}(\tau_*\mathcal{F}) & \xleftarrow[\psi]{} & H^i_{\tilde{E}}(\mathcal{G}) \\ \uparrow{\scriptstyle \delta_{\pi'}} & & \uparrow{\scriptstyle \delta_\pi} & & \uparrow{\scriptstyle \delta_\pi} \\ H^i_{E'}(\pi'_*\mathcal{F}) & \xleftarrow[\delta_\sigma]{} & H^i_E(\sigma_*\pi'_*\mathcal{F}) & \xleftarrow[\phi]{} & H^i_E(\pi_*\mathcal{G}) \end{array}$$

commutes.

Proof. The left square commutes by Lemma 1.10 and the right square commutes by Lemma 1.11. □

Lipman and Tessier describe a dual definition for pseudorationality in section 4 of [LT], whenever R admits a residual complex, which will be the case, for instance, when R is essentially of finite type over a field. By applying the exact contravariant functor $\mathrm{Hom}_R((-), E(R/m))$, where $E(R/m)$ is an injective hull of the residue field for R to the setup in Definition 1.8, they are led to:

THEOREM 1.13. *Let* (R, m) *be a d-dimensional local ring with canonical module* ω_R. *Then R is pseudorational if and only if it is normal, Cohen-Macaulay, analytically unramified, and if for any proper, birational map* $\pi\colon W \longrightarrow X = \mathrm{Spec}\, R$ *with W normal,*

$$\pi_* \omega_W = \omega_X$$

where ω_x *denotes the canonical sheaf of* $X = \mathrm{Spec}\, R$, *whose global sections are of course* ω_R, *and* ω_W *denotes the canonical sheaf of* W. *Equivalently, the global section map* $H^\circ(W, \omega_W) \xrightarrow{\tau} H^0(X, \omega_X)$ *is surjective.*

Remark 1.14. The map τ is obviously an isomorphism upon localization at any point in the nonsingular locus of $\mathrm{Spec}\, R$, so that both the maps τ and δ are nonzero.

As in Remark 1.9, if $\mathrm{Spec}\, R$ admits a desingularization $W \xrightarrow{\pi} \mathrm{Spec}\, R$, it is enough to check that the corresponding map of canonical sheaves is surjective for just this one map.

By comparing Theorem 1.13 with Theorem 1.7, it is now easy to see that pseudorationality is equivalent to rational singularities for rings essentially of finite type over a field of characteristic zero (e.g. for local rings of algebraic varieties over $\mathbb{C}$).

2. A characterization of F-rationality. Throughout this section we let (R, m) be a local ring of $\dim d > 0$ and $x_1, \ldots, x_d$ a system of parameters for R. We abbreviate the product $\Pi_{i=1}^d x_i$ simply by x.

We recall that the local cohomology modules with support in m, denoted by $H^i_m(R)$, can be described as a limit of Koszul cohomology. In particular, for any fixed system of parameters, $x_1, \ldots, x_d$,

$$H^d_m(R) \cong \varinjlim_t \frac{R}{(x_1^t, \ldots, x_d^t)R},$$

where the maps are given by multiplication by $x = \Pi_{i=1}^d x_i$ as follows:

$$\frac{R}{(x_1^t, \ldots, x_d^t)} \longrightarrow \frac{R}{(x_1^{t+1}, \ldots, x_d^{t+1})R}$$

$$z + (x_1^t, \ldots, x_d^t)R \mapsto xz + (x_1^{t+1}, \ldots, x_d^{t+1})R$$

For a local ring (R, m) of dimension d, we will make the identification

$$H_m^d(R) = \varinjlim_t \frac{R}{(x_1^t, \ldots, x_d^t)}$$

throughout, choosing the system of parameters $x_1, \ldots, x_d$ as convenient. With this identification, it will be convenient to denote an element $\eta \in H_m^d(R)$ by

$$[z + (x_1^t, \ldots, x_d^t)]$$

(for sufficiently large t). That is, η is represented by $z \mod (x_1^t, \ldots, x_d^t) \in \frac{R}{(x_1^t, \ldots, x_d^t)}$ in the direct limit system defining $H_m^d(R)$. Note that η is equally well written $[xz + (x_1^{t+1}, \ldots, x_d^{t+1})]$. Of course, since any s.o.p. works in the definition of $H_m^d(R)$, we may change notation, setting $x_i = x_i^t$, and therefore we lose no generality by assuming that a given $\eta \in H_m^d(R)$ is represented by $[z + (x_1, \ldots, x_d)]$.

Discussion 2.1. When R has characteristic p, the Frobenius endomorphism

$$F\colon\ R \to R$$

sending r to r^p naturally induces a map of local cohomology modules

$$F\colon\ H_m^i(R) \to H_m^i(R).$$

When $i = d$, the explicit map on elements is easily described: for $\eta = [z + (x_1^t, \ldots, x_d^t)] \in H_m^d(R) = \varinjlim_t \frac{R}{(x_1^t, \ldots, x_d^t)}$, we have $F(\eta) = [z^p + (x_1^{pt}, \ldots, x_d^{pt})]$. Of course, iterating Frobenius we therefore have:

$$F^e\colon\ H_m^d(R) \to H_m^d(R)$$
$$\eta = [z + (x_1^t, \ldots, x_d^t)] \mapsto \eta^q = [z^q + (x_1^{qt}, \ldots, x_d^{qt})].$$

Some readers may prefer to think of the local cohomology $H_m^d(R)$ as the $(d-1)^{th}$Čech cohomology $H^{d-1}(\mathcal{U}, \mathcal{O}_{R|U})$ for the scheme $\mathcal{U} = \operatorname{Spec} R - \{m\}$. That is, $H_m^d(R)$ can be thought of as the cokernel of the final map in the Čech complex for the open cover of $\operatorname{Spec} R - \{m\}$ given by $\operatorname{Spec} R_{x_i}$ as x_i runs through the s.o.p. $x_1, \ldots, x_d$:

$$\Pi_{i=1}^d R_{\frac{x}{x_i}} \to R_x.$$

Under the natural identification, $\eta = [z + (x_1^t, \ldots, x_d^t)] \in \varinjlim_t \frac{R}{(x_1^t, \ldots, x_d^t)}$ is easily seen to be represented by the equivalence class $\frac{z}{x^t} \in R_x$ in the cokernel, or equally well (of course) by $\frac{xz}{x^{t+1}}$.

The action of Frobenius on $H^d_m(R)$ is then easily interpreted by considering the action of Frobenius on the sections of $\mathcal{O}_R$. In particular, because the action of F on the sections of $\mathcal{O}_R$ simply raises them to the p^{th} power, we see that for $\eta \in H^d_m(R)$ represented by $\frac{z}{x^t}$ in R_x, $F(\eta)$ is represented by $\frac{z^p}{x^{pt}}$ in R_x. This agrees with our previous description of the action of Frobenius on $H^d_m(R) = \varinjlim_t \frac{R}{(x_1^t, \ldots, x_d^t)}$.

Definition 2.2. Let (R, m) be a local ring of char p and dimension d. We say that a submodule M of $H^d_m(R)$ is F-stable if $F(M) \subset M$.

Note that when M is an F-stable submodule of $H^d_m(R)$, we have a descending chain of submodules of $H^d_m(R)$,

$$M = F^0(M) \supset F(M) \supset F^2(M) \supset F^3(M) \supset \cdots,$$

where (in a slight abuse of notation) we denote by $F(M)$ the R submodule of $H^d_m(R)$ generated by $F(M)$. Because $H^d_m(R)$ satisfies the descending chain condition, this chain eventually stabilizes; that is, there exists a nonnegative integer e such that $F^e(M) = F^{e+f}(M)$ for all nonnegative integers f.

For any characteristic p local ring (R, m) of dimension $d > 0$, we can always construct F-stable submodules of $H^d_m(R)$. In particular, we state the following definition:

Definition 2.3. Let (R, m) be a local ring of dimension $d > 0$. for $\eta = [z + (x_1, \ldots, x_d)] \in H^d_m(R)$, the F-span N_η of η in $H^d_m(R)$ is the R submodule of $H^d_m(R)$ generated by

$$\{F^e(\eta) = [z^{p^e} + (x_1^{p^e}, \ldots, x_d^{p^e})]\}_{e=0}^{\infty}.$$

It is clear that the F-span of any $\eta \in H^d_m(R)$ is an F-stable submodule of $H^d_m(R)$.

Alternative interpretation 2.4. Let $R[F]$ denote the (noncommutative) subring of $End_{\mathbf{Grp}}R$ generated by R (acting by left multiplication) and by the Frobenius map F. The Frobenius action described in Discussion 2.1 makes $H^d_m(R)$ into a left $R[F]$ module. Interpreted in this context, an F-stable R module is simply a left $R[F]$ submodule of $H^d_m(R)$. The F-span of an element $\eta \in H^d_m(R)$ is just the cyclic $R[F]$ submodule of $H^d_m(R)$ generated by η.

PROPOSITION 2.5. *Let (R, m) be an excellent equidimensional local ring of characteristic p with system of parameters $x_1, \ldots, x_d$. Then $z \in (x_1, \ldots, x_d)^*$ if and only if there is some $c \in R^0$ that annihilates the F-span N_η of $\eta = [z + (x_1, \ldots, x_d)]$ in $H^d_m(R)$.*

Proof. If $z \in (x_1, \ldots, x_d)^*$, then there is some $c' \in R^0$ and some q_0 such that $c'z^q \in (x_1^q, \ldots, x_d^q)$ for all $q > q_0$. Let $c = x_1^{q_0}c'$. Clearly $c \in R^0$, and $cz^q \in (x_1^q, \ldots, x_d^q)$ for all $q = p^e \geq 1$. We conclude that c kills each $F^e(\eta) = [z^{p^e} = (x_1^{p^e}, \ldots, x_d^{p^e})]$ and hence annihilates N_η.

Conversely, suppose that $cN_\eta = 0$. If R is Cohen-Macaulay, the direct limit system defining $H^d_m(R)$ is injective, so we may immediately conclude that $c[z^q + (x_1^q, \ldots, x_d^q)] = 0$ implies that $cz^q \in (x_1^q, \ldots, x_d^q)$ for all q, and hence $z \in (x_1, \ldots, x_d)^*$.

However, some technical modifications are required when R is not Cohen-Macaulay. In this case, from the fact that c kills N_η, we may only conclude that for each q, there is some nonnegative integer t such that

$$(\dagger) \qquad cx^t z^q \in (x_1^{q+t}, \ldots, x_d^{q+t}).$$

In this case we can apply the colon capturing properties of tight closure (Theorem 1.5) to see that (†) implies that

$$cz^q \in (x_1^q, \ldots, x_d^q)^*.$$

Because R is excellent, it has a (weak) test element for R : there exists $d \in R^0$ and q_0 such that d works in *all* tight closure tests with $Q \geq q_0$ [HH3, theorem 6.1]. In particular, d "works" in each tight closure test for $cz^q \in (x_1^q, \ldots, x_d^q)^*$, so that for all $Q \geq q_0$, we have $d(cz^q)^Q \in (x_1^q, \ldots, x_d^q)^{[Q]}$. For $Q = q_0$ this yields

$$dc^{q_0}z^{qq_0} \in (x_1^{qq_0}, \ldots, x_d^{qq_0})$$

for all $q \gg 0$. Since q_0 is fixed and $dc^{q_0} \in R_0$, we deduce that $z \in (x_1, \ldots, x_d)^*$. ☐

Experts will observe that the condition of Proposition 2.5 that there exists some $c \in R_0$ that kills the F-span of η is simply the condition that η is in the tight closure of 0 in $H^d_m(R)$. This point of view is developed in [S2].

The following theorem shows that the apparent abundance of F-stable submodules suggested by Definition 2.3 is an illusion.

THEOREM 2.6. *Let (R, m) be an excellent local Cohen-Macaulay ring of dimension d and characteristic $p > 0$. The ring R is F-rational if and only if $H^d_m(R)$ has no proper nontrivial submodules stable under the action of Frobenius.*

Equivalently, R is F-rational if and only if $H^d_m(R)$ is a simple left $R[F]$ module.

Proof. Assume that R is not F-rational and choose a system of parameters $x_1, \ldots, x_d$ and an element $z \in (x_1, \ldots, x_d)^*$ but not in $(x_1, \ldots, x_d)$. The Cohen-Macaulayness of R ensures that the direct limit system defining $H^d_m(R)$ is injective, so that $\eta = [z + (x_1, \ldots, x_d)]$ is nonzero. Hence the F-span N_η of η is clearly a nonzero F-stable submodule of $H^d_m(R)$. Now Proposition 2.5 ensures that because $z \in (x_1, \ldots, x_d)^*$, this submodule is killed by some $c \in R^0$. Thus N_η can not be all of $H^d_m(R)$, since $H^d_m(R)$ has annihilator 0. We conclude that N_η is a nontrivial proper F-stable submodule of $H^d_m(R)$.

We now prove the converse. Since R is F-rational if and only if $\hat{R}$ is, and since the R submodules of $H^d_m(R)$ are exactly the same as the $\hat{R}$ submodules of $H^d_m(R) = H^d_{m\hat{R}}(\hat{R})$, the conclusion will follow for R if we can prove the theorem for $\hat{R}$. We henceforth assume that R is a complete local F-rational ring.

The F-rationality of R implies that R is normal and Cohen-Macaulay, so we may furthermore assume that R is a Cohen-Macaulay domain. Assume $H^d_m(R)$ has a proper nontrivial F-stable submodule M. Choosing any nonzero $\eta = [z + (x_1, \ldots, x_d)]$ in M, we see that the F-spat N_η of η is contained in M. We may therfore replace M by N_η and assume that our proper nontrivial F-stable submodule is of the form N_η for some $\eta = [z + (x_1, \ldots, x_d)] \in H^d_m(R)$.

Letting C be the cokernel of the inclusion map $N_\eta \hookrightarrow H^d_m(R)$, we have an exact sequence of R modules with DCC:

$$0 \to N_\eta \to H^d_m(R) \to C \to 0.$$

Applying the contravariant exact Matlis Dual functor $\mathrm{Hom}\,(*, E)$ (denoted by $*^\vee$), where E is an injective hull of the residue field of (R, m), we have an exact sequence of R modules with ACC:

$$0 \leftarrow N_\eta^\vee \leftarrow H^d_m(R)^\vee \leftarrow C^\vee \leftarrow 0.$$

Since $H^d_m(R)^\vee$ is a canonical module for R, it is a torsion-free R module of rank 1, its submodule $C^\vee$ is also torsion-free.

Thus, denoting the fraction field of R by K, we see that either $K \otimes C^\vee = 0$ or $K \otimes N_\eta^\vee = 0$. But in the former situation, the torsion-freeness of $C^\vee$ would be a contradiction unless $C^\vee = 0$, whence $C = \mathrm{Hom}\,(C^\vee, E)$ must also be 0. But this forces $N_\eta = H^d_m(R)$, contrary to the properness of $N_\eta \subset H^d_m(R)$.

On the other hand, if $K \otimes N_\eta^\vee = 0$, there is some $c \in R^0$ such that c annihilates $N_\eta^\vee$. but then c also annihilates $\mathrm{Hom}\,(N_\eta^\vee, E) = N_\eta$. An application of Proposition 2.5 then demonstrates that $z \in (x_1, \ldots, x_d)^*$. However, N_η was assumed to be nonzero, so that $[z + (x_1, \ldots, x_d)] \neq 0$ and thus z is not in $(x_1, \ldots, x_d)$, contrary to the assumption that R is F-rational. The theorem is proved. □

3. F-rational implies pseudo-rational. We can now prove the main theorem.

THEOREM 3.1. *Let (R, m) be an excellent local ring of characteristic p. If R is F-rational, then it is pseudorational.*

The proof is not difficult after an appeal to the characterization of F-rationality (theorem 2.6) proved in Section 3, given the "abstract nonsense" of Proposition 1.12.

Proof of Theorem 3.1. Because R is excellent, it is analytically unramified. Since excellent local F-rational rings are normal and Cohen-Macaulay, we need check only the map δ_π^d is injective for all proper birational maps $W \xrightarrow{\pi} \operatorname{Spec} R$, with W normal.

Now if $\dim R = 0$, R must be a field, and the result follows trivially, so we assume dimension $d > 0$.

Let W be any normal scheme mapping properly and birationally to $X = \operatorname{Spec} R$:

$$W \xrightarrow{\pi} X,$$

and let $E = \pi^{-1}(\{m\})$. We now prove that the kernel of

$$H_m^d(R) \xrightarrow{\delta_\pi^d} H_E^d(\mathcal{O}_W)$$

is an F-stable submodule of $H_m^d(R)$.

We have a commutative diagram

$$\begin{array}{ccc} {}^1W & \xrightarrow{\tilde{F}} & W \\ \pi\downarrow & & \pi\downarrow \\ {}^1X & \xrightarrow{F} & X \end{array}$$

where ${}^1W = W$ and the map $\tilde{F}\colon\ {}^1W \to W$ is the identity map on the underlying topological spaces, but the corresponding map of structure sheaves $\mathcal{O}_W \xrightarrow{\psi} \tilde{F}_*\mathcal{O}_{{}^1W}$ is the Frobenius map: for any open set $U \subset W$

$$\mathcal{O}_W(U) \to \tilde{F}_*\mathcal{O}_{{}^1W}(U) = \mathcal{O}_{{}^1W}(\tilde{F}^{-1}(U)) = \mathcal{O}_{{}^1W}(U)$$
$$r \mapsto r^p$$

Likewise, for the affine scheme $\operatorname{Spec} R$, the map ${}^1X \xrightarrow{F} X$ is the identity map on the underlying topological spaces, but the map of sheaves $\phi\colon\ \mathcal{O}_X \to F_*\mathcal{O}_{{}^1X}$ is given by the global section map $R \to {}^1R$ sending r to r^p, that is, the usual Frobenius endomorphism of R.

Of course, the map $\mathcal{O}_W \to \tilde{F}_*\mathcal{O}_{{}^1W}$ is a map of $\mathcal{O}_W$ modules, but the sheaf $\tilde{F}_*\mathcal{O}_{{}^1W}$ of $\mathcal{O}_W$ modules must be carefully interpreted. On any open set $U \subset W$, the $\mathcal{O}_W(U)$-module $\tilde{F}_*\mathcal{O}_{{}^1W}(U)$ is, *as an abelian group*, the same abelian group as $\mathcal{O}_W(U)$. However, the action of an element r in the *ring* $\mathcal{O}_W(U)$ on an element x

of the *group* $\tilde{F}_*\mathcal{O}_{{}^1W}(U) = \mathcal{O}_W(U)$ is defined by $r \cdot x = r^p x$. The same is true for the $\mathcal{O}_x$ module structure of $F_*\mathcal{O}_{{}^1X}$.

We wish to apply Proposition 1.12 to this situation, where ${}^1W = W$ plays the role of W' from the proposition, where the structure sheaf $\mathcal{O}_{{}^1W}$ plays the role of $\mathcal{F}$, where $\mathcal{O}_W$ plays the role of $\mathcal{G}$, and where the map between them is the obvious map $\psi\colon \mathcal{O}_W \to \tilde{F}_*\mathcal{O}_{{}^1W}$ defining the scheme map above. The pushdown of this map to the affine scheme $X = \operatorname{Spec} R$,

$$\pi_*\psi\colon \pi_*\mathcal{O}_W \to \pi_*\tilde{F}_*\mathcal{O}_{{}^1W},$$

corresponds to the map, here described by global sections,

$$R \to {}^1 R$$
$$r \mapsto r^p$$

Proposition 1.12 reveals the following commutative diagram:

$$\begin{array}{ccccc} H^d_E(\mathcal{O}_{{}^1W}) & \xleftarrow[\delta\tilde{F}]{} & H^d_E(\tilde{F}_*\mathcal{O}_{{}^1W}) & \xleftarrow[\tilde{F}]{} & H^d_E(\mathcal{O}_W) \\ \uparrow{\scriptstyle\delta_\pi} & & \uparrow{\scriptstyle\delta_\pi} & & \uparrow{\scriptstyle\delta_\pi} \\ H^d_m({}^1R) & \xleftarrow[\delta F]{} & H^d_m({}^1R) & \xleftarrow[F]{} & H^d_m(R). \end{array}$$

The bottom row in this diagram describes exactly the same action as the F-action described in Discussion 2.1: the map $H^d_m(R) \xrightarrow{F} H^d_m({}^1R)$ is the natural map induced by the map of R modules $R \xrightarrow{F} {}^1R$ whereas the map $H^d_m({}^1R) \xrightarrow{\delta_F} H^d_m({}^1R)$ is the map that reinterprets the second copy of ${}^1R$ as a *ring*, as opposed to an R module via Frobenius (as in the first copy). The top row in this diagram may be interpreted as giving a natural action of Frobenius on $H^d_e(\mathcal{O}_W)$.

Condensing the above diagram, we have a commutative diagram

$$\begin{array}{ccc} H^d_e(\mathcal{O}_W) & \xleftarrow[\tilde{F}]{} & H^d_E(\mathcal{O}_W) \\ \uparrow{\scriptstyle\delta_\pi} & & \uparrow{\scriptstyle\delta_\pi} \\ H^d_m(R) & \xleftarrow[F]{} & H^d_m(R). \end{array}$$

It is now easy to see that the kernel of δ_π is F-stable. For if $\eta \in H^d_m(R)$ is in the kernel of δ_π, then $\tilde{F}(\delta_\pi(\eta)) = 0 = \delta_\pi(F(\eta))$, so $F(\eta) \in \operatorname{kernel} \delta_\pi$ as well.

This completes the proof of Theorem 3.1, for if R is pseudorational but fails to be F-rational, the kernel of

$$H^d_m(R) \xrightarrow{\delta^d_\pi} H^d_E(\mathcal{O}_W)$$

is a nonzero F-stable submodule of $H^d_m(R)$. But δ_π is never the zero map, because it is dual to the natural map $\tau\colon\ \pi_*\omega_W \to \omega_X$ of the canonical sheaves (Remark 1.14), and therefore kernel δ_π must be a proper submodule as well. This contradicts Theorem 2.6, the theorem that ensures that for a local F-rational ring, $H^d_m(R)$ should have no proper, nontrivial F-stable submodules. □

An immediate consequence of Theorem 3.1 is a characteristic p analog of Boutôt's theorem that a direct summand of a rational singularity (in characteristic 0) is a rational singularity [B].

COROLLARY 3.2. *If a local excellent ring R of characteristic p is a direct summand, as an R module, of a regular (or even F-regular) overring, then R is pseudorational.*

Proof. It is easy to prove that a ring which is direct summand of an F-regular ring must itself be F-regular. In particular, such rings are F-rational, and therefore pseudorational. □

We also note an interesting corollary that gives a sufficient condition for a ring to be pseudorational in terms of its integral extensions.

COROLLARY 3.3. *If* (R, m) *is an excellent local domain of characteristic* $p \neq 0$ *such that every ideal generated by parameters (equivalently, some ideal generated by a full system of paramenters) is contracted from every module finite extension, then R is pseudorational.*

In particular, if R is a direct summand (as an R module) of every integral extension, then R is pseudorational.

Proof. This is an immediate consequence of the characterization of the tight closure of parameter ideals given in [S2]: $I^* = IR^+ \cap R$ for any parameter ideal I in an excellent local domain of prime characterisitc, where R^+ denotes the integral closure of R in an algebraic closure of its fraction field. □

It is possible that the condition that some ideal generated by a full system of parameters is contracted from every module-finite extension and characterizes pseudorational rings of characteristic $p \neq 0$.

4. Results in zero characteristic. Although tight closure is essentially a characteristic p phenomenon, several notions of tight closure exist for various classes of rings containing a field of characteristic zero [HH3]. However, the theory is not yet very well developed. It is not at all clear whether or not the various definitions are the same; and if not, it is not clear which of the definitions is the "right" one. Moreover, there has been very little progress towards defining tight closure in mixed characteristic, i.e., for rings containing $\mathbb{Z}$ but not $\mathbb{Q}$.

The goal of this section is to make sense of the property of F-rationality in characteristic zero for the "geometric case," that is, for finitely generated algebras over a field of characteristic zero, and then to give an analog of Theorem 3.1 in this case. We will accomplish this not by appealing to any of the various definitions of tight closure alluded to above, but rather by a straightforward reduction to characteristic p approach that bypasses the need to define tight closure for these rings. Our definition of "F-rational type" (Definition 4.1) is an analog of "F-pure type" as defined by Hochster and Roberts in [HR] and "F-contracted type" (or "F-injective type" in more recent parlance) as defined by Fedder in [F2]. It is possible that our definition is equivalent to the property that all parameter ideals are tightly closed with respect to some given definition of tight closure, a method of defining F-rationality explored in [HH3]. However, this issue remains unresolved, and the definition given here appears to be geometrically more natural.

We first indicate briefly the idea involved in the definition of F-rational type before recording the precise definition (Definition 4.1). Suppose that R is a finitely generated algebra over a field k of characterisitic zero. We replace the structural homomorphism $k \to R$ by a flat map of finite type $A \to R_A$ where A is a finitely generated $\mathbb{Z}$-subalgebra of k, R_A is some finitely generated A algebra and such that by tensoring over A with k we get back the structural homomorphism $k \to R$. The closed fibers of the map $A \to R_A$ are all finitely generated algebras over various *finite* fields. Because this map is flat, our geometric intuition indicates that if all the closed fibers (at least on some dense open set of $\operatorname{Spec} A$) have a particular "nice" property, then indeed, the fiber over the generic point (which is roughly the original ring R) should share this "nice" property.

We now proceed with the formal definition.

Definition 4.1. Let k be a field of characteristic 0 and let R be a finitely generated k-algebra. We say that R is of F-rational type if there exists a finitely generated $\mathbb{Z}$-algebra A contained in k, a finitely generated A-algebra R_A, and a flat map $A \hookrightarrow R_A$ such that:

(i) $(A \hookrightarrow R_A) \otimes_A k$ is isomorphic to $k \hookrightarrow R$, and

(ii) the ring $R_A \otimes_A \frac{A}{\mu}$ is F-rational for all maximal ideals μ in a dense open subset of $\operatorname{Spec} A$.

We note that whether or not the k-algebra R is F-rational, a finitely generated $\mathbb{Z}$-algebra A and a finitely generated A-algebra always exist such that condition (i) holds, In fact, if R has presentation $R = \frac{k[X_1, X, \ldots, X_N]}{(F_1, F_2, \ldots, F_M)}$, then we can let A be the finitely generated $\mathbb{Z}$ algebra to which all the elements of k have been adjoined which occur as coefficients of the $F_i's$ and let $R_A = \frac{A[X_1, X, \ldots, X_N]}{(F_1, F_2, \ldots, F_M)}$. The map $A \to R_A$ can then be replaced by a suitable localization so as to ensure the flatness required by condition (i). It is not difficult to check, then, that condition (ii) does not depend on the choice of A.

By localizing A and R_A further at an element of A, we may assume that condition (ii) holds for *all* maximal ideals of A. Furthermore, an additional application of the lemma of generic freeness allows us to assume, by localization of A and

R_A at an appropriate element of A, that any finite number of finitely generated R_A-modules or R_A-algebras are also A free (theorem 24.1, [M]).

We are led to the natural extension

Definition 4.2. Let X be a scheme of finite type over a field of characterisitic zero. We say that a point $x \in X$ has F-rational type if x has an open affine neighborhood defined by a ring R of F-rational type. We say that the scheme X has F-rational type if every $x \in X$ has F-rational type.

We now present the characteristic zero version of the main theorem:

THEOREM 4.3. *Let X be a scheme of finite type over a field of characteristic zero. If X has F-rational type, then X has rational singularities.*

Proof. Because both conditions are local, we assume X is the affine scheme $\operatorname{Spec} R$, where we have fixed a presentation

$$R = \frac{k[X_1, X_, \dots, X_M]}{(F_1, F_2, \dots, F_t)}$$

for R as a k-algebra. We thus think of $X = \operatorname{Spec} R$ as embedded in $\mathbb{A}^M_k$. If W is a desingularization for X, the W may be identified with $Bl_I R = Proj(R[It])$ for some ideal I defining the singular locus of R [H]. That is, $W \subset \mathbb{P}^N_R$ for some N.

We now choose a finitely generated $\mathbb{Z}$-algebra A contained in k such that A contains not only all the coefficients of the elements F_i occurring in the presentation for R, but also all the coefficients of some finite set elements generating the ideal I. Thus, we have $I_A \subset R_A$ such that $I_A R = I$. Not only does $R_A \otimes_A k = R$, but when $W_A = Proj(R_A[I_A t])$, we have that $W_A \otimes_A k = W$ as well.

By localizing A sufficiently, we may assume that all of $I_A, R_A, R/I_A$, and $R[It]$ are A-free. Hence both the maps $W_A \longrightarrow \operatorname{Spec} A$ and $\operatorname{Spec} R = X \longrightarrow \operatorname{Spec} A$ are flat. In fact, because the map $W_A \longrightarrow \operatorname{Spec} A$ is generically smooth, we may further localize A so as to assume that W_A is actually A-smooth.

For a given maximal ideal μ in $\operatorname{Spec} A$ we let $A/\mu = \bar{A}$ and we denote the corresponding closed fiber schemes and maps by $W_{\bar{A}} \xrightarrow{f\bar{A}} X_{\bar{A}} = \operatorname{Spec} R_{\bar{A}}$ and so forth. Note that $W_{\bar{A}} \to \operatorname{Spec} \bar{A}$ is smooth because it arises by base change from the smooth map $W_A \to A$.

The scheme W_A is a closed subscheme of $\mathbb{P}^M_{R_A} \subset \mathbb{A}^M_A \times \mathbb{P}^N_{S_A}$, where $S_A = A[X_1, \dots, X_M]$ defines the affine space into which X_A embeds, and the fibered product is taken over $\operatorname{Spec} A$. We can therefore define the relative canonical sheaf ω_{W_A} of W_A as

$$\omega_{W_A} = \mathcal{EXT}^d_{\mathcal{O}_{\mathbb{P}^N_{S_A}}}(-N-1))$$

and by yet further localization of A, we also assume that ω_{W_A} is a flat sheaf of $\mathcal{O}_A$-modules.

With this setup, the $\mathcal{O}_A$-flatness of $\mathcal{O}_{W_A}$ implies that for a given A-albegra B, the natural map

$$\begin{aligned} \mathcal{O}_B \otimes_{\mathcal{O}_A} \mathcal{EXT}^i_{\mathcal{O}_{\mathbb{P}^N_{S_A}}}(\mathcal{O}_{W_A}, \mathcal{O}_{\mathbb{P}^N_{S_A}}(-N-1)) \longrightarrow \\ \mathcal{EXT}^i_{(\mathcal{O}_{\mathbb{P}^N_{S_A}} \otimes \mathcal{O}_B)}(\mathcal{O}_{W_A} \otimes_{\mathcal{O}_A} \mathcal{O}_B, (\mathcal{O}_{\mathbb{P}^N_{S_A}}(-N-1) \otimes_{\mathcal{O}_A} \mathcal{O}_B)) \\ \cong \mathcal{EXT}^i_{\mathcal{O}_{\mathbb{P}^N_{S_B}}}(\mathcal{O}_{W_B}, (\mathcal{O}_{\mathbb{P}^N_{S_B}}(-N-1)) \end{aligned}$$

is an isomorphism.

Thus, for any field L such that $A \to L$, we have $L \otimes_A \omega_{W_A} \cong \omega_{W_L}$. In paricular, $\omega_{W_A} \otimes_{\mathcal{O}_A} \mathcal{O}_{\bar{A}} = \omega_{W_{\bar{A}}}$ and $\omega_{W_A} \otimes_{\mathcal{O}_A} \mathcal{O}_k = \omega_W$.

We now begin the descent to characteristic p by replacing the structural homomorphism $k \to R$ by the map $A \xrightarrow{\imath} R_A$ as described in the above discussion. We know that each closed fiber of $\imath$ is a finitely generated algebra over a field and is F-rational, thus Cohen-Macaulay and normal. Hence the generic fiber as well enjoys these properties, and so does $R_A \otimes_A k = R$. Certainly R is excellent, hence analytically unramified.

It remains to show that the natural map

$$f_* \omega_W \xrightarrow{\tau} \omega_X$$

is surjective. Because X is affine, it is enough to check that the map of global sections, denoted by $\omega_W(W) \xrightarrow{\tau} \omega_X(X) = \omega_R$, is surjective.

We consider the map $W_A \xrightarrow{f_A} \operatorname{Spec} R_A = X_A$ which induces the map of $\mathcal{O}_X$-modules $(f_A)_* \omega_{W_A} \xrightarrow{\tau_A} \omega_{X_A}$ and a global section map,

$$\omega_{W_A}(W_A) \xrightarrow{\tau_A} \omega_{R_A}.$$

By localization at a single element of $\operatorname{Spec} A$ we may assume that the cokernel of τ_A is A-free of some rank. Choose a maximal μ in $\operatorname{Spec} A$. After tensoring over A with $A/\mu = \bar{A}$, the map

$$\omega_{W_{\bar{A}}}(W_{\bar{A}}) \xrightarrow{\tau_{\bar{A}}} \omega_{R_{\bar{A}}}$$

has A/μ-free cokernel of the same rank. But $\omega_{A_{\bar{A}}}$is the canonical sheaf for the desingularization $W_{\bar{A}}$ of $\operatorname{Spec} R_{\bar{A}}$, and our assumption that R be F-rational type implies that $R_{\bar{A}}$ is F-rational. Therefore, $R_{\bar{A}}$ is psuedorational by Theorem 3.1, and $\tau_{\bar{A}}$ must be surjective, by 1.13, a contradiction. $\square$

5. Applications. In general, it is quite difficult to verify that a given variety has rational singularities. Work of Flenner [Fl], Watanabe [W1], Fedder [F1] [F2], and Fedder and Watanabe [FW] has addressed this issue in the case where the variety in question is the spectrum of a graded algebra over a field. In particular, Flenner [Fl] and Watanabe [W1] independently characterized rational singularities for graded rings with an isolated singularity at the irrelevant maximal ideal.

Theorem 4.3 provides a new tool for easily checking that certain varieties have rational singularities. We briefly indicate how this may be accomplished here.

Example 5.1. Consider the “ladder” of indeterminates:

$$\begin{array}{cccc} X_{11} & X_{12} & X_{13} & \\ X_{21} & X_{22} & X_{23} & Y_{13} \\ X_{31} & X_{32} & X_{33} & Y_{23} \\ & Y_{31} & Y_{32} & Y_{33} \end{array}$$

We let Δ_1 denote the determinant of the matrix (X_{ij}) and Δ_2 denote the determinant of the matrix (Y_{ij}) (setting $X_{22} = Y_{11}$, $X_{23} = Y_{12}$, $X_{32} = Y_{21}$, $X_{33} = Y_{22}$). The ring

$$R = \frac{k[X_{ij}, Y_{ij}]}{(\Delta_1, \Delta_2)}$$

is the coordinate ring for a complete intersection variety of dimension 12 sitting in $\mathbb{A}^{14}$. This is a simple example of what is known as a “ ladder determinantal variety.” Ladder determinantal varieties were first studied by Abhyankar [Abh] during his investigations of the singularities of Schubert varieties.

Theorem 4.3 can be used to demonstrate that the variety determined by Δ_1, Δ_2 in $\mathbb{A}^{17}_k$ (that is, Spec R) has rational singularities, when k is any field of characteristic zero.

The point is that the map

$$\mathbb{Z} \longrightarrow \frac{\mathbb{Z}[X_{ij}, Y_{ij}]}{(\Delta_1, \Delta_2)}$$

is a faithfully flat (indeed free) map. The fibers over the closed points all have the form

$$\frac{\mathbb{Z}/p\mathbb{Z}[X_{ij}, Y_{ij}]}{(\Delta_1, \Delta_2)}$$

for vaious prime numbers p. One then easily checks directly that these closed fiber rings are all F-rational by verifying by brute force that a particular linear system of parameters is tightly closed. Or one can use the criterion for F-rationality of graded algebras described in [HH3, theorem 7.8]. This implies that the generic fiber ring

$$\frac{\mathbb{Q}[X_{ij}, Y_{ij}]}{(\Delta_1, \Delta_2)}$$

has F-rational type, as does any ring of the same form where $\mathbb{Q}$ is replaced by some larger field, e.g. $\mathbb{C}$. Theorem 4.3 now implies that the variety $V(\Delta_1, \Delta_2) \subset \mathbb{A}_k^{17}$ has rational singularities.

A general method for checking F-rationality in rings of this type is developed in [GS], where it is shown that all complete intersection ladder determinantal varieties are F-regular and hence have rational singularities. The above is an easy special case. Conca and Herzog have generalized these results, using Theorem 3.1 to prove that all ladder determinantal varieties are rational singularities. Although it is possible that the particular Example 5.1 may be directly checked to have rational singularities, as far as the author knows, there is no proof that ladder determinantal varieties in general are rational singularities that does not use Theorem 3.1.

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