STABLE EQUIVALENCE AND GENERIC MODULES

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Abstract

Let Λ and Γ be finite dimensional algebras. It is shown that any stable equivalence $f : \overline{\text{mod}}\,\Lambda \to \overline{\text{mod}}\,\Gamma$ between the categories of finitely generated modules induces a bijection $M \mapsto M_f$ between the sets of isomorphism classes of generic modules over Λ and Γ such that the endolength of M_f is bounded by the endolength of M up to a scalar which depends only on f. Using Crawley-Boevey's characterization of tame representation type in terms of generic modules, one obtains as a consequence a new proof for the fact that a stable equivalence preserves tameness. This proof also shows that polynomial growth is preserved.

Generic modules were introduced by Crawley-Boevey in order to give a conceptual description of the representation type of a finite dimensional algebra [**1**]. In this note we study the relation between the generic modules over two finite dimensional algebras Λ and Γ under the assumption that there exists an equivalence $\overline{\text{mod}}\,\Lambda \to \overline{\text{mod}}\,\Gamma$ between the stable categories of finitely generated modules.

Recall that a Λ-module is *generic* if it is indecomposable and endofinite but not of finite length over Λ. A Λ-module M is *endofinite* if the *endolength* $\text{endol}(M)$, which is the length of M over its endomorphism ring $\text{End}_\Lambda(M)$, is finite.

The endolength of a module is *not* a categorical invariant. In other words, a Morita equivalence $F : \text{Mod}\,\Lambda \to \text{Mod}\,\Gamma$ between two module categories does not necessarily preserve the endolength. However, it is not hard to see that there exists a constant c_F such that

$$\text{endol}(FM) \leqslant c_F \cdot \text{endol}(M)$$

for every Λ-module M. The purpose of this paper is to show that this fact remains true for a stable equivalence.

Using Crawley-Boevey's characterization of tame representation type in terms of generic modules, we obtain in this way a simple proof for the fact that a stable equivalence preserves tame representation type [**2**]. In fact, we improve the results obtained in [**2**], and show that a stable equivalence preserves polynomial growth.

In [**2**], it was shown that an equivalence $f : \overline{\text{mod}}\,\Lambda \to \overline{\text{mod}}\,\Gamma$ extends to an equivalence $F : \overline{\text{Pinj}}\,\Lambda \to \overline{\text{Pinj}}\,\Gamma$ between the full subcategories of (not necessarily finitely generated) pure-injective modules in $\overline{\text{Mod}}\,\Lambda$ and $\overline{\text{Mod}}\,\Gamma$. Any endofinite module is pure-injective, and our main result states that the functor F controls the endolength, as follows.

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Theorem. *Let Λ and Γ be artin algebras, and suppose that $f : \overline{\text{mod}}\,\Lambda \to \overline{\text{mod}}\,\Gamma$ is an equivalence. Denote by $F : \overline{\text{Pinj}}\,\Lambda \to \overline{\text{Pinj}}\,\Gamma$ the equivalence between the full subcategories of pure-injective modules in $\overline{\text{Mod}}\,\Lambda$ and $\overline{\text{Mod}}\,\Gamma$ which extends f (see* [**2**]*). Then there exists a constant $c_f \in \mathbb{N}$ such that*

$$\text{endol}(FM) \leqslant c_f \cdot \text{endol}(M)$$

for every endofinite non-zero Λ-module M.

Corollary 1. *An equivalence $f : \overline{\text{mod}}\,\Lambda \to \overline{\text{mod}}\,\Gamma$ induces a bijection $M \mapsto M_f$ between the sets of isomorphism classes of indecomposable endofinite and non-injective modules over Λ and Γ such that* $\text{endol}(M_f) \leqslant c_f \cdot \text{endol}(M)$ *for all M.*

Before we can give the proof of the main theorem, we need some preparation. Let us start with some notation. We fix an artin algebra Λ, and denote by $\text{Mod}\,\Lambda$ the category of all Λ-modules. The full subcategory of finitely generated modules is denoted by $\text{mod}\,\Lambda$. The *stable category* $\overline{\text{Mod}}\,\Lambda$ has the same objects as $\text{Mod}\,\Lambda$, but $\overline{\text{Hom}}_\Lambda(M,N)$ is the group $\text{Hom}_\Lambda(M,N)$ of Λ-maps modulo the subgroup of maps which factor through some injective module. The stable category $\overline{\text{mod}}\,\Lambda$ of finitely generated modules and the stable category $\overline{\text{Pinj}}\,\Lambda$ of pure-injective modules are defined analogously.

We denote by $S_1,\dots,S_p$ a representative set of simple Λ-modules, and denote by $T_1,\dots,T_q$ a representative set of simple Γ-modules. Keeping the equivalence $f : \overline{\text{mod}}\,\Lambda \to \overline{\text{mod}}\,\Gamma$ fixed, we choose Λ-modules $U_1,\dots,U_q$ such that $f(U_j) \cong T_j$ in $\overline{\text{mod}}\,\Gamma$ for all j. Now let M and X be Λ-modules, and suppose that X has finite length. We denote by $[X,M]$ the length of $\text{Hom}_\Lambda(X,M)$ as an $\text{End}_\Lambda(M)$-module, and $[\![X,M]\!]$ denotes the length of $\overline{\text{Hom}}_\Lambda(X,M)$ as an $\overline{\text{End}}_\Lambda(M)$-module. The multiplicity of S_i in a composition series of X is denoted by $m_i(X)$. We note the following elementary fact.

Lemma 1. *Let M and X be Λ-modules, and suppose that X has finite length. Then*

$$[\![X,M]\!] \leqslant [X,M] \leqslant \sum_{i=1}^{p} m_i(X)[S_i,M].$$

Proof. The first inequality follows from the fact that the length of $\overline{\text{Hom}}_\Lambda(X,M)$ as an $\overline{\text{End}}_\Lambda(M)$-module equals its length as an $\text{End}_\Lambda(M)$-module. In order to prove the second inequality, observe that any exact sequence $0 \to X' \to X \to X'' \to 0$ of Λ-modules gives an exact sequence

$$0 \longrightarrow \text{Hom}_\Lambda(X'',M) \longrightarrow \text{Hom}_\Lambda(X,M) \longrightarrow \text{Hom}_\Lambda(X',M),$$

so $[X,M] \leqslant [X',M] + [X'',M]$. By induction on the length of X, we obtain the second inequality.

The next lemma is simple but extremely useful.

Lemma 2. *Let M be a Λ-module with no non-zero injective submodules, and let S be a simple Λ-module. Then* $\text{Hom}_\Lambda(S,M) = \overline{\text{Hom}}_\Lambda(S,M)$, *and therefore* $[S,M] = [\![S,M]\!]$.

Proof. Any non-zero map $S \to M$ which factors through some injective module also factors through the injective envelope $E(S)$ of S, and therefore induces a monomorphism $E(S) \to M$. This is impossible by our assumption on M. Thus $\mathrm{Hom}_\Lambda(S, M) = \overline{\mathrm{Hom}}_\Lambda(S, M)$.

Proof of the Theorem. Let M be an endofinite non-zero Λ-module, and let $N = FM$. We denote by I_N a maximal injective submodule of N, which exists by Zorn's lemma since direct limits of injective modules are injective. This gives a direct decomposition $N = N_I \coprod I_N$. Using the preceding lemmas, we obtain the following bound for the endolength of N_I:

$$\begin{aligned}
\mathrm{endol}(N_I) = [\Gamma, N_I] &\leqslant \sum_{j=1}^{q} m_j(\Gamma)[T_j, N_I] \\
&= \sum_{j=1}^{q} m_j(\Gamma)[\![T_j, N_I]\!] = \sum_{j=1}^{q} m_j(\Gamma)[\![U_j, M]\!] \\
&\leqslant \sum_{j=1}^{q} m_j(\Gamma)[U_j, M] \leqslant \sum_{j=1}^{q} m_j(\Gamma)\left(\sum_{i=1}^{p} m_i(U_j)[S_i, M]\right) \\
&= \sum_{i=1}^{p}\left(\sum_{j=1}^{q} m_j(\Gamma) m_i(U_j)\right)[S_i, M] \leqslant n_f \sum_{i=1}^{p}[S_i, M] \\
&\leqslant n_f[\Lambda/\mathrm{rad}\,\Lambda, M] \leqslant n_f[\Lambda, M] = n_f \cdot \mathrm{endol}(M),
\end{aligned}$$

where $n_f = \max\{\sum_{j=1}^{q} m_j(\Gamma)\, m_i(U_j) \mid 1 \leqslant i \leqslant p\}$. The endolength of any injective Γ-module I is bounded by the composition length $\ell(\Gamma)$ of Γ, since I is isomorphic to a direct factor of a product of copies of the injective cogenerator $D\Gamma$ (where D denotes the usual duality between $\mathrm{mod}\,\Gamma$ and $\mathrm{mod}\,\Gamma^{\mathrm{op}}$), and

$$\mathrm{endol}\left(\prod_{\kappa} D\Gamma\right) = \mathrm{endol}(D\Gamma) = \mathrm{endol}(\Gamma) = \ell(\Gamma)$$

for any cardinal $\kappa > 0$. Defining $c_f = n_f + \ell(\Gamma)$, we obtain

$$\mathrm{endol}(FM) = \mathrm{endol}(N_I) + \mathrm{endol}(I_N) \leqslant n_f \cdot \mathrm{endol}(M) + \ell(\Gamma) \leqslant c_f \cdot \mathrm{endol}(M).$$

This finishes the proof of the Theorem.

We are now in a position to prove that a stable equivalence preserves the representation type. To this end, for an algebra Λ, denote by $g_\Lambda(n)$ the number of isomorphism classes of generic Λ-modules of endolength n, and put $\gamma_\Lambda(n) = \sum_{i=1}^{n} g_\Lambda(i)$.

COROLLARY 2. *Let* $\overline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Gamma$ *be an equivalence. Then there exists a constant* $c \in \mathbb{N}$ *such that* $\gamma_\Lambda(n) \leqslant \gamma_\Gamma(cn)$ *for all* $n \geqslant 1$. *In particular, the following hold.*

(1) *If* Γ *is generically tame, that is,* $g_\Gamma(n) < \infty$ *for all* $n \geqslant 1$, *then* Λ *is generically tame.*
(2) *If* Γ *is generically of polynomial growth, that is, there exist* $k \geqslant 1$ *and* $d \geqslant 1$ *such that* $g_\Gamma(n) \leqslant dn^k$ *for all* $n \geqslant 1$, *then* Λ *is generically of polynomial growth.*
(3) *If* Γ *is generically domestic, that is, there exist only finitely many isomorphism classes of generic modules, then* Λ *is generically domestic.*

Proof. The formula $\gamma_\Lambda(n) \leqslant \gamma_\Gamma(cn)$ is an immediate consequence of Corollary 1. Observe that in the definition of generically tame algebras, one can replace $g_\Lambda(n)$ by $\gamma_\Lambda(n)$. Hence (1) follows from the formula $\gamma_\Lambda(n) \leqslant \gamma_\Gamma(cn)$.

Now assume that $g_\Gamma(n) \leqslant dn^k$ for some $k \geqslant 1$, $d \geqslant 1$, and all $n \geqslant 1$. Then

$$g_\Lambda(n) \leqslant \gamma_\Lambda(n) \leqslant \gamma_\Gamma(cn) = \sum_{i=1}^{cn} g_\Gamma(i) \leqslant \sum_{i=1}^{cn} di^k \leqslant (cn)d(cn)^k = (c^{k+1}d)n^{k+1},$$

for all $n \geqslant 1$. This implies (2).

An algebra Γ is generically domestic if and only if there exists $N \in \mathbb{N}$ such that $\gamma_\Gamma(n) \leqslant N$ for all $n \geqslant 1$. Hence (3) is also a consequence of the formula $\gamma_\Lambda(n) \leqslant \gamma_\Gamma(cn)$.

Suppose that Λ is a finite dimensional algebra over an algebraically closed field, and denote by $\mu_\Lambda(n)$ the minimal number of continuous one-parameter families which are needed to parametrize all but finitely many indecomposable Λ-modules of dimension n. In [**1**, Theorems 4.4 and 5.6], Crawley-Boevey has shown that $\mu_\Lambda(n)$ is finite for all $n \geqslant 1$ if and only if Λ is generically tame; moreover, in this case, $\mu_\Lambda(n) = \sum_{i|n} g_\Lambda(i)$. Therefore the 'generical' definition of the representation type coincides with the 'classical' one. More precisely, we have the following.

(i) Λ is of tame representation type (that is, $\mu_\Lambda(n) < \infty$ for all $n \geqslant 1$) if and only if Λ is generically tame.

(ii) Λ is of polynomial growth (that is, there exist $k \geqslant 1$ and $d \geqslant 1$ such that $\mu_\Lambda(n) \leqslant dn^k$ for all $n \geqslant 1$) if and only if Λ is generically of polynomial growth. Indeed, the inequality $g_\Lambda(n) \leqslant \mu_\Lambda(n)$ shows that polynomial growth implies 'generical' polynomial growth. On the other hand, if $g_\Lambda(n) \leqslant dn^k$ for some $k \geqslant 1$, $d \geqslant 1$, and all $n \geqslant 1$, then

$$\mu_\Lambda(n) \leqslant \sum_{i=1}^{n} g_\Lambda(i) \leqslant \sum_{i=1}^{n} di^k \leqslant ndn^k = dn^{k+1}.$$

(iii) Λ is of domestic representation type (that is, there exists $N \in \mathbb{N}$ such that $\mu_\Lambda(n) \leqslant N$ for all $n \geqslant 1$) if and only if Λ is generically domestic.

References

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