Critical finite-size corrections for the Sherrington–Kirkpatrick spin glass

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Abstract. Starting from the effective Lagrangian of the Sherrington-Kirkpatrick spin glass near the critical temperature we investigate the finite-size effects for the free energy and the propagator above and at $T_c = 1$. We perform the calculation going up to the three-loop diagrams. For $T \to T_c$ the cubic vertex only gives the leading term, the inclusion of the quartic vertex gives the following terms. The numerical coefficients in these terms are established by the different methods of the partial summation of perturbation series. The comparison with numerical simulations is done.

1. Introduction

The Sherrington-Kirkpatrick (SK) model [1,2], despite its simplicity, has a very complicated structure of the phase space in the low temperature region [3]. The solution of the model can be obtained using the replica method. It finally turns out that this structure can be understood in terms of a branched diffusion process along a tree structure and within the concept of the ultrametricity [4–6]. The solution is marginally stable [7]. However, it is still unclear, whether the ultrametric solution, taking the hierarchical ansatz for the order parameter matrix, is the true minimum of the free energy or some metastable state. Neither is it known whether the ultrametric structure applies even in the finite-dimensional systems [8]. Analytical calculations [9] as well as Monte Carlo simulations [10–12] are devoted to this question.

One of the methods how to investigate the problem is to study the finite-size effects for a variety of physical quantities. In fact, within the fully connected model we only rescale the landscape function when changing the number of spins, while its form remains the same. Thus, we can investigate different features of the landscape function observing the finite-size effects for different quantities and different ranges of the number of spins N. To this end, we use the effective-Lagrangian formulation of the SK model [13], the order parameter matrix $\varphi_{(\alpha\beta)}$ being the field variable. It is common use to refer to this formulation as the replica field theory for the SK model.

The numerical simulations of the finite-size effects were already done long ago [14], as well as the exact computation of the partition function for small clusters [15]. The definite conclusions on the structure of the phase space, i.e. on the *properties of*

the landscape function is still lacking, though. The main problem in making firm statements is the small size and small number of realizations of the disorder, which can be reached by numerical work.

The investigation of the question whether the landscape function is ultrametric or not would require computation of the finite-size corrections in the spin-glass phase with non-trivial replica-symmetry breaking state. Until now the problem remains to be unresolved because of the presence of the complicated set of zero modes [9].

The task posed in this paper will be less ambitious: to compute the finite-size effects at the critical temperature. In this way we investigate the onset of the spin-glass ordering, the main aim being to investigate the capacity of the replica field theory to give quantitative results, comparable with numerical simulations.

2. Above the critical temperature

Being in the paramagnetic phase, the computation of the finite-size corrections is straightforward. Since the landscape function has a unique minimum and the fluctuations are Gaussian, the saddlepoint method for computing the free energy gives us directly the finite-size corrections to thermodynamic quantities in the form of power series in $1/N$. Successive terms of this expansion can be calculated perturbatively.

The Sherrington–Kirkpatrick model near the critical temperature $T_c = 1$ is described by the effective Lagrangian [13]

$$\mathcal{L}[\varphi] = \tau \sum_{(\alpha\beta)} \varphi^2_{(\alpha\beta)} - \mathcal{L}_1[\varphi] - \mathcal{L}_2[\varphi] \tag{1}$$

$$\mathcal{L}_1[\varphi] = \frac{1}{6} \sum_{\alpha\beta\gamma} \varphi_{(\alpha\beta)}\varphi_{(\beta\gamma)}\varphi_{(\gamma\alpha)} \tag{2}$$

$$\mathcal{L}_2[\varphi] = \frac{1}{12} \sum_{\alpha\beta} \varphi^4_{(\alpha\beta)} + \frac{1}{8} \sum_{\alpha\beta\gamma\delta} \varphi_{(\alpha\beta)}\varphi_{(\beta\gamma)}\varphi_{(\gamma\delta)}\varphi_{(\delta\alpha)} - \frac{1}{4} \sum_{\alpha\beta\gamma} \varphi^2_{(\alpha\beta)}\varphi^2_{(\alpha\gamma)} \tag{3}$$

where $\tau = 1 - 1/T$ and the replica summations over $\alpha, \beta, \ldots = 1, 2, 3, \ldots, n$ are unrestricted except that $\varphi_{(\alpha\alpha)} = 0$ and in the first term, $(\alpha\beta)$ means ordered pairs of indices. In this notation the free energy in the thermodynamic limit is zero, i.e. the free energy computed from this Lagrangian is directly the finite-size correction:

$$F = -\frac{T}{N} \ln \int \left(\prod_{(\alpha\beta)} \sqrt{N/2\pi T^2} \, d\varphi_{(\alpha\beta)} \right) \exp(-N\mathcal{L}[\varphi]). \tag{4}$$

The corrections to the internal energy can be computed using the analogue of the Bray and Moore formula [4]

$$E = -\frac{1}{2T} \int_0^1 (1 - q^2(x)) \, dx \tag{5}$$

where $q_{(\alpha\beta)} = \langle \varphi_{(\alpha\beta)} \rangle$, computed at $N = \infty$, is the mean-field value of the order parameter. For N finite, simple algebra generalizes the above formula to the following form:

$$E = -\frac{1}{2T} - \frac{1}{Tn} \sum_{(\alpha\beta)} \left\langle \varphi_{(\alpha\beta)} \left(\varphi_{(\alpha\beta)} - T^2 \frac{\partial \mathcal{L}_T}{\partial \varphi_{(\alpha\beta)}} \right) \right\rangle \qquad n \to 0 \tag{6}$$

where $\mathcal{L}_T$ is the full effective Lagrangian at the temperature T.

The ultrametric structure of the paramagnetic phase is the simplest possible: all different replicas have the same distance, which simplifies the form of the Green function. The fact that there is a unique minimum of Lagrangian above the critical temperature reduces the number of the independent components of the Green function to one number, so

$$G_{\alpha\beta;\gamma\delta} = \langle \varphi_{(\alpha\beta)}\varphi_{(\gamma\delta)} \rangle = \bar{G}_1(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}). \tag{7}$$

From the following calculations it will be clear that for the comparison with the numerical simulation data available for the internal energy we need only the $\bar{G}_1$ element of the Green function. Thus, the quantities of interest will be the free energy and the Green function.

Performing the perturbative expansion for F and $\bar{G}_1$ up to three-loop and two-loop diagrams respectively figures $1(a\text{-}f)$ and $2(a\text{-}e)$ we finally obtain the following
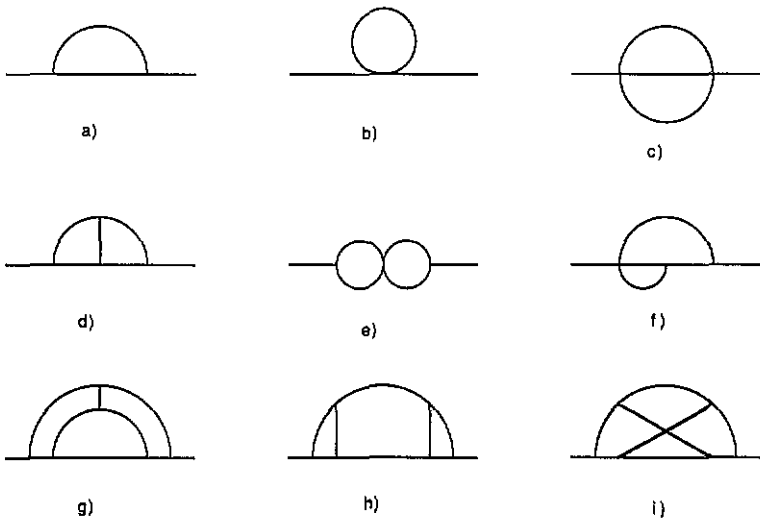

Figure 1. The irreducible diagrams for the diagonal element of the Green function up to three loops.
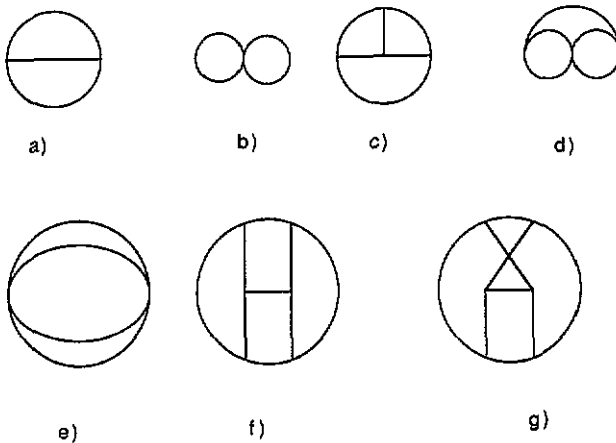

Figure 2. The irreducible diagrams for the free energy up to four loops.

formulae:

$$\bar{G}_1 = \frac{1}{2\tau N} - \frac{1}{(2\tau)^4}(2+2\tau)\frac{1}{N^2} + \frac{1}{(2\tau)^7}\left(16+110\tau+\frac{31}{6}\tau^2\right)\frac{1}{N^3} + O\left(\frac{1}{N^4}\right) \tag{8}$$

$$F = -\frac{1}{4}\ln(2\tau)\frac{1}{N} - \frac{1}{(2\tau)^3}\left(\frac{1}{6}+\frac{\tau}{4}\right)\frac{1}{N^2} + \frac{1}{(2\tau)^6}\left(\frac{2}{3}+\frac{11\tau}{2}+\frac{31\tau^2}{6}\right)\frac{1}{N^3} + O\left(\frac{1}{N^4}\right). \tag{9}$$

We can see that the relation $\bar{G}_1 = -2dF/d\tau$ holds, as it should from the thermodynamics of the system.

3. At the critical temperature

As usual, each term of the finite-size corrections diverges when approaching the critical point and the most diverging terms stem from the diagrams with three-leg vertices only. It means that at the critical point we have to sum the series by some means, regrouping the diagrams not according to the number of loops, but according to the number of four-leg vertices in otherwise totally three-leg-vertex diagram. So, in order to get the leading term we would need the pure φ^3 theory. Such a theory with real Lagrangian is possible in our replica field theory (RFT), because the integrals which would diverge in usual field theory are convergent due to the strange property of RFT, that the number of field components is in some sense $-\frac{1}{2}$.

The behaviour of the finite-size corrections in the limit $\tau \to 0$ can be described alternatively by introducing the auxiliary Lagrangian

$$\tilde{\mathscr{L}}[\varphi] = \frac{1}{2}\sum_{(\alpha\beta)}\varphi^2_{(\alpha\beta)} - x^{1/2}\mathscr{L}_1[\varphi] \tag{10}$$

and sending $x \to \infty$ at the end of the calculations. The physical meaning of the formally introduced parameter x will be discussed later.

The partition function an the averages corresponding to $\tilde{\mathscr{L}}$ are

$$\tilde{Z} = \int \left(\prod_{(\alpha\beta)} x^{1/6}\frac{d\varphi_{(\alpha\beta)}}{\sqrt{2\pi}}\right)\exp(-\tilde{\mathscr{L}}[\varphi]) \tag{11}$$

$$\langle A \rangle_- = \frac{1}{\tilde{Z}}\int \left(\prod_{(\alpha\beta)} x^{1/6}\frac{d\varphi_{(\alpha\beta)}}{\sqrt{2\pi}}\right)A\exp(-\tilde{\mathscr{L}}[\varphi]). \tag{12}$$

Simple change of variables and expanding in powers of $\mathscr{L}_2$ gives us the following expansion for the free energy and the Green function at the critical point

$$F = \frac{\ln N}{12N} + \frac{F_{(-1)}}{N} + \frac{F_{(-4/3)}}{N^{4/3}} + O(N^{-5/3}) \tag{13}$$

$$\bar{G}_1 = \frac{G_{(-2/3)}}{N^{2/3}} + \frac{G_{(-1)}}{N} + O(N^{-4/3}). \tag{14}$$

We can compare the result for the free energy with the well known leading term of the finite-size corrections at the critical temperature for the K-component theory with φ^L interaction, which is $-K(L-2)\ln N/2LN$. Having $L=3$ in the SK model, it goes out that $K = -1/2$ effectively, as we have already noted.

The non-trivial problem is to establish the coefficients $F_{(p)}$, $G_{(p)}$ corresponding to respective powers N^p in the expansions (13), (14). They can be found from the $x \to \infty$ behaviour of certain averages, whose $x \to 0$ behaviour can be calculated perturbatively. So we have for the Green function

$$G_{(-2/3)} = \lim_{x \to \infty} x^{1/3} \langle \varphi^2_{(\alpha\beta)} \rangle_{\sim} = \lim_{x \to \infty} x^{1/3} f(x) \tag{15}$$

$$G_{(-1)} = \lim_{x \to \infty} x \langle \varphi^2_{(\alpha\beta)} \mathscr{L}_2[\varphi] \rangle_{\sim} = -\lim_{x \to \infty} x \bar{f}(x) \tag{16}$$

and for the free energy similarly

$$F_{(-1)} = -\lim_{x \to \infty} \ln \tilde{Z} = -\frac{1}{12} \lim_{x \to \infty} \ln \frac{f_F(x)}{x} \tag{17}$$

$$F_{(-4/3)} = -\lim_{x \to \infty} x^{2/3} \langle \mathscr{L}_2[\varphi] \rangle_{\sim} = -\lim_{x \to \infty} x^{2/3} \bar{f}_F(x). \tag{18}$$

The variable x, introduced formally in this calculation in order to have Gaussian fluctuations, i.e. to be able to work perturbatively, can be interpreted by two ways. First, it plays the role of temperature, identifying $x^{-1/3}$ with $2\tau N^{1/3}$. Second, it is the effective interaction constant, keeping the temperature fixed. Obviously, these points of view are equivalent. At fixed N, the values of $x = 0^+$, $\pm\infty$, 0^- correspond to the temperatures $T = \infty$, T_c, 0 respectively. The point $x = 0$ connects, then, the high-temperature phase with the ground state, symmetry-broken phase and the power series for the four functions f will have zero radius of convergence. The way out from this trouble can be to forget it and to sum the series by using some prescription, or, better, to turn to self-consistent description, supposing that the self-energy itself is free of obstacles like those for the Green function (supposing that it is meromorphic, if not analytic on the real axis). Free energy can be then computed by two ways, corresponding to two interpretations of the x variable. It means either using the Helmholtz equation of thermodynamics or the well known 'Feynman trick', computing of the free energy from the known Green function by integrating by coupling constant. In our case it means that the following relations hold:

$$f(x) = 1 - \frac{x}{f_F(x)} \frac{df_F(x)}{dx} \tag{19}$$

$$-\ln \frac{f_F(x)}{x} = \int_0^1 \frac{f(y) - 1}{y} dy + \int_1^x \frac{f(y)}{y} dy \tag{20}$$

for the leading terms and

$$\bar{f}(x) = 8\bar{f}_F + 12x \frac{d\bar{f}_F(x)}{dx} \tag{21}$$

$$\bar{f}_F(x) = \frac{1}{12} x^{-2/3} \int_0^x y^{-1/3} \bar{f}(y) dy \tag{22}$$

for the next leading ones.

Having these equations, we can compute one quantity taking the presciption for the complementary one, and also check the compatibility of different prescriptions.

When computing the perturbation expansion for the f's, we use the values already quoted in the equations (8) and (9). Moreover, we compute the three- and four-loop contributions for $f(x)$ and $f_F(x)$, respectively.

The irreducible diagrams of those used in (8) and (9) are in figures 1 and 2. The result is

$$f(x) = 1 - 2x + 16x^2 - 172x^3 + O(x^4) \tag{23}$$

$$\bar{f}(x) = 1 - 55x + O(x^2) \tag{24}$$

for the Green function and

$$\ln f_F(x) = 2x - 8x^2 + \tfrac{172}{3}x^3 + O(x^4) \tag{25}$$

$$\bar{f}_F(x) = \tfrac{1}{8} - \tfrac{11}{4}x + O(x^2) \tag{26}$$

for the free energy. We see that the f's obey the relations (19)-(22). But the proper utility of the relations comes out when dealing with the $x \to \infty$ limit.

4. Numerical values of the coefficients: without self-consistency

Assuming that the coefficients of the finite-size expansions (13) and (14) are non-zero and finite, we can deduce the $x \to \infty$ limit behaviour of the functions $f(x)$, $\bar{f}(x)$, $f_F(x)$ and $\bar{f}_F(x)$ and try various prescriptions for analytic continuation from the neighbourhood of 0 to the neighbourhood of ∞. There are various methods of perturbation-series analysis, dealing with large-coupling-constant regime [16] and [17]. Our problem is slightly different because we are interested directly with infinite-coupling-constant regime.

The formulation 'without self-consistency' in the heading of this section means that we forget the fact that the functions f are not analytic in the origin and we will approximate them by functions which do not respect this non-analyticity. Generally we will speak of 'interpolation formula' (IF) and 'residuum formula' (RF) results. We mean by an interpolation formula any function which meets the requirements both at $x \to 0^+$ and $x \to \infty$. More specifically, knowing that $f(x) \sim x^{-1/\gamma}$ for $x \to \infty$, we try to use Padé approximants for $f^{-\gamma}$. This is the simplest way how to cope with the analytic continuation. The disadvantage of this approach is that the approximations for f obtained in this way are manifestly analytic at the origin.

The residuum formula needs further explanation. If we have the power series expansion of the kind

$$f(x) = \sum_{k=0}^{\infty} g(k)(-x)^k \tag{27}$$

we can use the fact that

$$f(x) = \frac{1}{2i} \int_c dz \frac{g(z)}{\sin \pi z} x^z \tag{28}$$

where the integration path goes around all the non-negative zeros of $\sin \pi z$. Then, for $x \to \infty$ the path can be deformed so that

$$f(x) \simeq -\pi x^{s_0} \operatorname{Res} \frac{g(s_0)}{\sin \pi s_0} \tag{29}$$

where s_0 is the pole of $g(z)$ with the greatest real part. In fact, (28) defines a linear transform connecting the functions $f(x)$ and $g(z)$. We see two reasons why it can be better to look for approximations for g instead of f. First, even when f is not analytic at the origin, the corresponding g can have a quite simple form (for the example of

an exactly solvable case see [16]). Second, the behaviour of f at zero and infinity translates into behaviour of g at the small integer arguments and near the rightmost pole. If the pole is small (~ 1) in absolute value, these regions are close one to another. Thus, we hope that the approximate analytic continuation of g from several small integers to the neighbourhood of the pole might give better results than the continuation from 0 to ∞ which is necessary for the function f itself.

The problem of choosing the prescription for the function $g(z)$ remains open. In this work we were led by known result for a toy model with ϕ^4 potential [16]. How it works, we can see first at the $F_{(-4/3)}$ coefficient. There are two terms only in the expansion of $\bar{f}_F$ so the prescription should have two free parameters too. The simplest interpolation formula of this type is

$$\bar{f}_F(x) = a(1+bx)^{-2/3} \tag{30}$$

and it gives

$$F_{(-4/3)} = -\frac{33^{-2/3}}{8} = -0.012\,14.\ldots \tag{31}$$

On the other hand, the residuum formula with the prescription

$$g(s) = \frac{ab^s}{s+\frac{2}{3}} \tag{32}$$

returns somewhat different result:

$$F_{(-4/3)} = -\frac{\pi}{6\sqrt{3}\,55^{2/3}} = -0.020\,90.\ldots \tag{33}$$

Unfortunately, the relation (21) gives us $G_{(-1)} = 0$ for both the residuum and the interpolation formula, because the term behaving like x^{-1} is lacking in the above prescriptions.

Proceeding with the $G_{(-1)}$ coefficient we find by the residuum formula with the prescription

$$g(s) = \frac{ab^s}{s+1} \tag{34}$$

the value

$$G_{(-1)} = -\frac{\ln 110}{110} = -0.042\,73.\ldots \tag{35}$$

The interpolation formula, using the prescription

$$\bar{f}(x) = a(1+bx)^{-1} \tag{36}$$

gives

$$G_{(-1)} = -\tfrac{1}{55} = -0.018\,181.\ldots \tag{37}$$

Applying the relation (22) together with the latter prescription we get an interesting result:

$$F_{(-4/3)} = -\frac{\pi}{6\sqrt{3}\,55^{2/3}} \tag{38}$$

i.e. exactly the same number as the one we received taking the residuum formula for the $\bar{f}_F$ function.

Now we turn to the coefficient $F_{(-1)}$. In this case the residuum formula is of no use, because the function g should have the pole at positive integer number. The physical origin is the fact that the quantity corresponding to the function f_F is the partition function, which scales exponentially with the temperature. So we concentrate on the interpolation formula. We present the result in table 1, the prescription being on the left- and the result for $F_{(-1)}$ on the right-hand side. The prescriptions on the first and second lines correspond to the two-loop and three-loop approximations, respectively. The remaining ones take the four-loop approximation for $f_F(x)$ (equation (25)).

Leaving all the discussions to the last section, we proceed with the (most interesting) coefficient $G_{(-2/3)}$. Using the interpolation formula, we obtain the results in table 2. Again, the first and second lines correspond to the one-loop and two-loop approximations, respectively. The remaining ones take the three-loop approximation for $f(x)$ (equation (23)).

The one-loop result can be analytically integrated using the relation (20), so that we obtain

$$F_{(-1)} = 0.0636. \ldots \tag{39}$$

When using the residuum formula, we deal with the one-loop approximation first, obtaining the results of table 3. As a by-product, we have the value of the coefficient $b = 8$ (in both cases). We will fix it in the following.

Different prescriptions give the results of table 4 in the two-loop approximation and of table 5 in the three-loop one.

Before we start with the self-consistency, we make several observations concerning the results just obtained. First, we see that the interpolation formula and the residuum one give rather different results, the residuum-formula results being generally larger (in absolute value). For the $F_{(-1)}$ coefficient there is even the difference in the sign. So, the question of the reliability of the two approaches arise. The observation of the

Table 1.

$f_F(x)$	$F_{(-1)}$
$1 + ax$	$-0.0577\ldots$
$\dfrac{1 + ax^2}{1 + bx}$	$-0.1342\ldots$
$(1 + ax + bx^2 + cx^3)^{1/3}$	$-0.1156\ldots$
$\dfrac{a + bx + cx^2}{a + x}$	$-0.0121\ldots$
$\dfrac{1 + cx^3}{1 + ax + bx^2}$	$-0.1676\ldots$

Table 2.

$f(x)$	$G_{(-2/3)}$
$(1 + ax)^{-1/3}$	$0.5503\ldots$
$\left(\dfrac{1 + ax^2}{1 + bx}\right)^{-1/3}$	$0.4642\ldots$
$\left(\dfrac{1 + ax + bx^2}{1 + cx}\right)^{-1/3}$	$0.6728\ldots$
$\left(\dfrac{1 + ax + bx^3}{1 + cx^2}\right)^{-1/3}$	$0.4070\ldots$
$\left(\dfrac{1 + ax^3}{1 + bx + cx^2}\right)^{-1/3}$	$0.4376\ldots$

Table 3.

$g(z)$	$G_{(-2/3)}$
$ab^z/(z + \frac{1}{3})$	$0.6046\ldots$
$ab^z\Gamma(z + 1)/(z + \frac{1}{3})$	$0.8187\ldots$

Table 4.

$g(z)$	$G_{(-2/3)}$
$(a+bz+cz^2)\dfrac{8^z\Gamma(z+1)}{z+\frac{1}{3}}$	0.79596...
$(a+bz)/(1+cz)\dfrac{8^z\Gamma(z+1)}{z+\frac{1}{3}}$	0.8187...
$1/(a+bz+cz^2)\dfrac{8^z\Gamma(z+1)}{(z+\frac{1}{3})}$	0.7935...
$(a+bz+cz^2)\dfrac{8^z}{z+\frac{1}{3}}$	0.7054...
$(a+bz)/(1+cz)\dfrac{8^z}{z+\frac{1}{3}}$	0.6046...
$1/(a+bz+cz^2)\dfrac{8^z}{(z+\frac{1}{3})}$	0.6682...

Table 5.

$g(z)$	$G_{(-2/3)}$
$(a+bz+cz^2+dz^3)\dfrac{8^z\Gamma(z+1)}{z+\frac{1}{3}}$	0.805 17...
$(a+bz+cz^2)/(1+dz)\dfrac{8^z\Gamma(z+1)}{z+\frac{1}{3}}$	0.801 79...
$(a+bz)/(1+cz+dz^2)\dfrac{8^z\Gamma(z+1)}{z+\frac{1}{3}}$	0.806 28...
$a/(1+bz+cz^2+dz^3)\dfrac{8^z\Gamma(z+1)}{z+\frac{1}{3}}$	0.844 06...
$(a+bz+cz^2+dz^3)\dfrac{8^z}{z+\frac{1}{3}}$	0.693 94...
$(a+bz+cz^2)/(1+dz)\dfrac{8^z}{z+\frac{1}{3}}$	0.695 53...
$(a+bz)/(1+cz+dz^2)\dfrac{8^z}{z+\frac{1}{3}}$	0.548 88...
$a/(1+bz+cz^2+dz^3)\dfrac{8^z}{z+\frac{1}{3}}$	0.752 06...

variation of the results when varying the prescription shows that the interpolation formula is in deep troubles as soon as we go beyond the lowest approximation (one-loop for the Green function and two-loop for the free energy). We can attribute this to the feature already mentioned several times, that the f's have zero radius of convergence. Because the interpolation prescriptions given manifestly analytic behaviour, they are essentially wrong. As for the $F_{(-1)}$ coefficient, we suppose the result (39) to be better, because the IF prescriptions (39) for f_F imply approximating a linear function (at $x \to \infty$) by an infinite series (around $x = 0$), which is somewhat ill-defined procedure.

On the other hand, the expansion (27) can be understood as a formal one (we neglect the convergence of anything), so we bypass the problem of non-analyticity in the origin. This fact makes the residuum formula more reliable. At the first sight there is a significant difference in the RF result depending on whether there is the Γ-function or not. But this difference is not so big as the spread in the results from IF and, moreover, diminishes when we got to higher orders, on the contrary to the IF. This phenomenon can be interpreted even in the way that the prescription with the Γ-function corresponds to making the Borel transform first, then summing and inverting the transform. The difference between these two procedures should diminish when going to higher orders. As for our feeling, we suppose the RF prescription with the Γ-function to be the best one.

5. Self-consistency

There are two conditions for the use of a self-consistent calculation, i.e. these based on the Dyson equation. First, the quantity of interest should be the Green function and second, there should be some Lagrangian corresponding to the procedure. These requirements limit our possibilities to the computation of the $G_{(-2/3)}$ coefficient. It implies to solve the equation for the function $f(x)$, taking into account only the cubic term in the Lagrangian, the quartic being omitted at all.

The task is to solve the usual Dyson equation

$$f(x) = 1 - \sigma(xf^3(x)) \qquad (40)$$

together with the approximate (up to three-loops) formula for the self-energy functional:

$$\sigma(H) = 2H - 4H^2 + 4H^3. \qquad (41)$$

Note that the latter consideration holds in the paramagnetic phase only, i.e. when we approach the critical point from above, x going to $+\infty$.

Making the limit $x \to \infty$ in the above equations, we arrive to the following equation for the coefficient in question:

$$\sigma(G^3_{(-2/3)}) = 1. \qquad (42)$$

This is an algebraic equation, so we could proceed directly on. But we can do more. The self-consistent approach has a built-in symmetry-breaking feature, nevertheless it does not tell us *a priori* anything more about the nature of the symmetry broken state. But, in our case we are left with the φ^3 part of the interaction Lagrangian only and we know that the low-temperature phase of this Lagrangian remains to be replica-symmetric [13], [3]. So the Green function as well as the self-energy has three independent components only [18]. The finiteness of the number of the scalar equations corresponding to (matrix) Dyson equation provides us with the hope that by eliminaton we can arrive at a single equation only for the diagonal term of the Green function, which has the same form (40) as in the paramagnetic phase, and the effective self-energy $\sigma(H)$ is memomorphic function, i.e. we can guess its behaviour using rational functions.

In order to put the information of the low-temperature phase into the self-energy, we compute first the behaviour of $f(x)$ for the temperature approaching absolute zero. According to what we have said about the meaning of the parameter x this equivalent to the limit $x \to 0^-$. Simple calculation gives us $f(x) \simeq 1/4x$ and we obtain for the self-energy

$$\sigma(H) \simeq 2H^{1/2} \qquad \text{for } H \to \infty. \qquad (43)$$

Now we are ready to write table 6 attributing to the different forms of $\sigma(H)$ the results obtained for the coefficient in the leading finite-size correction for the Green function.

Table 6.

$\sigma(H)$	$G_{(-2/3)}$
$2H$	$0.793\,70\ldots$
$2H - 4H^2$	$0.74 + 0.27i$
$2H - 4H^2 + 4H^3$	$0.917\,30\ldots$
$2H^{1/2}$	$0.629\,96\ldots$
$2H(1 + H)^{-1/2}$	$0.861\,95\ldots$
$2H(1 + aH + H^2)^{-1/4}$	$0.949\,37\ldots$
$2H(1 + aH + bH^2 + cH^3)^{-1/6}$	$1.024\,04\ldots$
$2H\left(\dfrac{1 + aH + bH^2 + H^3}{1 + H}\right)^{-1/4}$	$1.059\,24\ldots$
$2H\left(\dfrac{1 + aH + bH^2 + H^3}{1 + H^2}\right)^{-1/2}$	$1.156\,14\ldots$
$2H(1 + aH + bH^2 + H^4)^{-1/3}$	$0.969\,86\ldots$

We proceed in analogy with the interpolation formula scheme, but now it is not crucial to fit the behaviour in *both* regions $H \to 0$ and $H \to \infty$, so we will use even the functions obeying only one of the two requirements. Even though we see some variations from one prescription to another, within the same level of approximation (the same number of loops involved) the results are nearly constant. (The complex result in the two-loop, $H \to 0$ only prescription on the second line is due the fact that even the power series for the self-energy converges poorly). This fact encourages us to rely fairly well on this method. In fact, within this approach we use all the physical information we can deduce from the calculations of the perturbation series performed here. The only weak point is the prescription for the self-energy. But, as we can see, this freedom causes the spread of about 15% only, which we consider to be a reasonably small value. What remains now is the comparison with (numerical) experiment. We devote the next section to it.

6. Comparison with the simulations and discussion

To our knowledge nothing has been published on the finite-size corrections for the Sherrington-Kirkpatrick model directly at the critical temperature. We have performed the Monte Carlo simulations for the SK model with the binary distribution of couplings (the leading term in finite-size corrections is the same as in the case of the Gaussian distribution) up to size $N = 1024$. The data for the internal energy are shown in figure 3. The power $-2/3$ in the leading term for the finite-size correction for the Green function is well confirmed. Making use of the equation (6) we deduce for the leading term of the finite-size corrections for the internal energy

$$E - E_{N=\infty} = \frac{G_{(-2/3)}}{2N^{2/3}} + O(N^{-1}) \qquad (44)$$

and the coefficient obtained from the data is then

$$G_{(-2/3)} = 0.876 \pm 0.008. \qquad (45)$$

Comparison with the theoretical values from different methods shows that the closest
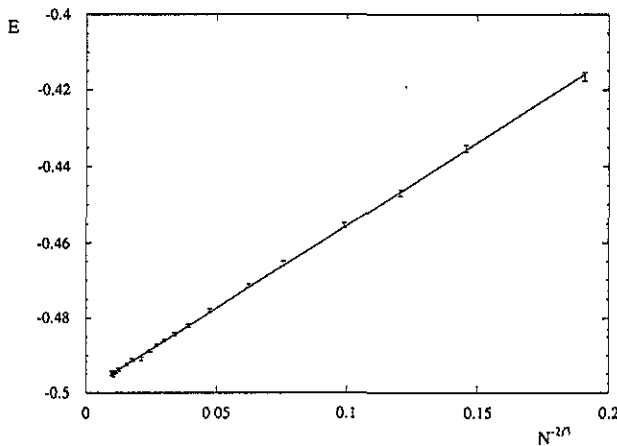

Figure 3. The dependence of the internal energy of the Sherrington–Kirkpatrick model at $T \approx 1$ on the number of spins N. The line is the least squares fit and has the slope 0.438.

value is obtained by the residuum formula non-self-consistent method. But all the results except these from the interpolation formula lie within 25% error from the experimental value.

As for the energy, the $\ln N/12N$ leading term is confirmed by the data seen in figure 4. The data were obtained from the Monte Carlo simulations for the internal energy by the integration of the Helmholtz equation, starting at $T = \infty$. Unfortunately, the next leading term cannot be read from the data because of big statistical errors. The errors bars themselves are difficult to establish, so we have left the graph without them. Even if we had reliable result for the next leading term, we would not be able to compare it with the theoretical predictions, because the results of table 1 and equation (39) from various prescriptions differ too much (neither the sign is well established, even though we suppose that (39) is more close to the correct result).
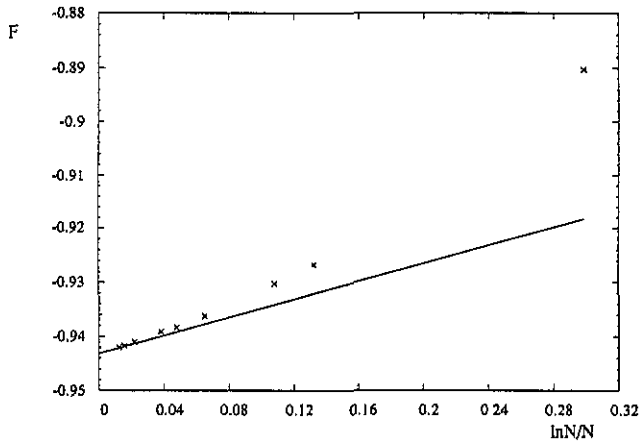

Figure 4. The size dependence of the free energy of the Sherrington-Kirkpatrick model at $T = 1$. The straight line is the function $F = -1/4 - \ln 2 + \ln N/12N$.

When comparing the results from various methods and different number of loops involved in the approximation, we see that even the lowest order gives sensible results, even though the quantitative agreement is not excellent. Going to higher orders, there is a serious difference between the methods involved. Looking first at the interpolation formula, we observe larger and larger spread of possible values of the coefficient, depending on the particular prescription used. This can be simply understood, because when summing a divergent series, we can obtain an arbitrary result, depending on the regrouping of the terms in the series. On the other hand the residuum formula provides us with a more firm outcomes, suggesting that it corresponds to some physically founded approximation, whose nature is not clear at this stage, though.

A direct physical interpretation is attributed to the self-consistent approach. That is why we rely on this way of computing most and even though the numerical results are still scattered, we take as the most probable result the value extracted from the three-loop self-consistent calculation:

$$G_{(-2/3)} = 0.95 \pm 0.15. \tag{46}$$

We guess the error from the observed dispersion of the results for different prescriptions.

Finally, we see that the analysis of the perturbation series gives us reasonable results, if we do not rely on one method only and if we explore as much as possible

the space of possible prescriptions. On the other hand, it is clear that taking an appropriate prescription we can receive *arbitrary* result. Thus, only 'good' prescriptions should be chosen, i.e. those based on some physical arguments. In our case, such an argument was the presence of the singularity at $x = 0$ of the f's. Nevertheless, this information still leaves much space for different prescriptions, whose physical meaning is unclear.

Regardless of the problems of physically unfounded approximations, the agreement with the numerical-simulation result is good. The main conclusion we can draw from our results is the possibility to get reasonable quantitative results even in the one-loop approximation. This encourages us to make similar calculations even in the spin-glass phase, where the computation of the diagrams is highly non-trivial.

The second methodological experience from the present work is that the residuum-formula method is significantly superior to the interpolation-formula one, once we go to higher orders in the perturbation series. Moreover, the best way for obtaining the numerical value of the finite-size corrections was the self-consistent method. The remaining freedom in choosing the prescription for the self-energy do not affect the results much. While the first-order calculation gives the order of magnitude correctly, going to higher orders we obtain agreement with 15% from the numerical-simulation value. We could hope to obtain better results by pushing the calculations to even higher orders of the perturbation series or, going from the opposite side, making the expansion around the zero-temperature result, obtaining in this way further information about the behaviour of the self-energy $\sigma(H)$ in the limit $H \to \infty$. But the latter would mean to go into the spin-glass phase already and we leave it for a future study.

Acknowledgments

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