GROTHENDIECK DUALITY ON FORMAL SCHEMES

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Abstract. We give several related versions of global Grothendieck Duality for unbounded complexes on noetherian formal schemes. The proofs, based on a non-trivial adaptation of Deligne's method for the special case of ordinary schemes, are reasonably self-contained, modulo the Special Adjoint Functor Theorem. (Also described is an alternative approach, inspired by Neeman and based on recent results about "Brown Representability.") A section on applications and examples illustrates how these theorems synthesize a number of different duality-related results (local duality, formal duality, residue theorems, ...).

The final version of this paper will include a flat-base-change theorem.

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1. Preliminaries and main theorems.

We begin with some notation and terminology. Let X be a ringed space, i.e., a topological space together with a sheaf of commutative rings $\mathcal{O}_X$. Let $\mathcal{A}(X)$ be the category of $\mathcal{O}_X$-modules, and $\mathcal{A}_{\mathrm{qc}}(X)$ (resp. $\mathcal{A}_{\mathrm{c}}(X)$, resp. $\mathcal{A}_{\vec{\mathrm{c}}}(X)$) the full subcategory of $\mathcal{A}(X)$ whose objects are the quasi-coherent (resp. coherent, resp. $\varinjlim$'s of coherent) $\mathcal{O}_X$-modules.[1] Let $\mathbf{K}(X)$ be the homotopy category of $\mathcal{A}(X)$-complexes, and let $\mathbf{D}(X)$ be the corresponding derived category, obtained from $\mathbf{K}(X)$ by adjoining an inverse for every quasi-isomorphism (= homotopy class of maps of complexes inducing homology isomorphisms). For any full subcategory $\mathcal{A}_{\dots}(X)$ of $\mathcal{A}(X)$, we denote by $\mathbf{D}_{\dots}(X)$ the full subcategory of $\mathbf{D}(X)$ whose objects are those complexes whose homology sheaves all lie in $\mathcal{A}_{\dots}(X)$, and by $\mathbf{D}^+_{\dots}(X)$ (resp. $\mathbf{D}^-_{\dots}(X)$) the full subcategory of $\mathbf{D}_{\dots}(X)$ whose objects are those complexes $\mathcal{F} \in \mathbf{D}_{\dots}(X)$ such that the homology $H^m(\mathcal{F})$ vanishes for all $m \ll 0$ (resp. $m \gg 0$).

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[1] The symbol $\varinjlim$ will always denote the direct limit of a system indexed by a small ordered set in which any two elements have an upper bound. More general direct limits will be referred to as *colimits*.

The full subcategory $\mathcal{A}_{\dots}(X)$ of $\mathcal{A}(X)$ is *plump* if it contains 0 and for every exact sequence $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M} \to \mathcal{M}_3 \to \mathcal{M}_4$ in $\mathcal{A}(X)$ with $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$ in $\mathcal{A}_{\dots}(X)$, $\mathcal{M}$ is in $\mathcal{A}_{\dots}(X)$ too. If $\mathcal{A}_{\dots}(X)$ is plump then it is abelian, and has a derived category $\mathbf{D}(\mathcal{A}_{\dots}(X))$. For example, $\mathcal{A}_{\mathrm{c}}(X)$ is plump [GD, p. 113, (5.3.5)]. If $\mathfrak{X}$ is a locally noetherian formal scheme,[2] then $\mathcal{A}_{\vec{c}}(\mathfrak{X}) \subset \mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$ (Corollary 3.1.5)—with equality when $\mathfrak{X}$ is an ordinary scheme, i.e., when $\mathcal{O}_{\mathfrak{X}}$ has discrete topology [GD, p. 319, (6.9.9)]—and both of these are plump subcategories of $\mathcal{A}(\mathfrak{X})$, see Proposition 3.2.2.

Let $\mathbf{K}_1$, $\mathbf{K}_2$ be triangulated categories with respective translation functors T_1, T_2 [H1, p. 20]. A (covariant) Δ-*functor* is a pair (F, Θ) consisting of an additive functor $F\colon \mathbf{K}_1 \to \mathbf{K}_2$ together with an isomorphism of functors $\Theta : FT_1 \xrightarrow{\sim} T_2F$ such that for every triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T_1A$ in $\mathbf{K}_1$, the corresponding diagram

$$FA \xrightarrow{Fu} FB \xrightarrow{Fv} FC \xrightarrow{\Theta\circ Fw} T_2FA$$

is a triangle in $\mathbf{K}_2$. Explicit reference to Θ is often suppressed—but one should keep it in mind. (For example, if $\mathcal{A}_{\dots}(X) \subset \mathcal{A}(X)$ is plump, then each of $\mathbf{D}_{\dots}(X)$ and $\mathbf{D}^{\pm}_{\dots}(X)$ carries a unique triangulation for which the translation is the restriction of that on $\mathbf{D}(X)$ and such that inclusion into $\mathbf{D}(X)$ together with $\Theta :=$identity is a Δ-functor; in other words, they are all *triangulated subcategories* of $\mathbf{D}(X)$.) Compositions of Δ-functors, and morphisms between Δ-functors, are defined in the natural way.[3] A Δ-functor $(G, \Psi)\colon \mathbf{K}_2 \to \mathbf{K}_1$ is a *right* Δ-*adjoint* of (F, Θ) if G is a right adjoint of F and the resulting functorial map $FG \to \mathbf{1}$ (or, equivalently, $\mathbf{1} \to GF$) is a morphism of Δ-functors.

We use $\mathbf{R}$ to denote right-derived functors, constructed e.g., via K-injective resolutions [Sp, p. 138, Thm. 4.5].[4] For a map f of ringed spaces, the left-derived functor of f^*, constructed via K-flat resolutions, is denoted by $\mathbf{L}f^*$ [Sp, p. 147, 6.7]. Each derived functor in this paper comes equipped, implicitly, with a Θ making it into a Δ-functor (modulo obvious modifications for contravariance), cf. [L4, Example (2.2.4)].[5] Conscientious readers may verify that such morphisms between derived functors as occur in this paper are in fact morphisms of Δ-functors.

1.1. The **first main result** of this paper, global Grothendieck Duality for a map $f\colon \mathfrak{X} \to \mathfrak{Y}$ of quasi-compact formal schemes with $\mathfrak{X}$ noetherian, is that, $\boldsymbol{j}\colon \mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X})) \to \mathbf{D}(\mathfrak{X})$ being the natural functor, *the* Δ-*functor* $\mathbf{R}f_*\circ\boldsymbol{j}$ *has a right* Δ-*adjoint.*

A more elaborate—but readily shown equivalent—statement is:

Theorem 1. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a map of quasi-compact formal schemes, with $\mathfrak{X}$ noetherian, and let $\boldsymbol{j}\colon \mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X})) \to \mathbf{D}(\mathfrak{X})$ be the natural functor. Then there exists a Δ-functor $f^{\times}\colon \mathbf{D}(\mathfrak{Y}) \to \mathbf{D}\left(\mathcal{A}_{\vec{c}}(\mathfrak{X})\right)$ together with a morphism of Δ-functors*

[2] Basic properties of formal schemes can be found in [GD, Chap. 1,§10].

[3] See also [De, §§0,1] for the multivariate case, where signs come into play—and Δ-functors are called "exact" functors.

[4] A complex F in an abelian category $\mathcal{A}$ is K-injective if for each exact $\mathcal{A}$-complex G, the abelian-group complex $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(G, F)$ is again exact. In particular, any bounded-below complex of injectives is K-injective. If every complex E admits a K-injective resolution $E \to I(E)$ (i.e., a quasi-isomorphism into a K-injective complex $I(E)$), then every functor Γ has a right-derived functor $\mathbf{R}\Gamma$ satisfying $\mathbf{R}\Gamma(E) = \Gamma(I(E))$. For example, $\mathbf{R}\mathrm{Hom}^{\bullet}_{\mathcal{A}}(E', E) = \mathrm{Hom}^{\bullet}_{\mathcal{A}}(E', I(E))$.

[5] We do not know, for instance, whether $\mathbf{L}f^*$—which is defined only up to isomorphism—can always be chosen so as to commute with translation, i.e., so that $\Theta =$ Identity will do.

$\tau : \mathbf{R}f_*\boldsymbol{j}f^\times \to \mathbf{1}$ *such that for all* $\mathcal{G} \in \mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))$ *and* $\mathcal{F} \in \mathbf{D}(\mathfrak{Y})$, *the composed map* (*in the derived category of abelian groups*)

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}_{\vec{c}}(\mathfrak{X})}(\mathcal{G}, f^\times\mathcal{F}) &\xrightarrow{\text{natural}} \mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}(\mathfrak{Y})}(\mathbf{R}f_*\mathcal{G}, \mathbf{R}f_*f^\times\mathcal{F}) \\ &\xrightarrow{\text{via } \tau} \mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}(\mathfrak{Y})}(\mathbf{R}f_*\mathcal{G}, \mathcal{F})\end{aligned}$$

is an isomorphism.

Here we think of the $\mathcal{A}_{\vec{c}}(\mathfrak{X})$-complexes $\mathcal{G}$ and $f^\times\mathcal{F}$ as objects in both $\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))$ and $\mathbf{D}(\mathfrak{X})$. But as far as we know, the natural map $\mathrm{Hom}_{\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))} \to \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}$ need not always be an isomorphism. It *is* when $\mathfrak{X}$ is *properly algebraic*, i.e., the J-adic completion of a proper B-scheme with B a noetherian ring and J a B-ideal: then $\boldsymbol{j}$ induces an equivalence of categories $\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X})) \to \mathbf{D}_{\vec{c}}(\mathfrak{X})$, see Corollary 3.3.4. So for properly algebraic $\mathfrak{X}$, we can replace $\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))$ in Theorem 1 by $\mathbf{D}_{\vec{c}}(\mathfrak{X})$, and let $\mathcal{G}$ be any $\mathcal{A}(\mathfrak{X})$-complex with $\mathcal{A}_{\vec{c}}(\mathfrak{X})$-homology.

We prove Theorem 1 in §4, adapting the argument of Deligne in [H1, Appendix] (see also [De, §1.1.12]) to the category $\mathcal{A}_{\vec{c}}(\mathfrak{X})$, which for our present purposes and capabilities is the appropriate generalization to formal schemes of the category of quasi-coherent sheaves on an ordinary noetherian scheme. For this adaptation what is needed, mainly, is the plumpness of $\mathcal{A}_{\vec{c}}(\mathfrak{X})$ in $\mathcal{A}(\mathfrak{X})$, a non-obvious fact mentioned above. In addition, we need some facts on "boundedness" of certain derived functors in order to extend the argument to unbounded complexes. (See section 3.4, which makes use of techniques from [Sp].)

There is another elegant approach to Duality (on a quasi-compact separated ordinary scheme X) due to Neeman [N1], in which "Brown Representability" is formulated so as to apply directly to the existence of right adjoints for a Δ-functor F on $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))$—a necessary and sufficient condition being that F commute with coproducts. In Deligne's approach the "Special Adjoint Functor Theorem" is used to ensure existence of right adjoints for certain functors on $\mathcal{A}_{\mathrm{qc}}(X)$, and then these right adjoints are applied to injective resolutions of complexes ... Both approaches require a small collection of category-generators—coherent sheaves for $\mathcal{A}_{\mathrm{qc}}(X)$ in Deligne's, and perfect complexes for $\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))$ in Neeman's. Lack of knowledge about perfect complexes over formal schemes discouraged us from pursuing Neeman's strategy. Recently, however, (after this paper was essentially written), Franke showed [Fe] that Brown Repesentability holds for the derived category $\mathbf{D}(\mathcal{A})$ of an arbitrary Grothendieck category (as does the closely related existence of K-injective resolutions for all $\mathcal{A}$-complexes). Consequently Theorem 1 follows from the fact that $\mathcal{A}_{\vec{c}}(\mathfrak{X})$ is a Grothendieck category (straightforward to see once we know it to be abelian), together with the fact that $\mathbf{R}f_*\circ\boldsymbol{j}$ commutes with coproducts (Proposition 3.5.2).

1.2. Two other, possibly more useful, generalizations—from ordinary schemes to formal schemes—of global Grothendieck Duality are stated below in Theorem 2 and treated in detail in §6. To describe them, and related results, we need some preliminaries about *torsion functors.*

1.2.1. Once again let $(X, \mathcal{O}_X)$ be a ringed space. For any $\mathcal{O}_X$-ideal $\mathcal{J}$, set

$$\varGamma_{\mathcal{J}}\mathcal{M} := \varinjlim_{n>0} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{M}) \qquad \big(\mathcal{M} \in \mathcal{A}(X)\big),$$

and regard $\varGamma_{\mathcal{J}}$ as a subfunctor of the identity functor on $\mathcal{O}_X$-modules. If $\mathcal{N} \subset \mathcal{M}$ then $\varGamma_{\mathcal{J}}\mathcal{N} = \varGamma_{\mathcal{J}}\mathcal{M} \cap \mathcal{N}$; and it follows formally that the functor $\varGamma_{\mathcal{J}}$ is idempotent ($\varGamma_{\mathcal{J}}\varGamma_{\mathcal{J}}\mathcal{M} = \varGamma_{\mathcal{J}}\mathcal{M}$) and left exact [St, p. 138, Proposition 1.7].

Set $\mathcal{A}_{\mathcal{J}}(X) := \varGamma_{\mathcal{J}}(\mathcal{A}(X))$, the full subcategory of $\mathcal{A}(X)$ whose objects are the *$\mathcal{J}$-torsion sheaves,* i.e., the $\mathcal{O}_X$-modules $\mathcal{M}$ such that $\varGamma_{\mathcal{J}}\mathcal{M} = \mathcal{M}$. Since $\varGamma_{\mathcal{J}}$ is an idempotent subfunctor of the identity functor, therefore it is right-adjoint to the inclusion $i = i_{\mathcal{J}} \colon \mathcal{A}_{\mathcal{J}}(X) \hookrightarrow \mathcal{A}(X)$. Moreover, $\mathcal{A}_{\mathcal{J}}(X)$ is closed under $\mathcal{A}(X)$-colimits: if F is any functor into $\mathcal{A}_{\mathcal{J}}(X)$ such that iF has a colimit $\mathcal{M} \in \mathcal{A}(X)$, then the corresponding functorial map from iF to the constant functor with value $\mathcal{M}$ comes from a functorial map from F to the constant functor with value $\varGamma_{\mathcal{J}}\mathcal{M}$, whence the inclusion $\varGamma_{\mathcal{J}}\mathcal{M} \hookrightarrow \mathcal{M}$ has a right inverse, so that it is an isomorphism, and thus $\mathcal{M} \in \mathcal{A}_{\mathcal{J}}(X)$. In particular, if the domain of a functor G into $\mathcal{A}_{\mathcal{J}}(X)$ is a small category, then iG does have a colimit, which is also a colimit of G; and so $\mathcal{A}_{\mathcal{J}}(X)$ has small colimits, i.e., it is small-cocomplete.

Submodules and quotient modules of $\mathcal{J}$-torsion sheaves are also $\mathcal{J}$-torsion sheaves. When $\mathcal{J}$ is (locally) *finitely-generated,* if $\mathcal{N} \subset \mathcal{M}$ are $\mathcal{O}_X$-modules such that $\mathcal{N}$ and $\mathcal{M}/\mathcal{N}$ are $\mathcal{J}$-torsion sheaves then $\mathcal{M}$ is a $\mathcal{J}$-torsion sheaf too; and hence $\mathcal{A}_{\mathcal{J}}(X)$ is plump in $\mathcal{A}(X)$.[6] In this case, the stalk of $\varGamma_{\mathcal{J}}\mathcal{M}$ at $x \in X$ is

$$(\varGamma_{\mathcal{J}}\mathcal{M})_x = \varinjlim_{n>0} \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathcal{J}_x^n,\, \mathcal{M}_x).$$

Let X be a locally noetherian scheme and $Z \subset X$ a closed subset, the support of $\mathcal{O}_X/\mathcal{J}$ for some quasi-coherent $\mathcal{O}_X$-ideal $\mathcal{J}$. The functor $\varGamma'_Z := \varGamma_{\mathcal{J}}$ does not depend on the quasi-coherent ideal $\mathcal{J}$ determining Z. It is a subfunctor of the left-exact functor $\varGamma_Z$ which associates to each $\mathcal{O}_X$-module $\mathcal{M}$ its subsheaf of sections supported in Z. If $\mathcal{M}$ is quasi-coherent, then $\varGamma'_Z(\mathcal{M}) = \varGamma_Z(\mathcal{M})$. (And, for any complex $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(X)$, the derived-category map $\mathbf{R}\varGamma'_Z\mathcal{E} \to \mathbf{R}\varGamma_Z\mathcal{E}$ induced by the inclusion $\varGamma'_Z \hookrightarrow \varGamma_Z$ is an isomorphism [AJL, p. 25, Corollary (3.2.4)]—so for such $\mathcal{E}$ we will usually identify $\mathbf{R}\varGamma'_Z\mathcal{E}$ and $\mathbf{R}\varGamma_Z\mathcal{E}$.) Let $\mathcal{A}_Z(X)$ $(= \mathcal{A}_{\mathcal{J}}(X))$ be the plump subcategory of $\mathcal{A}(X)$ whose objects are the *Z-torsion sheaves,* i.e., $\mathcal{O}_X$-modules $\mathcal{M}$ such that $\varGamma'_Z\mathcal{M} = \mathcal{M}$; and $\mathcal{A}_{\mathrm{qc}Z}(X) := \mathcal{A}_{\mathrm{qc}}(X) \cap \mathcal{A}_Z(X)$ the plump subcategory of $\mathcal{A}(X)$ whose objects are the quasi-coherent $\mathcal{O}_X$-modules supported in Z.

For a locally noetherian formal scheme $\mathfrak{X}$ with ideal of definition $\mathscr{J}$, set $\varGamma'_{\mathfrak{X}} := \varGamma_{\mathscr{J}}$, a left-exact functor which depends only on the sheaf of topological rings $\mathcal{O}_{\mathfrak{X}}$, not on the choice of $\mathscr{J}$, because locally any ideal of definition contains a power of any other [GD, p.409, (10.5.1)]—in fact $\varGamma'_{\mathfrak{X}}\mathcal{M}$ is the $\mathcal{O}_{\mathfrak{X}}$-submodule of $\mathcal{M}$ whose sections are those of $\mathcal{M}$ which are annihilated locally by an open ideal. Say that $\mathcal{M}$ is a *torsion sheaf* if $\varGamma'_{\mathfrak{X}}\mathcal{M} = \mathcal{M}$. Denote by $\mathcal{A}_{\mathrm{t}}(\mathfrak{X})$ $(= \mathcal{A}_{\mathcal{J}}(\mathfrak{X}))$ the plump subcategory of $\mathcal{A}(\mathfrak{X})$ whose objects are the torsion sheaves; and by $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) := \mathcal{A}_{\mathrm{qc}}(\mathfrak{X}) \cap \mathcal{A}_{\mathrm{t}}(\mathfrak{X})$ the full (in fact plump, see Corollary 5.1.3) subcategory of $\mathcal{A}(\mathfrak{X})$ whose objects are the quasi-coherent torsion sheaves. It holds that $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) \subset \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$, see Corollary 5.1.4. If $\mathfrak{X}$ is an ordinary locally noetherian scheme (i.e., $\mathscr{J} = 0$), then $\mathcal{A}_{\mathrm{t}}(\mathfrak{X}) = \mathcal{A}(\mathfrak{X})$ and $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) = \mathcal{A}_{\mathrm{qc}}(\mathfrak{X}) = \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$.

1.2.2. For any map $f \colon \mathfrak{X} \to \mathfrak{Y}$ of locally noetherian formal schemes there exist ideals of definition $\mathscr{I} \subset \mathcal{O}_{\mathfrak{Y}}$ and $\mathscr{J} \subset \mathcal{O}_{\mathfrak{X}}$ such that $\mathscr{I}\mathcal{O}_{\mathfrak{X}} \subset \mathscr{J}$ [GD, p. 416,

[6] Thus the subcategory $\mathcal{A}_{\mathcal{J}}(X)$ is a *hereditary torsion class* in $\mathcal{A}(X)$, in the sense of Dickson, see [St, pp. 139–141].

(10.6.10)]; and correspondingly there is a map of ordinary schemes (= formal schemes having (0) as ideal of definition) $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{J}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}/\mathcal{I})$ [GD, p. 410, (10.5.6)]. We say that f is *separated* (resp. *affine*, resp. *pseudo-proper*, resp. *pseudo-finite*) if for some—and hence any—such $\mathcal{I}, \mathcal{J}$ the corresponding scheme-map is separated (resp. affine, resp. proper, resp. finite), see [GD, §§10.15–10.16, p. 444 *ff.*] (keeping in mind [GD, p. 416, (10.6.10)(ii)]). Any affine or pseudo-proper map is separated. The map f is pseudo-finite $\Leftrightarrow$ it is pseudo-proper and affine $\Leftrightarrow$ it is pseudo-proper and has finite fibers [EGA, p. 136, (4.4.2)].

We say that f is *adic* if for some—and hence any—ideal of definition $\mathcal{I} \subset \mathcal{O}_{\mathcal{Y}}$, $\mathcal{I}\mathcal{O}_{\mathcal{X}}$ is an ideal of definition of $\mathcal{X}$ [GD, p. 436, (10.12.1)]. We say that f is *proper* (resp. *finite*) if f is pseudo-proper (resp. pseudo-finite) and adic [EGA, p. 119, (3.4.1)], [EGA, p. 148, (4.8.11)].

1.2.3. Here is our **second main result**, Torsion Duality for formal schemes. (See Theorem 6.1 and Corollary 6.1.4 for more elaborate statements.)

Theorem 2. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a separated map of noetherian formal schemes.*

(a) *The restriction of $\mathbf{R}f_*$ takes $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$ to $\mathbf{D}_{\mathrm{qct}}(\mathcal{Y})$, and it has a right Δ-adjoint*

$$f_{\mathrm{t}}^{\times}\colon \mathbf{D}(\mathcal{Y}) \to \mathbf{D}_{\mathrm{qct}}(\mathcal{X}).$$

(b) *There exists a Δ-functor $f^{\#}\colon \mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X})$ and a bifunctorial isomorphism*

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathcal{G}, f^{\#}\mathcal{F}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(\mathcal{Y})}(\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{G}, \mathcal{F}) \qquad \big(\mathcal{G} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X}),\ \mathcal{F} \in \mathbf{D}(\mathcal{Y})\big).$$

Remarks 1.2.4. (1) The functors $f^{\#}$ and $f_{\mathrm{t}}^{\times}$ are connected thus:

$$f^{\#} = \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{O}_{\mathcal{X}}, f_{\mathrm{t}}^{\times}), \qquad f_{\mathrm{t}}^{\times} = \mathbf{R}\varGamma'_{\mathcal{X}} f^{\#}.$$

(See Corollaries 6.1.4 and 6.1.5(a)).

(2) The proof of Theorem 2 is similar to that of Theorem 1, at least when the formal scheme $\mathcal{X}$ is separated (i.e., the unique formal-scheme map $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is separated), in which case the natural functor $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathcal{X})) \to \mathbf{D}_{\mathrm{qct}}(\mathcal{X})$ is an *equivalence of categories* (Proposition 5.3.1). (As mentioned before, we know the corresponding result with "$\vec{\mathrm{c}}$" in place of "qct" only for *properly algebraic* formal schemes.) In addition, replacing separatedness of $\mathcal{X}$ by separatedness of f takes a technical pasting argument.

(3) For an ordinary scheme X (having (0) as ideal of definition), $\varGamma'_X$ is just the identity functor of $\mathcal{A}(X)$, and $\mathbf{D}_{\mathrm{qct}}(X) = \mathbf{D}_{\mathrm{qc}}(X)$. In this case, Theorems 1 and 2 both reduce to the usual global (non-sheafified) version of Grothendieck Duality. In §2 we will describe how Theorem 2 generalizes and ties together various strands in the literature on local, formal, and global duality. In particular, the behavior of Theorem 2 vis-à-vis variable f gives compatibility between local and global duality, at least on an abstract level—i.e., without the involvement of differentials, residues, etc. (See Corollary 6.1.6.)

1.3. A SHEAFIFICATION OF THEOREM 2 WILL BE IN THE FINAL VERSION OF THIS PAPER.

In closing this introductory section, we wish to express our appreciation for illuminating conversations with Amnon Neeman and Amnon Yekutieli.

2. Applications and examples.

It was previously noted that Theorem 2 generalizes global Grothendieck Duality for noetherian schemes. This section illustrates further how Theorem 2 provides a common home for a number of different duality-related results (local duality, formal duality, residue theorems, ...). For a quick example, see Remark 2.3.7.

In §2.1 we review several forms of local duality. In §2.2 we sheafify these results, and put them into the context of Theorem 2. In particular, Proposition 2.1.6 is an abstract version of the Local Duality theorem of [HüK, p. 73, Theorem 3.4]; and Theorem 2.2.3 (Pseudo-finite Duality) globalizes it to formal schemes.

Paragraph 2.3 relates Theorems 1 and 2 to the central "Residue Theorems" in [L1] and [HüS] (but does not subsume those results).

Paragraph 2.4 indicates how the Formal Duality theorem of [H2, p. 48 ; Proposition (5.2)]; and the Local-Global Duality theorem in [L3, p. 188] can be deduced from Theorem 2.

2.1. (Local Duality.) All rings will be assumed commutative, unless otherwise indicated.

Let $\varphi\colon R\to S$ be a homomorphism of (commutative) rings, with S noetherian, let J be an S-ideal, and let Γ_J be the functor taking any S-module to its submodule of elements which are annihilated by some power of J. Let E and E' be complexes in $\mathbf{D}(S)$, the derived category of S-modules, and let $F\in\mathbf{D}(R)$. With $\underset{=}{\otimes}$ denoting derived tensor product in $\mathbf{D}(S)$ (defined via K-flat resolutions, [Sp, p. 147, Proposition 6.5]), there is a natural isomorphism $E\underset{=}{\otimes}\mathbf{R}\Gamma_J E'\xrightarrow{\,\sim\,}\mathbf{R}\Gamma_J(E\underset{=}{\otimes}E')$, see e.g., [AJL, p. 20, Corollary(3.1.2)]. Also, viewing $\mathbf{R}\mathrm{Hom}^\bullet_R(E',F)$ as a functor from $\mathbf{D}(S)^{\mathrm{op}}\times\mathbf{D}(R)$ to $\mathbf{D}(S)$, one has a canonical $\mathbf{D}(S)$-isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet_R(E\underset{=}{\otimes}E',F)\xrightarrow{\,\sim\,}\mathbf{R}\mathrm{Hom}^\bullet_S\big(E,\mathbf{R}\mathrm{Hom}^\bullet_R(E',F)\big),$$

see [Sp, p. 147; 6.6]. Thus, with $\varphi^{\#}_J\colon\mathbf{D}(R)\to\mathbf{D}(S)$ the functor given by

$$\varphi^{\#}_J(-):=\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_J S,-)\cong\mathbf{R}\mathrm{Hom}^\bullet_S\big(\mathbf{R}\Gamma_J S,\mathbf{R}\mathrm{Hom}^\bullet_R(S,-)\big),$$

there is a composed isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet_S(E,\varphi^{\#}_J F)\xrightarrow{\,\sim\,}\mathbf{R}\mathrm{Hom}^\bullet_R(E\underset{=}{\otimes}\mathbf{R}\Gamma_J S,F)\xrightarrow{\,\sim\,}\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_J E,F).$$

Application of homology H^0 yields the (rather trivial) *local duality isomorphism*

$$\mathrm{Hom}_{\mathbf{D}(S)}(E,\varphi^{\#}_J F)\xrightarrow{\,\sim\,}\mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\Gamma_J E,F). \tag{2.1.1}$$

"Non-trivial" versions of (2.1.1) include more information about $\varphi^{\#}_J$. For example, Greenlees-May duality [AJL, p. 4, $(0.3)_{\mathrm{aff}}$] gives a canonical isomorphism

$$\varphi^{\#}_J F\cong\mathbf{L}\Lambda_J\mathbf{R}\mathrm{Hom}^\bullet_R(S,F), \tag{2.1.2}$$

where Λ_J is the *J-adic completion functor,* and $\mathbf{L}$ denotes "left-derived." In particular, if R is noetherian, S is a finite R-module, and $F\in\mathbf{D}_{\mathrm{c}}(R)$ (i.e., each homology module of F is finitely generated), then as in [AJL, p. 6, Proposition (0.4.1)],

$$\varphi^{\#}_J F=\mathbf{R}\mathrm{Hom}^\bullet_R(S,F)\otimes_S\hat S\qquad(\hat S=J\text{-adic completion of }S). \tag{2.1.3}$$

More particularly, for $S=R$ and $\varphi=\mathrm{id}$ (the identity map) we get

$$\mathrm{id}^{\#}_J F=F\otimes_R\hat R\qquad\big(F\in\mathbf{D}_{\mathrm{c}}(R)\big).$$

Hence, *classical local duality* [H1, p. 278 (modulo Matlis dualization)] is just (2.1.1) when R is local, $\varphi = \mathrm{id}$, J is the maximal ideal of R, and F is a normalized dualizing complex—so that, as in Corollary 5.2.3, and by [H1, p. 276, Proposition 6.1],

$$\mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\Gamma_J E, F) = \mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\Gamma_J E, \mathbf{R}\Gamma_J F) = \mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\Gamma_J E, I)$$

where I is an R-injective hull of the residue field R/J.

For another example, let $S = R[[\mathbf{t}]]$ where $\mathbf{t} := (t_1, \dots, t_d)$ is a sequence of variables, and set $J := \mathbf{t}S$. The standard calculation (via Koszul complexes) gives an isomorphism $\mathbf{R}\Gamma_J S \cong \nu[-d]$ where ν is the free R-submodule of the localization $S_{t_1 \dots t_d}$ generated by those monomials $t_1^{n_1} \dots t_d^{n_d}$ with all exponents $n_i < 0$, the S-module structure being induced by that of $S_{t_1 \dots t_d}/S \supset \nu$. The *relative canonical module* $\omega_{R[[\mathbf{t}]]/R} := \mathrm{Hom}_R(\nu, R)$ is a *free, rank one, S-module.* There result, for finitely-generated R-modules F, functorial isomorphisms

$$\varphi^{\#}_{\mathbf{t}R[[\mathbf{t}]]} F \cong \mathrm{Hom}_R(\nu[-d], F) \cong \omega_{R[[\mathbf{t}]]/R}[d] \otimes_R F \cong R[[\mathbf{t}]] \otimes_R F[d]; \tag{2.1.4}$$

and when R is noetherian, the usual way-out argument [H1, p. 69, (ii)] yields the same for any $F \in \mathbf{D}^+_{\mathrm{c}}(R)$.

Next, we give a commutative-algebra analogue of Theorem 2 in §1, in the form of a "torsion" variant of the duality isomorphism (2.1.1). Proposition 2.2.1 will clarify the relation between the algebraic and formal-scheme contexts.

With $\varphi \colon R \to S$ and J an S-ideal as before, let $\mathcal{A}_J(S)$ be the category of J-torsion S-modules, i.e., S-modules M such that

$$M = \Gamma_J M := \{\, m \in M \mid J^n m = 0 \text{ for some } n > 0 \,\}.$$

The derived category of $\mathcal{A}_J(S)$ is equivalent to the full subcategory $\mathbf{D}_J(S)$ of $\mathbf{D}(S)$ with objects those S-complexes E whose homology lies in $\mathcal{A}_J(S)$, (or equivalently, such that the natural map $\mathbf{R}\Gamma_J E \to E$ is an isomorphism), and the functor $\mathbf{R}\Gamma_J$ is right-adjoint to the inclusion $\mathbf{D}_J(S) \hookrightarrow \mathbf{D}(S)$ (cf. Proposition 5.2.1 and its proof). Hence the functor $\varphi^{\times}_J \colon \mathbf{D}(R) \to \mathbf{D}_J(S)$ defined by

$$\varphi^{\times}_J(-) := \mathbf{R}\Gamma_J \mathbf{R}\mathrm{Hom}^{\bullet}_R(S, -) \cong \mathbf{R}\Gamma_J S \underset{=}{\otimes} \mathbf{R}\mathrm{Hom}^{\bullet}_R(S, -)$$

is right-adjoint to the natural composition $\mathbf{D}_J(S) \hookrightarrow \mathbf{D}(S) \to \mathbf{D}(R)$: in fact, for $E \in \mathbf{D}_J(S)$ and $F \in \mathbf{D}(R)$ there are natural isomorphisms

$$\mathbf{R}\mathrm{Hom}^{\bullet}_S(E, \varphi^{\times}_J F) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_S(E, \mathbf{R}\mathrm{Hom}^{\bullet}_R(S, F)) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_R(E, F). \tag{2.1.5}$$

Here is another interpretation of $\varphi^{\times}_J F$. For S-modules A and R-modules B set

$$\mathrm{Hom}_{R,J}(A, B) := \Gamma_J \mathrm{Hom}_R(A, B),$$

the S-module of R-homomorphisms α vanishing on $J^n A$ for some n (depending on α), i.e., *continuous* when A is J-adically topologized and B is discrete. If E is a K-flat S-complex and F is a K-injective R-complex, then $\mathrm{Hom}^{\bullet}_R(E, F)$ is a K-injective S-complex; and it follows for all $E \in \mathbf{D}(S)$ and $F \in \mathbf{D}(R)$ that

$$\mathbf{R}\mathrm{Hom}^{\bullet}_{R,J}(E, F) \cong \mathbf{R}\Gamma_J \mathbf{R}\mathrm{Hom}^{\bullet}_R(E, F).$$

Thus,

$$\varphi^{\times}_J F = \mathbf{R}\mathrm{Hom}^{\bullet}_{R,J}(S, F).$$

A *torsion version of local duality* is the natural isomorphism, derived from (2.1.5),

$$\mathrm{Hom}_{\mathbf{D}_J(S)}\big(E, \mathbf{R}\mathrm{Hom}^{\bullet}_{R,J}(S, F)\big) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(R)}(E, F) \qquad \big(E \in \mathbf{D}_J(S),\ F \in \mathbf{D}(R)\big).$$

Apropos of Remark 1.2.4(1), the functors $\varphi_J^\times$ and $\varphi_J^\#$ are related by

$$\mathbf{L}\Lambda_J\mathbf{R}\mathrm{Hom}^\bullet_R(S,F)\underset{(2.1.2)}{\cong}\varphi_J^\# F\cong\mathbf{L}\Lambda_J\varphi_J^\times F,$$

$$\mathbf{R}\Gamma_J\mathbf{R}\mathrm{Hom}^\bullet_R(S,F)\;=\;\varphi_J^\times F\cong\mathbf{R}\Gamma_J\varphi_J^\# F.$$

The first relation is the case $E=\mathbf{R}\Gamma_J S$ of (2.1.5), followed by Greenlees-May duality. The second results, e.g., from the sequence of natural isomorphisms, holding for $G\in\mathbf{D}_J(S)$, $E\in\mathbf{D}(S)$, and $F\in\mathbf{D}(R)$:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(S)}\big(G,\mathbf{R}\Gamma_J\mathbf{R}\mathrm{Hom}^\bullet_R(E,F)\big)&\cong\mathrm{Hom}_{\mathbf{D}(S)}\big(G,\mathbf{R}\mathrm{Hom}^\bullet_R(E,F)\big)\\&\cong\mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\Gamma_J S\underline{\underline{\otimes}}_S G\underline{\underline{\otimes}}_S E,F)\\&\cong\mathrm{Hom}_{\mathbf{D}(S)}\big(G,\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_J E,F)\big)\\&\cong\mathrm{Hom}_{\mathbf{D}(S)}\big(G,\mathbf{R}\Gamma_J\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_J E,F)\big),\end{aligned}$$

which entail that the natural map is an isomorphism

$$\mathbf{R}\Gamma_J\mathbf{R}\mathrm{Hom}^\bullet_R(E,F)\;\xrightarrow{\sim}\;\mathbf{R}\Gamma_J\mathbf{R}\mathrm{Hom}^\bullet_R(\mathbf{R}\Gamma_J E,F).$$

Local Duality theorems are often formulated, as in (c) of the following, in terms of modules and local cohomology ($H_J^\bullet := H^\bullet\mathbf{R}\Gamma_J$) rather than derived categories.

Proposition 2.1.6. *Let $\varphi: R\to S$ be a homomorphism of noetherian rings, let J be an S-ideal, and suppose that there exists a sequence $\mathbf{u}=(u_1,\dots,u_d)$ in J such that $S/\mathbf{u}S$ is R-finite. Then for any R-finite module F :*

(a) *$H^n\varphi_J^\# F=0$ for all $n<-d$, so that there is a natural $\mathbf{D}(S)$-map*

$$h\colon (H^{-d}\varphi_J^\# F)[d]\to\varphi_J^\# F.$$

(b) *If $\tau_F\colon \mathbf{R}\Gamma_J\varphi_J^\# F\to F$ corresponds in (2.1.1) to the identity map of $\varphi_J^\# F$,*[7] *and $\int=\int^d_{\varphi,J}(F)$ is the composed map*

$$\mathbf{R}\Gamma_J(H^{-d}\varphi_J^\# F)[d]\xrightarrow{\mathbf{R}\Gamma_J(h)}\mathbf{R}\Gamma_J\varphi_J^\# F\xrightarrow{\tau_F}F,$$

then $\big(H^{-d}\varphi_J^\# F,\int\big)$ represents the functor $\mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\Gamma_J E[d],F)$ of S-modules E.

(c) *If $J\subset\sqrt{\mathbf{u}S}$ then there is a bifunctorial isomorphism (with E, F as before):*

$$\mathrm{Hom}_S(E,H^{-d}\varphi_J^\# F)\;\xrightarrow{\sim}\;\mathrm{Hom}_R(H_J^d E,F).$$

Proof. If $\hat\varphi$ is the obvious map from R to the $\mathbf{u}$-adic completion $\hat S$ of S, then in $\mathbf{D}(R)$, $\varphi_J^\# F=\hat\varphi_J^\# F$ since $\mathbf{R}\Gamma_J S=\mathbf{R}\Gamma_J\hat S$. We may therefore assume in proving (a) that S is $\mathbf{u}$-adically complete, so that φ factors as $R\xrightarrow{\psi}R[[\mathbf{t}]]\xrightarrow{\chi}S$ with $\mathbf{t}=(t_1,\dots,t_d)$ a sequence of indeterminates and S finite over $R[[\mathbf{t}]]$. (ψ is the natural map, and $\chi(t_i)=u_i$.) In view of the easily verified transitivity relation $\varphi_J^\#=\chi_J^\#\circ\psi^\#_{\mathbf{t}R[[\mathbf{t}]]}$, (2.1.3) and (2.1.4) yield (a). Then (b) results from the natural isomorphisms

$$\mathrm{Hom}_S(E,H^{-d}\varphi_J^\# F)\underset{\text{via }h}{\xrightarrow{\sim}}\mathrm{Hom}_{\mathbf{D}(S)}(E[d],\varphi_J^\# F)\underset{(2.1.1)}{\xrightarrow{\sim}}\mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\Gamma_J E[d],F).$$

Finally, (c) follows from (b) because $H^i_J E=H^i_{\mathbf{u}S}E=0$ for all $i>d$ (as one sees from the usual calculation of $H^i_{\mathbf{u}S}E$ via Koszul complexes), so that the natural map is an isomorphism $\mathrm{Hom}_{\mathbf{D}(R)}(\mathbf{R}\Gamma_J E[d],F)\xrightarrow{\sim}\mathrm{Hom}_R(H^d_J E,F)$. □

[7] τ_F may be thought of as "evaluation at 1" : $\mathbf{R}\mathrm{Hom}^\bullet_{R,J}(S,F)\to F$.

2.1.7. Proposition 2.1.6 gives, in particular, that the functor $\mathrm{Hom}_R(H^d_J E, R)$ of S-modules E is representable. Under suitable conditions (for example, $\hat{S}$ a generic local complete intersection over $R[[\mathbf{t}]]$, and $J \subset \sqrt{\mathbf{u}S}\,$), Hübl and Kunz represent this functor by a *canonical* pair described explicitly via differential forms, residues, and certain trace maps [HüK, p. 73, Theorem 3.4]. For example, with $S = R[[\mathbf{t}]]$, $J = \mathbf{t}S$, and ν as in (2.1.4), the S-homomorphism from the module $\widehat{\Omega}^d_{S/R}$ of universally finite d-forms to the relative canonical module $\omega_{R[[\mathbf{t}]]/R} = \mathrm{Hom}_R(\nu, R)$ sending the form $dt_1 \dots dt_d$ to the R-homomorphism $\nu \to R$ which takes the monomial $t_1^{-1} \dots t_d^{-1}$ to 1 and all other monomials $t_1^{n_1} \dots t_d^{n_d}$ to 0, is clearly an isomorphism; and the resulting isomorphism $\widehat{\Omega}^d_{S/R}[d] \xrightarrow{\ \sim\ } \varphi_J^{\#}R$ *does not depend on the d-element sequence* $\mathbf{t}$ *generating* J—it corresponds under (2.1.1) to the *residue map*

$$\mathbf{R}\Gamma_J \widehat{\Omega}^d_{S/R}[d] = H^d_J \widehat{\Omega}^d_{S/R} \to R$$

(see, e.g., [L2, §2.7]). Thus $\mathrm{Hom}_R(H^d_J E, R)$ *is represented by* $\widehat{\Omega}^d_{S/R}$ *together with the residue map.* The general case reduces to this one via traces of differential forms.

2.2. (Formal sheafification of Local Duality). A slight variant of Theorem 2 in §1 is that *for any separated map* $f\colon \mathfrak{X} \to \mathfrak{Y}$ *of noetherian formal schemes, the functors* $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qct}}(\mathfrak{X}) \to \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y})$ *and* $\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathfrak{X}}\colon \mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{X}) \to \mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{Y})$ *have right* Δ*-adjoints.*

In brief, $\mathbf{R}\varGamma'_{\mathfrak{X}}$ takes $\mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{X})$ to $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ (Corollary 3.1.5 and Proposition 5.2.1), $\mathbf{R}f_*$ takes $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ to $\mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}) \subset \mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{Y})$ (Proposition 5.2.5 and Corollary 3.1.7), and the right adjoints $f_{\mathrm{t}}^{\times}$ (resp. $\mathbf{R}Q_{\mathfrak{X}}f^{\#} := \mathbf{R}Q_{\mathfrak{X}}\mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{O}_{\mathfrak{X}}, f_{\mathrm{t}}^{\times})$) are established by Theorem 6.1 (resp. by Corollary 6.1.4 and Proposition 3.2.3.)

For *affine* f, these results are closely related to the Local Duality isomorphisms (2.1.5) and (2.1.1). Recall that an *adic ring* is a pair (R, I) with R a ring and I an R-ideal such that with respect to the I-adic topology R is Hausdorff and complete. The topology on R having been specified, the corresponding affine formal scheme is then denoted by $\mathrm{Spf}(R)$.

Proposition 2.2.1. *Let* $\varphi\colon (R, I) \to (S, J)$ *be a continuous homomorphism of noetherian adic rings, and let* $\mathfrak{X} := \mathrm{Spf}(S) \xrightarrow{f} \mathrm{Spf}(R) =: \mathfrak{Y}$ *be the corresponding (affine) formal-scheme map. Let* $\kappa_{\mathfrak{X}}\colon \mathfrak{X} \to X := \mathrm{Spec}(S)$, $\kappa_{\mathfrak{Y}}\colon \mathfrak{Y} \to Y := \mathrm{Spec}(R)$ *be the completion maps, and let* $^{\sim} = {}^{\sim_S}$ *denote the standard exact functor from* S*-modules to quasi-coherent* $\mathcal{O}_X$*-modules. Then:*

(a) *The restriction of* $\mathbf{R}f_*$ *takes* $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ *to* $\mathbf{D}_{\mathrm{qct}}(\mathfrak{Y})$, *and this restricted functor has a right adjoint* $f_{\mathrm{t}}^{\times}\colon \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}) \to \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ *given by*

$$f_{\mathrm{t}}^{\times}\mathcal{F} := \kappa_{\mathfrak{X}}^{*}\big(\varphi_J^{\times}\mathbf{R}\Gamma(\mathfrak{Y}, \mathcal{F})\big)^{\sim} = \kappa_{\mathfrak{X}}^{*}\big(\mathbf{R}\mathrm{Hom}^{\bullet}_{R,J}\big(S, \mathbf{R}\Gamma(\mathfrak{Y}, \mathcal{F})\big)\big)^{\sim} \qquad \big(\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y})\big).$$

(b) *The restriction of* $\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathfrak{X}}$ *takes* $\mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{X})$ *to* $\mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{Y})$, *and this restricted functor has a right adjoint* $f_{\vec{\mathrm{c}}}^{\#}\colon \mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{Y}) \to \mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{X})$ *given by*

$$f_{\vec{\mathrm{c}}}^{\#}\mathcal{F} := \kappa_{\mathfrak{X}}^{*}\big(\varphi_J^{\#}\mathbf{R}\Gamma(\mathfrak{Y}, \mathcal{F})\big)^{\sim} = \kappa_{\mathfrak{X}}^{*}\big(\mathbf{R}\mathrm{Hom}^{\bullet}_{R}\big(\mathbf{R}\Gamma_J S, \mathbf{R}\Gamma(\mathfrak{Y}, \mathcal{F})\big)\big)^{\sim} \qquad \big(\mathcal{F} \in \mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{Y})\big).$$

(c) *There are natural isomorphisms*

$$\mathbf{R}\Gamma(\mathfrak{X}, f_{\mathrm{t}}^{\times}\mathcal{F}) \xrightarrow{\ \sim\ } \varphi_J^{\times}\mathbf{R}\Gamma(\mathfrak{Y}, \mathcal{F}) \qquad \big(\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y})\big),$$
$$\mathbf{R}\Gamma(\mathfrak{X}, f_{\vec{\mathrm{c}}}^{\#}\mathcal{F}) \xrightarrow{\ \sim\ } \varphi_J^{\#}\mathbf{R}\Gamma(\mathfrak{Y}, \mathcal{F}) \qquad \big(\mathcal{F} \in \mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{Y})\big).$$

Proof. The functor $^\sim$ induces an equivalence of categories $\mathbf{D}(S) \to \mathbf{D}_{\mathrm{qc}}(X)$, with quasi-inverse $\mathbf{R}\Gamma_X := \mathbf{R}\Gamma(X,-)$ ([BN, p. 225, Thm. 5.1], [AJL, p. 12, Prop. (1.3)]); and Proposition 3.3.1 below implies that $\kappa_{\mathcal{X}}^*\colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{X})$ is an equivalence, with quasi-inverse $(\mathbf{R}\Gamma_X\kappa_{\mathcal{X}*}-)^\sim = (\mathbf{R}\Gamma_{\mathcal{X}}-)^\sim$.[8] It follows that *the functor taking* $G \in \mathbf{D}(S)$ *to* $\kappa_{\mathcal{X}}^*\widetilde{G}$ *is an equivalence, with quasi-inverse* $\mathbf{R}\Gamma_{\mathcal{X}}\colon \mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{X}) \to \mathbf{D}(S)$, and similarly for $\mathcal{Y}$ and R. Moreover, there is an induced equivalence between $\mathbf{D}_J(S)$ and $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$ (see Proposition 5.2.4). In particular, (c) follows from (a) and (b).

Corresponding to (2.1.5) and (2.1.1) there are then functorial isomorphisms

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathcal{E}, f_{\mathrm{t}}^{\times}\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathcal{Y})}(\kappa_{\mathcal{Y}}^*(\mathbf{R}\Gamma_{\mathcal{X}}\mathcal{E})^{\sim_R}, \mathcal{F}) \qquad \big(\mathcal{E} \in \mathbf{D}_{\mathrm{qct}}(\mathcal{X}),\ \mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathcal{Y})\big),$$
$$\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathcal{E}, f_{\vec{\mathrm{c}}}^{\#}\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathcal{Y})}(\kappa_{\mathcal{Y}}^*(\mathbf{R}\Gamma_J\mathbf{R}\Gamma_{\mathcal{X}}\mathcal{E})^{\sim_R}, \mathcal{F}) \qquad \big(\mathcal{E} \in \mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{X}),\ \mathcal{F} \in \mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{Y})\big);$$

and it remains to demonstrate functorial isomorphisms

$$\kappa_{\mathcal{Y}}^*(\mathbf{R}\Gamma_{\mathcal{X}}\mathcal{E})^{\sim_R} \xrightarrow{\sim} \mathbf{R}f_*\mathcal{E} \qquad \big(\mathcal{E} \in \mathbf{D}_{\mathrm{qct}}(\mathcal{X})\big),$$
$$\kappa_{\mathcal{Y}}^*(\mathbf{R}\Gamma_J\mathbf{R}\Gamma_{\mathcal{X}}\mathcal{E})^{\sim_R} \xrightarrow{\sim} \mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}\mathcal{E} \qquad \big(\mathcal{E} \in \mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{X})\big),$$

of which the first is a special case of the second. To prove the second, let $E := \mathbf{R}\Gamma_{\mathcal{X}}\mathcal{E}$, let $Z := \mathrm{Spec}(S/J) \subset X$, and let $f_0\colon X \to Y$ be the scheme-map corresponding to φ. The desired isomorphism comes from the sequence of natural isomorphisms

$$\begin{aligned}
\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}\mathcal{E} &\cong \mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}\kappa_{\mathcal{X}}^*\widetilde{E} \\
&\cong \mathbf{R}f_*\kappa_{\mathcal{X}}^*\mathbf{R}\mathit{\Gamma}_Z\widetilde{E} && \text{(Proposition 5.2.4(b))} \\
&\cong \kappa_{\mathcal{Y}}^*\mathbf{R}f_{0*}\mathbf{R}\mathit{\Gamma}_Z\widetilde{E} && \text{(Corollary 5.2.6)} \\
&\cong \kappa_{\mathcal{Y}}^*\mathbf{R}f_{0*}(\mathbf{R}\Gamma_J E)^\sim && \text{([AJL, p. 9, (0.4.5)])} \\
&\cong \kappa_{\mathcal{Y}}^*(\mathbf{R}\Gamma_J E)^{\sim_R}.
\end{aligned}$$

(The last isomorphism—well-known for bounded-below E—can be checked via the equivalences $\mathbf{R}\Gamma_X$ and $\mathbf{R}\Gamma_Y$, which satisfy $\mathbf{R}\Gamma_Y\mathbf{R}f_{0*} \cong \mathbf{R}\Gamma_X$ (see [Sp, pp. 142–143, 5.15(b) and 5.17]). □

Theorem 2.2.3 below globalizes Proposition 2.1.6. But first some preparatory remarks are needed. Recall from 1.2.2 that a map $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian formal schemes is *pseudo-finite* if it is pseudo-proper and has finite fibers, or equivalently, if f is pseudo-proper and *affine.* Such an f corresponds locally to a homomorphism $\varphi\colon (R, I) \to (S, J)$ of noetherian adic rings such that $\varphi(I) \subset J$ and S/J is a finite R-module. This φ can be extended to a homomorphism from a power series ring $R[[\mathbf{t}]] := R[[t_1, t_2, \dots, t_e]]$ such that the images of the variables t_i together with $\varphi(I)$ generate J, and thereby S becomes a finite $R[[\mathbf{t}]]$-module. Pseudo-finiteness is preserved under arbitrary (noetherian) base change.

We say that a pseudo-finite map $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian formal schemes has relative dimension $\le d$ if each $y \in \mathcal{Y}$ has an affine neighborhood $\mathcal{U}$ such that the map $\varphi_{\mathcal{U}}\colon R \to S$ of adic rings corresponding to $f^{-1}\mathcal{U} \to \mathcal{U}$ has a continuous extension $R[[t_1, \dots, t_d]] \to S$ making S into a finite $R[[t_1, \dots, t_d]]$-module, or equivalently, there is a topologically nilpotent sequence $\mathbf{u} = (u_1, \dots, u_d)$ in S (i.e., $\lim_{n\to\infty} u_i^n = 0$ $(1 \le i \le d)$) such that $S/\mathbf{u}S$ is finitely generated as an R-module. The *relative dimension* $\dim f$ is defined to be the least among the integers d such that f has relative dimension $\le d$.

[8] In checking this note that $\kappa_{\mathcal{X}*}$ has an exact left adjoint, hence preserves K-injectivity.

For any separated map $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian formal schemes, we have the functor $f^{\#}\colon \mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X})$ of Corollary 6.1.4, commuting with open base change on $\mathcal{Y}$ (Theorem 1.3).

Lemma 2.2.2. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a pseudo-finite map of noetherian formal schemes. Then for all $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}^{+}(\mathcal{Y})$, it holds that $f^{\#}\mathcal{F} \in \mathbf{D}_{\mathrm{c}}^{+}(\mathcal{X})$.*

Proof. Since $f^{\#}$ commutes with open base change, the question is local, so we may assume that f corresponds to $\varphi\colon (R, I) \to (S, J)$ as above. Moreover, the transitivity isomorphism $(gf)^{\#} \cong f^{\#}g^{\#}$ in Corollary 6.1.4 allows us to assume that *either* $S = R[[t_1, \dots, t_d]]$ and φ is the natural map *or* S is a finite R-module and $J = IS$. In either case f is obtained by completing a proper map $f_0\colon X \to \mathrm{Spec}(R)$ along a closed subscheme $Z \subset f_0^{-1}\mathrm{Spec}(R/I)$. (In the first case, take X to be the projective space $\mathbb{P}_R^d \supset \mathrm{Spec}(R[t_1, \dots, t_d])$, and $Z := \mathrm{Spec}(R[t_1, \dots, t_d]/(I, t_1, \dots, t_d))$.) The conclusion is given then by Corollary 6.2.3. □

Theorem 2.2.3 (Pseudo-finite Duality). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a pseudo-finite map of noetherian formal schemes, and let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathcal{Y}}$-module. Then:*

(a) *$H^n f^{\#}\mathcal{F} = 0$ for all $n < -\dim f$.*

(b) *If $\dim f \le d$ and $\mathcal{X}$ is covered by affine open subsets with d-generated defining ideals, then with $f'_{\mathcal{X}*} := f_*\varGamma'_{\mathcal{X}}$ and, for $i \in \mathbb{Z}$ and $\mathcal{J}$ a defining ideal of $\mathcal{X}$,*

$$R^i f'_{\mathcal{X}*} := H^i\mathbf{R}f'_{\mathcal{X}*} = H^i\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathcal{X}} = \varinjlim_n H^i\mathbf{R}f_*\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{O}_{\mathcal{X}}/\mathcal{J}^n, -),$$ [9]

there is, for quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{E}$, a functorial isomorphism

$$f_*\mathcal{H}om_{\mathcal{X}}(\mathcal{E}, H^{-d}f^{\#}\mathcal{F}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{Y}}(R^d f'_{\mathcal{X}*}\mathcal{E}, \mathcal{F}).$$

(*Here $H^{-d}f^{\#}\mathcal{F}$ is coherent* (*Lemma 2.2.2*), *and vanishes* (*by* (a)) *unless $d = \dim f$.*)

Proof. Since $f^{\#}$ commutes with open base change we may assume that $\mathcal{Y}$ is affine and that f corresponds to a map $\varphi\colon (R, I) \to (S, J)$ as in Proposition 2.1.6. Then there is an isomorphism of functors

$$\mathbf{R}Q_{\mathcal{X}}f^{\#} \cong \kappa_{\mathcal{X}}^{*}\big(\varphi_J^{\#}\mathbf{R}\Gamma(\mathcal{Y}, -)\big)^{\sim},$$

both of these functors being right-adjoint to $\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathcal{X}}\colon \mathbf{D}_{\vec{c}}(\mathcal{X}) \to \mathbf{D}_{\vec{c}}(\mathcal{Y})$ (Proposition 2.2.1(b) and remarks about right adjoints preceding it). Since $f^{\#}\mathcal{F} \in \mathbf{D}_{\mathrm{c}}^{+}(\mathcal{X})$ (Lemma 2.2.2), therefore, by Corollary 3.3.4, the natural map is an isomorphism $\mathbf{R}Q_{\mathcal{X}}f^{\#}\mathcal{F} \xrightarrow{\sim} f^{\#}\mathcal{F}$; and so, since $\kappa_{\mathcal{X}}^{*}$ is exact, Proposition 2.1.6 gives (a).

Next, consider the presheaf map associating to each open $\mathcal{U} \subset \mathcal{Y}$ the natural composition (with $\mathcal{V} := f^{-1}\mathcal{U}$):

$$\begin{aligned}\mathrm{Hom}_{\mathcal{V}}(\mathcal{E}, H^{-d}f^{\#}\mathcal{F}) \underset{\text{by (a)}}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathbf{D}(\mathcal{V})}(\mathcal{E}[d], f^{\#}\mathcal{F}) &\underset{6.1.4}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathbf{D}(\mathcal{U})}(\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E}[d], \mathcal{F})\\ &\longrightarrow \mathrm{Hom}_{\mathcal{U}}(R^d f'_{\mathcal{X}*}\mathcal{E}, \mathcal{F}).\end{aligned}$$

To prove (b) by showing that the resulting sheaf map

$$f_*\mathcal{H}om_{\mathcal{X}}(\mathcal{E}, H^{-d}f^{\#}\mathcal{F}) \to \mathcal{H}om_{\mathcal{Y}}(R^d f'_{\mathcal{X}*}\mathcal{E}, \mathcal{F})$$

is an isomorphism, it suffices to show that $R^i f'_{\mathcal{X}*}\mathcal{E} = 0$ for all $i > d$, a local problem for which we can (and do) assume that f corresponds to $\varphi\colon R \to S$ as above.

---

[9] The equalities hold because $\mathcal{X}$ being noetherian, any $\varinjlim$ of flasque sheaves is f_*-acyclic, and $\varinjlim$ commutes with f_*.

Now $\mathbf{R}\varGamma'_{\mathfrak X}\mathcal E\in\mathbf D_{\mathrm{qct}}(\mathfrak X)$ (Proposition 5.2.1), so Proposition 5.2.4 for $X:=\operatorname{Spec}(S)$ and $Z:=\operatorname{Spec}(S/J)$ gives $\mathbf{R}\varGamma'_{\mathfrak X}\mathcal E\cong\kappa^*_{\mathfrak X}\mathcal E_0$ with $\mathcal E_0:=\kappa_{\mathfrak X*}\mathbf{R}\varGamma'_{\mathfrak X}\mathcal E\in\mathbf D^+_{\mathrm{qc}Z}(X)$. Since $\mathfrak X$ has, locally, a d-generated defining ideal, we can represent $\mathbf{R}\varGamma'_{\mathfrak X}\mathcal E$ locally by a $\varinjlim$ of Koszul complexes on d elements [AJL, p. 18, Lemma 3.1.1], whence $H^i\mathbf{R}\varGamma'_{\mathfrak X}\mathcal E=0$ for all $i>d$, and so, $\kappa_{\mathfrak X*}$ being exact, $H^i\mathcal E_0=0$. Since the map $f_0:=\operatorname{Spec}(\varphi)$ is affine, it follows that $H^i\mathbf{R}f_{0*}\mathcal E_0=0$, whereupon, $\kappa_{\mathfrak Y}$ being flat, Corollary 5.2.6 yields

$$R^if'_{\mathfrak X*}\mathcal E\cong H^i\mathbf{R}f_*\kappa^*_{\mathfrak X}\mathcal E_0\cong H^i\kappa^*_{\mathfrak Y}\mathbf{R}f_{0*}\mathcal E_0\cong\kappa^*_{\mathfrak Y}H^i\mathbf{R}f_{0*}\mathcal E_0=0\qquad(i>d),$$

as desired. (Alternatively, use Lemmas 3.4.2 and 5.1.4.) □

2.3. Our results provide a framework for "Residue Theorems" such as those appearing in [L1, pp. 87–88] and [HüS, pp. 750-752] (central theorems in those papers): roughly speaking, Theorems 1 and 2 in section 1 include both local and global duality, and Corollary 6.1.6 expresses the compatibility between these dualities. But the dualizing objects we deal with are determined only up to isomorphism. The Residue Theorems run deeper in that they include a *canonical realization* of dualizing data, via differential forms. (See the above remarks on the Hübl-Kunz treatment of local duality.) This extra dimension belongs properly to a theory of the "Fundamental Class" of a morphism, a canonical map from relative differential forms to the relative dualizing complex, which will be pursued in a separate paper.

2.3.1. Let us be more explicit, beginning with some remarks about "Grothendieck Duality with supports" for a map $f\colon X\to Y$ of noetherian schemes with respective closed subschemes $W\subset Y$ and $Z\subset f^{-1}W$. There is a natural equivalence of categories $\mathbf D(\mathcal A_{\mathrm{qc}}(X))\to\mathbf D_{\mathrm{qc}}(X)$ (see §3.3); and via this equivalence we regard the functor $f^\times\colon\mathbf D(Y)\to\mathbf D(\mathcal A_{\vec{\mathrm c}}(X))=\mathbf D(\mathcal A_{\mathrm{qc}}(X))$ of Theorem 1 as being right-adjoint to $\mathbf{R}f_*\colon\mathbf D_{\mathrm{qc}}(X)\to\mathbf D(Y)$.[10] The functor $\mathbf{R}\varGamma'_Z$ can be regarded as being right-adjoint to the inclusion $\mathbf D_Z(X)\hookrightarrow\mathbf D(X)$ (cf. Proposition 5.2.1(c)); and its restriction to $\mathbf D_{\mathrm{qc}}(X)$ agrees naturally with that of $\mathbf{R}\varGamma_Z$, both restrictions being right-adjoint to the inclusion $\mathbf D_{\mathrm{qc}Z}(X)\hookrightarrow\mathbf D_{\mathrm{qc}}(X)$. Similar statements hold for $W\subset Y$. Since $\mathbf{R}f_*(\mathbf D_{\mathrm{qc}Z}(X))\subset\mathbf D_W(Y)$,[11] we find that the functors $\mathbf{R}\varGamma_Zf^\times$ and $\mathbf{R}\varGamma_Zf^\times\mathbf{R}\varGamma'_W$ are both right-adjoint to $\mathbf{R}f_*\colon\mathbf D_{\mathrm{qc}Z}(X)\to\mathbf D(Y)$, so are isomorphic.

Define the *local integral* (a generalized residue map, cf. [HüK, §4]):

$$\rho(\mathcal G)\colon\mathbf{R}f_*\mathbf{R}\varGamma_Zf^\times\mathcal G\to\mathbf{R}\varGamma'_W\mathcal G\qquad\big(\mathcal G\in\mathbf D(Y)\big)$$

to be the natural composition

$$\mathbf{R}f_*\mathbf{R}\varGamma_Zf^\times\mathcal G\xrightarrow{\ \sim\ }\mathbf{R}f_*\mathbf{R}\varGamma_Zf^\times\mathbf{R}\varGamma'_W\mathcal G\to\mathbf{R}f_*f^\times\mathbf{R}\varGamma'_W\mathcal G\to\mathbf{R}\varGamma'_W\mathcal G.$$

Noting that for $\mathcal F\in\mathbf D_W(Y)$ there is a canonical isomorphism $\mathbf{R}\varGamma'_W\mathcal F\xrightarrow{\ \sim\ }\mathcal F$ (proof similar to that of Proposition 5.2.1(a)), we have then:

Proposition 2.3.2 (Duality with supports). *For any $\mathcal E\in\mathbf D_{\mathrm{qc}Z}(X)$, $\mathcal F\in\mathbf D_W(Y)$, the natural composition*

$$\begin{aligned}\operatorname{Hom}_{\mathbf D_{\mathrm{qc}Z}(X)}(\mathcal E,\mathbf{R}\varGamma_Zf^\times\mathcal F)&\longrightarrow\operatorname{Hom}_{\mathbf D_W(Y)}(\mathbf{R}f_*\mathcal E,\mathbf{R}f_*\mathbf{R}\varGamma_Zf^\times\mathcal F)\\&\xrightarrow{\rho(\mathcal F)}\operatorname{Hom}_{\mathbf D_W(Y)}(\mathbf{R}f_*\mathcal E,\mathcal F)\end{aligned}$$

[10] For ordinary schemes, this functor $f^\times$ is well-known, and usually denoted $f^!$ when f is proper. When f is an open immersion, the functors $f^\times$ and $f^!(=f^*)$ need not agree.

[11] Argue, for example, as in the proof of Proposition 5.2.5.

is an isomorphism.

This follows from adjointness of $\mathbf{R}f_*$ and $f^\times$, via the natural diagram

$$\begin{array}{ccc} \mathbf{R}f_*\mathbf{R}\varGamma_Z f^\times\mathcal{G} & \longrightarrow & \mathbf{R}f_* f^\times\mathcal{G} \\ {\scriptstyle\rho(\mathcal{G})}\big\downarrow & & \big\downarrow \\ \mathbf{R}\varGamma'_W\mathcal{G} & \longrightarrow & \mathcal{G} \end{array} \qquad \big(\mathcal{G}\in\mathbf{D}(Y)\big),$$

whose (obvious) commutativity is a cheap version of the Residue Theorem of [HüS, pp. 750-752].

Again, however, to be worthy of the name a Residue Theorem should involve *canonical realizations* of dualizing objects. For instance, when V is a proper d-dimensional variety over a field k and $v\in V$ is a closed point, taking $X=V$, $Z=\{v\}$, $W=Y=\mathrm{Spec}(k)$, $\mathcal{G}=k$, and setting $\omega_V := H^{-d}f^\times k$, we get an $\mathcal{O}_{V,v}$-module $\omega_{V,v}$ (commonly called "canonical", though defined only up to isomorphism) together with the k-linear map induced by $\rho(k)$:

$$H^d_v(\omega_{V,v})\to k,$$

a map whose truly-canonical realization via differentials and residues is indicated in [L1, p. 86, (9.5)].

2.3.3. Consider next the completion diagram

$$\begin{array}{ccc} X_{/Z} =: \mathcal{X} & \xrightarrow{\ \kappa_{\mathcal{X}}\ } & X \\ {\scriptstyle\hat f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y_{/W} =: \mathcal{Y} & \xrightarrow[\ \kappa_{\mathcal{Y}}\ ]{} & Y \end{array}$$

Duality with supports can be regarded more intrinsically—via $\hat f$ rather than f—as a special case of the Torsion-Duality Theorem 6.1, as follows.

First of all, the local integral ρ is completely determined by $\kappa_{\mathcal{Y}}^*(\rho)$: for $\mathcal{G}\in\mathbf{D}(\mathcal{Y})$, the natural map $\mathbf{R}\varGamma'_W\mathcal{G}\to\kappa_{\mathcal{Y}*}\kappa_{\mathcal{Y}}^*\mathbf{R}\varGamma'_W\mathcal{G}$ is an isomorphism (Proposition 5.2.4); and the same holds for $\mathbf{R}f_*\mathbf{R}\varGamma_Z f^\times\mathcal{G}\to\kappa_{\mathcal{Y}*}\kappa_{\mathcal{Y}}^*\mathbf{R}f_*\mathbf{R}\varGamma_Z f^\times\mathcal{G}$ since as above,

$$\mathbf{R}f_*\mathbf{R}\varGamma_Z f^\times\mathcal{G}\in\mathbf{R}f_*(\mathbf{D}_{\mathrm{qc}Z}(X))\subset\mathbf{D}_W(Y)$$

—and so $\rho=\kappa_{\mathcal{Y}*}\kappa_{\mathcal{Y}}^*(\rho)$. Furthermore $\kappa_{\mathcal{Y}}^*(\rho)$ is determined by the duality map $\tau_{\mathbf{t}}\colon\mathbf{R}\hat f_*\hat f_{\mathrm{t}}^\times\to\mathbf{1}$, as per the following natural commutative diagram, whose rows are isomorphisms:

$$\begin{array}{ccccccc} \kappa_{\mathcal{Y}}^*\mathbf{R}f_*\mathbf{R}\varGamma_Z f^\times\mathcal{G} & \underset{5.2.6}{\xrightarrow{\ \sim\ }} & \mathbf{R}\hat f_*\kappa_{\mathcal{X}}^*\mathbf{R}\varGamma_Z f^\times\kappa_{\mathcal{Y}*}\kappa_{\mathcal{Y}}^*\mathcal{G} & \underset{6.1.6}{\xrightarrow{\ \sim\ }} & \mathbf{R}\hat f_*\hat f_{\mathrm{t}}^\times\kappa_{\mathcal{Y}}^*\mathcal{G} & \underset{6.1.3}{\xrightarrow{\ \sim\ }} & \mathbf{R}\hat f_*\hat f_{\mathrm{t}}^\times\mathbf{R}\varGamma'_{\mathcal{Y}}\kappa_{\mathcal{Y}}^*\mathcal{G} \\ {\scriptstyle\kappa_{\mathcal{Y}}^*(\rho)}\big\downarrow & & & & & & \big\downarrow{\scriptstyle\tau_{\mathbf{t}}} \\ \kappa_{\mathcal{Y}}^*\mathbf{R}\varGamma'_W\mathcal{G} & & & \underset{5.2.4}{\xleftarrow{\qquad\qquad\sim\qquad\qquad}} & & & \mathbf{R}\varGamma'_{\mathcal{Y}}\kappa_{\mathcal{Y}}^*\mathcal{G} \end{array}$$

(To see that the natural map $\mathbf{R}\varGamma_Z f^\times\mathcal{G}\to\mathbf{R}\varGamma_Z f^\times\kappa_{\mathcal{Y}*}\kappa_{\mathcal{Y}}^*\mathcal{G}$ is an isomorphism, replace $\mathbf{R}\varGamma_Z f^\times$ by the isomorphic functor $\mathbf{R}\varGamma_Z f^\times\mathbf{R}\varGamma'_W$ and apply Proposition 5.2.4.)

Finally, we have isomorphisms (for $\mathcal{E}\in\mathbf{D}_{\mathrm{qc}Z}(X)$, $\mathcal{F}\in\mathbf{D}_W(Y)$),

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{E},\mathbf{R}\varGamma_Z f^{\times}\mathcal{F}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\kappa_{\mathfrak{X}}^*\mathcal{E},\,\kappa_{\mathfrak{X}}^*\mathbf{R}\varGamma_Z f^{\times}\kappa_{\mathfrak{Y}*}\kappa_{\mathfrak{Y}}^*\mathcal{F}) && (5.2.4)\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\kappa_{\mathfrak{X}}^*\mathcal{E},\,\hat{f}_{\mathrm{t}}^{\times}\kappa_{\mathfrak{Y}}^*\mathcal{F}) && (6.1.6)\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathfrak{Y})}(\mathbf{R}\hat{f}_*\kappa_{\mathfrak{X}}^*\mathcal{E},\,\kappa_{\mathfrak{Y}}^*\mathcal{F}) && (6.1)\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathfrak{Y})}(\kappa_{\mathfrak{Y}}^*\mathbf{R}f_*\mathcal{E},\,\kappa_{\mathfrak{Y}}^*\mathcal{F}) && (5.2.6)\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}f_*\mathcal{E},\,\mathcal{F}) && (5.2.4),
\end{aligned}$$

whose composition can be checked, via the preceding diagram, to be the same as the isomorphism of Proposition 2.3.2.

2.3.4. Proposition 2.3.5 expresses some homological consequences of the foregoing dualities, and furnishes a general context for [L1, pp. 87–88, Theorem (10.2)].

For any noetherian formal scheme $\mathfrak{X}$, $\mathcal{E}\in\mathbf{D}(\mathfrak{X})$, and $n\in\mathbb{Z}$, set

$$\mathrm{H}'^{n}_{\mathfrak{X}}(\mathcal{E}) := \mathrm{H}^n\mathbf{R}\Gamma(\mathfrak{X},\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{E}).$$

For instance, if $\mathfrak{X}$ arises through completion $\mathfrak{X}=X_{/Z}\xrightarrow{\kappa}X$, as above, then for $\mathcal{F}\in\mathbf{D}(X)$, Proposition 5.2.4 yields natural isomorphisms

$$\begin{aligned}
\mathbf{R}\Gamma(\mathfrak{X},\mathbf{R}\varGamma'_{\mathfrak{X}}\kappa^*\mathcal{F}) &= \mathbf{R}\Gamma(X,\kappa_*\mathbf{R}\varGamma'_{\mathfrak{X}}\kappa^*\mathcal{F})\\
&\cong \mathbf{R}\Gamma(X,\kappa_*\kappa^*\mathbf{R}\varGamma'_Z\mathcal{F}) \cong \mathbf{R}\Gamma(X,\mathbf{R}\varGamma'_Z\mathcal{F}),
\end{aligned}$$

and so if $\mathcal{F}\in\mathbf{D}_{\mathrm{qc}}(Y)$, then with $\mathrm{H}^{\bullet}_Z$ the usual (hyper)cohomology with supports in Z,

$$\mathrm{H}'^{n}_{\mathfrak{X}}(\kappa^*\mathcal{F})\cong\mathrm{H}^n_Z(\mathcal{F}).$$

Let $\mathcal{J}\subset\mathcal{O}_{\mathfrak{X}}$ be an ideal of definition. Writing $\Gamma_{\mathfrak{X}}$ for the functor $\Gamma(\mathfrak{X},-)$, we have a functorial map

$$\gamma(\mathcal{E})\colon \mathbf{R}(\Gamma_{\mathfrak{X}}\circ\varGamma'_{\mathfrak{X}})\mathcal{E}\to\mathbf{R}\Gamma_{\mathfrak{X}}\circ\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{E}\qquad\big(\mathcal{E}\in\mathbf{D}(\mathfrak{X})\big),$$

which is an *isomorphism* when $\mathcal{E}$ is bounded-below, since for any injective $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{I}$, $\varinjlim_i$ of the flasque modules $\mathcal{H}om(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^i,\mathcal{I})$ is $\Gamma_{\mathfrak{X}}$-acyclic. Whenever $\gamma(\mathcal{E})$ is an isomorphism, the induced homology maps are isomorphisms

$$\varinjlim_i \mathrm{Ext}^n(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^i,\mathcal{E})\xrightarrow{\sim}\mathrm{H}'^{n}_{\mathfrak{X}}(\mathcal{E}).$$

If $\mathcal{E}\in\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$, then $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{E}\in\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ (Proposition 5.2.1), and for any separated map $g\colon\mathfrak{X}\to\mathfrak{Y}$ of noetherian formal schemes, $\mathcal{G}\in\mathbf{D}(\mathfrak{Y})$, and $R:=\mathrm{H}^0(\mathfrak{Y},\mathcal{O}_{\mathfrak{Y}})$, there are natural maps

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{E},g_{\mathrm{t}}^{\times}\mathcal{G}) &\xrightarrow{\sim}\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{E},g_{\mathrm{t}}^{\times}\mathbf{R}\varGamma'_{\mathfrak{Y}}\mathcal{G}) && (\ 6.1.3)\\
&\xrightarrow{\sim}\mathrm{Hom}_{\mathbf{D}(\mathfrak{Y})}(\mathbf{R}g_*\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{E},\mathbf{R}\varGamma'_{\mathfrak{Y}}\mathcal{G}) && (\ 6.1)\\
&\longrightarrow\mathrm{Hom}_R(\mathrm{H}'^{n}_{\mathfrak{X}}\mathcal{E},\mathrm{H}'^{n}_{\mathfrak{Y}}\mathcal{G})
\end{aligned}\tag{2.3.4.1}$$

where the last map arises via the functor $\mathrm{H}^n\mathbf{R}\Gamma(\mathfrak{Y},-)$.

In particular, for $g=\hat{f}$ in the completion situation of §2.3.3, and for $\mathcal{E}=\kappa_{\mathfrak{X}}^*\mathcal{E}_0$, $\mathcal{G}=\kappa^*\mathcal{G}_0$ ($\mathcal{E}_0\in\mathbf{D}_{\mathrm{qc}}(X)$, $\mathcal{G}_0\in\mathbf{D}_{\mathrm{qc}}(Y)$), preceding considerations show that this composed map operates via Duality with Supports for f (Proposition 2.3.2), i.e., it can be identified with the natural composition

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{R}\varGamma_Z\mathcal{E}_0, \mathbf{R}\varGamma_Z f^{\times}\mathcal{G}_0) &\underset{2.3.2}{\xrightarrow{\ \sim\ }} \mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}f_*\mathbf{R}\varGamma_Z\mathcal{E}_0, \mathbf{R}\varGamma_W\mathcal{G}_0)\\ &\longrightarrow \mathrm{Hom}_{\mathrm{H}^0(Y,\mathcal{O}_Y)}(\mathrm{H}^n_Z\mathcal{E}_0, \mathrm{H}^n_W\mathcal{G}_0).\end{aligned}$$

Next, let R be a complete noetherian local ring topologized as usual by its maximal ideal I, let (S, J) be a noetherian adic ring, let $\varphi\colon (R, I) \to (S, J)$ be a continuous homomorphism, and let

$$\mathcal{Y} := \mathrm{Spf}(S) \xrightarrow{\ f\ } \mathrm{Spf}(R) =: \mathcal{V}$$

be the corresponding formal-scheme map. As before, $g\colon \mathcal{X} \to \mathcal{Y}$ is a separated map, and we set $h := fg$. Since the underlying space of $\mathcal{V}$ is a single point, at which the stalk of $\mathcal{O}_{\mathcal{V}}$ is just R, therefore the categories of $\mathcal{O}_{\mathcal{V}}$-modules and of R-modules are identical, and accordingly, for any $\mathcal{E} \in \mathbf{D}(\mathcal{X})$ we can identify $\mathbf{R}h_*\mathcal{E}$ with $\mathbf{R}\Gamma(\mathcal{X}, \mathcal{E}) \in \mathbf{D}(R)$.

Let K be an injective R-module, and $\mathcal{K}$ the corresponding injective $\mathcal{O}_{\mathcal{V}}$-module. There exist integers r, s such that $H^i(f^{\#}\mathcal{K}) = 0$ for all $i < -r$ (resp. $H^i(h^{\#}\mathcal{K}) = 0$ for all $i < -s$) (Corollary 6.1.4). Set $\omega_{\mathcal{Y}} := H^{-r}(f^{\#}\mathcal{K})$ (resp. $\omega_{\mathcal{X}} := H^{-s}(h^{\#}\mathcal{K})$).

Proposition 2.3.5. *In the above situation, via the map* (2.3.4.1), $\omega_{\mathcal{X}}$ *represents the functor* $\mathrm{Hom}_S(\mathrm{H}'^{s}_{\mathcal{X}}\mathcal{E}, \mathrm{H}'^{0}_{\mathcal{Y}}(f^{\#}\mathcal{K}))$ *of quasi-coherent* $\mathcal{O}_{\mathcal{X}}$*-modules* $\mathcal{E}$*. If* $\omega_{\mathcal{Y}}$ *is the only non-zero homology of* $f^{\#}\mathcal{K}$*, this functor is isomorphic to* $\mathrm{Hom}_S(\mathrm{H}'^{s}_{\mathcal{X}}\mathcal{E}, \mathrm{H}'^{r}_{\mathcal{Y}}\omega_{\mathcal{Y}})$.

Proof. There are natural maps

$$\mathrm{H}'^{r}_{\mathcal{Y}}(\omega_{\mathcal{Y}}) = \mathrm{H}'^{0}_{\mathcal{Y}}(\omega_{\mathcal{Y}}[r]) \xrightarrow{\ h\ } \mathrm{H}'^{0}_{\mathcal{Y}}(f^{\#}\mathcal{K}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{R,J}(S, K)$$

where the last isomorphism results from Proposition 2.2.1(a), in view of the identity $\mathbf{R}\varGamma'_{\mathcal{Y}}f^{\#} = f_{\mathrm{t}}^{\times}$ (Corollary 6.1.5(a)) and the natural isomorphisms

$$\mathbf{R}\Gamma(\mathcal{Y}, \kappa_{\mathcal{Y}}^*\widetilde{G}) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma(Y, \kappa_{\mathcal{Y}*}\kappa_{\mathcal{Y}}^*\widetilde{G}) \underset{5.2.4}{\xrightarrow{\ \sim\ }} \mathbf{R}\Gamma(Y, \widetilde{G}) \xrightarrow{\ \sim\ } G \qquad \big(G \in \mathbf{D}^+_J(S)\big),$$ [12]

applied to $G := \mathbf{R}\mathrm{Hom}^{\bullet}_{R,J}(S, \mathbf{R}\Gamma(\mathcal{V}, \mathcal{K}))$. In case $\omega_{\mathcal{Y}}$ is the only non-vanishing homology of $f^{\#}\mathcal{K}$, then h is an isomorphism too.

The assertions follow from the (easily checked) commutativity, for any quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$, of the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{X}}(\mathcal{E}, \omega_{\mathcal{X}}) & = & \mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathcal{E}[s], g^{\#}f^{\#}\mathcal{K}) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E}[s], g_{\mathrm{t}}^{\times}f^{\#}\mathcal{K}) \\
 & & \simeq\big\downarrow & & \big\downarrow{\scriptstyle (2.3.4.1)} \\
 & & \mathrm{Hom}_R(\mathbf{R}h_*\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E}[s], \mathcal{K}) & & \mathrm{Hom}_S\big(\mathrm{H}'^{0}_{\mathcal{X}}(\mathcal{E}[s]), \mathrm{H}'^{0}_{\mathcal{Y}}(f^{\#}\mathcal{K})\big) \\
 & & \big\| & & \big\downarrow\simeq \\
 & & \mathrm{Hom}_R(\mathrm{H}'^{s}_{\mathcal{X}}\mathcal{E}, \mathcal{K}) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_S\big(\mathrm{H}'^{s}_{\mathcal{X}}\mathcal{E}, \mathrm{Hom}_{R,J}(S, K)\big)
\end{array}$$

2.3.6. Now let us fit [L1, pp. 87–88, Theorem (10.2)] into the preceding setup.

The cited Theorem has both local and global components. The first deals with maps $\varphi\colon R \to S$ of local domains essentially of finite type over a perfect field k, with residue fields finite over k. To each such ring T one associates the canonical

[12] In fact $\mathbf{R}\Gamma(\mathcal{Y}, \kappa_{\mathcal{Y}}^*\widetilde{G}) \cong G$ for any $G \in \mathbf{D}(S)$, see Corollary 3.3.2 and the beginning of §3.3.

module ω_T of "regular" k-differentials of degree $\dim T$. Under mild restrictions on φ, the assertion is that the functor

$$\mathrm{Hom}_R\bigl(H^{\dim S}_{m_{\hat S}}G,\, H^{\dim R}_{m_R}\omega_R\bigr) \qquad (m := \text{maximal ideal})$$

of $\hat S$-modules G is represented by the completion $\widehat{\omega_S}$ together with a canonical map, the *relative residue*

$$\rho_\varphi\colon H^{\dim S}_{m_{\hat S}}\widehat{\omega_S} = H^{\dim S}_{m_S}\omega_S \to H^{\dim R}_{m_R}\omega_R.$$

This may be viewed as a consequence of *concrete* local duality over k (§2.1.7).

The global aspect concerns a proper map of k-varieties $g\colon V \to W$ which is equidimensional in codimension 1, a closed point $w \in W$, the fiber $E := g^{-1}(w)$, and the completion $\widehat V := V_{/E}$. The assertion is that the functor

$$\mathrm{Hom}_R\bigl(H^{\dim V}\mathbf{R}\varGamma'_{\widehat V}\mathcal G,\, H^{\dim R}_{m_R}\omega_R\bigr) \qquad (R := \mathcal O_{W,w})$$

of coherent $\mathcal O_{\widehat V}$-modules $\mathcal G$ is represented by the completion $\widehat{\omega_V}$ along E of the canonical sheaf ω_V of regular differentials, together with a canonical map

$$\theta\colon H^{\dim V}\mathbf{R}\varGamma'_{\widehat V}\widehat{\omega_V} = H^{\dim V}\mathbf{R}\varGamma_E\omega_V \to H^{\dim R}_{m_R}\omega_R.$$

Moreover, the local and global representations are *compatible* in the sense that if $v \in E$ is any closed point and $\varphi_v\colon R \to S := \mathcal O_{V,v}$ is the canonical map, then the residue $\rho_v := \rho_{\varphi_v}$ factors as the natural map $H^{\dim S}_{m_S}\omega_S \to H^{\dim V}\mathbf{R}\varGamma_E\omega_V$ followed by θ. This compatibility determines θ uniquely if the ρ_v $(v \in E)$ are given [L1, p. 95, (10.6)]; and of course conversely.

Basically, all this—*without the explicit description of the ω's and the maps ρ_v via differentials and residues*—is contained in Proposition 2.3.5, as follows.

In the completion situation of §2.3.3, take X and Y to be finite-type schemes over an artinian local ring R, of respective pure dimensions s and r, let $W = \{w\}$ with w a closed point of Y, write g in place of f, and assume that $Z \subset g^{-1}W$ is proper over R (which is so, e.g., if g is proper and Z is closed). Let K be an injective hull of the residue field of R, and let $\mathcal K$ be the corresponding injective sheaf on $\mathrm{Spec}(R) = \mathrm{Spf}(R)$. With $f\colon Y \to \mathrm{Spec}(R)$ the canonical map, and $h = fg$, define the *dualizing sheaves*

$$\omega_X := H^{-s}h^!\mathcal K, \qquad \omega_Y := H^{-r}f^!\mathcal K,$$

where $h^!$ is the Grothendieck duality functor (compatible with open immersions, and equal to $h^\times$ when h is proper), and similarly for $f^!$. It is well-known (for example via a local factorization of h as smooth$\circ$finite) that $h^!\mathcal K$ has coherent homology, vanishing in all degrees $< -s$; and similarly $f^!\mathcal K$ has coherent homology, vanishing in all degrees $< -r$.

Let

$$\hat f\colon \mathscr Y := \mathrm{Spf}(\mathcal O_{W,w}) \to \mathrm{Spf}(R) =: \mathscr V$$

be the completion of f. We may assume, after compactifying f and g—which does not affect $\hat f$ or $\hat g$ (see [Lü]), that f and g are proper maps. Then, Corollary 6.2.3 shows that $\hat h^{\#}\mathcal K = \kappa_{\mathscr X}^*h^!\mathcal K$, so that, $\kappa_{\mathscr X}$ being flat, we see that

$$\kappa_{\mathscr X}^*\omega_X = \omega_{\mathscr X} \tag{2.3.6.1}$$

where $\omega_{\mathscr X}$ is as in Proposition 2.3.5; and similarly $\kappa_{\mathscr Y}^*\omega_Y = \omega_{\mathscr Y}$.

Once again, some form of the theory of the Fundamental Class will enable us to represent ω_X by means of regular differential forms; and then both the local and global components of the cited Theorem (10.2) become special cases of Proposition 2.3.5 (modulo some technicalities [L1, p. 89, Lemma (10.3)] which allow a weakening of the condition that $\omega_{\mathscr{Y}}$ be the only non-vanishing homology of $\hat{f}^{\#}\mathcal{K}$).

As for the local-global compatibility, consider quite generally a pair of maps

$$\mathscr{X}_1 \xrightarrow{q} \mathscr{X} \xrightarrow{p} \mathscr{Y}$$

of noetherian formal schemes. In the above situation, for instance, we could take p to be $\hat{g}$, $\mathscr{X}_1$ to be the completion of X at a closed point $v \in Z$, and q to be the natural map. Theorem 2 gives us the adjunction

$$\mathbf{D}_{\mathrm{qct}}(\mathscr{X}) \underset{p_{\mathrm{t}}^{\times}}{\overset{\mathbf{R}p_*}{\rightleftarrows}} \mathbf{D}_{\mathrm{qct}}(\mathscr{Y}).$$

The natural isomorphism $\mathbf{R}(pq)_* \xrightarrow{\sim} \mathbf{R}p_*\mathbf{R}q_*$ gives rise then to an adjoint isomorphism $q_{\mathrm{t}}^{\times}p_{\mathrm{t}}^{\times} \xrightarrow{\sim} (pq)_{\mathrm{t}}^{\times}$; and for $\mathcal{E} \in \mathbf{D}_{\mathrm{qct}}(\mathscr{Y})$ the natural map $\mathbf{R}(pq)_*(pq)_{\mathrm{t}}^{\times}\mathcal{E} \to \mathcal{E}$ factors as

$$\mathbf{R}(pq)_*(pq)_{\mathrm{t}}^{\times}\mathcal{E} \xrightarrow{\sim} \mathbf{R}p_*\mathbf{R}q_*q_{\mathrm{t}}^{\times}p_{\mathrm{t}}^{\times}\mathcal{E} \to \mathbf{R}p_*p_{\mathrm{t}}^{\times}\mathcal{E} \to \mathcal{E}.$$

This factorization contains the compatibility between the above maps θ and ρ_v, as one sees by interpreting them as homological derivatives of maps of the type $\mathbf{R}p_*p_{\mathrm{t}}^{\times}\mathcal{E} \to \mathcal{E}$ (with $\mathcal{E} := \mathbf{R}\varGamma'_{\mathscr{Y}}\hat{f}^{\#}\mathcal{K}$). Details are left to the reader.

Remark 2.3.7. In the preceding situation, suppose further that $Y = \mathrm{Spec}(R)$ (with R artinian) and $f =$ identity, so that $h = g\colon X \to Y$ is a finite-type separated map, X being of pure dimension s, and $\kappa_{\mathscr{X}}\colon \mathscr{X} \to X$ is the completion of X along a closed subset Z proper over Y. Again, K is an injective R-module, $\mathcal{K}$ is the corresponding $\mathcal{O}_Y$-module, and $\omega_X := H^{-s}g^!\mathcal{K}$ is a "dualizing sheaf" on X. Now Proposition 2.3.5 is just the instance $i = s$ of the canonical isomorphisms, for $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(\mathscr{X})$, $i \in \mathbb{Z}$ (and with $\mathrm{H}'^{\bullet}_{\mathscr{X}} := \mathrm{H}^{\bullet}\mathbf{R}\Gamma(\mathscr{X}, \mathbf{R}\varGamma'_{\mathscr{X}})$, see §2.3.4, and $\hat{g} := g \circ \kappa_{\mathscr{X}}$):

$$\mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(\mathcal{E}[i], \hat{g}^{\#}\mathcal{K}) \underset{\text{Thm. 2}}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathbf{D}(\mathscr{Y})}(\mathbf{R}\hat{g}_*\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{E}[i], \mathcal{K}) \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{H}'^{i}_{\mathscr{X}}\mathcal{E}, K) =: (\mathrm{H}'^{i}_{\mathscr{X}}\mathcal{E})\check{}.$$

If X is Cohen-Macaulay then all the homology of $g^!\mathcal{K}$ other than ω_X vanishes, so all the homology of $\hat{g}^{\#}\mathcal{K} \cong \kappa_{\mathscr{X}}^*g^!\mathcal{K}$ other than $\omega_{\mathscr{X}} = \kappa_{\mathscr{X}}^*\omega_X$ vanishes (see (2.3.6.1)), and the preceding composed isomorphism becomes

$$\mathrm{Ext}_{\mathscr{X}}^{s-i}(\mathcal{E}, \omega_{\mathscr{X}}) \xrightarrow{\sim} (\mathrm{H}'^{i}_{\mathscr{X}}\mathcal{E})\check{}.$$

In particular, when $Z = X$ (so that $\mathscr{X} = X$) this is the usual duality isomorphism

$$\mathrm{Ext}_{X}^{s-i}(\mathcal{E}, \omega_{X}) \xrightarrow{\sim} \mathrm{H}^{i}(X, \mathcal{E})\check{}.$$

If X is Gorenstein and $\mathcal{F}$ is a locally free $\mathcal{O}_{\mathscr{X}}$-module of finite rank, then ω_X is invertible; and taking $\mathcal{E} := \mathcal{H}om_{\mathscr{X}}(\mathcal{F}, \omega_{\mathscr{X}}) = \check{\mathcal{F}} \otimes \omega_{\mathscr{X}}$ we get the isomorphism

$$\mathrm{H}^{s-i}(\mathscr{X}, \mathcal{F}) \xrightarrow{\sim} \big(\mathrm{H}'^{i}_{\mathscr{X}}(\check{\mathcal{F}} \otimes \omega_{\mathscr{X}})\big)\check{},$$

which generalizes the Formal Duality theorem [H2, p. 48, Proposition (5.2)].

2.4. Both [H2, p. 48; Proposition (5.2)] (Formal Duality) and the Theorem in [L3, p. 188] (Local-Global Duality) are contained in Proposition 2.4.1, see [AJL, §5.3].

Let R be a noetherian ring, discretely topologized, and set

$$Y := \mathrm{Spec}(R) = \mathrm{Spf}(R) =: \mathcal{Y}.$$

Let $g\colon X \to Y$ be a finite-type separated map, let $Z \subset X$ be *proper* over Y, let $\kappa\colon \mathcal{X} = X_{/Z} \to X$ be the completion of X along Z, and set $\hat{g} := g \circ \kappa\colon \mathcal{X} \to \mathcal{Y}$.

Assume that R has a *residual complex* $\mathcal{R}$ [H1, p. 304]. Then the corresponding quasi-coherent $\mathcal{O}_Y$-complex $\mathcal{R}_Y := \widetilde{\mathcal{R}}$ is a *dualizing complex,* and $\mathcal{R}_X := g^!\mathcal{R}_Y$ is a dualizing complex on X [V, p. 396, Corollary 3]. For any $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$ set

$$\mathcal{F}' := \mathbf{R}\mathcal{H}om^{\bullet}_X(\mathcal{F}, \mathcal{R}_X) \in \mathbf{D}_{\mathrm{c}}(X),$$

so that $\mathcal{F} \cong \mathcal{F}'' = \mathbf{R}\mathcal{H}om^{\bullet}_X(\mathcal{F}', \mathcal{R}_X)$.

Proposition 2.4.1. *In the preceding situation, and with* $\Gamma_Z(-) := \Gamma(X, \mathit{\Gamma}_Z(-))$, *there is a functorial isomorphism*

$$\mathbf{R}\Gamma(\mathcal{X}, \kappa^*\mathcal{F}) \cong \mathbf{R}\mathrm{Hom}^{\bullet}_A(\mathbf{R}\Gamma_Z\mathcal{F}', \mathcal{R}) \qquad \big(\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)\big).$$

Proof. Replacing g by a compactification ([Lü]) doesn't affect $\mathcal{X}$ or $\mathbf{R}\Gamma_Z$, so we may assume g proper, and then Corollary 6.2.3 gives an isomorphism $\kappa^*\mathcal{R}_X \cong \hat{g}^{\#}\mathcal{R}_Y$. Now just compose the chain of functorial isomorphisms

$$\begin{aligned}
\mathbf{R}\Gamma(\mathcal{X}, \kappa^*\mathcal{F}) &\cong \mathbf{R}\Gamma\big(\mathcal{X}, \kappa^*\mathbf{R}\mathcal{H}om^{\bullet}_X(\mathcal{F}', \mathcal{R}_X)\big) && \text{(see above)}\\
&\cong \mathbf{R}\Gamma\big(\mathcal{X}, \mathbf{R}\mathcal{H}om^{\bullet}_{\mathcal{X}}(\kappa^*\mathcal{F}', \kappa^*\mathcal{R}_X)\big) && \text{(Lemma 2.4.2)}\\
&\cong \mathbf{R}\mathrm{Hom}^{\bullet}_{\mathcal{X}}(\kappa^*\mathcal{F}', \hat{g}^{\#}\mathcal{R}_Y)) && \text{(see above)}\\
&\cong \mathbf{R}\mathrm{Hom}^{\bullet}_{\mathcal{Y}}(\mathbf{R}\hat{g}_*\mathbf{R}\mathit{\Gamma}'_{\mathcal{X}}\kappa^*\mathcal{F}', \mathcal{R}_Y) && \text{(Theorem 2)}\\
&\cong \mathbf{R}\mathrm{Hom}^{\bullet}_Y(\mathbf{R}g_*\mathbf{R}\mathit{\Gamma}_Z\mathcal{F}', \mathcal{R}_Y) && \text{(Proposition 5.2.4)}\\
&\cong \mathbf{R}\mathrm{Hom}^{\bullet}_Y(\widetilde{\mathbf{R}\Gamma_Z\mathcal{F}'}, \mathcal{R}_Y) && \text{[AJL, footnote, §5.3]}\\
&\cong \mathbf{R}\mathrm{Hom}^{\bullet}_A(\mathbf{R}\Gamma_Z\mathcal{F}', \mathcal{R}) && \text{[AJL, p. 9, (0.4.4)].}
\end{aligned}$$

□

Lemma 2.4.2. *Let X be a locally noetherian scheme, and let $\kappa\colon \mathcal{X} \to X$ be its completion along some closed subset Z. Then for $\mathcal{G} \in \mathbf{D}_{\mathrm{qc}}(X)$ of finite injective dimension and for $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$, the natural map is an isomorphism*

$$\kappa^*\mathbf{R}\mathcal{H}om^{\bullet}_X(\mathcal{F}, \mathcal{G}) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^{\bullet}_{\mathcal{X}}(\kappa^*\mathcal{F}, \kappa^*\mathcal{G}).$$

Proof. By [H1, p. 134, Proposition 7.20] we may assume that $\mathcal{G}$ is a bounded complex of quasi-coherent injective $\mathcal{O}_X$-modules, vanishing, say, in all degrees $> n$.

When $\mathcal{F}$ is bounded-above the assertion is well-known, easily proved by localizing to the affine case and applying way-out reasoning [H1, p. 68, Proposition 7.1] to reduce to the trivial case $\mathcal{F} = \mathcal{O}_X^m$ $(0 < m \in \mathbb{Z})$. To do the same for unbounded $\mathcal{F}$ we must first show, for fixed $\mathcal{G}$, that the contravariant functor $\mathbf{R}\mathcal{H}om^{\bullet}_{\mathcal{X}}(\kappa^*\mathcal{F}, \kappa^*\mathcal{G})$ is bounded-above.[13] In fact we will show that if $H^i\mathcal{F} = 0$ for all $i < i_0$ then $H^j\mathbf{R}\mathcal{H}om^{\bullet}_{\mathcal{X}}(\kappa^*\mathcal{F}, \kappa^*\mathcal{G}) = 0$ for all $j > n - i_0$, or equivalently,

$$H^j\kappa_*\mathbf{R}\mathcal{H}om^{\bullet}_{\mathcal{X}}(\kappa^*\mathcal{F}, \kappa^*\mathcal{G}) = H^j\mathbf{R}\mathcal{H}om^{\bullet}_X(\mathcal{F}, \kappa_*\kappa^*\mathcal{G}) = 0 \qquad (j > n - i_0).$$

[13] This step is missing from the proof of [AJL, p. 36, Lemma (5.3.3)(b)]. The argument for (2) on p. 8 of *loc. cit* suffers a similar—though more easily remedied—deficiency.

The homology in question is the sheaf associated to the presheaf which assigns

$$\mathrm{Hom}_{\mathbf{D}(U)}\big(\mathcal{F}|_U[-j], (\kappa_*\kappa^*\mathcal{G})|_U\big) = \mathrm{Hom}_{\mathbf{D}(U)}\big(\mathcal{F}|_U[-j], \mathbf{R}Q_U(\kappa_*\kappa^*\mathcal{G})|_U\big)$$

to each affine open subset $U = \mathrm{Spec}(A)$ in X. (See beginning of §3.1 and of §3.3). Let $\mathcal{U} := \kappa^{-1}U$, and $\hat{A} := \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$, so that $\kappa|_{\mathcal{U}}$ factors naturally as

$$\mathcal{U} = \mathrm{Spf}(\hat{A}) \xrightarrow{\kappa_1} U_1 := \mathrm{Spec}(\hat{A}) \xrightarrow{k} \mathrm{Spec}(A) = U.$$

The functors $\mathbf{R}Q_U k_*$ and $k_*\mathbf{R}Q_{U_1}$ are both right-adjoint to the natural composition $\mathbf{D}_{\mathrm{qc}}(U) \xrightarrow{k^*} \mathbf{D}_{\mathrm{qc}}(U_1) \hookrightarrow \mathbf{D}(U_1)$, and so are isomorphic. Thus there are natural isomorphisms

$$\mathbf{R}Q_U(\kappa_*\kappa^*\mathcal{G})|_U = \mathbf{R}Q_U k_*\kappa_{1*}\kappa_1^* k^*(\mathcal{G}|_U) \xrightarrow{\ \sim\ } k_*\mathbf{R}Q_{U_1}\kappa_{1*}\kappa_1^* k^*(\mathcal{G}|_U) \underset{3.3.1}{\xrightarrow{\ \sim\ }} k_*k^*(\mathcal{G}|_U)$$

and the presheaf becomes

$$U \mapsto \mathrm{Hom}_{\mathbf{D}(U)}\big(\mathcal{F}|_U[-j], k_*k^*(\mathcal{G}|_U)\big).$$

The equivalence of categories $\mathbf{D}_{\mathrm{qc}}(U) \cong \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(U)) = \mathbf{D}(A)$ indicated at the beginning of §3.3 yields an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(U)}\big(\mathcal{F}|_U[-j], k_*k^*(\mathcal{G}|_U)\big) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(A)}\big(F[-j], G \otimes_A \hat{A}\big)$$

where F is a complex of A-modules with $\mathrm{H}^i F = 0$ for $i < i_0$, and both G and $G \otimes_A \hat{A}$ are complexes of injective A-modules vanishing in all degrees $> n$ (the latter since $\hat{A}$ is A-flat). Hence the presheaf vanishes, and the conclusion follows. □

3. Direct limits of coherent sheaves on formal schemes.

In this section we establish, for a locally noetherian formal scheme $\mathcal{X}$, properties of $\mathcal{A}_{\vec{c}}(\mathcal{X})$ needed in §4 to adapt Deligne's proof of global Grothendieck Duality to the formal context. The basic result, Proposition 3.2.2, is that $\mathcal{A}_{\vec{c}}(\mathcal{X})$ is *plump* (see opening remarks in §1), hence abelian, and so (being closed under $\varinjlim$) cocomplete, i.e., it has arbitrary small colimits. This enables us to speak about $\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathcal{X}))$, and to apply standard adjoint functor theorems to colimit-preserving functors on $\mathcal{A}_{\vec{c}}(\mathcal{X})$. (See e.g., Proposition 3.2.3, Grothendieck Duality for the identity map of $\mathcal{X}$).

The preliminary paragraph 3.1 sets up an equivalence of categories which allows us to reduce local questions about the (globally defined) category $\mathcal{A}_{\vec{c}}(\mathcal{X})$ to corresponding questions about quasi-coherent sheaves on ordinary noetherian schemes. Paragraph 3.3 extends this equivalence to derived categories. As one immediate application, Corollary 3.3.4 asserts that the natural functor $\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathcal{X})) \to \mathbf{D}_{\vec{c}}(\mathcal{X})$ is an equivalence of categories when $\mathcal{X}$ is *properly algebraic,* i.e., the J-adic completion of a proper B-scheme with B a noetherian ring and J a B-ideal. This will yield a stronger version of Grothendieck Duality on such formal schemes—for $\mathbf{D}_{\vec{c}}(\mathcal{X})$ rather than $\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathcal{X}))$, see Corollary 4.1.1. We do not know whether such global results hold over arbitrary noetherian formal schemes.

Paragraph 3.4 establishes "boundedness" for some derived functors, a condition which allows us to apply them freely to unbounded complexes, as illustrated, e.g., in Paragraph 3.5.

3.1. For X a noetherian (ordinary) scheme, $\mathcal{A}_{\vec{c}}(X) = \mathcal{A}_{\mathrm{qc}}(X)$ [GD, p. 319, (6.9.9)]. The inclusion $j_X : \mathcal{A}_{\mathrm{qc}}(X) \to \mathcal{A}(X)$ has a right adjoint $Q_X : \mathcal{A}(X) \to \mathcal{A}_{\mathrm{qc}}(X)$, the "quasi-coherator," necessarily left exact [I, p. 187, Lemme 3.2]. (See Proposition 3.2.3 and Corollary 5.1.5 for generalizations to formal schemes.)

Proposition 3.1.1. *Let A be a noetherian adic ring with ideal of definition I, let $f_0\colon X \to \mathrm{Spec}(A)$ be a proper map, $Z := f_0^{-1}\mathrm{Spec}(A/I)$, and $\kappa\colon \mathcal{X} = X_{/Z} \to X$ the formal completion of X along Z. Let $Q := Q_X$ be as above. Then κ^* induces equivalences of categories from $\mathcal{A}_{\mathrm{qc}}(X)$ to $\mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X})$ and from $\mathcal{A}_{\mathrm{c}}(X)$ to $\mathcal{A}_{\mathrm{c}}(\mathcal{X})$, both with quasi-inverse $Q\kappa_*$.*

Proof. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ and any $i \geq 0$, the canonical map is an *isomorphism*

$$\mathrm{H}^i(X,\mathcal{G}) \xrightarrow{\sim} \mathrm{H}^i(X,\kappa_*\kappa^*\mathcal{G}) = \mathrm{H}^i(\mathcal{X},\kappa^*\mathcal{G}).\tag{3.1.2}$$[14]

To see this, let $(\mathcal{G}_\lambda)$ be the family of coherent submodules of $\mathcal{G}$, ordered by inclusion. Then, X and $\mathcal{X}$ being noetherian, one checks that (3.1.2) is the composition of the sequence of natural isomorphisms

$$\begin{aligned}
\mathrm{H}^i(X,\mathcal{G}) &\xrightarrow{\sim} \mathrm{H}^i(X,\varinjlim_\lambda \mathcal{G}_\lambda) && \text{[GD, p. 319, (6.9.9)]}\\
&\xrightarrow{\sim} \varinjlim_\lambda \mathrm{H}^i(X,\mathcal{G}_\lambda)\\
&\xrightarrow{\sim} \varinjlim_\lambda \mathrm{H}^i(\mathcal{X},\kappa^*\mathcal{G}_\lambda) && \text{[EGA, p. 125, (4.1.7)]}\\
&\xrightarrow{\sim} \mathrm{H}^i(\mathcal{X},\varinjlim_\lambda \kappa^*\mathcal{G}_\lambda)\\
&\xrightarrow{\sim} \mathrm{H}^i(\mathcal{X},\kappa^*\varinjlim_\lambda \mathcal{G}_\lambda) \xrightarrow{\sim} \mathrm{H}^i(\mathcal{X},\kappa^*\mathcal{G}).
\end{aligned}$$

Next, for any $\mathcal{G}$ and $\mathcal{H}$ in $\mathcal{A}_{\mathrm{qc}}(X)$ the natural map is an *isomorphism*

$$\mathrm{Hom}_X(\mathcal{G},\mathcal{H}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{X}}(\kappa^*\mathcal{G},\kappa^*\mathcal{H})\tag{3.1.3}$$

For, with $\mathcal{G}_\lambda$ as above, (3.1.3) factors as the sequence of natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_X(\mathcal{G},\mathcal{H}) &\xrightarrow{\sim} \varprojlim_\lambda \mathrm{Hom}_X(\mathcal{G}_\lambda,\mathcal{H})\\
&\xrightarrow{\sim} \varprojlim_\lambda \mathrm{H}^0\big(X,\mathcal{H}om_X(\mathcal{G}_\lambda,\mathcal{H})\big)\\
&\xrightarrow{\sim} \varprojlim_\lambda \mathrm{H}^0\big(\mathcal{X},\kappa^*\mathcal{H}om_X(\mathcal{G}_\lambda,\mathcal{H})\big) && \big(\text{see (3.1.2)}\big)\\
&\xrightarrow{\sim} \varprojlim_\lambda \mathrm{H}^0\big(\mathcal{X},\mathcal{H}om_{\mathcal{X}}(\kappa^*\mathcal{G}_\lambda,\kappa^*\mathcal{H})\big)\\
&\xrightarrow{\sim} \varprojlim_\lambda \mathrm{Hom}_{\mathcal{X}}(\kappa^*\mathcal{G}_\lambda,\kappa^*\mathcal{H})\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{X}}(\varinjlim_\lambda \kappa^*\mathcal{G}_\lambda,\kappa^*\mathcal{H}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{X}}(\kappa^*\mathcal{G},\kappa^*\mathcal{H}).
\end{aligned}$$

Finally, we show the equivalence of the following conditions, for $\mathcal{F} \in \mathcal{A}(\mathcal{X})$:

(1) The functorial map $\alpha(\mathcal{F})\colon \kappa^*Q\kappa_*\mathcal{F} \to \mathcal{F}$ (adjoint to the canonical map $Q\kappa_*\mathcal{F} \to \kappa_*\mathcal{F}$) is an isomorphism.
(2) There exists $\mathcal{G} \in \mathcal{A}_{\mathrm{qc}}(X)$ together with an isomorphism $\kappa^*\mathcal{G} \xrightarrow{\sim} \mathcal{F}$.
(3) $\mathcal{F} \in \mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X})$.

[14] The equality holds because κ_* transforms any flasque resolution of $\kappa^*\mathcal{G}$ into one of $\kappa_*\kappa^*\mathcal{G}$.

Clearly (1) ⇒ (2); and (2) ⇒ (3) because $\varinjlim_\lambda \kappa^*\mathcal{G}_\lambda \xrightarrow{\sim} \kappa^*\mathcal{G}$ ($\mathcal{G}_\lambda$ as before).

Since κ^* commutes with $\varinjlim$ and induces an equivalence of categories from $\mathcal{A}_{\mathrm{c}}(X)$ to $\mathcal{A}_{\mathrm{c}}(\mathfrak{X})$ [EGA, p. 150, (5.1.6)], we see that (3) ⇒ (2).

For $\mathcal{G} \in \mathcal{A}_{\mathrm{qc}}(X)$, let $\beta(\mathcal{G})\colon \mathcal{G} \to Q\kappa_*\kappa^*\mathcal{G}$ be the canonical map (the unique one whose composition with $Q\kappa_*\kappa^*\mathcal{G} \to \kappa_*\kappa^*\mathcal{G}$ is the canonical map $\mathcal{G} \to \kappa_*\kappa^*\mathcal{G}$). Then for any $\mathcal{H} \in \mathcal{A}_{\mathrm{qc}}(X)$ we have the natural commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{H},\mathcal{G}) & \xrightarrow{\text{via }\beta} & \mathrm{Hom}(\mathcal{H}, Q\kappa_*\kappa^*\mathcal{G}) \\
\simeq\downarrow & & \downarrow\simeq \\
\mathrm{Hom}(\kappa^*\mathcal{H},\kappa^*\mathcal{G}) & \xrightarrow{\sim} & \mathrm{Hom}(\mathcal{H},\kappa_*\kappa^*\mathcal{G})
\end{array}$$

where the left vertical arrow is an isomorphism by (3.1.3), the right one is an isomorphism because Q is right-adjoint to $\mathcal{A}_{\mathrm{qc}}(X) \hookrightarrow \mathcal{A}(X)$, and the bottom arrow is an isomorphism because κ_* is right-adjoint to κ^*; so "via β" is an isomorphism for all $\mathcal{H}$, whence $\beta(\mathcal{G})$ *is an isomorphism.* The implication (2) ⇒ (1) follows now from the easily checked fact that $\alpha(\kappa^*\mathcal{G})\circ\kappa^*\beta(\mathcal{G})$ is the identity map of $\kappa^*\mathcal{G}$.

We see also that $Q\kappa_*(\mathcal{A}_{\mathrm{c}}(\mathfrak{X})) \subset \mathcal{A}_{\mathrm{c}}(X)$, since by [EGA, p. 150, (5.1.6)] every $\mathcal{F} \in \mathcal{A}_{\mathrm{c}}(\mathfrak{X})$ is isomorphic to $\kappa^*\mathcal{G}$ for some $\mathcal{G} \in \mathcal{A}_{\mathrm{c}}(X)$, and $\beta(\mathcal{G})$ is an isomorphism.

Thus we have the functors $\kappa^*\colon \mathcal{A}_{\mathrm{qc}}(X) \to \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$ and $Q\kappa_*\colon \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X}) \to \mathcal{A}_{\mathrm{qc}}(X)$, both of which preserve coherence, and the functorial isomorphisms

$$\alpha(\mathcal{F})\colon \kappa^*Q\kappa_*\mathcal{F} \xrightarrow{\sim} \mathcal{F} \qquad \big(\mathcal{F} \in \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})\big),$$
$$\beta(\mathcal{G})\colon \mathcal{G} \xrightarrow{\sim} Q\kappa_*\kappa^*\mathcal{G} \qquad \big(\mathcal{G} \in \mathcal{A}_{\mathrm{qc}}(X)\big).$$

Proposition 3.1.1 results. □

Since κ^* is right-exact, we deduce:

Corollary 3.1.4. *For any affine noetherian formal scheme $\mathfrak{X}$, $\mathcal{F} \in \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$ iff $\mathcal{F}$ is a cokernel of a map of free $\mathcal{O}_{\mathfrak{X}}$-modules (i.e., direct sums of copies of $\mathcal{O}_{\mathfrak{X}}$).*

Corollary 3.1.5. *For any locally noetherian formal scheme $\mathfrak{X}$, $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X}) \subset \mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$, i.e., any $\varinjlim$ of coherent $\mathcal{O}_{\mathfrak{X}}$-modules is quasi-coherent.*

Proof. Being local, the assertion follows from Corollary 3.1.4. □

Corollary 3.1.6 (cf. [Y, 3.4, 3.5])**.** *Let $\mathfrak{X}$ be a locally noetherian formal scheme, and let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules.*

(a) *The kernel, cokernel, and image of any morphism $\mathcal{F} \to \mathcal{G}$ are quasi-coherent.*

(b) *$\mathcal{F}$ is coherent iff $\mathcal{F}$ is locally finitely generated.*

(c) *If $\mathcal{F}$ is coherent and $\mathcal{G}$ is a sub- or quotient module of $\mathcal{F}$ then $\mathcal{G}$ is coherent.*

(d) *If $\mathcal{F}$ is coherent then $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is quasi-coherent; and if also $\mathcal{G}$ is coherent then $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is coherent.*

Proof. The questions are local, so we may assume that $\mathfrak{X} = \mathrm{Spf}(A)$ (A noetherian adic), and, by Corollary 3.1.4, that $\mathcal{F}$ and $\mathcal{G}$ are in $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$. Then, κ^* being exact, Proposition 3.1.1 with $X := \mathrm{Spec}(A)$ and $f_0 :=$ identity reduces the problem to noting that the corresponding statements about coherent and quasi-coherent sheaves on X are true. (These statements, and others, are in [GD, p. 217, Cor. (2.2.2) and p. 228, §(2.7.1)]. Note also that if F is coherent then $\mathcal{H}om_{\mathfrak{X}}(\kappa^*F,\kappa^*G) \cong \kappa^*\mathcal{H}om_X(F,G)$.) □

Corollary 3.1.7. *If $\mathcal{X}$ is a locally noetherian formal scheme then any $\mathcal{F} \in \mathcal{A}_{\vec{c}}(\mathcal{X})$ is the $\varinjlim$ of its coherent $\mathcal{O}_{\mathcal{X}}$-submodules.*

Proof. Note that by Corollary 3.1.6(a) and (b) the sum of any two coherent submodules of $\mathcal{F}$ is again coherent. By definition, $\mathcal{F} = \varinjlim_\mu \mathcal{F}_\mu$ with $\mathcal{F}_\mu$ coherent, and from Corollary 3.1.6(a) and (b) it follows that the canonical image of $\mathcal{F}_\mu$ is a coherent submodule of $\mathcal{F}$, whence the conclusion. □

Corollary 3.1.8. *For any affine noetherian formal scheme $\mathcal{X}$, any $\mathcal{F} \in \mathcal{A}_{\vec{c}}(\mathcal{X})$, and any $i > 0$,*

$$\mathrm{H}^i(\mathcal{X}, \mathcal{F}) = 0.$$

Proof. Taking f_0 in Proposition 3.1.1 to be the identity map, we have $\mathcal{F} \cong \kappa^*\mathcal{G}$ with $\mathcal{G}$ quasi-coherent; and so by (3.1.2), $\mathrm{H}^i(\mathcal{X}, \mathcal{F}) \cong \mathrm{H}^i(\mathrm{Spec}(A), \mathcal{G}) = 0$. □

3.2. Proposition 3.1.1 will now be used to show, for locally noetherian formal schemes $\mathcal{X}$, that $\mathcal{A}_{\vec{c}}(\mathcal{X}) \subset \mathcal{A}(\mathcal{X})$ is plump, and that this inclusion has a right adjoint, extending to derived categories.

Lemma 3.2.1. *Let $\mathcal{X}$ be a locally noetherian formal scheme, let $\mathcal{F} \in \mathcal{A}_{\mathrm{c}}(\mathcal{X})$, and let $(\mathcal{G}_\alpha, \gamma_{\alpha\beta} \colon \mathcal{G}_\beta \to \mathcal{G}_\alpha)_{\alpha,\beta\in\Omega}$ be a directed system in $\mathcal{A}_{\mathrm{c}}(\mathcal{X})$. Then for every $q \geq 0$ the natural map is an isomorphism*

$$\varinjlim_{\alpha} \mathrm{Ext}^q(\mathcal{F}, \mathcal{G}_\alpha) \xrightarrow{\sim} \mathrm{Ext}^q(\mathcal{F}, \varinjlim_{\alpha} \mathcal{G}_\alpha).$$

Proof. For any $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{M}$, let $\mathrm{E}(\mathcal{M})$ denote the standard spectral sequence

$$\mathrm{E}_2^{pq}(\mathcal{M}) := \mathrm{H}^p\big(\mathcal{X}, \mathcal{E}xt^q(\mathcal{F}, \mathcal{M})\big) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{M}).$$

It suffices that the natural map of spectral sequences be an isomorphism

$$\varinjlim \mathrm{E}(\mathcal{G}_\alpha) \xrightarrow{\sim} \mathrm{E}(\varinjlim \mathcal{G}_\alpha) \qquad (\varinjlim := \varinjlim_{\alpha}),$$

and for that we need only check out the E_2^{pq} terms, i.e., show that the natural maps

$$\varinjlim \mathrm{H}^p\big(\mathcal{X}, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G}_\alpha)\big) \to \mathrm{H}^p\big(\mathcal{X}, \varinjlim \mathcal{E}xt^q(\mathcal{F}, \mathcal{G}_\alpha)\big) \to \mathrm{H}^p\big(\mathcal{X}, \mathcal{E}xt^q(\mathcal{F}, \varinjlim \mathcal{G}_\alpha)\big)$$

are isomorphisms. The first one is, because $\mathcal{X}$ is noetherian. So we need only show that the natural map is an isomorphism

$$\varinjlim \mathcal{E}xt^q(\mathcal{F}, \mathcal{G}_\alpha) \xrightarrow{\sim} \mathcal{E}xt^q(\mathcal{F}, \varinjlim \mathcal{G}_\alpha).$$

For this localized question we may assume that $\mathcal{X} = \mathrm{Spf}(A)$ with A a noetherian adic ring. By Proposition 3.1.1 (with f_0 the identity map of $X := \mathrm{Spec}(A)$) there is a coherent $\mathcal{O}_X$-module F and a directed system $(G_\alpha, g_{\alpha\beta} \colon G_\beta \to G_\alpha)_{\alpha,\beta\in\Omega}$ of coherent $\mathcal{O}_X$-modules such that $\mathcal{F} = \kappa^*F$, $\mathcal{G}_\alpha = \kappa^*G_\alpha$, and $\gamma_{\alpha,\beta} = \kappa^*g_{\alpha,\beta}$. Then the well-known natural isomorphisms (see [EGA, (Chapter 0), p. 61, Prop. (12.3.5)]—or the proof of Corollary 3.3.2 below)

$$\begin{aligned}\varinjlim \mathcal{E}xt^q_{\mathcal{X}}(\mathcal{F}, \mathcal{G}_\alpha) &\xrightarrow{\sim} \varinjlim \kappa^*\mathcal{E}xt^q_X(F, G_\alpha) \xrightarrow{\sim} \kappa^*\varinjlim \mathcal{E}xt^q_X(F, G_\alpha)\\ &\xrightarrow{\sim} \kappa^*\mathcal{E}xt^q_X(F, \varinjlim G_\alpha) \xrightarrow{\sim} \mathcal{E}xt^q_{\mathcal{X}}(\kappa^*F, \kappa^*\varinjlim G_\alpha) \xrightarrow{\sim} \mathcal{E}xt^q_{\mathcal{X}}(\mathcal{F}, \varinjlim \mathcal{G}_\alpha)\end{aligned}$$

give the desired conclusion. □

Proposition 3.2.2. *Let $\mathfrak{X}$ be a locally noetherian formal scheme. If*

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F} \to \mathcal{F}_3 \to \mathcal{F}_4$$

is an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules and if $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ are all in $\mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$ (resp. $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$) then $\mathcal{F} \in \mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$ (resp. $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$). Thus $\mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$ and $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$ are plump—hence abelian—subcategories of $\mathcal{A}(\mathfrak{X})$, and both $\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$ and its subcategory $\mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{X})$ are triangulated subcategories of $\mathbf{D}(\mathfrak{X})$. Furthermore, $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$ is closed under arbitrary small $\mathcal{A}(\mathfrak{X})$-colimits.

Proof. Part of the $\mathcal{A}_{\mathrm{qc}}$ case is covered by Corollary 3.1.6(a), and all of it by [Y, Proposition 3.5]. At any rate, since every quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module is locally in $\mathcal{A}_{\vec{\mathrm{c}}} \subset \mathcal{A}_{\mathrm{qc}}$ (see Corollaries 3.1.4 and 3.1.5), it suffices to treat the $\mathcal{A}_{\vec{\mathrm{c}}}$ case.

Let us first show that the kernel $\mathcal{K}$ of an $\mathcal{A}_{\vec{\mathrm{c}}}$ map

$$\psi\colon \varinjlim_{\beta}\mathcal{H}_\beta = \mathcal{H} \to \mathcal{G} = \varinjlim_{\alpha}\mathcal{G}_\alpha \qquad (\mathcal{G}_\alpha, \mathcal{H}_\beta \in \mathcal{A}_{\mathrm{c}}(\mathfrak{X}))$$

is itself in $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$. It will suffice to do so for the kernel $\mathcal{K}_\beta$ of the composition

$$\psi_\beta\colon \mathcal{H}_\beta \xrightarrow{\text{natural}} \mathcal{H} \xrightarrow{\psi} \mathcal{G},$$

since $\mathcal{K} = \varinjlim_{\beta}\mathcal{K}_\beta$. By the case $q = 0$ of Corollary 3.2.1, there is an α such that ψ_β factors as

$$\mathcal{H}_\beta \xrightarrow{\psi_{\beta\alpha}} \mathcal{G}_\alpha \xrightarrow{\text{natural}} \mathcal{G};$$

and then with $\mathcal{K}_{\beta\alpha'}$ $(\alpha' > \alpha)$ the (coherent) kernel of the composed map

$$\mathcal{H}_\beta \xrightarrow{\psi_{\beta\alpha}} \mathcal{G}_\alpha \xrightarrow{\text{natural}} \mathcal{G}_{\alpha'}$$

we have $\mathcal{K}_\beta = \varinjlim_{\alpha'}\mathcal{K}_{\beta\alpha'} \in \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$.

Similarly, we find that $\operatorname{coker}(\psi) \in \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$. Being closed under small direct sums, then, $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$ is closed under arbitrary small $\mathcal{A}(\mathfrak{X})$-colimits [M1, Corollary 2, p. 109].

Consideration of the exact sequence

$$0 \longrightarrow \operatorname{coker}(\mathcal{F}_1 \to \mathcal{F}_2) \longrightarrow \mathcal{F} \longrightarrow \ker(\mathcal{F}_3 \to \mathcal{F}_4) \longrightarrow 0$$

now reduces the original question to where $\mathcal{F}_1 = \mathcal{F}_4 = 0$. Since $\mathcal{F}_3$ is the $\varinjlim$ of its coherent submodules (Corollary 3.1.7) and $\mathcal{F}$ is the $\varinjlim$ of the inverse images of those submodules, we need only show that each such inverse image is in $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$. Thus we may assume $\mathcal{F}_3$ coherent (and $\mathcal{F}_2 = \varinjlim_{\alpha}\mathcal{G}_\alpha$ with $\mathcal{G}_\alpha$ coherent).

The exact sequence $0 \to \mathcal{F}_2 \to \mathcal{F} \to \mathcal{F}_3 \to 0$ represents an element

$$\eta \in \operatorname{Ext}^1(\mathcal{F}_3, \mathcal{F}_2) = \operatorname{Ext}^1(\mathcal{F}_3, \varinjlim_{\alpha}\mathcal{G}_\alpha);$$

and by Corollary 3.2.1, there is an α such that η is the natural image of an element $\eta_\alpha \in \operatorname{Ext}^1(\mathcal{F}_3, \mathcal{G}_\alpha)$, represented by an exact sequence $0 \to \mathcal{G}_\alpha \to \mathcal{F}_\alpha \to \mathcal{F}_3 \to 0$. Then $\mathcal{F}_\alpha$ is coherent, and by [M2, p. 66, Lemma 1.4], we have an isomorphism

$$\mathcal{F} \xrightarrow{\sim} \mathcal{F}_2 \oplus_{\mathcal{G}_\alpha} \mathcal{F}_\alpha.$$

Thus $\mathcal{F}$ is the cokernel of a map in $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$, and so as above, $\mathcal{F} \in \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$. □

Proposition 3.2.3. *On a locally noetherian formal scheme $\mathfrak{X}$, the inclusion functor $j_{\mathfrak{X}}\colon \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X}) \to \mathcal{A}(\mathfrak{X})$ has a right adjoint $Q_{\mathfrak{X}}\colon \mathcal{A}(\mathfrak{X}) \to \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$; and $\mathbf{R}Q_{\mathfrak{X}}$ is right-adjoint to the natural functor $\mathbf{D}(\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})) \to \mathbf{D}(\mathfrak{X})$. In particular, if $\kappa\colon \mathfrak{X} \to X$ is as in Proposition 3.1.1 then $Q_{\mathfrak{X}} \cong \kappa^* Q_X \kappa_*$ and $\mathbf{R}Q_{\mathfrak{X}} \cong \kappa^*\mathbf{R}Q_X\kappa_*$.*

Proof. Since $\mathcal{A}_{\vec{c}}(\mathfrak{X})$ has a small family of (coherent) generators, and is closed under arbitrary small $\mathcal{A}(\mathfrak{X})$-colimits, the existence of $Q_{\mathfrak{X}}$ follows from the Special Adjoint Functor Theorem ([F, p. 90] or [M1, p. 126, Corollary]).[15]

In an abelian category $\mathcal{A}$, a complex J is, by definition, K-injective if for each exact $\mathcal{A}$-complex G, the complex $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(G, J)$ is exact too. Since $j_{\mathfrak{X}}$ is exact, it follows that its right adjoint $Q_{\mathfrak{X}}$ transforms K-injective $\mathcal{A}(\mathfrak{X})$-complexes into K-injective $\mathcal{A}_{\vec{c}}(\mathfrak{X})$-complexes, whence the derived functor $\mathbf{R}Q_{\mathfrak{X}}$ is right-adjoint to the natural functor $\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X})) \to \mathbf{D}(\mathfrak{X})$ (see [Sp, p. 129, Proposition 1.5(b)]).

The next assertion is a corollary of Proposition 3.1.1: any $\mathcal{M} \in \mathcal{A}_{\vec{c}}(\mathfrak{X})$ is isomorphic to $\kappa^*\mathcal{G}$ for some $\mathcal{G} \in \mathcal{A}_{\mathrm{qc}}(X)$, and then for any $\mathcal{N} \in \mathcal{A}(\mathfrak{X})$ there are natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{X}}(j_{\mathfrak{X}}\mathcal{M}, \mathcal{N}) &\cong \mathrm{Hom}_{\mathfrak{X}}(j_{\mathfrak{X}}\kappa^*\mathcal{G}, \mathcal{N}) \\ &\cong \mathrm{Hom}_X(j_X\mathcal{G}, \kappa_*\mathcal{N}) \cong \mathrm{Hom}_{\mathcal{A}_{\mathrm{qc}}(X)}(\mathcal{G}, Q_X\kappa_*\mathcal{N}) \\ &\cong \mathrm{Hom}_{\mathcal{A}_{\vec{c}}(\mathfrak{X})}(\kappa^*\mathcal{G}, \kappa^*Q_X\kappa_*\mathcal{N}) \cong \mathrm{Hom}_{\mathcal{A}_{\vec{c}}(\mathfrak{X})}(\mathcal{M}, \kappa^*Q_X\kappa_*\mathcal{N}).\end{aligned}$$

Moreover, since κ_* has an exact left adjoint (viz. κ^*), therefore, as above, κ_* transforms K-injective $\mathcal{A}(\mathfrak{X})$-complexes into K-injective $\mathcal{A}(X)$-complexes, and it follows at once that $\mathbf{R}Q_{\mathfrak{X}} \cong \kappa^*\mathbf{R}Q_X\kappa_*$. □

3.3. Proposition 3.2.3 applies in particular to any noetherian scheme X. In this case, j_X induces an *equivalence of categories* $\boldsymbol{j}_X\colon \mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X)) \cong \mathbf{D}_{\mathrm{qc}}(X)$, with quasi-inverse $\mathbf{R}Q_X|_{\mathbf{D}_{\mathrm{qc}}(X)}$. (See [H1, p. 133, Cor. 7.19] for bounded-below complexes, and [BN, p. 230, Corollary 5.5] or [AJL, p. 12, Proposition (1.3)] for the general case.) We do not know if such an equivalence, with "$\vec{c}$" in place of "qc," holds for arbitrary noetherian formal schemes. The next result will at least take care of the "properly algebraic" case, see Corollary 3.3.4.

Proposition 3.3.1. *In Proposition 3.1.1, the functor* $\kappa^*\colon \mathbf{D}(X) \to \mathbf{D}(\mathfrak{X})$ *induces equivalences from* $\mathbf{D}_{\mathrm{qc}}(X)$ *to* $\mathbf{D}_{\vec{c}}(\mathfrak{X})$ *and from* $\mathbf{D}_{\mathrm{c}}(X)$ *to* $\mathbf{D}_{\mathrm{c}}(\mathfrak{X})$, *both with quasi-inverse* $\mathbf{R}Q\kappa_*$ (*where* $\mathbf{R}Q$ *stands for* $\boldsymbol{j}_X \circ \mathbf{R}Q_X$).

Proof. Since κ^* is exact, Proposition 3.1.1 implies that $\kappa^*(\mathbf{D}_{\mathrm{qc}}(X)) \subset \mathbf{D}_{\vec{c}}(\mathfrak{X})$ and $\kappa^*(\mathbf{D}_{\mathrm{c}}(X)) \subset \mathbf{D}_{\mathrm{c}}(\mathfrak{X})$. So it will be enough to show that:

(1) If $\mathcal{F} \in \mathbf{D}_{\vec{c}}(\mathfrak{X})$ then the functorial $\mathbf{D}(\mathfrak{X})$-map $\kappa^*\mathbf{R}Q\kappa_*\mathcal{F} \to \mathcal{F}$ adjoint to the natural map $\mathbf{R}Q\kappa_*\mathcal{F} \to \kappa_*\mathcal{F}$ is an isomorphism.

(2) If $\mathcal{G} \in \mathbf{D}_{\mathrm{qc}}(X)$ then the natural map $\mathcal{G} \xrightarrow{\sim} \mathbf{R}Q\kappa_*\kappa^*\mathcal{G}$ is an isomorphism.

(3) If $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(\mathfrak{X})$ then $\mathbf{R}Q\kappa_*\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(X)$.

Since $\mathbf{D}_{\vec{c}}(\mathfrak{X})$ is triangulated (Corollary 3.2.2), we can use way-out reasoning [H1, p. 68, Proposition 7.1 and p. 73, Proposition 7.3] to reduce to where $\mathcal{F}$ or $\mathcal{G}$ is a single sheaf. (For bounded-below complexes we just need the obvious facts that κ^* and the restriction of $\mathbf{R}Q\kappa_*$ to $\mathbf{D}_{\vec{c}}(\mathfrak{X})$ are both bounded-below (= way-out right) functors. For unbounded complexes, we need those functors to be bounded-above as well, which is clear for the exact functor κ^*, and will be shown for $\mathbf{R}Q\kappa_*|_{\mathbf{D}_{\vec{c}}(\mathfrak{X})}$ in Proposition 3.4.4 below.)

[15] It follows that $\mathcal{A}_{\vec{c}}(\mathfrak{X})$ is closed under *all* $\mathcal{A}(\mathfrak{X})$-colimits (not necessarily small): if F is any functor into $\mathcal{A}_{\vec{c}}(X)$ and $\mathcal{F} \in \mathcal{A}(X)$ is a colimit of $j_{\mathfrak{X}} \circ F$, then $Q_{\mathfrak{X}}\mathcal{F}$ is a colimit of F, and the natural map is an isomorphism $\mathcal{F} \xrightarrow{\sim} j_{\mathfrak{X}}Q_{\mathfrak{X}}\mathcal{F}$. (Proof: exercise, given in dual form in [F, p. 80].)

Any $\mathcal{F}\in\mathcal{A}_{\vec{c}}(\mathfrak{X})$ is isomorphic to $\kappa^*\mathcal{G}$ for some $\mathcal{G}\in\mathcal{A}_{\mathrm{qc}}(X)$; and one checks that the natural composed map $\kappa^*\mathcal{G}\to\kappa^*\mathbf{R}Q\kappa_*\kappa^*\mathcal{G}\to\kappa^*\mathcal{G}$ is the identity, whence (2) ⇒ (1). Moreover, if $\mathcal{F}\in\mathcal{A}_{\mathrm{c}}(\mathfrak{X})$ then $\mathcal{G}\cong Q\kappa_*\mathcal{F}\in\mathcal{A}_{\mathrm{c}}(X)$, whence (2) ⇒ (3).

Now a map $\varphi:\mathcal{G}_1\to\mathcal{G}_2$ in $\mathbf{D}^+_{\mathrm{qc}}(X)$ is an isomorphism iff

(∗): the induced map $\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{E}[-n],\mathcal{G}_1)\to\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{E}[-n],\mathcal{G}_2)$ is an isomorphism for every $\mathcal{E}\in\mathcal{A}_{\mathrm{c}}(X)$ and every $n\in\mathbb{Z}$.

(For, if $\mathcal{V}$ is the vertex of a triangle with base φ, then (∗) says that for all $\mathcal{E}$, n, $\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{E}[-n],\mathcal{V})=0$; but if φ is not an isomorphism, i.e., $\mathcal{V}$ has non-vanishing homology, say $H^n(\mathcal{V})\neq 0$ and $H^i(\mathcal{V})=0$ for all $i<n$, then the inclusion map into $H^n(\mathcal{V})$ of any coherent non-zero submodule $\mathcal{E}$ gives a non-zero map $\mathcal{E}[-n]\to\mathcal{V}$.) So for (2) it's enough to check that the natural composition

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{E}[-n],\mathcal{G})&\longrightarrow\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{E}[-n],\mathbf{R}Q\kappa_*\kappa^*\mathcal{G})\\&\xrightarrow{\ \sim\ }\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{E}[-n],\kappa_*\kappa^*\mathcal{G})\xrightarrow{\ \sim\ }\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\kappa^*\mathcal{E}[-n],\kappa^*\mathcal{G})\end{aligned}$$

is the *isomorphism* $\mathrm{Ext}^n_X(\mathcal{E},\mathcal{G})\xrightarrow{\ \sim\ }\mathrm{Ext}^n_{\mathfrak{X}}(\kappa^*\mathcal{E},\kappa^*\mathcal{G})$ in the following consequence of (3.1.2):

Corollary 3.3.2. *With $\kappa\colon\mathfrak{X}\to X$ as in Proposition 3.1.1, for any $\mathcal{L}\in\mathbf{D}_{\mathrm{qc}}(X)$ the natural map $\mathbf{R}\Gamma(X,\mathcal{L})\to\mathbf{R}\Gamma(\mathfrak{X},\kappa^*\mathcal{L})$ is an isomorphism. In particular, for $\mathcal{E}\in\mathbf{D}^-_{\mathrm{c}}(X)$ and $\mathcal{G}\in\mathbf{D}^+_{\mathrm{qc}}(X)$ the natural map $\mathrm{Ext}^n_X(\mathcal{E},\mathcal{G})\to\mathrm{Ext}^n_{\mathfrak{X}}(\kappa^*\mathcal{E},\kappa^*\mathcal{G})$ is an isomorphism.*

Proof. After "way-out" reduction to the case where $\mathcal{L}\in\mathcal{A}_{\mathrm{qc}}(X)$ (the $\mathbf{R}\Gamma$'s are bounded, by Corollary 3.4.3(a) below), the first assertion is given by (3.1.2). To get the second assertion, take $\mathcal{L}:=\mathbf{R}\mathcal{H}om^{\bullet}_X(\mathcal{E},\mathcal{G})$ (which is in $\mathbf{D}^+_{\mathrm{qc}}(X)$, [H1, p. 92, Proposition 3.3]), so that $\kappa^*\mathcal{L}\cong\mathbf{R}\mathcal{H}om^{\bullet}_{\mathfrak{X}}(\kappa^*\mathcal{E},\kappa^*\mathcal{G})$ (as one sees easily after way-out reduction to where $\mathcal{E}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, and further reduction to where X is affine, so that $\mathcal{E}$ has a resolution by finite-rank free modules …). □

Definition 3.3.3. A formal scheme $\mathfrak{X}$ is said to be *properly algebraic* if there exist a noetherian ring B, a B-ideal J, a proper B-scheme X, and an isomorphism from $\mathfrak{X}$ to the J-adic completion of X.

Corollary 3.3.4. *On a properly algebraic formal scheme $\mathfrak{X}$ the natural functor $\boldsymbol{j}_{\mathfrak{X}}\colon\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))\to\mathbf{D}_{\vec{c}}(\mathfrak{X})$ is an equivalence of categories, with quasi-inverse $\mathbf{R}Q_{\mathfrak{X}}$; and therefore $\boldsymbol{j}_{\mathfrak{X}}\circ\mathbf{R}Q_{\mathfrak{X}}$ is right-adjoint to the inclusion $\mathbf{D}_{\vec{c}}(\mathfrak{X})\hookrightarrow\mathbf{D}(\mathfrak{X})$.*

Proof. If $\mathfrak{X}$ is properly algebraic, then with $A:=J$-adic completion of B and $I:=JA$, it holds that $\mathfrak{X}$ is the I-adic completion of $X\otimes_B A$, and so we may assume the hypotheses and conclusions of Proposition 3.1.1. We have also, as above, the equivalence of categories $\boldsymbol{j}_X:\mathbf{D}(\mathcal{A}_{\mathrm{qc}}(X))\to\mathbf{D}_{\mathrm{qc}}(X)$; and so the assertion follows from Propositions 3.3.1 and 3.2.3. □

Proposition 3.3.5. *For any map $g\colon\mathfrak{Z}\to\mathfrak{X}$ of locally noetherian formal schemes,*

$$\mathbf{L}g^*(\mathbf{D}_{\vec{c}}(\mathfrak{X}))\subset\mathbf{D}_{\mathrm{qc}}(\mathfrak{Z}).$$

If $\mathfrak{X}$ is properly algebraic, then

$$\mathbf{L}g^*(\mathbf{D}_{\vec{c}}(\mathfrak{X}))\subset\mathbf{D}_{\vec{c}}(\mathfrak{Z}).$$

Proof. The first assertion, being local on $\mathscr{X}$, follows from the second. Assuming $\mathscr{X}$ properly algebraic we may, as in the proof of Corollary 3.3.4, place ourselves in the situation of Proposition 3.1.1, so that any $\mathcal{G} \in \mathbf{D}_{\vec{c}}(\mathscr{X})$ is, by Corollary 3.3.4 and Proposition 3.1.1, isomorphic to $\kappa^*\mathcal{E}$ for some $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(X)$. By [AJL, p. 10, Proposition (1.1)]), $\mathcal{E}$ is isomorphic to a $\varinjlim$ of bounded-above quasi-coherent flat complexes (see the very end of the proof of *ibid.*); and therefore $\mathcal{G} \cong \kappa^*\mathcal{E}$ is isomorphic to a K-flat complex of $\mathcal{A}_{\vec{c}}(\mathscr{X})$-objects. Since $\mathbf{L}g^*$ agrees with g^* on K-flat complexes, and $g^*(\mathcal{A}_{\vec{c}}(\mathscr{X})) \subset \mathcal{A}_{\vec{c}}(\mathscr{Z})$, we are done. ☐

Remarks 3.3.6. (a) Let $\mathscr{X}$ be a properly algebraic formal scheme (necessarily noetherian) with ideal of definition $\mathscr{I}$, and set $I := \mathrm{H}^0(\mathscr{X}, \mathscr{I}) \subset A := \mathrm{H}^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$. Then *$A$ is a noetherian I-adic ring, and $\mathscr{X}$ is* $\mathrm{Spf}(A)$*-isomorphic to the I-adic completion of a proper A-scheme.* (The canonical map $\mathscr{X} \to \mathrm{Spf}(A)$ is given by [GD, p. 407, (10.4.6)].)

Indeed, with B, J and X as in Definition 3.3.3, [EGA, p. 125, Theorem (4.1.7)] implies that the topological ring

$$A = \varprojlim_{n>0} \mathrm{H}^0(\mathscr{X}, \mathcal{O}_{\mathscr{X}}/\mathscr{I}^n\mathcal{O}_{\mathscr{X}}) = \varprojlim_{n>0} \mathrm{H}^0(X, \mathcal{O}_X/I^n\mathcal{O}_X)$$

is the J-adic completion of the noetherian B-algebra $A_0 := H^0(X, \mathcal{O}_X)$, and that the J-adic and I-adic topologies on A are the same; and then $\mathscr{X}$ is the I-adic completion of $X \otimes_{A_0} A$.

(b) It follows that a quasi-compact formal scheme $\mathscr{X}$ is properly algebraic iff so is each of its connected components.

(c) While (a) provides a less relaxed characterization of properly algebraic formal schemes than Definition 3.3.3, Corollary 3.3.8 below provides a more relaxed one.

Lemma 3.3.7. *Let X be a locally noetherian scheme, $\mathcal{I}_1 \subset \mathcal{I}_2$ two quasi-coherent $\mathcal{O}_X$-ideals, Z_i the support of $\mathcal{O}_X/\mathcal{I}_i$, and $\mathscr{X}_i$ the completion $X_{/Z_i}$ $(i = 1, 2)$. Suppose that $\mathcal{I}_1\mathcal{O}_{\mathscr{X}_2}$ is an ideal of definition of $\mathscr{X}_2$. Then $\mathscr{X}_2$ is a union of connected components of $\mathscr{X}_1$ (with the induced formal-subscheme structure).*

Proof. We need only show that Z_2 is open in Z_1. Locally we have a noetherian ring A and A-ideals $I \subset J$ equal to their own radicals such that with $\hat{A}$ the J-adic completion, $J^n\hat{A} \subset I\hat{A}$ for some $n > 0$; and we want the natural map $A/I \twoheadrightarrow A/J$ to be *flat.*[16] It suffices that the localization $(A/I)_{1+J} \to (A/J)_{1+J} = A/J$ by the multiplicatively closed set $1 + J$ be an isomorphism, i.e., that its kernel $J(A/I)_{1+J}$ be nilpotent (hence (0), since A/I is reduced.) But this is so because the natural map $A_{1+J} \to \hat{A}$ is faithfully flat, and therefore $J^nA_{1+J} \subset IA_{1+J}$. ☐

Corollary 3.3.8. *Let A be a noetherian ring, let I be an A-ideal, and let $\hat{A}$ be the I-adic completion of A. Let $f_0 \colon X \to \mathrm{Spec}(A)$ be a separated finite-type scheme-map, let Z be a closed subscheme of $f_0^{-1}(\mathrm{Spec}(A/I))$, let $\mathscr{X} = X_{/Z}$ be the completion*

[16] If A is a ring, and L a finitely generated A-ideal such that the natural surjection $A \to A/L$ is flat, then $L/L^2 = \mathrm{Tor}_1^A(A/L, A/L) = 0$, whence $(1 - \ell)L = (0)$ for some $\ell \in L$, whence $\ell = \ell^2$ and $L = \ell A$, so that $A \cong L \times A/L$ and $\mathrm{Spec}(A/L) \hookrightarrow \mathrm{Spec}(A)$ is open.

of X along Z, and let $f\colon \mathcal{X} \to \mathrm{Spf}(\hat{A})$ be the formal-scheme map induced by f_0:

$$\begin{array}{ccc} \mathcal{X} := X_{/Z} & \longrightarrow & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_0} \\ \mathrm{Spf}(\hat{A}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

If f is proper (see §1.2.2) then $\mathcal{X}$ is properly algebraic.

Proof. Consider a compactification of f_0 (see [Lü, Theorem 3.2]):

$$X \underset{\text{open}}{\hookrightarrow} \overline{X} \xrightarrow[\text{proper}]{\bar{f}_0} \mathrm{Spec}(A).$$

Since f is proper, therefore Z is proper over $\mathrm{Spec}(A)$, hence closed in $\overline{X}$. Thus we may replace f_0 by $\bar{f}_0$, i.e., we may assume f_0 proper. Since f, being proper, is adic, Lemma 3.3.7, with $Z_2 := Z$ and $Z_1 := f_0^{-1}(\mathrm{Spec}(A/I))$, shows that $\mathcal{X}$ is a union of connected components of the properly algebraic formal scheme $X_{/Z_1}$. Conclude by Remark 3.3.6(b). □

3.4. To deal with unbounded complexes we need the following boundedness results on certain derived functors. (See, e.g., Propositions 3.5.1 and 3.5.3 below.)

3.4.1. Refer to §1.2.2 for the definitions of separated, resp. affine, maps.

A formal scheme $\mathcal{X}$ is *separated* if the natural map $f_{\mathcal{X}}\colon \mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is separated, i.e., for some—hence any—ideal of definition $\mathcal{J}$, the scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{J})$ is separated. For example, any locally noetherian affine formal scheme is separated.

A locally noetherian formal scheme $\mathcal{X}$ is affine if and only if the map $f_{\mathcal{X}}$ is affine, i.e., for some—hence any—ideal of definition $\mathcal{J}$, the scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{J})$ is affine. Hence the intersection $\mathcal{V} \cap \mathcal{V}'$ of any two affine open subsets of a separated locally noetherian formal scheme $\mathcal{Y}$ is again affine. In other words, the inclusion $\mathcal{V} \hookrightarrow \mathcal{Y}$ is an affine map. More generally, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a map of locally noetherian formal schemes, if $\mathcal{Y}$ is separated, and if $\mathcal{V}$ and $\mathcal{V}'$ are affine open subsets of $\mathcal{Y}$ and $\mathcal{X}$ respectively, then $f^{-1}\mathcal{V} \cap \mathcal{V}'$ is affine [GD, p. 282, (5.8.10)].

Lemma 3.4.2. *If $g\colon \mathcal{X} \to \mathcal{Y}$ is an affine map of locally noetherian formal schemes, then every $\mathcal{M} \in \mathcal{A}_{\vec{c}}(\mathcal{X})$ is g_*-acyclic, i.e., $R^i g_*\mathcal{M} = 0$ for all $i > 0$. More generally, if $\mathcal{G} \in \mathbf{D}_{\vec{c}}(\mathcal{X})$ and $e \in \mathbb{Z}$ are such that $H^i(\mathcal{G}) = 0$ for all $i \ge e$, then $H^i(\mathbf{R}g_*\mathcal{G}) = 0$ for all $i \ge e$.*

Proof. $R^i g_*\mathcal{M}$ is the sheaf associated to the presheaf $\mathcal{U} \mapsto \mathrm{H}^i(g^{-1}(\mathcal{U}), \mathcal{M})$, ($\mathcal{U}$ open in $\mathcal{Y}$) [EGA, Chap. 0, (12.2.1)]. If $\mathcal{U}$ is affine then so is $g^{-1}(\mathcal{U}) \subset \mathcal{X}$, and Corollary 3.1.8 gives $\mathrm{H}^i(g^{-1}(\mathcal{U}), \mathcal{M}) = 0$ for all $i > 0$.

Now consider in $\mathbf{K}(\mathcal{X})$ a quasi-isomorphism $\mathcal{G} \to I$ where I is a "special" inverse limit of injective resolutions I_{-e} of the truncations $\tau_{\ge e}\mathcal{G}$, so that $H^i(\mathbf{R}g_*\mathcal{G})$ is the sheaf associated to the presheaf $\mathcal{U} \mapsto \mathrm{H}^i(\Gamma(g^{-1}\mathcal{U}, I))$, see [Sp, p. 134, 3.13]. If C_{-e} is the kernel of the split surjection $I_{-e} \to I_{1-e}$ then $C_{-e}[e]$ is an injective resolution of $H^e(\mathcal{G}) \in \mathcal{A}_{\vec{c}}(\mathcal{X})$, and so for any affine open $\mathcal{U} \subset \mathcal{Y}$ and $i > e$, $\mathrm{H}^i(\Gamma(g^{-1}\mathcal{U}, C_{-e})) = 0$. Applying [Sp, p. 126, Lemma], one finds then that for $i \ge e$ the natural map $\mathrm{H}^i(\Gamma(g^{-1}\mathcal{U}, I)) \to \mathrm{H}^i(\Gamma(g^{-1}\mathcal{U}, I_{-e}))$ is an isomorphism. Consequently if $H^i(\mathcal{G}) = 0$ for all $i \ge e$ (whence $I_{-e} \cong \tau_{\ge e}\mathcal{G} = 0$ in $\mathbf{D}(\mathcal{X})$) then $\mathrm{H}^i(\Gamma(g^{-1}\mathcal{U}, I)) = 0$. □

Proposition 3.4.3. *Let $\mathcal{X}$ be a noetherian formal scheme. Then:*

(a) *The functor $\mathbf{R}\Gamma(\mathcal{X},-)$ is bounded-above on $\mathbf{D}_{\vec{c}}(\mathcal{X})$. In other words, there is an integer $e \ge 0$ such that if $\mathcal{G} \in \mathbf{D}_{\vec{c}}(\mathcal{X})$ and $H^i(\mathcal{G}) = 0$ for all $i \ge i_0$ then $\mathrm{H}^i(\mathbf{R}\Gamma(\mathcal{X},-)) = 0$ for all $i \ge i_0 + e$.*

(b) *For any formal-scheme map $f\colon \mathcal{X} \to \mathcal{Y}$ with $\mathcal{Y}$ quasi-compact, the functor $\mathbf{R}f_*$ is bounded-above on $\mathbf{D}_{\vec{c}}(\mathcal{X})$, i.e., there is an integer $e \ge 0$ such that if $\mathcal{G} \in \mathbf{D}_{\vec{c}}(\mathcal{X})$ and $H^i(\mathcal{G}) = 0$ for all $i \ge i_0$ then $H^i(\mathbf{R}f_*\mathcal{G}) = 0$ for all $i \ge i_0 + e$.*

Proof. Let us prove (b). (The proof of (a) is the same, *mutatis mutandis.*) Suppose first that $\mathcal{X}$ is separated. Since $\mathcal{Y}$ has a finite affine open cover and $\mathbf{R}f_*$ commutes with open base change, we may assume that $\mathcal{Y}$ itself is affine. Let $n(\mathcal{X})$ be the least positive integer n such that there exists a finite affine open cover $\mathcal{X} = \cup_{i=1}^n \mathcal{X}_i$, and let us show by induction on $n(\mathcal{X})$ that $e := n(\mathcal{X}) - 1$ *will do.*

The case $n(\mathcal{X}) = 1$ is covered by Lemma 3.4.2. So assume that $n := n(\mathcal{X}) \ge 2$, let $\mathcal{X} = \cup_{i=1}^n \mathcal{X}_i$ be an affine open cover, and let $u_1\colon \mathcal{X}_1 \hookrightarrow \mathcal{X}$, $u_2\colon \cup_{i=2}^n \mathcal{X}_i \hookrightarrow \mathcal{X}$, $u_3\colon \cup_{i=2}^n (\mathcal{X}_1 \cap \mathcal{X}_i) \hookrightarrow \mathcal{X}$ be the respective inclusion maps. Note that $\mathcal{X}_1 \cap \mathcal{X}_i$ is affine because $\mathcal{X}$ is separated, see §3.4.1. So by the inductive hypothesis, the assertion holds for the maps $f_i := f \circ u_i$ $(i = 1, 2, 3)$. Apply the Δ-functor $\mathbf{R}f_*$ to the "Mayer-Vietoris" triangle

$$\mathcal{G} \longrightarrow \mathbf{R}u_{1*}u_1^*\mathcal{G} \oplus \mathbf{R}u_{2*}u_2^*\mathcal{G} \longrightarrow \mathbf{R}u_{3*}u_3^*\mathcal{G} \xrightarrow{+1}$$

(derived from the standard exact sequence

$$0 \to \mathcal{E} \to u_{1*}u_1^*\mathcal{E} \oplus u_{2*}u_2^*\mathcal{E} \to u_{3*}u_3^*\mathcal{E} \to 0$$

where $\mathcal{G} \to \mathcal{E}$ is a K-injective resolution) to get the $\mathbf{D}(\mathcal{Y})$-triangle

$$\mathbf{R}f_*\mathcal{G} \longrightarrow \mathbf{R}f_{1*}u_1^*\mathcal{G} \oplus \mathbf{R}f_{2*}u_2^*\mathcal{G} \longrightarrow \mathbf{R}f_{3*}u_3^*\mathcal{G} \xrightarrow{+1}$$

whose associated long exact homology sequence yields the assertion for f.

The general case can now be disposed of with a similar Mayer-Vietoris induction on the least number of *separated* open subsets needed to cover $\mathcal{X}$. □

Proposition 3.4.4. *Let X be a separated noetherian scheme, $Z \subset X$ a closed subscheme, and $\kappa = \kappa_{\mathcal{X}}\colon \mathcal{X} = X_{/Z} \to X$ the completion map. Then the functor $\mathbf{R}Q_X\kappa_*$ is bounded-above on $\mathbf{D}_{\vec{c}}(\mathcal{X})$.*

Proof. Let $n(X)$ be the least number of affine open subschemes needed to cover X. When X is affine, Q_X is the sheafification of the global section functor, and since κ_* is exact and, being right adjoint to the *exact* functor κ^*, preserves K-injectivity, we find that for any $\mathcal{F} \in \mathbf{D}(\mathcal{X})$, $\mathbf{R}Q_X\kappa_*\mathcal{F}$ is the sheafification of the complex $\mathbf{R}\Gamma(X, \kappa_*\mathcal{F}) = \mathbf{R}\Gamma(\mathcal{X}, \mathcal{F})$. Thus Proposition 3.4.3(a) yields the desired result for $n(X) = 1$.

Proceed by induction when $n(X) > 1$, using a "Mayer-Vietoris" argument as in the proof of Proposition 3.4.3. The enabling points are that if $v\colon V \hookrightarrow X$ is an open immersion with $n(V) < n(X)$, giving rise to the natural commutative diagram

$$\begin{array}{ccc} V_{/Z\cap V} =: \mathcal{V} & \xrightarrow{\kappa_{\mathcal{V}}} & V \\ {\scriptstyle\hat{v}}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ \mathcal{X} & \xrightarrow[\kappa_{\mathcal{X}}]{} & X \end{array}$$

then there are natural isomorphisms, for $\mathcal{F} \in \mathbf{D}_{\vec{c}}(\mathfrak{X})$ and $v_*^{\mathrm{qc}}\colon \mathcal{A}_{\mathrm{qc}}(V) \to \mathcal{A}_{\mathrm{qc}}(X)$ the restriction of v_*:

$$\mathbf{R}Q_X\kappa_{\mathfrak{X}*}\mathbf{R}\hat{v}_*\hat{v}^*\mathcal{F} \cong \mathbf{R}Q_X\mathbf{R}v_*\kappa_{\mathcal{V}*}\hat{v}^*\mathcal{F} \cong \mathbf{R}v_*^{\mathrm{qc}}\mathbf{R}Q_V\kappa_{\mathcal{V}*}\hat{v}^*\mathcal{F},$$

and the functor $\mathbf{R}Q_V\kappa_{\mathcal{V}*}\hat{v}^*$ is bounded-above, by the inductive hypothesis on $n(V) < n(X)$, as is $\mathbf{R}v_*^{\mathrm{qc}}$, by the proof of [AJL, p. 12, Proposition (1.3)]. □

3.5. Here are some examples of how boundedness is used.

Proposition 3.5.1. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper map of noetherian formal schemes. Then*

$$\mathbf{R}f_*\mathbf{D}_{\mathrm{c}}(\mathfrak{X}) \subset \mathbf{D}_{\mathrm{c}}(\mathfrak{Y}) \quad \text{and} \quad \mathbf{R}f_*\mathbf{D}_{\vec{c}}(\mathfrak{X}) \subset \mathbf{D}_{\vec{c}}(\mathfrak{Y}).$$

Proof. For any coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$, $\mathbf{R}f_*\mathcal{M} \in \mathbf{D}_{\mathrm{c}}(\mathfrak{Y})$ [EGA, p. 119, (3.4.2)]. Since $\mathfrak{X}$ is noetherian, the homology functors $H^i\mathbf{R}f_*$ commute with $\varinjlim$ on $\mathcal{O}_{\mathfrak{X}}$-modules, whence $\mathbf{R}f_*\mathcal{N} \in \mathbf{D}_{\vec{c}}(\mathfrak{Y})$ for all $\mathcal{N} \in \mathcal{A}_{\vec{c}}(\mathfrak{X})$. $\mathbf{R}f_*$ being bounded on $\mathbf{D}_{\vec{c}}(\mathfrak{X})$ (Proposition 3.4.3(b)), way-out reasoning [H1, p. 74, (iii)] completes the proof. □

Proposition 3.5.2. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a map of quasi-compact formal schemes, with $\mathfrak{X}$ noetherian. Then the functor $\mathbf{R}f_*|_{\mathbf{D}_{\vec{c}}(\mathfrak{X})}$ commutes with small direct sums, i.e., for any small family $(\mathcal{E}_\alpha)$ in $\mathbf{D}_{\vec{c}}(\mathfrak{X})$ the natural map*

$$\oplus_\alpha(\mathbf{R}f_*\mathcal{E}_\alpha) \to \mathbf{R}f_*(\oplus_\alpha\mathcal{E}_\alpha)$$

is a $\mathbf{D}(\mathfrak{Y})$-isomorphism.

Proof. It suffices to look at the induced homology maps in each degree, i.e., setting $R^if_* := H^i\mathbf{R}f_*$ $(i \in \mathbb{Z})$, we need to show that *the natural map*

$$\oplus_\alpha(R^if_*\mathcal{E}_\alpha) \xrightarrow{\;\sim\;} R^if_*(\oplus_\alpha\mathcal{E}_\alpha).$$

is an isomorphism.

For any $\mathcal{F} \in \mathbf{D}_{\vec{c}}(\mathfrak{X})$ and any integer $e \geq 0$, the vertex $\mathcal{G}$ of a triangle based on the natural map t_{i-e} from $\mathcal{F}$ to the truncation $\mathcal{F}^{\geq i-e}$ (see (4.1.3)) satisfies $H^j(\mathcal{G}) = 0$ for all $j \geq i-e-1$; so if e is the integer in Proposition 3.4.3(b), then $R^{i-1}f_*\mathcal{G} = R^if_*\mathcal{G} = 0$, and the map induced by t_{i-e} is an *isomorphism*

$$R^if_*\mathcal{F} \xrightarrow{\;\sim\;} R^if_*\mathcal{F}^{\geq i-e}.$$

We can therefore replace each $\mathcal{E}_\alpha$ by $\mathcal{E}_\alpha^{\geq i-e}$, i.e., we may assume that the $\mathcal{E}_\alpha$ are uniformly bounded below.

We may assume further that each complex $\mathcal{E}_\alpha$ is injective, hence f_*-acyclic (i.e., the canonical map is an *isomorphism* $f_*\mathcal{E}_\alpha \xrightarrow{\;\sim\;} \mathbf{R}f_*\mathcal{E}_\alpha$). Since $\mathfrak{X}$ is noetherian the R^if_* commute with direct sums, and so each component of $\oplus_\alpha\mathcal{E}_\alpha$ is an f_*-acyclic $\mathcal{O}_{\mathfrak{X}}$-module. This implies that the bounded-below complex $\oplus_\alpha\mathcal{E}_\alpha$ is itself f_*-acyclic. Thus in the natural commutative diagram

$$\begin{array}{ccc} \oplus_\alpha(f_*\mathcal{E}_\alpha) & \xrightarrow{\;\sim\;} & f_*(\oplus_\alpha\mathcal{E}_\alpha) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \oplus_\alpha(\mathbf{R}f_*\mathcal{E}_\alpha) & \longrightarrow & \mathbf{R}f_*(\oplus_\alpha\mathcal{E}_\alpha) \end{array}$$

the top and both sides are isomorphisms, whence so is the bottom. □

The following Proposition generalizes [EGA, p. 92, Theorem (4.1.5)].

Proposition 3.5.3. *Let $f_0 : X \to Y$ be a proper map of locally noetherian schemes, $W \subset Y$ a closed subset, $Z := f_0^{-1}W$, $\kappa_{\mathcal{Y}} : \mathcal{Y} = Y_{/W} \to Y$ and $\kappa_{\mathcal{X}} : \mathcal{X} = X_{/Z} \to X$ the respective (flat) completion maps, and $f : \mathcal{X} \to \mathcal{Y}$ the map induced by f_0. Then for $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(X)$ there is a natural functorial isomorphism*

$$\theta_{\mathcal{E}} : \kappa_{\mathcal{Y}}^* \mathbf{R} f_{0*} \mathcal{E} \xrightarrow{\sim} \mathbf{R} f_* \kappa_{\mathcal{X}}^* \mathcal{E}.$$

Proof. Let $\theta_{\mathcal{E}}$ be the map adjoint to the natural composition

$$\mathbf{R} f_{0*} \mathcal{E} \longrightarrow \mathbf{R} f_{0*} \kappa_{\mathcal{X}*} \kappa_{\mathcal{X}}^* \mathcal{E} \xrightarrow{\sim} \kappa_{\mathcal{Y}*} \mathbf{R} f_* \kappa_{\mathcal{X}}^* \mathcal{E}.$$

To verify that $\theta_{\mathcal{E}}$ is an isomorphism we may assume Y affine, say $Y = \mathrm{Spec}(A)$, and then $W = \mathrm{Spec}(A/I)$ for some A-ideal I. Let $\hat{A}$ be the I-adic completion of A, so that there is a natural cartesian diagram

$$\begin{array}{ccc} X \otimes_A \hat{A} =: X_1 & \xrightarrow{k_X} & X \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_0} \\ \mathrm{Spec}(\hat{A}) =: Y_1 & \xrightarrow[k_Y]{} & Y \end{array}$$

Here k_Y is flat, and the natural map is an isomorphism $k_Y^* \mathbf{R} f_{0*} \mathcal{E} \xrightarrow{\sim} \mathbf{R} f_{1*} k_X^* \mathcal{E}$: since $\mathbf{R} f_{0*}$ (resp. $\mathbf{R} f_{1*}$) is bounded-above on $\mathbf{D}_{\mathrm{qc}}(X)$ (resp. $\mathbf{D}_{\mathrm{qc}}(X_1)$), see Proposition 3.4.3(b), way-out reasoning reduces this assertion to the well-known case where $\mathcal{E}$ is a single quasi-coherent $\mathcal{O}_X$-module. Simple considerations show then that we can replace f_0 by f_1 and $\mathcal{E}$ by $k_X^* \mathcal{E}$; in other words, we can assume $A = \hat{A}$.

From Proposition 3.5.1 it follows that $\mathbf{R} f_{0*} \mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(Y)$ and $\mathbf{R} f_* \kappa_{\mathcal{X}}^* \mathcal{E} \in \mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{Y})$. Recalling the equivalences in Proposition 3.3.1, we see that any $\mathcal{F} \in \mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{Y})$ is isomorphic to $\kappa_{\mathcal{Y}}^* \mathcal{F}_0$ for some $\mathcal{F}_0 \in \mathbf{D}_{\mathrm{qc}}(Y)$ (so that $\mathbf{L} f_0^* \mathcal{F}_0 \in \mathbf{D}_{\mathrm{qc}}(X)$), and that there is a sequence of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{Y}}(\mathcal{F}, \kappa_{\mathcal{Y}}^* \mathbf{R} f_{0*} \mathcal{E}) &\xrightarrow{\sim} \mathrm{Hom}_Y(\mathcal{F}_0, \mathbf{R} f_{0*} \mathcal{E}) \\ &\xrightarrow{\sim} \mathrm{Hom}_X(\mathbf{L} f_0^* \mathcal{F}_0, \mathcal{E}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{X}}(\kappa_{\mathcal{X}}^* \mathbf{L} f_0^* \mathcal{F}_0, \kappa_{\mathcal{X}}^* \mathcal{E}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{X}}(\mathbf{L} f^* \kappa_{\mathcal{Y}}^* \mathcal{F}_0, \kappa_{\mathcal{X}}^* \mathcal{E}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Y}}(\mathcal{F}, \mathbf{R} f_* \kappa_{\mathcal{X}}^* \mathcal{E}). \end{aligned}$$

The conclusion follows. □

4. Global Grothendieck Duality on formal schemes.

Theorem 4.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a map of quasi-compact formal schemes, with $\mathcal{X}$ noetherian, and let $\boldsymbol{j} : \mathbf{D}(\mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X})) \to \mathbf{D}(\mathcal{X})$ be the natural functor. Then the Δ-functor $\mathbf{R} f_* \circ \boldsymbol{j}$ has a right Δ-adjoint. In fact there is a bounded-below Δ-functor $f^\times : \mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X}))$ and a map of Δ-functors $\tau : \mathbf{R} f_* \boldsymbol{j} f^\times \to \mathbf{1}$ such that for all $\mathcal{G} \in \mathbf{D}(\mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X}))$ and $\mathcal{F} \in \mathbf{D}(\mathcal{Y})$, the composed map (in the derived category of abelian groups)*

$$\begin{aligned} \mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X})}(\mathcal{G}, f^\times \mathcal{F}) &\xrightarrow{\text{natural}} \mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}(\mathcal{Y})}(\mathbf{R} f_* \boldsymbol{j} \mathcal{G}, \mathbf{R} f_* \boldsymbol{j} f^\times \mathcal{F}) \\ &\xrightarrow{\text{via } \tau} \mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}(\mathcal{Y})}(\mathbf{R} f_* \boldsymbol{j} \mathcal{G}, \mathcal{F}) \end{aligned}$$

is an isomorphism.

With Corollary 3.3.4 this gives:

Corollary 4.1.1. *If $\mathcal{X}$ is properly algebraic then the restriction of $\mathbf{R}f_*$ to $\mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{X})$ has a right Δ-adjoint (also to be denoted $f^\times$ when no confusion results).*

Remarks. 1. Recall that over any abelian category $\mathcal{A}$ in which each complex $\mathcal{F}$ has a K-injective resolution $\rho(\mathcal{F})$, we can set

$$\mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathcal{G},\mathcal{F}) := \mathrm{Hom}^\bullet_{\mathcal{A}}\big(\mathcal{G},\rho(\mathcal{F})\big) \qquad \big(\mathcal{G},\mathcal{F}\in\mathbf{D}(\mathcal{A})\big);$$

and there are natural isomorphisms

$$\mathrm{H}^i\mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}}(\mathcal{G},\mathcal{F}) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}\big(\mathcal{G},\mathcal{F}[i]\big) \qquad (i\in\mathbb{Z}).$$

2. Application of homology to the second assertion in the Theorem reveals that it is equivalent to the first one.

3. We do not know in general (when $\mathcal{X}$ is not properly algebraic) that the functor $\boldsymbol{j}$ is fully faithful—$\boldsymbol{j}$ has a right adjoint (identity)$^\times \cong \mathbf{R}Q_{\mathcal{X}}$ (see Proposition 3.2.3), but it may be that for some $\mathcal{E}\in\mathcal{A}_{\vec{\mathrm{c}}}(\mathcal{X})$ the natural map $\mathcal{E}\to\mathbf{R}Q_{\mathcal{X}}\boldsymbol{j}\mathcal{E}$ is not an isomorphism.

4. For *proper* f it is customary to write $f^!$ instead of $f^\times$.

5. Theorem 4.1 includes the case where $\mathcal{X}$ and $\mathcal{Y}$ are ordinary noetherian schemes. The next Corollary relates the formal situation to the ordinary one.

Corollary 4.1.2. *Let A be a noetherian adic ring with ideal of definition I, set $Y := \mathrm{Spec}(A)$ and $W := \mathrm{Spec}(A/I) \subset Y$. Let $f_0\colon X\to Y$ be a proper map and set $Z := f_0^{-1}W$, so that there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} := X_{/Z} & \xrightarrow{\kappa_{\mathcal{X}}} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_0} \\ \mathcal{Y} := \mathrm{Spf}(A) & \xrightarrow[\kappa_{\mathcal{Y}}]{} & Y \end{array}$$

with $\kappa_{\mathcal{X}}$ and $\kappa_{\mathcal{Y}}$ the respective (flat) completion maps, and f the (proper) map induced by f_0. Then the map adjoint to the natural composition

$$\mathbf{R}f_*\kappa_{\mathcal{X}}^*f_0^!\kappa_{\mathcal{Y}*} \xrightarrow{3.5.3} \kappa_{\mathcal{Y}}^*\mathbf{R}f_*f_0^!\kappa_{\mathcal{Y}*} \longrightarrow \kappa_{\mathcal{Y}}^*\kappa_{\mathcal{Y}*} \longrightarrow \mathbf{1}$$

is an isomorphism of functors—from $\mathbf{D}(\mathcal{Y})$ to $\mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{X})$, see Corollary 4.1.1—

$$\kappa_{\mathcal{X}}^*f_0^!\kappa_{\mathcal{Y}*} \xrightarrow{\ \sim\ } f^!.$$

Proof. For any $\mathcal{E}\in\mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{X})$ set $\mathcal{E}_0 := \mathbf{R}Q_X\kappa_{\mathcal{X}*}\mathcal{E}\in\mathbf{D}_{\mathrm{qc}}(X)$. Using Proposition 3.3.1 we have then for any $\mathcal{F}\in\mathbf{D}(\mathcal{Y})$ the sequence of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{X}}(\mathcal{E},\kappa_{\mathcal{X}}^*f_0^!\kappa_{\mathcal{Y}*}\mathcal{F}) &\xrightarrow{\ \sim\ } \mathrm{Hom}_X(\mathcal{E}_0, f_0^!\kappa_{\mathcal{Y}*}\mathcal{F})\\ &\xrightarrow{\ \sim\ } \mathrm{Hom}_Y(\mathbf{R}f_{0*}\mathcal{E}_0,\kappa_{\mathcal{Y}*}\mathcal{F})\\ &\xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{Y}}(\kappa_{\mathcal{Y}}^*\mathbf{R}f_{0*}\mathcal{E}_0,\mathcal{F})\\ &\xrightarrow[3.5.3]{\ \sim\ } \mathrm{Hom}_{\mathcal{Y}}(\mathbf{R}f_*\kappa_{\mathcal{X}}^*\mathcal{E}_0,\mathcal{F}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{Y}}(\mathbf{R}f_*\mathcal{E},\mathcal{F}). \end{aligned}$$

Thus $\kappa_{\mathcal{X}}^*f_0^!\kappa_{\mathcal{Y}*}$ is right-adjoint to $\mathbf{R}f_*|_{\mathbf{D}_{\vec{\mathrm{c}}}(\mathcal{X})}$, whence the conclusion. □

Proof of Theorem 4.1. 1. Following Deligne [H1, p. 417, top], we begin by considering for $\mathcal{M}\in\mathcal{A}(\mathcal{X})$ the functorial flasque *Godement resolution*

$$0\to\mathcal{M}\to G^0(\mathcal{M})\to G^1(\mathcal{M})\to\cdots.$$

Here, with $G^{-2}(\mathcal{M}) := 0$, $G^{-1}(\mathcal{M}) := \mathcal{M}$, and for $i \geq 0$, $K^i(\mathcal{M})$ the cokernel of $G^{i-2}(\mathcal{M}) \to G^{i-1}(\mathcal{M})$, the sheaf $G^i(\mathcal{M})$ is specified inductively by

$$G^i(\mathcal{M})\big(\mathcal{U}\big) := \prod_{x\in\mathcal{U}} K^i(\mathcal{M})_x \qquad (\mathcal{U} \text{ open in } \mathcal{X}).$$

One shows by induction on i that all the functors G^i and K^i (from $\mathcal{A}(\mathcal{X})$ to itself) are *exact.* Moreover, for $i \geq 0$, $G^i(\mathcal{M})$, being flasque, is f_**-acyclic,* i.e.,

$$R^j f_* G^i(\mathcal{M}) = 0 \quad \text{for all } j > 0.$$

The category $\mathcal{A}_{\vec{c}}(\mathcal{X})$ has small colimits (Proposition 3.2.2), and is generated by its coherent members, of which there exists a small set containing representatives of every isomorphism class. So the Special Adjoint Functor Theorem ([F, p. 90] or [M1, p. 126, Corollary]) guarantees that a right-exact functor F from $\mathcal{A}_{\vec{c}}$ into an abelian category $\mathcal{A}'$ has a right adjoint iff F is *continuous* in the sense that it commutes with filtered direct limits, i.e., for any small directed system $(\mathcal{M}_\alpha,\, \varphi_{\alpha\beta}\colon \mathcal{M}_\beta \to \mathcal{M}_\alpha)$ in $\mathcal{A}_{\vec{c}}$, with $\varinjlim_\alpha \mathcal{M}_\alpha = (\mathcal{M},\, \varphi_\alpha\colon \mathcal{M}_\alpha \to \mathcal{M})$ it holds that

$$\big(F(\mathcal{M}), F(\varphi_\alpha)\big) = \varinjlim_\alpha \big(F(\mathcal{M}_\alpha), F(\varphi_{\alpha\beta})\big).$$

Accordingly, for constructing right adjoints we need to replace the restrictions of G^i and K^i to $\mathcal{A}_{\vec{c}}(\mathcal{X})$ by continuous functors.

Lemma 4.1.3. *Let $\mathcal{X}$ be a locally noetherian formal scheme and let G be a functor from $\mathcal{A}_{\mathrm{c}}(\mathcal{X})$ to a category $\mathcal{A}'$ in which direct limits exist for all small directed systems. Let $j\colon \mathcal{A}_{\mathrm{c}}(\mathcal{X}) \hookrightarrow \mathcal{A}_{\vec{c}}(\mathcal{X})$ be the inclusion functor. Then:*

(a) *There exists a continuous functor $G_{\vec{c}}\colon \mathcal{A}_{\vec{c}}(\mathcal{X}) \to \mathcal{A}'$ and an isomorphism of functors $\varepsilon\colon G \xrightarrow{\sim} G_{\vec{c}} \circ j$ such that for any map of functors $\psi\colon G \to F\circ j$ with F continuous, there is a unique map of functors $\psi_{\vec{c}}\colon G_{\vec{c}} \to F$ such that ψ factors as*

$$G \xrightarrow{\varepsilon} G_{\vec{c}} \circ j \xrightarrow{\text{via } \psi_{\vec{c}}} F\circ j\,.$$

(b) *Assume that $\mathcal{A}'$ is abelian, and has exact filtered direct limits* (*i.e., satisfies Grothendieck's axiom* AB5). *Then if G is exact, so is $G_{\vec{c}}$.*

Proof. (a) For $\mathcal{M} \in \mathcal{A}_{\vec{c}}(\mathcal{X})$, let $(\mathcal{M}_\alpha)$ be the directed system of coherent $\mathcal{O}_{\mathcal{X}}$-submodules of $\mathcal{M}$, and set

$$G_{\vec{c}}(\mathcal{M}) := \varinjlim_\alpha G(\mathcal{M}_\alpha).$$

For any $\mathcal{A}_{\vec{c}}(\mathcal{X})$-map $\nu\colon \mathcal{M} \to \mathcal{N}$ and any α, there exists a coherent submodule $\mathcal{N}_\beta \subset \mathcal{N}$ such that $\nu|_{\mathcal{M}_\alpha}$ factors as $\mathcal{M}_\alpha \to \mathcal{N}_\beta \hookrightarrow \mathcal{N}$ (Corollary 3.1.7 and Lemma 3.2.1, with $q = 0$); and the resulting composition

$$\nu'_\alpha\colon G(\mathcal{M}_\alpha) \to G(\mathcal{N}_\beta) \to G_{\vec{c}}(\mathcal{N})$$

does not depend on the choice of $\mathcal{N}_\beta$. We define the map

$$G_{\vec{c}}(\nu)\colon G_{\vec{c}}(\mathcal{M}) = \varinjlim_\alpha G(\mathcal{M}_\alpha) \to G_{\vec{c}}(\mathcal{N})$$

to be the unique one whose composition with $G(\mathcal{M}_\alpha) \to G_{\vec{c}}(\mathcal{M})$ is ν'_α for all α. Verification of the rest of assertion (a) is straightforward.

(b) Let $0 \to \mathcal{M} \to \mathcal{N} \xrightarrow{\pi} \mathcal{Q} \to 0$ be an exact sequence in $\mathcal{A}_{\vec{c}}(\mathcal{X})$. Let $(\mathcal{N}_\beta)$ be the filtered system of coherent submodules of $\mathcal{N}$, so that $\mathcal{N} = \varinjlim \mathcal{N}_\beta$ (Corollary 3.1.7). Then $(\mathcal{M}\cap\mathcal{N}_\beta)$ is a filtered system of coherent $\mathcal{O}_{\mathcal{X}}$-modules whose $\varinjlim$

is $\mathcal{M}$, and $(\pi\mathcal{N}_\beta)$ is a filtered system of coherent $\mathcal{O}_{\mathfrak{X}}$-modules whose $\varinjlim$ is $\mathcal{Q}$ (see Corollary 3.1.6). The exactness of $G_{\vec{c}}$ is then made apparent by application of $\varinjlim_\beta$ to the system of exact sequences

$$0 \to G(\mathcal{M} \cap \mathcal{N}_\beta) \to G(\mathcal{N}_\beta) \to G(\pi\mathcal{N}_\beta) \to 0. \qquad \square$$

Now for $\mathcal{M} \in \mathcal{A}_{\vec{c}}(\mathfrak{X})$, the $\varinjlim$ of the system of Godement resolutions of all the coherent submodules $\mathcal{M}_\alpha \subset \mathcal{M}$ is a functorial resolution

$$0 \to \mathcal{M} \to G^0_{\vec{c}}(\mathcal{M}) \to G^1_{\vec{c}}(\mathcal{M}) \to \cdots;$$

and similarly we see that $K^i_{\vec{c}}(\mathcal{M})$ is the cokernel of $G^{i-2}_{\vec{c}}(\mathcal{M}) \to G^{i-1}_{\vec{c}}(\mathcal{M})$. By (b) above, the continuous functors $G^i_{\vec{c}}$ and $K^i_{\vec{c}}$ are exact; and $G^i_{\vec{c}}(\mathcal{M}) = \varinjlim G^i(\mathcal{M}_\alpha)$ is f_*-acyclic since $G^i(\mathcal{M}_\alpha)$ is, and—$\mathfrak{X}$ being noetherian—the functors R^jf_* commute with $\varinjlim$. Proposition 3.4.3(b) implies then that there is an integer $e \ge 0$ such that for all $\mathcal{M} \in \mathcal{A}_{\vec{c}}(\mathfrak{X})$, $K^e_{\vec{c}}(\mathcal{M})$ is f_*-acyclic. So if we define the exact functors $\mathcal{D}^i\colon \mathcal{A}_{\vec{c}}(\mathfrak{X}) \to \mathcal{A}(\mathfrak{X})$ by

$$\mathcal{D}^i(\mathcal{M}) = \begin{cases} G^i_{\vec{c}}(\mathcal{M}) & (0 \le i < e) \\ K^e_{\vec{c}}(\mathcal{M}) & (i = e) \\ 0 & (i > e) \end{cases}$$

then for $\mathcal{M} \in \mathcal{A}_{\vec{c}}(\mathfrak{X})$, each $\mathcal{D}^i(\mathcal{M})$ is f_*-acyclic and the natural sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{\delta(\mathcal{M})} \mathcal{D}^0(\mathcal{M}) \xrightarrow{\delta^0(\mathcal{M})} \mathcal{D}^1(\mathcal{M}) \xrightarrow{\delta^1(\mathcal{M})} \mathcal{D}^2(\mathcal{M}) \longrightarrow \cdots \longrightarrow \mathcal{D}^e(\mathcal{M}) \longrightarrow 0$$

is exact. In short, the sequence $\mathcal{D}^0 \to \mathcal{D}^1 \to \mathcal{D}^2 \to \cdots \to \mathcal{D}^e \to 0$ is an *exact, continuous, f_*-acyclic, finite resolution of the inclusion functor* $\mathcal{A}_{\vec{c}}(\mathfrak{X}) \hookrightarrow \mathcal{A}(\mathfrak{X})$.

2. We have then a Δ-functor $(\mathcal{D}^\bullet, \mathrm{Id})\colon \mathbf{K}(\mathcal{A}_{\vec{c}}(\mathfrak{X})) \to \mathbf{K}(\mathfrak{X})$ which assigns an f_*-acyclic resolution to each $\mathcal{A}_{\vec{c}}(\mathfrak{X})$-complex $\mathcal{G} = (\mathcal{G}^p)_{p\in\mathbb{Z}}$:

$$(\mathcal{D}^\bullet\mathcal{G})^m := \bigoplus_{p+q=m} \mathcal{D}^q(\mathcal{G}^p) \qquad (m \in \mathbb{Z},\ 0 \le q \le e),$$

the differential $(\mathcal{D}^\bullet\mathcal{G})^m \to (\mathcal{D}^\bullet\mathcal{G})^{m+1}$ being defined on $\mathcal{D}^q(\mathcal{G}^p)$ $(p + q = m)$ to be $d' + (-1)^p d''$ where $d'\colon \mathcal{D}^q(\mathcal{G}^p) \to \mathcal{D}^q(\mathcal{G}^{p+1})$ comes from the differential in $\mathcal{G}$ and $d'' = \delta^q(\mathcal{G}^p)\colon \mathcal{D}^q(\mathcal{G}^p) \to \mathcal{D}^{q+1}(\mathcal{G}^p)$.

It is elementary to check that the natural map $\delta(\mathcal{G})\colon \mathcal{G} \to \mathcal{D}^\bullet\mathcal{G}$ is a *quasi-isomorphism.* The canonical maps are $\mathbf{D}(\mathfrak{Y})$*-isomorphisms*

$$f_*\mathcal{D}^\bullet(\mathcal{G}) \xrightarrow{\ \sim\ } \mathbf{R}f_*\mathcal{D}^\bullet(\mathcal{G}) \xleftarrow[\mathbf{R}f_*\delta(\mathcal{G})]{\ \sim\ } \mathbf{R}f_*\mathcal{G}, \tag{4.1.3}$$

i.e., the natural map $\alpha^i\colon H^i\big(f_*\mathcal{D}^\bullet(\mathcal{G})\big) \to H^i\big(\mathbf{R}f_*\mathcal{D}^\bullet(\mathcal{G})\big)$ is an isomorphism for all $i \in \mathbb{Z}$: this holds for bounded-below $\mathcal{G}$ because $\mathcal{D}^\bullet(\mathcal{G})$ is a complex of f_*-acyclic objects; and for arbitrary $\mathcal{G}$ since for any $n \in \mathbb{Z}$, with $\mathcal{G}^{\ge n}$ denoting the truncation

$$\cdots \to 0 \to 0 \to \operatorname{coker}(\mathcal{G}^{n-1} \to \mathcal{G}^n) \to \mathcal{G}^{n+1} \to \mathcal{G}^{n+2} \to \cdots \tag{4.1.3}$$

there is a natural commutative diagram

$$\begin{array}{ccc} H^i\big(f_*\mathcal{D}^\bullet(\mathcal{G})\big) & \xrightarrow{\ \alpha^i\ } & H^i\big(\mathbf{R}f_*\mathcal{D}^\bullet(\mathcal{G})\big) \\ {\scriptstyle\beta^i_n}\big\downarrow & & \big\downarrow{\scriptstyle\gamma^i_n} \\ H^i\big(f_*\mathcal{D}^\bullet(\mathcal{G}^{\ge n})\big) & \xrightarrow[\alpha^i_n]{} & H^i\big(\mathbf{R}f_*\mathcal{D}^\bullet(\mathcal{G}^{\ge n})\big) \end{array}$$

in which, when $n \ll i$, β_n^i is an isomorphism (since $\mathcal{G}$ and $\mathcal{G}^{\geq n}$ are identical in all degrees $> n$), γ_n^i is an isomorphism (by Proposition 3.4.3(b) applied to the mapping cone of the natural map $\mathcal{D}^\bullet(\mathcal{G}) \xrightarrow{\sim} \mathcal{G} \longrightarrow \mathcal{G}^{\geq n} \xrightarrow{\sim} \mathcal{D}^\bullet(\mathcal{G}^{\geq n})$), and α_n^i is an isomorphism (since $\mathcal{G}^{\geq n}$ is bounded below).

Thus we have realized $\mathbf{R}f_* \circ \boldsymbol{j}$ at the homotopy level, via the functor $\mathcal{C}^\bullet := f_*\mathcal{D}^\bullet$; and our task is now to find a right adjoint at this level.

3. Each functor $\mathcal{C}^p = f_*\mathcal{D}^p \colon \mathcal{A}_{\vec{c}}(\mathfrak{X}) \to \mathcal{A}(\mathfrak{Y})$ is exact, since $R^1f_*(\mathcal{D}^p(\mathcal{M})) = 0$ for all $\mathcal{M} \in \mathcal{A}_{\vec{c}}(\mathfrak{X})$. $\mathcal{C}^p$ is continuous, since $\mathcal{D}^p$ is and, $\mathfrak{X}$ being noetherian, f_* commutes with $\varinjlim$. As before, the Special Adjoint Functor Theorem yields that $\mathcal{C}^p$ *has a right adjoint* $\mathcal{C}_p \colon \mathcal{A}(\mathfrak{Y}) \to \mathcal{A}_{\vec{c}}(\mathfrak{X})$.

For each $\mathcal{A}(\mathfrak{Y})$-complex $\mathcal{F} = (\mathcal{F}^p)_{p\in\mathbb{Z}}$ let $\mathcal{C}_\bullet\mathcal{F}$ be the $\mathcal{A}_{\vec{c}}(\mathfrak{X})$-complex with

$$(\mathcal{C}_\bullet\mathcal{F})^m := \prod_{p-q=m} \mathcal{C}_q\mathcal{F}^p \qquad (m \in \mathbb{Z}, 0 \leq q \leq e),$$

and with differential $(\mathcal{C}_\bullet\mathcal{F})^m \to (\mathcal{C}_\bullet\mathcal{F})^{m+1}$ the unique map making the following diagram commute for all r, s with $r - s = m+1$:

$$\begin{array}{ccc} \prod\limits_{p-q=m} \mathcal{C}_q\mathcal{F}^p & \longrightarrow & \prod\limits_{p-q=m+1} \mathcal{C}_q\mathcal{F}^p \\ \big\downarrow & & \big\downarrow \\ \mathcal{C}_s\mathcal{F}^{r-1} \oplus \mathcal{C}_{s+1}\mathcal{F}^r & \xrightarrow[d_\prime + (-1)^r d_{\prime\prime}]{} & \mathcal{C}_s\mathcal{F}^r \end{array}$$

where the vertical arrows come from projections, where $d_\prime \colon \mathcal{C}_s\mathcal{F}^{r-1} \to \mathcal{C}_s\mathcal{F}^r$ corresponds to the differential in $\mathcal{F}$, and where, with $\delta_s \colon \mathcal{C}_{s+1} \to \mathcal{C}_s$ corresponding by adjunction to $f_*(\delta^s) \colon \mathcal{C}^s \to \mathcal{C}^{s+1}$,

$$d_{\prime\prime} := (-1)^s\delta_s(\mathcal{F}^r) \colon \mathcal{C}_{s+1}\mathcal{F}^r \to \mathcal{C}_s\mathcal{F}^r.$$

This construction leads naturally to a Δ-functor $(\mathcal{C}_\bullet, \mathrm{Id}) \colon \mathbf{K}(\mathfrak{Y}) \to \mathbf{K}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))$. The adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{A}_{\vec{c}}(\mathfrak{X})}(\mathcal{M}, \mathcal{C}_p\mathcal{N}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}(\mathfrak{Y})}(\mathcal{C}^p\mathcal{M}, \mathcal{N}) \qquad \big(\mathcal{M} \in \mathcal{A}_{\vec{c}}(\mathfrak{X}),\ \mathcal{N} \in \mathcal{A}(\mathfrak{Y})\big)$$

applied componentwise produces an isomorphism of complexes of abelian groups

$$\mathrm{Hom}^\bullet_{\mathcal{A}_{\vec{c}}(\mathfrak{X})}(\mathcal{G}, \mathcal{C}_\bullet\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{A}(\mathfrak{Y})}(\mathcal{C}^\bullet\mathcal{G}, \mathcal{F}) \tag{4.1.4}$$

for all $\mathcal{A}_{\vec{c}}(\mathfrak{X})$-complexes $\mathcal{G}$ and $\mathcal{A}(\mathfrak{Y})$-complexes $\mathcal{F}$.

4. The isomorphism (4.1.4) suggests that we use $\mathcal{C}_\bullet$ to construct $f^\times$, as follows. Recall that a complex $\mathcal{J} \in \mathbf{K}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))$ is K-injective iff for each exact complex $\mathcal{G} \in \mathbf{K}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))$, the complex $\mathrm{Hom}^\bullet_{\mathcal{A}_{\vec{c}}(\mathfrak{X})}(\mathcal{G}, \mathcal{J})$ is exact too. By (4.1.3), $\mathcal{C}^\bullet\mathcal{G}$ is exact if $\mathcal{G}$ is; so it follows from (4.1.4) that *if $\mathcal{F}$ is K-injective in* $\mathbf{K}(\mathfrak{Y})$ *then $\mathcal{C}_\bullet\mathcal{F}$ is K-injective in* $\mathbf{K}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))$. Thus if $\mathbf{K_I}(-) \subset \mathbf{K}(-)$ is the full subcategory of all K-injective complexes, then we have a Δ-functor $(\mathcal{C}_\bullet, \mathrm{Id}) \colon \mathbf{K_I}(\mathfrak{Y}) \to \mathbf{K_I}(\mathcal{A}_{\vec{c}}(\mathfrak{X}))$. Associating a K-injective resolution to each complex in $\mathcal{A}(\mathfrak{Y})$ leads to a Δ-functor $(\rho, \Theta) \colon \mathbf{D}(\mathfrak{Y}) \to \mathbf{K_I}(\mathfrak{Y})$.[17] This ρ is bounded below: an $\mathcal{A}(\mathfrak{Y})$-complex $\mathcal{E}$ such that $H^i(\mathcal{E}) = 0$ for all $i < n$ is quasi-isomorphic to the truncated complex $\mathcal{E}^{\geq n}$, which

[17] In fact (ρ, Θ) is an equivalence of Δ-categories, see [L4, §1.7]. But note that Θ need not be the identity morphism, i.e., one may not be able to find a complete family of K-injective resolutions commuting with translation. For example, we do not know that every periodic complex has a periodic K-injective resolution.

is in turn quasi-isomorphic to an injective complex $\mathcal{F}$ which vanishes in all degrees below n. (Such an $\mathcal{F}$ is K-injective.) Finally, one can define $f^{\times}$ to be the composition of the functors

$$\mathbf{D}(\mathcal{Y}) \xrightarrow{\rho} \mathbf{K_I}(\mathcal{Y}) \xrightarrow{\mathcal{C}_\bullet} \mathbf{K_I}(\mathcal{A}_{\vec{c}}(\mathcal{X})) \xrightarrow{\text{natural}} \mathbf{D}(\mathcal{A}_{\vec{c}}(\mathcal{X})),$$

and check, via (4.1.3) and (4.1.4) that Theorem 4.1 is satisfied. (This involves some tedium with respect to Δ-details.) □

5. Torsion sheaves.

Refer to §1.2 for notation and first sorites regarding torsion sheaves.

In Paragraphs 5.1 and 5.2 we develop basic properties of quasi-coherent torsion sheaves and their derived categories on locally noetherian formal schemes—see for example Propositions 5.2.1, 5.2.4, 5.2.5, and 5.2.7. (There is some overlap here with §4 in [Y].) Such properties will be used in section 6 to prove Theorem 2 of section 1, and some of its consequences. In particular, in Paragraph 5.3 we establish, for a separated noetherian scheme $\mathcal{X}$, an *equivalence of categories* $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathcal{X})) \xrightarrow{\approx} \mathbf{D}_{\mathrm{qct}}(\mathcal{X})$, thereby enabling the use of $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$—rather than $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathcal{X}))$—in Theorem 6.1 (≅ Theorem 2).

5.1. This paragraph treats categories of quasi-coherent torsion sheaves on locally noetherian formal schemes.

Proposition 5.1.1. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of noetherian formal schemes, and let $\mathcal{M} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$. Then $f_*\mathcal{M} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{Y})$. Moreover, if f is pseudo-proper (see* §1.2.2) *and $\mathcal{M}$ is coherent then $f_*\mathcal{M}$ is coherent.*

Proof. Let $\mathcal{J} \subset \mathcal{O}_{\mathcal{X}}$ and $\mathcal{I} \subset \mathcal{O}_{\mathcal{Y}}$ be ideals of definition such that $\mathcal{I}\mathcal{O}_{\mathcal{X}} \subset \mathcal{J}$, and let

$$\mathcal{X}_{[n]} := (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{J}^n) \xrightarrow{f_{[n]}} (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}/\mathcal{I}^n) =: \mathcal{Y}_{[n]} \qquad (n > 0)$$

be the scheme-maps induced by f, so that if j_n and i_n are the canonical closed immersions then $fj_n = i_n f_{[n]}$. Let $\mathcal{M}_n := \mathcal{H}om(\mathcal{O}_{\mathcal{X}}/\mathcal{J}^n, \mathcal{M})$, so that

$$\mathcal{M} = \varGamma'_{\mathcal{X}}\mathcal{M} = \varinjlim_n \mathcal{M}_n = \varinjlim_n j_{n*}j_n^*\mathcal{M}_n.$$

Since $\mathcal{J}^n$ is a coherent $\mathcal{O}_{\mathcal{X}}$-ideal [GD, p. 427], therefore $\mathcal{M}_n$ is quasi-coherent (Corollary 3.1.6(d)), and it is straightforward to check that $i_{n*}f_{[n]*}j_n^*\mathcal{M}_n \in \mathcal{A}_{\mathrm{qct}}(\mathcal{Y})$. Thus, $\mathcal{X}$ being noetherian, and by Corollary 5.1.3 below,

$$f_*\mathcal{M} = f_*\varinjlim_n \mathcal{M}_n \cong \varinjlim_n f_*j_{n*}j_n^*\mathcal{M}_n = \varinjlim_n i_{n*}f_{[n]*}j_n^*\mathcal{M}_n \in \mathcal{A}_{\mathrm{qct}}(\mathcal{Y}).$$

When f is pseudo-proper then every $f_{[n]}$ is proper; and if $\mathcal{M} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ is coherent then so is $f_*\mathcal{M}$, because for some n, $f_*\mathcal{M} = f_*j_{n*}j_n^*\mathcal{M} = i_{n*}f_{[n]*}j_n^*\mathcal{M}$. □

Proposition 5.1.2. *Let Z be a closed subset of a locally noetherian scheme X, and let $\kappa\colon \mathcal{X} \to X$ be the completion of X along Z. Then the functors κ^* and κ_* restrict to inverse isomorphisms between the categories $\mathcal{A}_Z(X)$ and $\mathcal{A}_{\mathrm{t}}(\mathcal{X})$, and between the categories $\mathcal{A}_{\mathrm{qc}Z}(X)$ and $\mathcal{A}_{\mathrm{qct}}(\mathcal{X})$; and if $\mathcal{M} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ is coherent, then so is $\kappa_*\mathcal{M}$.*

Proof. Let $\mathcal{J}$ be a quasi-coherent $\mathcal{O}_X$-ideal such that the support of $\mathcal{O}_X/\mathcal{J}$ is Z. Applying $\varinjlim_n$ to the natural isomorphisms

$$\kappa^*\mathcal{H}om_X(\mathcal{O}_X/\mathcal{J}^n,\mathcal{N}) \xrightarrow{\sim} \mathcal{H}om_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^n\mathcal{O}_{\mathfrak{X}},\kappa^*\mathcal{N}) \qquad (\mathcal{N}\in\mathcal{A}(X),\ n>0)$$

we get a functorial isomorphism $\kappa^*\varGamma'_Z \xrightarrow{\sim} \varGamma'_{\mathfrak{X}}\kappa^*$, and hence $\kappa^*(\mathcal{A}_Z(X))\subset\mathcal{A}_t(\mathfrak{X})$. Applying $\varinjlim_n$ to the natural isomorphisms

$$\mathcal{H}om_X(\mathcal{O}_X/\mathcal{J}^n,\kappa_*\mathcal{M}) \xrightarrow{\sim} \kappa_*\mathcal{H}om_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^n\mathcal{O}_{\mathfrak{X}},\mathcal{M}) \qquad (\mathcal{M}\in\mathcal{A}(\mathfrak{X}),\ n>0)$$

we get a functorial isomorphism $\varGamma'_Z\kappa_* \xrightarrow{\sim} \kappa_*\varGamma'_{\mathfrak{X}}$, and hence $\kappa_*(\mathcal{A}_t(\mathfrak{X}))\subset\mathcal{A}_Z(X)$.

As κ is a pseudo-proper map of locally noetherian formal schemes ((0) being an ideal of definition of X), we see as in the proof of Lemma 5.1.1 that for $\mathcal{M}\in\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$, $\kappa_*\mathcal{M}$ is a $\varinjlim$ of quasi-coherent $\mathcal{O}_X$-modules, so is itself quasi-coherent, and $\kappa_*\mathcal{M}$ is coherent whenever $\mathcal{M}$ is.[18]

Finally, examining stalks (see §1.2) we find that the natural transformations $1\to\kappa_*\kappa^*$ and $\kappa^*\kappa_*\to 1$ induce isomorphisms

$$\varGamma'_Z\mathcal{N} \xrightarrow{\sim} \kappa_*\kappa^*\varGamma'_Z\mathcal{N} \qquad \big(\mathcal{N}\in\mathcal{A}(X)\big),$$
$$\kappa^*\kappa_*\varGamma'_{\mathfrak{X}}\mathcal{M} \xrightarrow{\sim} \varGamma'_{\mathfrak{X}}\mathcal{M} \qquad \big(\mathcal{M}\in\mathcal{A}(\mathfrak{X})\big).$$

□

Corollary 5.1.3. *If $\mathfrak{X}$ is a locally noetherian formal scheme then $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ is plump in $\mathcal{A}(\mathfrak{X})$ and closed under small $\mathcal{A}(\mathfrak{X})$-colimits.*[19]

Proof. The assertions are local, and so, since $\mathcal{A}_{\mathrm{t}}(\mathfrak{X})$ is plump (§1.2.1), Proposition 5.1.2 (where κ^* commutes with $\varinjlim$) enables reduction to well-known facts about $\mathcal{A}_{\mathrm{qc}Z}(X)\subset\mathcal{A}(X)$ with X an affine noetherian (ordinary) scheme. □

Lemma 5.1.4. *Let $\mathfrak{X}$ be a locally noetherian formal scheme. If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module then $\varGamma'_{\mathfrak{X}}\mathcal{M}\in\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ is the $\varinjlim$ of its coherent submodules. In particular, $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})\subset\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$.*

Proof. Let $\mathcal{J}$ be an ideal of definition of $\mathfrak{X}$. For any positive integer n, let $\mathfrak{X}_{[n]}$ be the scheme $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^n)$, let $j_n\colon\mathfrak{X}_{[n]}\to\mathfrak{X}$ be the canonical closed immersion, and let $\mathcal{M}_n:=\mathcal{H}om(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^n,\mathcal{M})\subset\varGamma'_{\mathfrak{X}}(\mathcal{M})$, so that $\mathcal{M}_n\in\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ (Corollary 3.1.6(d)). Then the quasi-coherent $\mathcal{O}_{\mathfrak{X}_{[n]}}$-module $j_n^*\mathcal{M}_n$ is the $\varinjlim$ of its coherent submodules [GD, p. 319, (6.9.9)], hence so is $\mathcal{M}_n=j_{n*}j_n^*\mathcal{M}_n$ (since j_n^* and j_{n*} preserve both $\varinjlim$ and coherence [GD, p. 115, (5.3.13) and (5.3.15)]), and hence so is $\varGamma'_{\mathfrak{X}}\mathcal{M}=\varinjlim_n\mathcal{M}_n$. That $\varinjlim_n\mathcal{M}_n\in\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ results from Corollary 5.1.3. □

Corollary 5.1.5. *If $\mathfrak{X}$ is a locally noetherian formal scheme then the inclusion functor $j^{\mathrm{t}}_{\mathfrak{X}}\colon\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})\hookrightarrow\mathcal{A}(\mathfrak{X})$ has a right adjoint $Q^{\mathrm{t}}_{\mathfrak{X}}$.*

Proof. In view of Corollary 5.1.3 and Lemma 5.1.4, one could just apply the Special Adjoint Functor theorem. More specifically, since $\varGamma'_{\mathfrak{X}}$ is right-adjoint to the inclusion $\mathcal{A}_{\mathrm{t}}(\mathfrak{X})\hookrightarrow\mathcal{A}(\mathfrak{X})$, and $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})\subset\mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$ (Corollary 3.1.5), it follows from Lemma 5.1.4 that the restriction of $\varGamma'_{\mathfrak{X}}$ to $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$ is right-adjoint to $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})\hookrightarrow\mathcal{A}_{\vec{\mathrm{c}}}(X)$; and by Proposition 3.2.3, $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})\hookrightarrow\mathcal{A}(\mathfrak{X})$ has a right adjoint $Q_{\mathfrak{X}}$; so $Q^{\mathrm{t}}_{\mathfrak{X}}:=\varGamma'_{\mathfrak{X}}\circ Q_{\mathfrak{X}}$ is right-adjoint to $j^{\mathrm{t}}_{\mathfrak{X}}$. (Similarly, $Q_{\mathfrak{X}}\circ\varGamma'_{\mathfrak{X}}$ is right-adjoint to $j^{\mathrm{t}}_{\mathfrak{X}}$.) □

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[18] The noetherian assumption in Lemma 5.1.1 is needed only for commutativity of f_* with $\varinjlim$, a condition clearly satisfied by $f=\kappa$ in the present situation.

[19] Actually, $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ is closed under *all* $\mathcal{A}(\mathfrak{X})$-colimits—see footnote under Proposition 3.2.3.

Remark. For an ordinary noetherian scheme X we have $Q^{\mathrm{t}}_X = Q_X$ (see §3.1). More generally, if $\kappa\colon \mathcal{X} \to X$ is as in Proposition 5.1.2, then $Q^{\mathrm{t}}_{\mathcal{X}} = \kappa^* \varGamma_Z Q_X \kappa_*$. Hence Proposition 5.1.1 (applied to open immersions $\mathcal{X} \hookrightarrow \mathcal{Y}$ with $\mathcal{X}$ affine) lets us construct the functor $Q^{\mathrm{t}}_{\mathcal{Y}}$ for any noetherian formal scheme $\mathcal{Y}$ by mimicking the construction for ordinary schemes (cf. [I, p. 187, Lemme 3.2].)

5.2. The preceding results carry over to derived categories.

From Corollary 5.1.3 it follows that on a locally noetherian formal scheme $\mathcal{X}$, $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$ is a triangulated subcategory of $\mathbf{D}(X)$, closed under direct sums.

Proposition 5.2.1. *Let $\mathcal{X}$ be a locally noetherian formal scheme. Set $\mathcal{A}_{\mathrm{t}} := \mathcal{A}_{\mathrm{t}}(\mathcal{X})$, the category of torsion $\mathcal{O}_{\mathcal{X}}$-modules, and let $\boldsymbol{j}\colon \mathbf{D}(\mathcal{A}_{\mathrm{t}}) \to \mathbf{D}(\mathcal{X})$ be the natural functor. Then*:

(a) *An $\mathcal{O}_{\mathcal{X}}$-complex $\mathcal{E}$ is in $\mathbf{D}_{\mathrm{t}}(\mathcal{X})$ iff the natural map $\boldsymbol{j}\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E} \to \mathcal{E}$ is a $\mathbf{D}(\mathcal{X})$-isomorphism.*

(b) *If $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ then $\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{A}_{\mathrm{t}})$.*

(c) *The functor $\boldsymbol{j}$ and its right adjoint $\mathbf{R}\varGamma'_{\mathcal{X}}$ induce quasi-inverse equivalences between $\mathbf{D}(\mathcal{A}_{\mathrm{t}})$ and $\mathbf{D}_{\mathrm{t}}(\mathcal{X})$ and between $\mathbf{D}_{\mathrm{qc}}(\mathcal{A}_{\mathrm{t}})$ and $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$.*

Proof. (a) For $\mathcal{F} \in \mathbf{D}(\mathcal{A}_{\mathrm{t}})$ (e.g., $\mathcal{F} := \mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E}$), any complex isomorphic to $\boldsymbol{j}\mathcal{F}$ is clearly in $\mathbf{D}_{\mathrm{t}}(\mathcal{X})$.

Suppose conversely that $\mathcal{E} \in \mathbf{D}_{\mathrm{t}}(\mathcal{X})$. The assertion that $\boldsymbol{j}\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E} \cong \mathcal{E}$ is local, so we may assume that $\mathcal{X} = \mathrm{Spf}(A)$ where $A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a noetherian adic ring, and that $\mathcal{J}$ is generated by a finite sequence in A. Then $\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E} \cong \mathcal{K}^\bullet_\infty \otimes \mathcal{E}$, where $\mathcal{K}^\bullet_\infty$ is a bounded flat complex—a $\varinjlim$ of Koszul complexes on powers of the generators of $\mathcal{J}$—see [AJL, p. 18, Lemma 3.1.1]. So $\mathbf{R}\varGamma'_{\mathcal{X}}$ is a bounded functor, and the usual way-out argument reduces the question to where $\mathcal{E}$ is a single torsion sheaf. But then it is immediate from the construction of $\mathcal{K}^\bullet_\infty$ that $\mathcal{K}^\bullet_\infty \otimes \mathcal{E} = \mathcal{E}$.

(b) Again, we can assume that $\mathcal{X} = \mathrm{Spf}(A)$ and $\mathbf{R}\varGamma'_{\mathcal{X}}$ is bounded, and since $\mathcal{A}_{\mathrm{qc}}(\mathcal{X})$ is plump (Corollary 3.1.6) we can reduce to where $\mathcal{E}$ is a single quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module, though it is better to assume only that $\mathcal{E} \in \mathbf{D}^+_{\mathrm{qc}}(\mathcal{X})$, for then we may also assume $\mathcal{E}$ injective, so that

$$\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E} \cong \varGamma'_{\mathcal{X}}\mathcal{E} = \varinjlim_{n>0} \mathcal{H}om(\mathcal{O}/\mathcal{J}^n, \mathcal{E}).$$

From Corollary 3.1.6(d) it follows that $\mathcal{H}om(\mathcal{O}/\mathcal{J}^n, \mathcal{E}) \in \mathbf{D}_{\mathrm{qct}}(\mathcal{X})$—for this assertion another way-out argument reduces us again to where $\mathcal{E}$ is a single quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module—and since homology commutes with $\varinjlim$ and $\mathcal{A}_{\mathrm{qct}}$ is closed under $\varinjlim$ (Corollary 5.1.3), therefore $\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E} \in \mathbf{D}_{\mathrm{qct}}(\mathcal{X})$.

Assertion (c) results now from the following simple lemma. □

Lemma 5.2.2. *Let $\mathcal{A}$ be an abelian category, let $j\colon \mathcal{A}_\flat \to \mathcal{A}$ be the inclusion of a plump subcategory such that j has a right adjoint $\varGamma$, and let $\boldsymbol{j}\colon \mathbf{D}(\mathcal{A}_\flat) \to \mathbf{D}(\mathcal{A})$ be the derived-category extension of j. Suppose that every $\mathcal{A}$-complex has a K-injective resolution, so that the derived functor $\mathbf{R}\varGamma\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A}_\flat)$ exists. Then $\mathbf{R}\varGamma$ is right-adjoint to $\boldsymbol{j}$. Furthermore, the following conditions are equivalent:*

1. *$\boldsymbol{j}$ induces an equivalence of categories from $\mathbf{D}(\mathcal{A}_\flat)$ to $\mathbf{D}_\flat(\mathcal{A})$, with quasi-inverse $\mathbf{R}_\flat\varGamma := \mathbf{R}\varGamma|_{\mathbf{D}_\flat(\mathcal{A})}$.*
2. *For every $\mathcal{E} \in \mathbf{D}_\flat(\mathcal{A})$ the natural map $\boldsymbol{j}\mathbf{R}\varGamma\mathcal{E} \to \mathcal{E}$ is an isomorphism.*

(3) *The functor* $\mathbf{R}_\flat\varGamma$ *is bounded, and for* $\mathcal{E}_0\in\mathcal{A}_\flat$ *the natural map* $\boldsymbol{j}\mathbf{R}\varGamma\mathcal{E}_0\to\mathcal{E}_0$ *is a* $\mathbf{D}(\mathcal{A})$*-isomorphism.*

When these conditions hold, every $\mathcal{A}_\flat$*-complex has a K-injective resolution.*

Proof. Since $\varGamma$ has an exact left adjoint, it takes K-injective $\mathcal{A}$-complexes to K-injective $\mathcal{A}_\flat$-complexes, whence there is a bifunctorial isomorphism in the derived category of abelian groups

$$\mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}}(\boldsymbol{j}\mathcal{G},\mathcal{E})\ \xrightarrow{\ \sim\ }\ \mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{A}_\flat}(\mathcal{G},\mathbf{R}\varGamma\mathcal{E})\qquad\big(\mathcal{G}\in\mathbf{D}(\mathcal{A}_\flat),\ \mathcal{E}\in\mathbf{D}(\mathcal{A})\big).$$

(To see this, one can assume $\mathcal{E}$ to be K-injective, and then drop the $\mathbf{R}$'s ...) Apply homology H^0 to this isomorphism to get adjointness of $\boldsymbol{j}$ and $\mathbf{R}\varGamma$.

The implications (1) ⇒ (3) ⇒ (2) are straightforward. For (2) ⇒ (1), one needs that for $\mathcal{G}\in\mathbf{D}(\mathcal{A}_\flat)$ the natural map $\mathcal{G}\to\mathbf{R}\varGamma\boldsymbol{j}\mathcal{G}$ is an isomorphism, or equivalently (look at homology), that the corresponding map $\boldsymbol{j}\mathcal{G}\to\boldsymbol{j}\mathbf{R}\varGamma\boldsymbol{j}\mathcal{G}$ is an isomorphism. But the composition of this last map with the isomorphism $\boldsymbol{j}\mathbf{R}\varGamma\boldsymbol{j}\mathcal{G}\xrightarrow{\ \sim\ }\boldsymbol{j}\mathcal{G}$ (given by (2)) is the identity, whence the conclusion.

Finally, if $\mathcal{G}$ is an $\mathcal{A}_\flat$-complex and $j\mathcal{G}\to\mathcal{J}$ is a K-injective $\mathcal{A}$-resolution, then as before $\varGamma\mathcal{J}$ is a K-injective $\mathcal{A}_\flat$-complex; and (1) implies that the natural composition

$$\mathcal{G}\to\varGamma j\mathcal{G}\to\varGamma\mathcal{J}\ \ (\cong\mathbf{R}\varGamma\boldsymbol{j}\mathcal{G})$$

is a $\mathbf{D}(\mathcal{A}_\flat)$-isomorphism, hence an $\mathcal{A}_\flat$-K-injective resolution. □

Corollary 5.2.3. *For any* $\mathcal{E}\in\mathbf{D}_{\mathrm{t}}(\mathfrak{X})$ *and* $\mathcal{F}\in\mathbf{D}(\mathfrak{X})$ *the natural map* $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{F}\to\mathcal{F}$ *induces an isomorphism*

$$\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{F})\ \xrightarrow{\ \sim\ }\ \mathbf{R}\mathcal{H}om^\bullet(\mathcal{E},\mathcal{F}).$$

Proof. Consideration of homology presheaves shows it sufficient that for each affine open $\mathfrak{U}\subset\mathfrak{X}$, the natural map

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{U})}\big(\mathcal{E}|_{\mathfrak{U}},(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{F})|_{\mathfrak{U}}\big)\to\mathrm{Hom}_{\mathbf{D}(\mathfrak{U})}\big(\mathcal{E}|_{\mathfrak{U}},\mathcal{F}|_{\mathfrak{U}}\big)$$

be an isomorphism. But since $\mathbf{R}\varGamma'_{\mathfrak{X}}$ commutes with restriction to $\mathfrak{U}$,[20] that is a direct consequence of Proposition 5.2.1(c) (with $\mathfrak{X}$ replaced by $\mathfrak{U}$). □

Proposition 5.2.4. *Let* Z *be a closed subset of a locally noetherian scheme* X*, and let* $\kappa\colon\mathfrak{X}\to X$ *be the completion of* X *along* Z*. Then*:

(a) *The exact functors* κ^* *and* κ_* *restrict to inverse isomorphisms between the categories* $\mathbf{D}_Z(X)$ *and* $\mathbf{D}_{\mathrm{t}}(\mathfrak{X})$*, and between the categories* $\mathbf{D}_{\mathrm{qc}Z}(X)$ *and* $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$*; and if* $\mathcal{M}\in\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ *has coherent homology, then so does* $\kappa_*\mathcal{M}$*.*

(b) *There are natural derived-category isomorphisms*

$$\mathbf{R}\varGamma'_Z\kappa_*\mathcal{E}\ \xrightarrow{\ \sim\ }\ \kappa_*\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{E}\qquad\big(\mathcal{E}\in\mathbf{D}(\mathfrak{X})\big),$$
$$\kappa^*\mathbf{R}\varGamma'_Z\mathcal{F}\ \xrightarrow{\ \sim\ }\ \mathbf{R}\varGamma'_{\mathfrak{X}}\kappa^*\mathcal{F}\qquad\big(\mathcal{F}\in\mathbf{D}(X)\big).$$

Proof. The assertions in (a) follow at once from Proposition 5.1.2.

Since κ_* has an exact left adjoint (viz. κ^*), therefore κ_* transforms K-injective $\mathcal{A}(\mathfrak{X})$-complexes into K-injective $\mathcal{A}(X)$-complexes, and so the first isomorphism in (b) results from the isomorphism $\varGamma'_Z\kappa_*\xrightarrow{\ \sim\ }\kappa_*\varGamma'_{\mathfrak{X}}$ in the proof of Proposition 5.1.2.

[20] If $i\colon\mathfrak{U}\hookrightarrow\mathfrak{X}$ is the inclusion then clearly $\varGamma'$ commutes with i^*; and furthermore i^* preserves K-injectivity since it has the exact left adjoint "extension by 0."

We have then the natural composed map

$$\kappa^*\mathbf{R}\varGamma'_Z\mathcal{F} \to \kappa^*\mathbf{R}\varGamma'_Z\kappa_*\kappa^*\mathcal{F} \xrightarrow{\ \sim\ } \kappa^*\kappa_*\mathbf{R}\varGamma'_{\mathfrak{X}}\kappa^*\mathcal{F} \to \mathbf{R}\varGamma'_{\mathfrak{X}}\kappa^*\mathcal{F}.$$

Showing this map to be an isomorphism is a local problem, so we can assume that $X = \mathrm{Spec}(A)$ where A is a noetherian adic ring. Let $K^\bullet_\infty$ be the usual $\varinjlim$ of Koszul complexes on powers of a finite system of generators of an ideal of definition of A (see [AJL, §3.1]); and let $\widetilde{K^\bullet_\infty}$ be the corresponding quasi-coherent complex on $\mathrm{Spec}(A)$, so that the complex $\mathcal{K}^\bullet_\infty$ in the proof of Proposition 5.2.1(a) is just $\kappa^*\widetilde{K^\bullet_\infty}$. Then one checks via [AJL, p. 18, Lemma (3.1.1)] that the map in question is isomorphic to the natural isomorphism of complexes

$$\kappa^*(\widetilde{K^\bullet_\infty} \otimes_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{\ \sim\ } \kappa^*\widetilde{K^\bullet_\infty} \otimes_{\mathcal{O}_{\mathfrak{X}}} \kappa^*\mathcal{F}. \qquad \square$$

Proposition 5.2.5. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of noetherian formal schemes. Then $\mathbf{R}f_*|_{\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})}$ is bounded, and*

$$\mathbf{R}f_*\big(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})\big) \subset \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}).$$

Moreover, if f is pseudo-proper and $\mathcal{F} \in \mathbf{D}_{\mathrm{t}}(\mathfrak{X})$ has coherent homology, then so does $\mathbf{R}f_*\mathcal{F} \in \mathbf{D}_{\mathrm{t}}(\mathfrak{Y})$.

Proof. Since $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}) \subset \mathbf{D}_{\vec{\mathrm{c}}}(\mathfrak{X})$ (Lemma 5.1.4), the boundedness assertion is given by Proposition 3.4.3(b). (It is clear that $\mathbf{R}f_*$ is bounded-below.) It suffices then for the next assertion (by the usual way-out arguments [H1, p. 73, Proposition 7.3]) to show for any $\mathcal{M} \in \mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$ that $\mathbf{R}f_*\mathcal{M} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y})$.

Let $\mathcal{E}$ be an injective resolution of $\mathcal{M}$, let $\mathcal{J}$ be an ideal of definition of $\mathfrak{X}$, and let $\mathcal{E}_n$ be the flasque complex $\mathcal{E}_n := \mathcal{H}om(\mathcal{O}/\mathcal{J}^n, \mathcal{E})$. Then by Proposition 5.2.1(a), $\mathcal{M} \cong \mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{M} \cong \varinjlim_n \mathcal{E}_n$. Since $\mathfrak{X}$ is noetherian, $\varinjlim$'s of flasque sheaves are f_*-acyclic and $\varinjlim$ commutes with f_*; so with notation as in the proof of Proposition 5.1.1,

$$\mathbf{R}f_*\mathcal{M} \cong \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{M} \cong f_*\varinjlim_n \mathcal{E}_n \cong \varinjlim_n f_*j_{n*}j_n^*\mathcal{E}_n \cong \varinjlim_n i_{n*}f_{[n]*}j_n^*\mathcal{E}_n.$$

Since $\mathcal{E} \in \mathbf{D}^+_{\mathrm{qc}}(\mathfrak{X})$, therefore

$$j_{n*}j_n^*\mathcal{E}_n = \mathcal{H}om(\mathcal{O}/\mathcal{J}^n, \mathcal{E}) \in \mathbf{D}_{\mathrm{qc}}(\mathfrak{X}),$$

as we see by way-out reduction to where $\mathcal{E}$ is a single quasi-coherent sheaf and then by Corollary 3.1.6(d); and hence $j_n^*\mathcal{E}_n \in \mathbf{D}_{\mathrm{qc}}(\mathfrak{X}_{[n]})$ (see [GD, p. 115, (5.3.15)]). Now $j_n^*\mathcal{E}_n$ is a flasque bounded-below $\mathcal{O}_{\mathfrak{X}_{[n]}}$-complex, so by way-out reduction to (for example) [Ke, p. 643, corollary 11],

$$f_{[n]*}j_n^*\mathcal{E}_n \cong \mathbf{R}f_{[n]*}j_n^*\mathcal{E}_n \in \mathbf{D}_{\mathrm{qc}}(\mathfrak{Y}_{[n]});$$

and finally, in view of Corollary 5.1.3,

$$\mathbf{R}f_*\mathcal{M} \cong i_{n*}\varinjlim_n f_{[n]*}j_n^*\mathcal{E}_n \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}).$$

For the last assertion, we reduce as before to showing for each coherent torsion $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ and each $p \ge 0$ that $R^pf_*\mathcal{M}$ is a coherent $\mathcal{O}_{\mathfrak{Y}}$-module. With $\mathcal{E}$ and $\mathcal{E}_n$ as above, for $n \gg 0$ the inclusion map $\mathcal{E}_n \hookrightarrow \mathcal{E}$ induces homology isomorphisms in all degrees $\le p+1$, that is, the third vertex of a triangle T based on this inclusion is exact in all degrees $\le p$. $\mathcal{E}$ and $\mathcal{E}_n$ being flasque, it follows from the exact homology sequence of the triangle $\mathbf{R}f_*T$ that

$$R^pf_*\mathcal{M} = H^pf_*\mathcal{E} \cong H^pf_*\mathcal{E}_n \cong H^pf_*j_{n*}j_n^*\mathcal{E}_n \cong H^pi_{n*}f_{[n]*}j_n^*\mathcal{E}_n \cong i_{n*}H^pf_{[n]*}j_n^*\mathcal{E}_n.$$

Arguing as before, we see that the flasque complex $j_n^*\mathcal{E}_n$ has coherent homology; and since $f_{[n]}$ is proper, we conclude that $R^pf_*\mathcal{M}$ is indeed coherent. □

Corollary 5.2.6 (cf. Corollary 3.5.3)**.** *Let $f_0\colon X \to Y$ be a map of locally noetherian schemes, let $W \subset Y$ and $Z \subset f_0^{-1}W$ be closed subsets, let $\kappa_{\mathscr{Y}}\colon \mathscr{Y} = Y_{/W} \to Y$ and $\kappa_{\mathscr{X}}\colon \mathscr{X} = X_{/Z} \to X$ be the respective (flat) completion maps, and let $f\colon \mathscr{X} \to \mathscr{Y}$ be the map induced by f_0. Then for $\mathcal{E} \in \mathbf{D}_{\mathrm{qc}Z}(X)$ the map adjoint to the natural composition*

$$\mathbf{R}f_{0*}\mathcal{E} \longrightarrow \mathbf{R}f_{0*}\kappa_{\mathscr{X}*}\kappa_{\mathscr{X}}^*\mathcal{E} \xrightarrow{\ \sim\ } \kappa_{\mathscr{Y}*}\mathbf{R}f_*\kappa_{\mathscr{X}}^*\mathcal{E}$$

is a functorial isomorphism.

$$\theta_{\mathcal{E}}\colon \kappa_{\mathscr{Y}}^*\mathbf{R}f_{0*}\mathcal{E} \xrightarrow{\ \sim\ } \mathbf{R}f_*\kappa_{\mathscr{X}}^*\mathcal{E}.$$

Proof. $\theta_{\mathcal{E}}$ is the composition of the natural maps

$$\kappa_{\mathscr{Y}}^*\mathbf{R}f_{0*}\mathcal{E} \to \kappa_{\mathscr{Y}}^*\mathbf{R}f_{0*}\kappa_{\mathscr{X}*}\kappa_{\mathscr{X}}^*\mathcal{E} \xrightarrow{\ \sim\ } \kappa_{\mathscr{Y}}^*\kappa_{\mathscr{Y}*}\mathbf{R}f_*\kappa_{\mathscr{X}}^*\mathcal{E} \to \mathbf{R}f_*\kappa_{\mathscr{X}}^*\mathcal{E}.$$

By Proposition 5.2.4, the first map and (in view of Proposition 5.2.5) the third map are both isomorphisms. □

Proposition 5.2.7. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be an adic map of locally noetherian formal schemes. Then*:

(a) $\mathbf{L}f^*(\mathbf{D}_{\mathrm{t}}(\mathscr{Y})) \subset \mathbf{D}_{\mathrm{t}}(\mathscr{X})$.

(b) $\mathbf{L}f^*(\mathbf{D}_{\mathrm{qct}}(\mathscr{Y})) \subset \mathbf{D}_{\mathrm{qct}}(\mathscr{X})$.

(c) *If $\mathscr{X}$ is noetherian then there is a functorial isomorphism*

$$\mathbf{R}\varGamma'_{\mathscr{Y}}\mathbf{R}f_*\mathcal{G} \xrightarrow{\ \sim\ } \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{G} \qquad \big(\mathcal{G} \in \mathbf{D}^+(\mathscr{X}) \text{ or } \mathcal{G} \in \mathbf{D}_{\mathrm{qc}}(\mathscr{X})\big).$$

Proof. (a) Let $\mathscr{I}$ be an ideal of definition of $\mathscr{Y}$, so that $\mathscr{I}\mathcal{O}_{\mathscr{X}}$ is an ideal of definition of $\mathscr{X}$. Let $\mathcal{F} \in \mathbf{D}_{\mathrm{t}}(\mathscr{Y})$. To show that $\mathbf{L}f^*\mathcal{F} \in \mathbf{D}_{\mathrm{t}}(\mathscr{X})$ we may assume that $\mathcal{F}$ is K-injective and, by Proposition 5.2.1(a), that the canonical map is a $\mathbf{D}(\mathscr{Y})$-isomorphism

$$\varGamma'_{\mathscr{X}}\mathcal{F} = \varinjlim_n \mathcal{H}om^\bullet(\mathcal{O}_{\mathscr{Y}}/\mathscr{I}^n, \mathcal{F}) \xrightarrow{\ \sim\ } \mathcal{F}.$$

Let $x \in \mathscr{X}$, set $y := f(x)$, and let P_x be a flat resolution of the $\mathcal{O}_{\mathscr{Y},y}$-module $\mathcal{O}_{\mathscr{X},x}$. Then for any i the stalk at x of the homology $H^i\mathbf{L}f^*\mathcal{F}$ is

$$\mathrm{H}^i\big(P_x \otimes_{\mathcal{O}_{\mathscr{Y},y}} \mathcal{F}_y\big) = \varinjlim_n \mathrm{H}^i\big(P_x \otimes_{\mathcal{O}_{\mathscr{Y},y}} \mathrm{Hom}^\bullet_{\mathcal{O}_{\mathscr{Y},y}}(\mathcal{O}_{\mathscr{Y},y}/\mathscr{I}_y^n, \mathcal{F}_y)\big).$$

Hence each element of the stalk is annihilated by a power of $\mathscr{I}\mathcal{O}_{\mathscr{X},x}$, and (a) results.

(b) Since $\mathbf{D}_{\mathrm{qct}}(\mathscr{Y}) = \mathbf{D}_{\vec{\mathrm{c}}}(\mathscr{Y}) \cap \mathbf{D}_{\mathrm{t}}(\mathscr{Y})$ (Corollary 3.1.5 and Lemma 5.1.4), (b) follows from (a) and Proposition 3.3.5.

(c) First of all, $\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{G} \in \mathbf{D}_{\mathrm{t}}(\mathscr{Y})$. Indeed, Propositions 5.2.1 and 5.2.5 give

$$\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}\big(\mathbf{D}_{\mathrm{qc}}(\mathscr{X})\big) \subset \mathbf{R}f_*\big(\mathbf{D}_{\mathrm{qct}}(\mathscr{X})\big) \subset \mathbf{D}_{\mathrm{qct}}(\mathscr{Y}),$$

taking care of the case $\mathcal{G}\in\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$; while for arbitrary $\mathcal{G}\in\mathbf{D}^+(\mathcal{X})$, which may be assumed to be bounded-below and injective, the complex $\mathcal{G}_n := \mathcal{H}om^\bullet(\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n\mathcal{O}_{\mathcal{X}},\mathcal{G})$ (with $\mathcal{I}$ as above) is flasque, as is $\varinjlim_n \mathcal{G}_n$, so

$$\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{G}\cong f_*\varinjlim_n \mathcal{G}_n \cong \varinjlim_n f_*\mathcal{G}_n\in\mathbf{D}_{\mathrm{t}}(\mathcal{Y}).$$

It suffices then to note, using (a) and Corollary 5.2.3, that for any $\mathcal{E}\in\mathbf{D}_{\mathrm{t}}(\mathcal{Y})$ the natural maps are isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{Y}}(\mathcal{E},\mathbf{R}\varGamma'_{\mathcal{Y}}\mathbf{R}f_*\mathcal{G}) &\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{Y}}(\mathcal{E},\mathbf{R}f_*\mathcal{G})\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{X}}(\mathbf{L}f^*\mathcal{E},\mathcal{G})\\ &\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{X}}(\mathbf{L}f^*\mathcal{E},\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{G})\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{Y}}(\mathcal{E},\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{G}).\end{aligned}$$

□

5.3. From the following key Proposition 5.3.1 there will result, for complexes with quasi-coherent torsion homology, a stronger version of the Duality Theorem 4.1, see Section 6.

Recall from Corollary 5.1.5 that the inclusion functor $j^{\mathrm{t}}_{\mathcal{X}}\colon\mathcal{A}_{\mathrm{qct}}(\mathcal{X})\hookrightarrow\mathcal{A}(\mathcal{X})$ has a right adjoint $Q^{\mathrm{t}}_{\mathcal{X}}$.

Proposition 5.3.1. *For a separated noetherian formal scheme $\mathcal{X}$, the extension of $j^{\mathrm{t}}_{\mathcal{X}}$ induces an* equivalence of categories

$$\boldsymbol{j}^{\mathrm{t}}_{\mathcal{X}}\colon\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathcal{X}))\xrightarrow{\approx}\mathbf{D}_{\mathrm{qct}}(\mathcal{X}),$$

with bounded quasi-inverse $\mathbf{R}Q^{\mathrm{t}}_{\mathcal{X}}|_{\mathbf{D}_{\mathrm{qct}}(\mathcal{X})}$.

Proof. The proof will use an induction, to begin which we need:

Lemma 5.3.2. *Let X be a separated noetherian scheme, and let $Z\subset X$ be a closed subset. Then the natural functors*

$$\mathbf{D}\big(\mathcal{A}_{\mathrm{qc}Z}(X)\big)\xrightarrow{\boldsymbol{j}_1}\mathbf{D}_{\mathrm{qc}}\big(\mathcal{A}_Z(X)\big)\xrightarrow{\boldsymbol{j}_2}\mathbf{D}_{\mathrm{qc}Z}(X)$$

are both equivalences of categories.

Proof. The proof that $\boldsymbol{j}_2$ is an equivalence, being quite similar to that of Proposition 5.2.1(c), is left to the reader.

It will be enough then to show that $\boldsymbol{j}:=\boldsymbol{j}_2\circ\boldsymbol{j}_1$ is an equivalence. We write $\mathcal{A}_{\dots}$ for $\mathcal{A}_{\dots}(X)$, $\mathbf{D}_{\dots}$ for $\mathbf{D}_{\dots}(X)$, and so on. Now $\boldsymbol{j}$ factors naturally as

$$\mathbf{D}(\mathcal{A}_{\mathrm{qc}Z})\xrightarrow{\boldsymbol{j}'}\mathbf{D}_Z(\mathcal{A}_{\mathrm{qc}})\xrightarrow{\boldsymbol{j}''}\mathbf{D}_{\mathrm{qc}Z}.$$

Here $\boldsymbol{j}''$ is induced by the natural equivalence of categories $\boldsymbol{j}_X\colon\mathbf{D}(\mathcal{A}_{\mathrm{qc}})\to\mathbf{D}_{\mathrm{qc}}$ (see e.g., [AJL, p. 12, Proposition (1.3)]), whose quasi-inverse $\mathbf{R}Q_X$ takes $\mathbf{D}_{\mathrm{qc}Z}$ to $\mathbf{D}_Z(\mathcal{A}_{\mathrm{qc}})$, since $\boldsymbol{j}\mathbf{R}Q_X\cong\mathbf{1}$; and therefore $\boldsymbol{j}''$ is an equivalence. The inclusion $\mathcal{A}_{\mathrm{qc}Z}\hookrightarrow\mathcal{A}_{\mathrm{qc}}$ has the right adjoint $\varGamma^{\mathrm{qc}}_Z:=\varGamma'_Z|_{\mathcal{A}_{\mathrm{qc}}}=\varGamma_Z|_{\mathcal{A}_{\mathrm{qc}}}$, and Lemma 5.2.2 gives that the derived functor $\mathbf{R}\varGamma^{\mathrm{qc}}_Z\colon\mathbf{D}(\mathcal{A}_{\mathrm{qc}})\to\mathbf{D}(\mathcal{A}_{\mathrm{qc}Z})$ induces a right adjoint of $\boldsymbol{j}'$. (Note that every $\mathcal{A}_{\mathrm{qc}}$-complex has an $\mathcal{A}_{\mathrm{qc}}$-K-injective resolution [AJL, p. 12, Corollary (1.3.1)].) By [AJL, p. 22, Proposition (3.1.7)] for the first, and by an obvious analogue of Proposition 5.2.1(a) for the second, the natural maps

$$\boldsymbol{j}''\boldsymbol{j}'\mathbf{R}\varGamma^{\mathrm{qc}}_Z\mathcal{E}\xrightarrow{\sim}\boldsymbol{j}_2\mathbf{R}\varGamma'_Z\boldsymbol{j}''\mathcal{E}\xrightarrow{\sim}\boldsymbol{j}''\mathcal{E}\qquad\big(\mathcal{E}\in\mathbf{D}_Z(\mathcal{A}_{\mathrm{qc}})\big)$$

are isomorphisms, whence the natural map is an isomorphism $\boldsymbol{j}'\mathbf{R}\varGamma^{\mathrm{qc}}_Z\mathcal{E}\xrightarrow{\sim}\mathcal{E}$, so that by Lemma 5.2.2, $\boldsymbol{j}'$ is an equivalence, and therefore so is $\boldsymbol{j}$. □

Corollary 5.3.3. *With $Z \subset X$ as above, let $\kappa\colon \mathcal{X} \to X$ be the formal completion along Z. Then the natural functors*

$$\mathbf{D}\big(\mathcal{A}_{\mathrm{qct}}(\mathcal{X})\big) \xrightarrow{\hat{\boldsymbol{j}}_1} \mathbf{D}_{\mathrm{qc}}\big(\mathcal{A}_{\mathrm{t}}(\mathcal{X})\big) \xrightarrow{\hat{\boldsymbol{j}}_2} \mathbf{D}_{\mathrm{qct}}(\mathcal{X})$$

are both equivalences of categories. Moreover, the equivalence $\boldsymbol{j}^{\mathrm{t}}_{\mathcal{X}} := \hat{\boldsymbol{j}}_2 \circ \hat{\boldsymbol{j}}_1$ is quasi-inverse to the bounded functor $\mathbf{R}Q^{\mathrm{t}}_{\mathcal{X}}|_{\mathbf{D}_{\mathrm{qct}}}$.

Proof. By Lemma 5.2.2, $\mathbf{R}Q^{\mathrm{t}}_{\mathcal{X}}\colon \mathbf{D} \to \mathbf{D}(\mathcal{A}_{\mathrm{qct}})$ is right-adjoint to $\boldsymbol{j}^{\mathrm{t}}_{\mathcal{X}}$, and its restriction to $\mathbf{D}_{\mathrm{qct}}$ is bounded. That $\hat{\boldsymbol{j}}_2$ is an equivalence is an instance of Proposition 5.2.1(c). That $\hat{\boldsymbol{j}}_1$ is an equivalence reduces via the isomorphisms in Proposition 5.1.2 to the corresponding statement for $\boldsymbol{j}_1$ in Lemma 5.3.2. □

The rest of the proof of Proposition 5.3.1 is similar in spirit to that of [AJL, p. 12, Proposition (1.3)] (which is the case where $\mathcal{X}$ is an ordinary scheme), *mutatis mutandis,* namely substitute "$\mathcal{X}$" for "X," "qct" for "qc," "Q^{t}" for "Q," and recall that for a map $v\colon \mathcal{V} \to \mathcal{X}$ of noetherian formal schemes we have $v_*(\mathcal{A}_{\mathrm{qct}}(\mathcal{V})) \subset \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ (Proposition 5.1.1), and furthermore if v is affine then $v_*|_{\mathcal{A}_{\mathrm{qct}}(\mathcal{V})}$ is exact (Lemmas 5.1.4 and Lemma 3.4.2).

According to Lemma 5.2.2 it suffices to show that the functor $\mathbf{R}Q^{\mathrm{t}}_{\mathcal{X}}$ is bounded on $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$ and that for each $\mathcal{E} \in \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ the natural map $\eta\colon \boldsymbol{j}^{\mathrm{t}}_{\mathcal{X}}\mathbf{R}Q^{\mathrm{t}}_{\mathcal{X}}\mathcal{E} \to \mathcal{E}$ is an isomorphism. We proceed by induction on $n(\mathcal{X})$, the least among natural numbers n such that $\mathcal{X}$ can be covered by n affine open subsets, the case $n(\mathcal{X}) = 1$ being settled by Corollary 5.3.3.

The argument for boundedness proceeds as in *loc. cit.* As for η, we need to say more because the reference in *loc. cit* to [I] does not apply here.

Assuming then that $n := n(\mathcal{X}) > 1$, let $\mathcal{X} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_n$, with each $\mathcal{X}_i$ an affine open subset. Set $\mathcal{U} := \mathcal{X}_1$, $\mathcal{V} := \mathcal{X}_2 \cup \ldots \cup \mathcal{X}_n$, $\mathcal{W} := \mathcal{U} \cap \mathcal{V}$, and let $u\colon \mathcal{U} \hookrightarrow \mathcal{X}$, $v\colon \mathcal{V} \hookrightarrow \mathcal{X}$, $w\colon \mathcal{W} \hookrightarrow \mathcal{X}$ be the inclusion maps. Then $n(\mathcal{U}) = 1$, $n(\mathcal{V}) < n$, and $n(\mathcal{W}) < n$ (because $\mathcal{X}$ being separated, $\mathcal{X}_1 \cap \mathcal{X}_i$ is affine, see § 3.4.1). So we may assume that Proposition 5.3.1 holds for the formal schemes $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$. The inductive hypothesis and Lemma 5.2.2 give that every complex in $\mathbf{K}(\mathcal{A}_{\mathrm{qct}}(\mathcal{V}))$ has a K-injective resolution, so that the functor $v^{\mathrm{qct}}_* := v_*|_{\mathcal{A}_{\mathrm{qct}}(\mathcal{V})}\colon \mathcal{A}_{\mathrm{qct}}(\mathcal{V}) \to \mathcal{A}_{\mathrm{qct}}(\mathcal{X})$ has a derived functor $\mathbf{R}v^{\mathrm{qct}}_*$. (Similar statements hold here and below for $\mathcal{U}$ and $\mathcal{W}$.)

For the remainder of this proof we write "Q" for "Q^{t}" and "$\boldsymbol{j}$" for "$\boldsymbol{j}^{\mathrm{t}}$." As in *loc. cit.* there is a natural functorial isomorphism $\mathbf{R}Q_{\mathcal{X}}\mathbf{R}v_* \xrightarrow{\sim} \mathbf{R}v^{\mathrm{qct}}_*\mathbf{R}Q_{\mathcal{V}}$.

One proves by induction on $n(\mathcal{V})$ that the map $\boldsymbol{j}_{\mathcal{X}}\mathbf{R}v^{\mathrm{qct}}_*\mathcal{E} \to \mathbf{R}v_*\boldsymbol{j}_{\mathcal{V}}\mathcal{E}$ adjoint to the canonical map $v^*\boldsymbol{j}_{\mathcal{X}}\mathbf{R}v^{\mathrm{qct}}_*\mathcal{E} = \boldsymbol{j}_{\mathcal{V}}v^*\mathbf{R}v^{\mathrm{qct}}_*\mathcal{E} \to \boldsymbol{j}_{\mathcal{V}}\mathcal{E}$ is an isomorphism for every $\mathcal{E} \in \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathcal{V}))$. Indeed, when $n(\mathcal{V}) = 1$ the map $v\colon \mathcal{V} \to \mathcal{X}$ is affine (§3.4.1), so v^{qct}_* is exact, and hence

$$\boldsymbol{j}_{\mathcal{X}}\mathbf{R}v^{\mathrm{qct}}_*\mathcal{E} \cong \boldsymbol{j}_{\mathcal{X}}v^{\mathrm{qct}}_*\mathcal{E} \cong v_*\boldsymbol{j}_{\mathcal{V}}\mathcal{E} \cong \mathbf{R}v_*\boldsymbol{j}_{\mathcal{V}}\mathcal{E},$$

where the last isomorphism follows, again, from Lemmas 5.1.4 and 3.4.2; and then for $n(\mathcal{V}) > 1$ the argument is analogous to the "Mayer-Vietoris" induction in the last paragraph of the proof of [AJL, p. 12, Proposition (1.3)]. It is a purely formal exercise to show that the following functorial diagram commutes:

$$\begin{array}{ccc} \boldsymbol{j}_{\mathcal{X}}\mathbf{R}Q_{\mathcal{X}}\mathbf{R}v_* & \xrightarrow{\ \sim\ } & \boldsymbol{j}_{\mathcal{X}}\mathbf{R}v^{\mathrm{qct}}_*\mathbf{R}Q_{\mathcal{V}} \\ {\scriptstyle \eta_v}\big\downarrow & & \big\downarrow{\scriptstyle \simeq} \\ \mathbf{R}v_* & \xleftarrow{\ \eta_{\mathcal{V}}\ } & \mathbf{R}v_*\boldsymbol{j}_{\mathcal{V}}\mathbf{R}Q_{\mathcal{V}} \end{array}$$

Here $\eta_{\mathcal{V}}$ is an isomorphism by the inductive hypothesis, so η_v *is an isomorphism.*

Now, given $\mathcal{E} \in \mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$, apply $\boldsymbol{j}_{\mathfrak{X}}\mathbf{R}Q_{\mathfrak{X}}$ to the Mayer-Vietoris triangle

$$\mathcal{E} \to \mathbf{R}u_*u^*\mathcal{E} \oplus \mathbf{R}v_*v^*\mathcal{E} \to \mathbf{R}w_*w^*\mathcal{E} \xrightarrow{+} \mathcal{E}[1]$$

to get the map of triangles

$$\begin{array}{ccccccc}
\boldsymbol{j}_{\mathfrak{X}}\mathbf{R}Q_{\mathfrak{X}}\mathcal{E} & \longrightarrow & \boldsymbol{j}_{\mathfrak{X}}\mathbf{R}Q_{\mathfrak{X}}\mathbf{R}u_*u^*\mathcal{E} \oplus \boldsymbol{j}_{\mathfrak{X}}\mathbf{R}Q_{\mathfrak{X}}\mathbf{R}v_*v^*\mathcal{E} & \longrightarrow & \boldsymbol{j}_{\mathfrak{X}}\mathbf{R}Q_{\mathfrak{X}}\mathbf{R}w_*w^*\mathcal{E} & \xrightarrow{+} \\
{\scriptstyle\eta}\big\downarrow & & {\scriptstyle\eta_u\oplus}\big\downarrow{\scriptstyle\eta_v} & & \big\downarrow{\scriptstyle\eta_w} & \\
\mathcal{E} & \longrightarrow & \mathbf{R}u_*u^*\mathcal{E} \oplus \mathbf{R}v_*v^*\mathcal{E} & \longrightarrow & \mathbf{R}w_*w^*\mathcal{E} & \xrightarrow{+}
\end{array}$$

Since as above, η_u, η_v, and η_w are isomorphisms, therefore so is η. □

6. Duality for torsion sheaves.

Paragraph 6.1 contains the proof of Theorem 2 (section 1), i.e., of two essentially equivalent forms of Torsion Duality on formal schemes—Theorem 6.1 and Corollary 6.1.4. The rest of the paragraph deals with numerous relations among the functors which have been introduced, and with compatibilities between dualizing functors occurring before and after completion of maps of ordinary schemes.

More can be said for complexes with coherent homology, thanks to Greenlees-May duality. This is done in paragraph 6.2.

Paragraph 6.3 discusses additional relations involving $\mathbf{R}\varGamma'_{\mathfrak{X}}\colon \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(\mathfrak{X})$ and its right adjoint $\mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{O}_{\mathfrak{X}}, -)$ on a locally noetherian formal scheme $\mathfrak{X}$.

Theorem 6.1. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a separated map of noetherian formal schemes.*

(a) *The Δ-functor $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qct}}(\mathfrak{X}) \xrightarrow{5.2.5} \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}) \hookrightarrow \mathbf{D}(\mathfrak{Y})$ has a right Δ-adjoint. In fact there is a bounded-below Δ-functor $f_{\mathrm{t}}^{\times}\colon \mathbf{D}(\mathfrak{Y}) \to \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and a map of Δ-functors $\tau_{\mathrm{t}}\colon \mathbf{R}f_*f_{\mathrm{t}}^{\times} \to \mathbf{1}$ such that for all $\mathcal{G} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and $\mathcal{F} \in \mathbf{D}(\mathfrak{Y})$, the composed map (in the derived category of abelian groups)*

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^\bullet_{\mathfrak{X}}(\mathcal{G}, f_{\mathrm{t}}^{\times}\mathcal{F}) &\xrightarrow{\text{natural}} \mathbf{R}\mathrm{Hom}^\bullet_{\mathfrak{Y}}(\mathbf{R}f_*\mathcal{G}, \mathbf{R}f_*f_{\mathrm{t}}^{\times}\mathcal{F}) \\ &\xrightarrow{\text{via }\tau_{\mathrm{t}}} \mathbf{R}\mathrm{Hom}^\bullet_{\mathfrak{Y}}(\mathbf{R}f_*\mathcal{G}, \mathcal{F})\end{aligned}$$

is an isomorphism.

(b) *If $g\colon \mathfrak{Y} \to \mathfrak{Z}$ is another separated map of noetherian formal schemes then there is a natural isomorphism $(gf)_{\mathrm{t}}^{\times} \xrightarrow{\sim} f_{\mathrm{t}}^{\times}g_{\mathrm{t}}^{\times}$.*

Proof. Assertion (b) follows from (a), which easily implies that $(gf)_{\mathrm{t}}^{\times}$ and $f_{\mathrm{t}}^{\times}g_{\mathrm{t}}^{\times}$ are both right-adjoint to the restriction of $\mathbf{R}(gf)_* = \mathbf{R}g_*\mathbf{R}f_*$ to $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$.

As for (a), assuming first that $\mathfrak{X}$ is separated, we can replace $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ by the *equivalent* category $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}))$ (Proposition 5.3.1). The inclusion $i\colon \mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) \hookrightarrow \mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$ has the right adjoint $\varGamma'_{\mathfrak{X}}$. (To check that $\varGamma'_{\mathfrak{X}}(\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})) \subset \mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$, see Lemma 5.1.4 and Corollary 3.1.5.) So for all $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$-complexes $\mathcal{G}'$ and $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$-complexes $\mathcal{F}'$ there is a natural isomorphism of abelian-group complexes

$$\mathrm{Hom}^\bullet_{\mathcal{A}_{\mathrm{qct}}}(\mathcal{G}', \varGamma'_{\mathfrak{X}}\mathcal{F}') \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{A}_{\vec{\mathrm{c}}}}(i\mathcal{G}', \mathcal{F}').$$

Note that if $\mathcal{F}'$ is K-injective over $\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X})$ then $\varGamma'_{\mathfrak{X}}\mathcal{F}'$ is K-injective over $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$, because $\varGamma'_{\mathfrak{X}}$ has an exact left adjoint. Combining this isomorphism with the isomorphism (4.1.4) in the proof of Theorem 4.1, we can conclude just as in part 4 at the end of that proof, with the functor $f_{\mathrm{t}}^{\times}$ defined to be the composition

$$\mathbf{D}(\mathscr{Y}) \xrightarrow{\rho} \mathbf{K_I}(\mathscr{Y}) \xrightarrow{\mathcal{C}_\bullet} \mathbf{K_I}(\mathcal{A}_{\vec{c}}(\mathscr{X})) \xrightarrow{\Gamma'_{\mathscr{X}}} \mathbf{K_I}(\mathcal{A}_{\mathrm{qct}}(\mathscr{X})) \xrightarrow{\text{natural}} \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathscr{X})).$$

(What we have in mind here is simply that the natural functor $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathscr{X})) \to \mathbf{D}(\mathcal{A}_{\vec{c}}(\mathscr{X}))$ has a right adjoint. That is easily seen to be true once one knows the existence of K-injective resolutions in $\mathbf{D}(\mathcal{A}_{\vec{c}}(\mathscr{X}))$; but we don't know how to prove the latter other than by quoting the generalization to arbitrary Grothendieck categories [Fe, Theorem 2]. The preceding argument avoids this issue.

One could also apply Brown Representability directly, as in the proof of Theorem 1 described in the Introduction.) Now suppose only that the map f is separated. If $\mathscr{Y}$ is affine, $\mathscr{X}$ is separated and the preceding argument holds. For arbitrary noetherian $\mathscr{Y}$, the existence of a bounded-below right adjoint for $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qct}}(\mathscr{X}) \to \mathbf{D}(\mathscr{Y})$ results then by an obvious induction from the following Mayer-Vietoris pasting argument. Finally, apply homology to the assertion about the $\mathbf{R}\mathrm{Hom}^\bullet$'s to reduce it to $f_{\mathrm{t}}^{\times}$ being a right adjoint.

To reduce clutter, we will abuse notation—but only in the rest of the proof of Theorem 6.1—by writing "$f^\times$" in place of "$f_{\mathrm{t}}^{\times}$."

Lemma 6.1.1. *Let $f\colon \mathscr{X} \to \mathscr{Y} = \mathscr{Y}_1 \cup \mathscr{Y}_2$ ($\mathscr{Y}_i$ open in $\mathscr{Y}$) be a map of formal schemes, with $\mathscr{X}$ noetherian. Consider the commutative diagrams*

$$\begin{array}{ccccc} \mathscr{X}_{12} := \mathscr{X}_1 \cap \mathscr{X}_2 & \xrightarrow{q_i} & \mathscr{X}_i & \xrightarrow{x_i} & \mathscr{X} \\ {\scriptstyle f_{12}}\big\downarrow & & {\scriptstyle f_i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathscr{Y}_{12} := \mathscr{Y}_1 \cap \mathscr{Y}_2 & \xrightarrow[p_i]{} & \mathscr{Y}_i & \xrightarrow[y_i]{} & \mathscr{Y} \end{array} \qquad (i = 1, 2)$$

where $\mathscr{X}_i := f^{-1}\mathscr{Y}_i$ and all the horizontal arrows represent inclusions. Suppose that for $i = 1, 2, 12$, the functor $\mathbf{R}f_{i*}\colon \mathbf{D}_{\mathrm{qct}}(\mathscr{X}_i) \to \mathbf{D}(\mathscr{Y}_i)$ has a right adjoint $f_i^\times$. Then $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qct}}(\mathscr{X}) \to \mathbf{D}(\mathscr{Y})$ has a right adjoint $f^\times$; and with the inclusions $y_{12} := y_i \circ p_i$, $x_{12} := x_i \circ q_i$, there is for each $\mathcal{F} \in \mathbf{D}(\mathscr{Y})$ a natural $\mathbf{D}(\mathscr{X})$-triangle

$$f^\times\mathcal{F} \to \mathbf{R}x_{1*}f_1^\times y_1^*\mathcal{F} \oplus \mathbf{R}x_{2*}f_2^\times y_2^*\mathcal{F} \xrightarrow{\lambda_{\mathcal{F}}} \mathbf{R}x_{12*}f_{12}^\times y_{12}^*\mathcal{F} \to (f^\times\mathcal{F})[1]\,.$$

Remark. If we expect $f^\times$ to exist, and the natural maps $x_i^*f^\times \to f_i^\times y_i^*$ to be isomorphisms, then there should be such a triangle—the Mayer-Vietoris triangle of $f^\times\mathcal{F}$. This suggests we first define $\lambda_{\mathcal{F}}$, then let $f^\times\mathcal{F}$ be the vertex of a triangle based on $\lambda_{\mathcal{F}}$, and verify ...

Proof. There are natural maps

$$\tau_1\colon \mathbf{R}f_{1*}f_1^\times \to \mathbf{1}, \qquad \tau_2\colon \mathbf{R}f_{2*}f_2^\times \to \mathbf{1}, \qquad \tau_{12}\colon \mathbf{R}f_{12*}f_{12}^\times \to \mathbf{1}.$$

For $i = 1, 2$, define the "base-change" map $\beta_i\colon q_i^*f_i^\times \to f_{12}^\times p_i^*$ to be adjoint under Theorem 6.1 to the map of functors

$$\mathbf{R}f_{12*}q_i^*f_i^\times \xrightarrow[\text{natural}]{\sim} p_i^*\mathbf{R}f_{i*}f_i^\times \xrightarrow{\tau_i} p_i^*.$$

This β_i corresponds to a functorial map $\beta_i'\colon f_i^\times \to \mathbf{R}q_{i*}f_{12}^\times p_i^*$, from which we obtain a functorial map

$$\mathbf{R}x_{i*}f_i^\times y_i^* \longrightarrow \mathbf{R}x_{i*}\mathbf{R}q_{i*}f_{12}^\times p_i^*y_i^* \xrightarrow{\sim} \mathbf{R}x_{12*}f_{12}^\times y_{12}^*\,,$$

and hence a natural map, for any $\mathcal{F} \in \mathbf{D}(\mathcal{Y})$:

$$\check{D}^0(\mathcal{F}) := \mathbf{R}x_{1*}f_1^{\times}y_1^*\mathcal{F} \oplus \mathbf{R}x_{2*}f_2^{\times}y_2^*\mathcal{F} \xrightarrow{\lambda_{\mathcal{F}}} \mathbf{R}x_{12*}f_{12}^{\times}y_{12}^*\mathcal{F} =: \check{D}^1(\mathcal{F}).$$

Embed this map in a triangle $\check{D}(\mathcal{F})$, and denote the third vertex by $f^{\times}(\mathcal{F})$:

$$\check{D}(\mathcal{F})\colon\ f^{\times}\mathcal{F} \to \check{D}^0(\mathcal{F}) \xrightarrow{\lambda_{\mathcal{F}}} \check{D}^1(\mathcal{F}) \to (f^{\times}\mathcal{F})[1].$$

Since $\check{D}^0(\mathcal{F})$ and $\check{D}^1(\mathcal{F})$ are in $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$ (see Proposition 5.2.5), therefore so is $f^{\times}\mathcal{F}$ (Corollary 5.1.3).

This is the triangle in Lemma 6.1.1. Of course we must still show that this $f^{\times}$ is functorial, and right-adjoint to $\mathbf{R}f_*$. (Then by uniqueness of adjoints such a triangle will exist no matter which right adjoint $f^{\times}$ is used.)

Let us next construct a map

$$\tau_{\mathcal{F}}\colon \mathbf{R}f_*f^{\times}\mathcal{F} \to \mathcal{F} \qquad \big(\mathcal{F} \in \mathbf{D}(\mathcal{Y})\big).$$

Set

$$\check{C}^0(\mathcal{F}) = \mathbf{R}y_{1*}y_1^*\mathcal{F} \oplus \mathbf{R}y_{2*}y_2^*\mathcal{F}, \qquad \check{C}^1(\mathcal{F}) = \mathbf{R}y_{12*}y_{12}^*\mathcal{F}.$$

We have then the Mayer-Vietoris $\mathbf{D}(\mathcal{Y})$-triangle

$$\check{C}(\mathcal{F})\colon\ \mathcal{F} \to \check{C}^0(\mathcal{F}) \xrightarrow{\mu_{\mathcal{F}}} \check{C}^1(\mathcal{F}) \to \mathcal{F}[1],$$

arising from the usual exact sequence (Čech resolution)

$$0 \to \mathcal{F} \to y_{1*}y_1^*\mathcal{F} \oplus y_{2*}y_2^*\mathcal{F} \to y_{12*}y_{12}^*\mathcal{F} \to 0,$$

where $\mathcal{F}$ may be taken to be K-injective.

Checking commutativity of the following natural diagram is a purely category-theoretic exercise (cf. [L4, Lemma (4.8.1.2)] :

$$\begin{array}{ccc}
\mathbf{R}f_*\check{D}^0(\mathcal{F}) & \xrightarrow{\mathbf{R}f_*\lambda_{\mathcal{F}}} & \mathbf{R}f_*\check{D}^1(\mathcal{F}) \\
\Vert & & \Vert \\
\mathbf{R}f_*(\mathbf{R}x_{1*}f_1^{\times}y_1^*\mathcal{F} \oplus \mathbf{R}x_{2*}f_2^{\times}y_2^*\mathcal{F}) & & \mathbf{R}f_*\mathbf{R}x_{12*}f_{12}^{\times}y_{12}^*\mathcal{F} \\
{\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\
\mathbf{R}y_{1*}\mathbf{R}f_{1*}f_1^{\times}y_1^*\mathcal{F} \oplus \mathbf{R}y_{2*}\mathbf{R}f_{2*}f_2^{\times}y_2^*\mathcal{F} & & \mathbf{R}y_{12*}\mathbf{R}f_{12*}f_{12}^{\times}y_{12}^*\mathcal{F} \\
{\scriptstyle\tau_1\oplus\tau_2}\downarrow & & \downarrow{\scriptstyle\tau_{12}} \\
\mathbf{R}y_{1*}y_1^*\mathcal{F} \oplus \mathbf{R}y_{2*}y_2^*\mathcal{F} & & \mathbf{R}y_{12*}y_{12}^*\mathcal{F} \\
\Vert & & \Vert \\
\check{C}^0(\mathcal{F}) & \xrightarrow[\mu_{\mathcal{F}}]{} & \check{C}^1(\mathcal{F})
\end{array}$$

This commutative diagram extends to a map $\check{\tau}_{\mathcal{F}}$ of triangles:

$$\begin{array}{ccccccc}
\mathbf{R}f_*f^{\times}\mathcal{F} & \longrightarrow & \mathbf{R}f_*\check{D}^0(\mathcal{F}) & \longrightarrow & \mathbf{R}f_*\check{D}^1(\mathcal{F}) & \longrightarrow & \mathbf{R}f_*f^{\times}\mathcal{F}[1] \\
{\scriptstyle\tau_{\mathcal{F}}}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\tau_{\mathcal{F}}[1]} \\
\mathcal{F} & \longrightarrow & \check{C}^0(\mathcal{F}) & \longrightarrow & \check{C}^1(\mathcal{F}) & \longrightarrow & \mathcal{F}[1]
\end{array}$$

The map $\tau_{\mathcal{F}}$ is not necessarily unique. But the next Lemma will show, for fixed $\mathcal{F}$, that *the pair* $(f^{\times}\mathcal{F}, \tau_{\mathcal{F}})$ *represents the functor* $\mathrm{Hom}_{\mathbf{D}(\mathcal{Y})}(\mathbf{R}f_*\mathcal{E}, \mathcal{F})$ $(\mathcal{E} \in \mathbf{D}_{\mathrm{qct}}(\mathcal{X}))$.

It follows formally that one can make $f^\times$ into a functor and $\tau\colon \mathbf{R}f_*f^\times \to \mathbf{1}$ into a morphism of functors in such a way that the pair $(f^\times, \tau)$ is a right adjoint for $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qct}}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y})$ (cf. [M1, p. 83, Corollary 2]); and that there is a unique isomorphism of functors $\Theta\colon f^\times T_2 \xrightarrow{\sim} T_1 f^\times$ (where T_1 and T_2 are the respective translations on $\mathbf{D}_{\mathrm{qct}}(\mathcal{X})$ and $\mathbf{D}(\mathcal{Y})$) such that $(f^\times, \Theta)$ is a Δ-functor Δ-adjoint to $\mathbf{R}f_*$ (cf. [L4, Proposition (3.3.8)]). That will complete the proof of Theorem 6.1. ☐

Lemma 6.1.2. *For $\mathcal{E} \in \mathbf{D}_{\mathrm{qct}}(\mathcal{X})$, and with $f^\times\mathcal{F}$, $\tau_{\mathcal{F}}$ as above, the composition*

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}_{\mathrm{qct}}(\mathcal{X})}(\mathcal{E},\, f^\times\mathcal{F}) &\xrightarrow{\text{natural}} \mathrm{Hom}_{\mathbf{D}(\mathcal{Y})}(\mathbf{R}f_*\mathcal{E}, \mathbf{R}f_*f^\times\mathcal{F})\\ &\xrightarrow{\text{via } \tau_{\mathcal{F}}} \mathrm{Hom}_{\mathbf{D}(\mathcal{Y})}(\mathbf{R}f_*\mathcal{E}, \mathcal{F})\end{aligned}$$

is an isomorphism.

Proof. In the following diagram, to save space we have written $H_{\mathcal{X}}$ for $\mathrm{Hom}_{\mathbf{D}_{\mathrm{qct}}(\mathcal{X})}$, $H_{\mathcal{Y}}$ for $\mathrm{Hom}_{\mathbf{D}(\mathcal{Y})}$, and f_* for $\mathbf{R}f_*$:

$$\begin{array}{ccccc}
H_{\mathcal{X}}\big(\mathcal{E}, (\check{D}^0\mathcal{F})[-1]\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, f_*\big((\check{D}^0\mathcal{F})[-1]\big)\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, (\check{C}^0\mathcal{F})[-1]\big)\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
H_{\mathcal{X}}\big(\mathcal{E}, (\check{D}^1\mathcal{F})[-1]\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, f_*\big((\check{D}^1\mathcal{F})[-1]\big)\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, (\check{C}^1\mathcal{F})[-1]\big)\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
H_{\mathcal{X}}\big(\mathcal{E}, f^\times\mathcal{F}\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, f_*f^\times\mathcal{F}\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, \mathcal{F}\big)\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
H_{\mathcal{X}}\big(\mathcal{E}, \check{D}^0\mathcal{F}\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, f_*\check{D}^0\mathcal{F}\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, \check{C}^0\mathcal{F}\big)\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
H_{\mathcal{X}}\big(\mathcal{E}, \check{D}^1\mathcal{F}\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, f_*\check{D}^1\mathcal{F}\big) & \longrightarrow & H_{\mathcal{Y}}\big(f_*\mathcal{E}, \check{C}^1\mathcal{F}\big)
\end{array}$$

The first column maps to the second via functoriality of f_*, and the second to the third via the above triangle map $\check{\tau}_{\mathcal{F}}$; so the diagram commutes. The columns are exact [H1, p. 23, Prop. 1.1 b)], and thus if each of the first two and last two rows is shown to compose to an isomorphism, then the same holds for the middle row, proving Lemma 6.1.2.

Let's look at the fourth row. With notation as in Lemma 6.1.1 (and again, with all the appropriate $\mathbf{R}$'s omitted), we want the left column in the following natural diagram to compose to an isomorphism:

$$\begin{array}{ccc}
H_{\mathcal{X}}(\mathcal{E}, x_{i*}f_i^\times y_i^*\mathcal{F}) & \xrightarrow{\ \sim\ } & H_{\mathcal{X}_i}(x_i^*\mathcal{E}, f_i^\times y_i^*\mathcal{F})\\
\big\downarrow & & \big\downarrow\\
H_{\mathcal{Y}}(f_*\mathcal{E}, f_*x_{i*}f_i^\times y_i^*\mathcal{F}) & & H_{\mathcal{Y}_i}(f_{i*}x_i^*\mathcal{E}, f_{i*}f_i^\times y_i^*\mathcal{F})\\
\simeq\big\downarrow & & \big\downarrow\simeq\\
H_{\mathcal{Y}}(f_*\mathcal{E}, y_{i*}f_{i*}f_i^\times y_i^*\mathcal{F}) & \xrightarrow{\ \sim\ } & H_{\mathcal{Y}_i}(y_i^*f_*\mathcal{E}, f_{i*}f_i^\times y_i^*\mathcal{F})\\
\text{via } \tau_i\big\downarrow & & \big\downarrow\text{via } \tau_i\\
H_{\mathcal{Y}}(f_*\mathcal{E}, y_{i*}y_i^*\mathcal{F}) & \xrightarrow{\ \sim\ } & H_{\mathcal{Y}_i}(y_i^*f_*\mathcal{E}, y_i^*\mathcal{F})
\end{array}$$

Here the horizontal arrows represent adjunction isomorphisms. Checking that the diagram commutes is a straightforward category-theoretic exercise. By hypothesis, the right column composes to an isomorphism. Hence so does the left one.

The argument for the fifth row is similar. Using the (easily checked) fact that the morphism $f_*\check{D}^0 \to \check{C}^0$ is Δ-functorial, we find that the first row is, up to isomorphism, the same as the fourth row with $\mathcal{F}[-1]$ in place of $\mathcal{F}$, so it too composes to an isomorphism. Similarly, isomorphism for the second row follows from that for the fifth. ☐

Remark 6.1.3. For a locally noetherian formal scheme $\mathscr{Y}$, $\mathbf{R}\varGamma'_{\mathscr{Y}}$ is a right adjoint $\mathbf{1}_{\mathrm{t}}^{\times}$ of the inclusion $\mathbf{D}_{\mathrm{qct}}(\mathscr{Y}) \hookrightarrow \mathbf{D}(\mathscr{Y})$ (see Proposition 5.2.1(c)). Hence in Theorem 6.1 (with $\mathscr{Y}$ noetherian), we have $f_{\mathrm{t}}^{\times} \cong f_{\mathrm{t}}^{\times} \circ \mathbf{R}\varGamma'_{\mathscr{Y}}$. Moreover, if $f\colon \mathscr{X} \to \mathscr{Y}$ is a map of quasi-compact formal schemes with $\mathscr{X}$ *properly algebraic,* and $f^{\times}$ is the right adjoint given by Corollary 4.1.1, then $f_{\mathrm{t}}^{\times} := \mathbf{R}\varGamma'_{\mathscr{X}} \circ f^{\times}$ is a right adjoint for the restriction of $\mathbf{R}f_*$ to $\mathbf{D}_{\mathrm{qct}}(\mathscr{X})$.

Corollary 6.1.4. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a separated map of noetherian formal schemes. Let $\mathbf{\Lambda}_{\mathscr{X}}\colon \mathbf{D}(\mathscr{X}) \to \mathbf{D}(\mathscr{X})$ be the Δ-functor*

$$\mathbf{\Lambda}_{\mathscr{X}}(-) := \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}, -),$$

and let $f^{\#}\colon \mathbf{D}(\mathscr{Y}) \to \mathbf{D}(\mathscr{X})$ be the Δ-functor

$$f^{\#} := \mathbf{\Lambda}_{\mathscr{X}} f_{\mathrm{t}}^{\times}.$$

(a) *The functor $f^{\#}$ is bounded-below, and there is a map of Δ-functors*

$$\tau^{\#}\colon \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}} f^{\#} \to \mathbf{1}$$

such that for all $\mathcal{G} \in \mathbf{D}_{\mathrm{qc}}(\mathscr{X})$ and $\mathcal{F} \in \mathbf{D}(\mathscr{Y})$, the composed map

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^{\bullet}_{\mathscr{X}}(\mathcal{G}, f^{\#}\mathcal{F}) &\xrightarrow{\text{natural}} \mathbf{R}\mathrm{Hom}^{\bullet}_{\mathscr{Y}}(\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{G}, \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}} f^{\#}\mathcal{F}) \\ &\xrightarrow{\text{via } \tau^{\#}} \mathbf{R}\mathrm{Hom}^{\bullet}_{\mathscr{Y}}(\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{G}, \mathcal{F})\end{aligned}$$

is an isomorphism. *In particular, with $\boldsymbol{j}\colon \mathbf{D}(\mathcal{A}_{\vec{\mathrm{c}}}(\mathscr{X})) \to \mathbf{D}(\mathscr{X})$ the natural functor, $\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}\boldsymbol{j}\colon \mathbf{D}(\mathcal{A}_{\vec{\mathrm{c}}}(\mathscr{X})) \to \mathbf{D}(\mathscr{Y})$ has the bounded-below right adjoint $\mathbf{R}Q_{\mathscr{X}} f^{\#}$.*

(b) *If $g\colon \mathscr{Y} \to \mathscr{Z}$ is another separated map of noetherian formal schemes then there is a natural isomorphism $(gf)^{\#} \xrightarrow{\sim} f^{\#}g^{\#}$.*

Proof. (a) We know from Theorem 6.1 that $f_{\mathrm{t}}^{\times}$ is bounded-below. For $\mathbf{\Lambda}_{\mathscr{X}}$—and hence $f^{\#}$—to be bounded-below, it suffices that the complex $\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}$ be homologically bounded-above, which it is, being locally isomorphic to the bounded complex $\mathcal{K}^{\bullet}_{\infty}$ in the proof of Proposition 5.2.1(a).

Now there are natural functorial isomorphisms, for $(\mathcal{E}, \mathcal{F} \in \mathbf{D}(\mathscr{X}))$,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{E}, \mathcal{F}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(\mathcal{E} \underset{=}{\otimes} \mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}, \mathcal{F}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathscr{X})}\big(\mathcal{E}, \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}, \mathcal{F})\big)\end{aligned} \tag{6.1.4.1}$$

(Whether the natural map $\mathcal{E} \underset{=}{\otimes} \mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}} \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{E}$ is an isomorphism is a local question, dealt with e.g., in [AJL, p. 20, Corollary (3.1.2)]). By Proposition 5.2.1, $\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{G} \in \mathbf{D}_{\mathrm{qct}}(\mathscr{X})$, and hence (a) follows from Theorem 6.1.

Since $\mathbf{R}Q_{\mathscr{X}}$ is right-adjoint to $\boldsymbol{j}$ (Proposition 3.2.3), the last assertion results.

(b) From Propositions 5.2.5 and 5.2.1(a) we see that for $\mathcal{G} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, $\mathbf{R}\varGamma'_{\mathcal{Y}}\mathbf{R}f_*\mathcal{G} \cong \mathbf{R}f_*\mathcal{G}$, whence the functors $f_{\mathrm{t}}^{\times}\mathbf{\Lambda}_{\mathcal{Y}}$ and $f_{\mathrm{t}}^{\times}$ are both right-adjoint to $\mathbf{R}f_*|_{\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})}$, and so are isomorphic. Using Theorem 6.1(b), we have then functorial isomorphisms

$$(gf)^{\#} = \mathbf{\Lambda}_{\mathfrak{X}}(gf)_{\mathrm{t}}^{\times} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathfrak{X}}f_{\mathrm{t}}^{\times}g_{\mathrm{t}}^{\times} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathfrak{X}}f_{\mathrm{t}}^{\times}\mathbf{\Lambda}_{\mathcal{Y}}g_{\mathrm{t}}^{\times} = f^{\#}g^{\#}. \qquad \square$$

Here are some "identities" involving the dualizing functors $f^{\times}$ (Theorem 4.1), $f_{\mathrm{t}}^{\times}$ (Theorem 6.1), and $f^{\#} := \mathbf{\Lambda}_{\mathfrak{X}}f_{\mathrm{t}}^{\times}$ (Corollary 6.1.4).

Note that $\mathbf{\Lambda}_{\mathfrak{X}}$ is right-adjoint to $\mathbf{R}\varGamma'_{\mathfrak{X}}$, see (6.1.4.1). Simple arguments show that the natural maps are isomorphisms $\mathbf{\Lambda}_{\mathfrak{X}} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathfrak{X}}\mathbf{\Lambda}_{\mathfrak{X}}$, $\mathbf{R}\varGamma'_{\mathfrak{X}} \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{\Lambda}_{\mathfrak{X}}$, see (b) and (d) in Remark 6.3.1(i).

Corollary 6.1.5. *With the notation of Corollary 6.1.4:*

(a) *There are natural isomorphisms*

$$\begin{aligned} \mathbf{R}\varGamma'_{\mathfrak{X}}f^{\#} &\xrightarrow{\sim} f_{\mathrm{t}}^{\times}, \qquad & f^{\#} &\xrightarrow{\sim} \mathbf{\Lambda}_{\mathfrak{X}}f_{\mathrm{t}}^{\times}, \\ \mathbf{R}\varGamma'_{\mathfrak{X}}f_{\mathrm{t}}^{\times} &\xrightarrow{\sim} f_{\mathrm{t}}^{\times}, \qquad & f^{\#} &\xrightarrow{\sim} \mathbf{\Lambda}_{\mathfrak{X}}f^{\#}. \end{aligned}$$

(b) *The natural functorial maps* $\mathbf{R}\varGamma'_{\mathcal{Y}} \to \mathbf{1} \to \mathbf{\Lambda}_{\mathcal{Y}}$ *induce isomorphisms*

$$\begin{aligned} f_{\mathrm{t}}^{\times}\mathbf{R}\varGamma'_{\mathcal{Y}} &\xrightarrow{\sim} f_{\mathrm{t}}^{\times} \xrightarrow{\sim} f_{\mathrm{t}}^{\times}\mathbf{\Lambda}_{\mathcal{Y}}, \\ f^{\#}\mathbf{R}\varGamma'_{\mathcal{Y}} &\xrightarrow{\sim} f^{\#} \xrightarrow{\sim} f^{\#}\mathbf{\Lambda}_{\mathcal{Y}}. \end{aligned}$$

(c) *There are natural pairs of maps*

$$\begin{aligned} f_{\mathrm{t}}^{\times} &\xrightarrow{\alpha_1} \mathbf{R}\varGamma'_{\mathfrak{X}}\boldsymbol{j}f^{\times} \xrightarrow{\alpha_2} f_{\mathrm{t}}^{\times}, \\ f^{\#} &\xrightarrow{\beta_1} \mathbf{\Lambda}_{\mathfrak{X}}\boldsymbol{j}f^{\times} \xrightarrow{\beta_2} f^{\#}, \end{aligned}$$

each of which composes to an identity map. If $\mathfrak{X}$ is properly algebraic then all of these maps are isomorphisms.

(d) *If f is* adic, *the functorial isomorphism* $\mathbf{R}\varGamma'_{\mathcal{Y}}\mathbf{R}f_*\boldsymbol{j} \xrightarrow{\sim} \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathfrak{X}}\boldsymbol{j}$ *of Proposition 5.2.7 induces an isomorphism between the respective left adjoints*

$$f^{\times}\mathbf{\Lambda}_{\mathcal{Y}} \xrightarrow{\sim} \mathbf{R}Q_{\mathfrak{X}}f^{\#}.$$

Proof. (a) The second isomorphism (first row) is the identity map. Proposition 5.2.1 yields the third. The first is the composition

$$\mathbf{R}\varGamma'_{\mathfrak{X}}f^{\#} = \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{\Lambda}_{\mathfrak{X}}f_{\mathrm{t}}^{\times} \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}}f_{\mathrm{t}}^{\times} \xrightarrow{\sim} f_{\mathrm{t}}^{\times}.$$

The fourth is the composition

$$f^{\#} = \mathbf{\Lambda}_{\mathfrak{X}}f_{\mathrm{t}}^{\times} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathfrak{X}}\mathbf{\Lambda}_{\mathfrak{X}}f_{\mathrm{t}}^{\times} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathfrak{X}}f^{\#}.$$

(b) The first isomorphism was given in Remark 6.1.3. For the second, check that $f_{\mathrm{t}}^{\times}$ and $f_{\mathrm{t}}^{\times}\mathbf{\Lambda}_{\mathcal{Y}}$ are both right-adjoint to $\mathbf{R}f_*|_{\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})}$ ... (Or, consider the composition $f_{\mathrm{t}}^{\times} \xrightarrow{\sim} f_{\mathrm{t}}^{\times}\mathbf{R}\varGamma'_{\mathcal{Y}} \xrightarrow{\sim} f_{\mathrm{t}}^{\times}\mathbf{R}\varGamma'_{\mathcal{Y}}\mathbf{\Lambda}_{\mathcal{Y}} \xrightarrow{\sim} f_{\mathrm{t}}^{\times}\mathbf{\Lambda}_{\mathcal{Y}}$.) Then apply $\mathbf{\Lambda}_{\mathfrak{X}}$ to the first row to get the second row.

(c) With $\boldsymbol{k}\colon \mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})) \to \mathbf{D}(\mathcal{A}_{\vec{\mathrm{c}}}(\mathfrak{X}))$ the natural functor, let $\alpha\colon \boldsymbol{k}\mathbf{R}Q^{\mathrm{t}}_{\mathfrak{X}}f_{\mathrm{t}}^{\times} \to f^{\times}$ be adjoint to $\mathbf{R}f_*\boldsymbol{j}\boldsymbol{k}\mathbf{R}Q^{\mathrm{t}}_{\mathfrak{X}}f_{\mathrm{t}}^{\times} \overset{5.3.1}{=} \mathbf{R}f_*f_{\mathrm{t}}^{\times} \xrightarrow{\tau_{\mathrm{t}}} \mathbf{1}$. By Corollary 5.2.3, $\boldsymbol{j}(\alpha)\colon f_{\mathrm{t}}^{\times} \to \boldsymbol{j}f^{\times}$ factors naturally as

$$f_{\mathrm{t}}^{\times} \xrightarrow{\alpha_1} \mathbf{R}\varGamma'_{\mathfrak{X}}\boldsymbol{j}f^{\times} \to \boldsymbol{j}f^{\times}.$$

Let α_2 be the map adjoint to the natural composition $\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}\boldsymbol{j}f^\times \to \mathbf{R}f_*\boldsymbol{j}f^\times \to \mathbf{1}$. One checks that $\tau_{\mathrm{t}} \circ \mathbf{R}f_*(\alpha_2\alpha_1) = \tau_{\mathrm{t}}$ (τ_{t} as in Theorem 6.1), whence $\alpha_2\alpha_1 =$ identity.

The pair β_1, β_2 is obtained from α_1, α_2 by application of the functor $\boldsymbol{\Lambda}_{\mathscr{X}}$—see Corollary 5.2.3. (Symmetrically, the pair α_1, α_2 is obtained from β_1, β_2 by application of the functor $\mathbf{R}\varGamma'_{\mathscr{X}}$.)

When $\mathscr{X}$ is properly algebraic, the functor $\boldsymbol{j}$ is fully faithful (Corollary 3.3.4); and it follows that $\mathbf{R}\varGamma'_{\mathscr{X}}\boldsymbol{j}f^\times$ and $f^\times_{\mathrm{t}}$ are both right-adjoint to $\mathbf{R}f_*|_{\mathbf{D}_{\mathrm{qct}}(\mathscr{X})}$.

(d) Straightforward. □

The next two corollaries deal with compatibilities between formal (local) and algebraic (global) duality.

Corollary 6.1.6. *Let $f_0\colon X \to Y$ be a map of noetherian schemes, $W \subset Y$ and $Z \subset f_0^{-1}W$ closed subsets, $\kappa_{\mathscr{Y}}\colon \mathscr{Y} = Y_{/W} \to Y$ and $\kappa_{\mathscr{X}}\colon \mathscr{X} = X_{/Z} \to X$ the respective (flat) completion maps, and $f\colon \mathscr{X} \to \mathscr{Y}$ the map induced by f_0.*

$$\begin{array}{ccc} \mathscr{X} := X_{/Z} & \xrightarrow{\kappa_{\mathscr{X}}} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_0} \\ \mathscr{Y} := Y_{/W} & \xrightarrow[\kappa_{\mathscr{Y}}]{} & Y \end{array}$$

Let τ_{t} be the functorial composition

$$\mathbf{R}f_*\kappa_{\mathscr{X}}^*\mathbf{R}\varGamma_Z f_0^\times\kappa_{\mathscr{Y}*} \underset{5.2.6}{\xrightarrow{\sim}} \kappa_{\mathscr{Y}}^*\mathbf{R}f_{0*}\mathbf{R}\varGamma_Z f_0^\times\kappa_{\mathscr{Y}*} \longrightarrow \kappa_{\mathscr{Y}}^*\mathbf{R}f_{0*}f_0^\times\kappa_{\mathscr{Y}*} \longrightarrow \kappa_{\mathscr{Y}}^*\kappa_{\mathscr{Y}*} \longrightarrow \mathbf{1}.$$

Then for all $\mathcal{E} \in \mathbf{D}_{\mathrm{qct}}(\mathscr{X})$ and $\mathcal{F} \in \mathbf{D}(\mathscr{Y})$, the composed map

$$\begin{aligned} \alpha(\mathcal{E},\mathcal{F})\colon \mathrm{Hom}_{\mathscr{X}}(\mathcal{E}, \kappa_{\mathscr{X}}^*\mathbf{R}\varGamma_Z f_0^\times\kappa_{\mathscr{Y}*}\mathcal{F}) &\longrightarrow \mathrm{Hom}_{\mathscr{X}}(\mathbf{R}f_*\mathcal{E}, \mathbf{R}f_*\kappa_{\mathscr{X}}^*\mathbf{R}\varGamma_Z f_0^\times\kappa_{\mathscr{Y}*}\mathcal{F}) \\ &\underset{\text{via } \tau_{\mathrm{t}}}{\longrightarrow} \mathrm{Hom}_{\mathscr{X}}(\mathbf{R}f_*\mathcal{E}, \mathcal{F}) \end{aligned}$$

is an isomorphism.

In particular, if f_0—and hence f—is separated, then the map adjoint to τ_{t} is an isomorphism of functors (from $\mathbf{D}(\mathscr{Y})$ to $\mathbf{D}_{\mathrm{qct}}(\mathscr{X})$):

$$\kappa_{\mathscr{X}}^*\mathbf{R}\varGamma_Z f_0^\times\kappa_{\mathscr{Y}*} \xrightarrow{\sim} f_{\mathrm{t}}^\times.$$

Proof. For any $\mathcal{E} \in \mathbf{D}_{\mathrm{qct}}(\mathscr{X})$, set $\mathcal{E}_0 := \kappa_{\mathscr{X}*}\mathcal{E} \in \mathbf{D}_{\mathrm{qc}Z}(X)$ (see Proposition 5.2.4). By Proposition 5.2.4 and [AJL, p. 7, Lemma (0.4.2)], there are natural isomorphisms

$$\mathrm{Hom}_{\mathscr{X}}(\mathcal{E}, \kappa_{\mathscr{X}}^*\mathbf{R}\varGamma_Z\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_X(\mathcal{E}_0, \mathbf{R}\varGamma_Z\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_X(\mathcal{E}_0, \mathcal{G}) \qquad \big(\mathcal{G} \in \mathbf{D}_{\mathrm{qc}}(X)\big).$$

(In other words, $\kappa_{\mathscr{X}}^*\mathbf{R}\varGamma_Z\mathcal{G} = (\kappa_{\mathscr{X}})^\times_{\mathrm{t}}\mathcal{G}$.) One checks then that the map $\alpha(\mathcal{E},\mathcal{F})$ factors as the sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{X}}(\mathcal{E}, \kappa_{\mathscr{X}}^*\mathbf{R}\varGamma_Z f_0^\times\kappa_{\mathscr{Y}*}\mathcal{F}) &\xrightarrow{\sim} \mathrm{Hom}_X(\mathcal{E}_0, f_0^\times\kappa_{\mathscr{Y}*}\mathcal{F}) \\ &\xrightarrow{\sim} \mathrm{Hom}_Y(\mathbf{R}f_{0*}\mathcal{E}_0, \kappa_{\mathscr{Y}*}\mathcal{F}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathscr{Y}}(\kappa_{\mathscr{Y}}^*\mathbf{R}f_{0*}\mathcal{E}_0, \mathcal{F}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathscr{Y}}(\mathbf{R}f_*\kappa_{\mathscr{X}}^*\mathcal{E}_0, \mathcal{F}) \qquad \text{(Corollary 5.2.6)} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathscr{Y}}(\mathbf{R}f_*\mathcal{E}, \mathcal{F}). \end{aligned}$$ □

Remark. Using Lemma 6.1.1, we find now that Theorem 6.1 and Corollary 6.1.4 hold for any map $f\colon \mathscr{X} \to \mathscr{Y}$—not necessarily separated—which fits *locally* (over $\mathscr{Y}$) into Corollary 6.1.6. We call such maps "locally algebraizable."

Corollary 6.1.7. *With hypotheses as in Corollary 6.1.6 (see preceding Remark):*

(a) *There are natural isomorphisms*

$$\mathbf{R}\varGamma'_{\mathfrak X}\kappa^*_{\mathfrak X}f_0^{\times}\kappa_{\mathcal Y*} = (\kappa_{\mathfrak X})^{\times}_{\mathrm t}f_0^{\times}\kappa_{\mathcal Y*} \xrightarrow{\ \sim\ } f^{\times}_{\mathrm t},$$
$$\mathbf{\Lambda}_{\mathfrak X}\kappa^*_{\mathfrak X}f_0^{\times}\kappa_{\mathcal Y*} = \kappa^{\#}_{\mathfrak X}f_0^{\times}\kappa_{\mathcal Y*} \xrightarrow{\ \sim\ } f^{\#},$$

and if f_0 is proper, $Y=\operatorname{Spec}(A)$ (A adic) and $Z=f_0^{-1}W$,

$$\kappa^*_{\mathfrak X}f_0^{\times}\kappa_{\mathcal Y*} \xrightarrow{\ \sim\ } f^{\times}.$$

(b) *The functor*

$$f^{\times}_{0,Z} := \mathbf{R}\varGamma_Z f_0^{\times}\colon \mathbf{D}(Y)\to\mathbf{D}_{\mathrm{qc}Z}(X)$$

is right-adjoint to $\mathbf{R}f_*|_{\mathbf{D}_{\mathrm{qc}Z}(X)}$; and there is an isomorphism

$$\kappa^*_{\mathfrak X}f^{\times}_{0,Z}\kappa_{\mathcal Y*} \xrightarrow{\ \sim\ } f^{\times}_{\mathrm t}.$$

(c) *The functor*

$$f^{\#}_{0,Z} := \mathbf{R}Q_X\mathbf{R}\mathcal{H}om^{\bullet}_X(\mathbf{R}\varGamma_Z\mathcal O_X\,,\,f_0^{\times}-)\colon \mathbf{D}(Y)\to\mathbf{D}_{\mathrm{qc}}(X)$$

is right-adjoint to $\mathbf{R}f_*\mathbf{R}\varGamma_Z|_{\mathbf{D}_{\mathrm{qc}}(X)}$; and when $\mathfrak X$ is properly algebraic—so that the categories $\mathbf{D}(\mathcal A_{\vec{\mathrm c}}(\mathfrak X))$ and $\mathbf{D}_{\vec{\mathrm c}}(\mathfrak X)$ are equivalent, see Corollary 3.3.4)—there is an isomorphism

$$\kappa^*_{\mathfrak X}f^{\#}_{0,Z}\kappa_{\mathcal Y*} \xrightarrow{\ \sim\ } \mathbf{R}Q_{\mathfrak X}f^{\#}.$$

Proof. (a) The first isomorphism combines Corollary 6.1.6 (in proving which we noted that $\kappa^*_{\mathfrak X}\mathbf{R}\varGamma_Z\mathcal G = (\kappa_{\mathfrak X})^{\times}_{\mathrm t}\mathcal G$ for $\mathcal G\in\mathbf{D}_{\mathrm{qc}}(X)$) an Proposition 5.2.4. The second follows from $f^{\#}=\mathbf{\Lambda}_{\mathfrak X}f^{\times}_{\mathrm t}$. The third is Corollary 4.1.2.

(b) The first assertion is easily checked; and the isomorphism is given by Corollary 6.1.6.

(c) The first assertion is easily checked. From Corollary 6.1.6 and Proposition 5.2.4 we get an isomorphism

$$\mathbf{R}\varGamma_Z f_0^{\times}\kappa_{\mathcal Y*} \xrightarrow{\ \sim\ } \kappa_{\mathfrak X*}f^{\times}_{\mathrm t}.$$

As in Corollary 5.2.3, the natural map is an isomorphism

$$\mathbf{R}\mathcal{H}om^{\bullet}_X(\mathbf{R}\varGamma_Z\mathcal O_X\,,\mathcal G) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^{\bullet}_X(\mathbf{R}\varGamma_Z\mathcal O_X\,,\mathbf{R}\varGamma_Z\mathcal G)\qquad\big(\mathcal G\in\mathbf{D}_{\mathrm{qc}}(X)\big).$$

When $\mathfrak X$ is properly algebraic, $\mathbf{R}Q_{\mathfrak X}\cong\kappa^*_{\mathfrak X}\mathbf{R}Q_X\kappa_{\mathfrak X*}$ (Proposition 3.2.3). So then there is a sequence of natural isomorphisms

$$\begin{aligned}
\kappa^*_{\mathfrak X}f^{\#}_{0,Z}\kappa_{\mathcal Y*} &=\!= \kappa^*_{\mathfrak X}\mathbf{R}Q_X\mathbf{R}\mathcal{H}om^{\bullet}_X(\mathbf{R}\varGamma_Z\mathcal O_X\,,\,f_0^{\times}\kappa_{\mathcal Y*}-)\\
&\xrightarrow{\ \sim\ } \kappa^*_{\mathfrak X}\mathbf{R}Q_X\mathbf{R}\mathcal{H}om^{\bullet}_X(\mathbf{R}\varGamma_Z\mathcal O_X\,,\,\mathbf{R}\varGamma_Z f_0^{\times}\kappa_{\mathcal Y*}-)\\
&\xrightarrow{\ \sim\ } \kappa^*_{\mathfrak X}\mathbf{R}Q_X\mathbf{R}\mathcal{H}om^{\bullet}_X(\mathbf{R}\varGamma_Z\mathcal O_X\,,\,\kappa_{\mathfrak X*}f^{\times}_{\mathrm t}-)\\
&\xrightarrow{\ \sim\ } \kappa^*_{\mathfrak X}\mathbf{R}Q_X\kappa_{\mathfrak X*}\mathbf{R}\mathcal{H}om^{\bullet}_{\mathfrak X}(\kappa^*_{\mathfrak X}\mathbf{R}\varGamma_Z\mathcal O_X\,,\,f^{\times}_{\mathrm t}-)\\
&\xrightarrow{\ \sim\ } \mathbf{R}Q_{\mathfrak X}\mathbf{R}\mathcal{H}om^{\bullet}_{\mathfrak X}(\mathbf{R}\varGamma'_{\mathfrak X}\mathcal O_{\mathfrak X},\,f^{\times}_{\mathrm t}-)\\
&=\!= \mathbf{R}Q_{\mathfrak X}f^{\#}.
\end{aligned}$$

□

6.2. The next Proposition is a special case of Greenlees-May Duality for formal schemes. It provides, in particular, information about the behavior of the duality functors $f^{\times}$ and $f^{\#}$ on complexes with coherent homology (Corollary 6.2.2, Corollary 6.2.3).

Proposition 6.2.1. *Let $\mathcal{X}$ be a locally noetherian formal scheme, and $\mathcal{E} \in \mathbf{D}(\mathcal{X})$. Then for all $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(\mathcal{X})$ the natural map $\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E} \to \mathcal{E}$ induces an isomorphism*

$$\mathbf{R}\mathcal{H}om^{\bullet}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E}, \mathcal{F}).$$

Proof. The canonical isomorphism (cf. (6.1.4.1))

$$\mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^{\bullet}\big(\mathcal{E}, \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{O}_{\mathcal{X}}, \mathcal{F})\big)$$

reduces the question to where $\mathcal{E} = \mathcal{O}_{\mathcal{X}}$. It suffices then—as in the proof of Corollary 5.2.3—that for affine $\mathcal{X} = \mathrm{Spf}(A)$, the natural map be an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathcal{O}_{\mathcal{X}}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{O}_{\mathcal{X}}, \mathcal{F}) \qquad \big(\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(\mathcal{X})\big).$$

Let I be an ideal of definition of the adic ring A, set $Z = \mathrm{Supp}(A/I)$, and let $\kappa\colon \mathcal{X} \to X := \mathrm{Spec}(A)$ be the completion map. Via the categorical equivalences in Proposition 3.3.1 and the isomorphism $\kappa^*\mathbf{R}\varGamma'_Z\mathcal{O}_X \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{O}_{\mathcal{X}}$ in Proposition 5.2.4, the question becomes whether for $\mathcal{F}_0 \in \mathbf{D}_{\mathrm{c}}(X)$ the natural map is an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{O}_X, \mathcal{F}_0) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{R}\varGamma_Z\mathcal{O}_X, \mathbf{R}Q_X\kappa_*\kappa^*\mathcal{F}_0).$$

But given the isomorphism $\kappa_*\kappa^*\mathcal{F}_0 \xrightarrow{\sim} \mathbf{L}\Lambda_Z\mathcal{F}_0$ of [AJL, p. 6, Proposition (0.4.1)], this is just one form of the Greenlees-May duality isomorphism (see [AJL, p. 5, Remark (0.4)(a)]). □

Corollary 6.2.2. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a separated adic map of noetherian formal schemes. Then for all $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(\mathcal{Y})$ the map corresponding to the natural composition $\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathcal{X}}\boldsymbol{j}f^{\times}\mathcal{F} \to \mathbf{R}f_*\boldsymbol{j}f^{\times}\mathcal{F} \to \mathcal{F}$ (see Corollary 6.1.4 and Theorem 4.1) is an isomorphism*

$$f^{\times}\mathcal{F} \xrightarrow{\sim} \mathbf{R}Q_{\mathcal{X}}f^{\#}\mathcal{F}.$$

Proof. By Proposition 6.2.1, $\mathcal{F} \cong \boldsymbol{\Lambda}_{\mathcal{Y}}\mathcal{F} := \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathcal{Y}}\mathcal{O}_{\mathcal{Y}}, \mathcal{F})$; so this Corollary is a special case of Corollary 6.1.5(d). □

Corollary 6.2.3. *In Corollary 6.1.6, suppose $Y = \mathrm{Spec}(A)$ (A adic) and that the the map f_0 is* proper. *Then with the customary notation $f_0^!$ for $f_0^{\times}$ we have, for any $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}^+(\mathcal{Y})$, a natural isomorphism*

$$\kappa_{\mathcal{X}}^*f_0^!\kappa_{\mathcal{Y}*}\mathcal{F} \xrightarrow{\sim} f^{\#}\mathcal{F} \in \mathbf{D}_{\mathrm{c}}^+(\mathcal{X}).$$

Proof. The natural map $f_0^!\mathbf{R}Q_Y\kappa_{\mathcal{Y}*} \to f_0^!\kappa_{\mathcal{Y}*}$ is an isomorphism of functors from $\mathbf{D}(\mathcal{Y})$ to $\mathbf{D}_{\mathrm{qc}}(X)$, both being right-adjoint to $\kappa_{\mathcal{Y}}^*\mathbf{R}f_{0*}$. Proposition 3.3.1 gives $\mathbf{R}Q_Y\kappa_{\mathcal{Y}*}\mathcal{F} \in \mathbf{D}_{\mathrm{c}}^+(Y)$; so by [V, p. 396, Lemma 1], $f_0^!\kappa_{\mathcal{Y}*}\mathcal{F} \in \mathbf{D}_{\mathrm{c}}^+(X)$.[21] Hence Proposition 6.2.1 and Corollary 6.1.7(a) yield isomorphisms

$$\kappa_{\mathcal{X}}^*f_0^!\kappa_{\mathcal{Y}*}\mathcal{F} \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{O}_{\mathcal{X}}, \kappa_{\mathcal{X}}^*f_0^!\kappa_{\mathcal{Y}*}\mathcal{F}) =: \boldsymbol{\Lambda}_{\mathcal{X}}\kappa_{\mathcal{X}}^*f_0^!\kappa_{\mathcal{Y}*}\mathcal{F} \xrightarrow{\sim} f^{\#}\mathcal{F}. \qquad □$$

6.3. More relations, involving the functors $\mathbf{R}\varGamma'_{\mathcal{X}}$ and $\boldsymbol{\Lambda}_{\mathcal{X}} := \mathbf{R}\mathcal{H}om^{\bullet}(\mathbf{R}\varGamma'_{\mathcal{X}}\mathcal{O}_{\mathcal{X}}, -)$ on a locally noetherian formal scheme $\mathcal{X}$, will now be summarized.

[21] For $\mathcal{G} \in \mathbf{D}_{\mathrm{c}}^+(Y)$ we have $f_0^!\mathcal{G} \in \mathbf{D}_{\mathrm{c}}^+(X)$: The question being local on X one reduces to where either X is the projective space $\mathbf{P}_Y^n$ for some n and f_0 is the projection—so that $f_0^!\mathcal{G} = f_0^*\mathcal{G} \otimes \Omega_{X/Y}^n[n] \in \mathbf{D}_{\mathrm{c}}^+(X)$—or f is a closed immersion and $f_{0*}f_0^!\mathcal{G} = \mathbf{R}\mathcal{H}om_Y^{\bullet}(f_{0*}\mathcal{O}_X, \mathcal{F}) \in \mathbf{D}_{\mathrm{c}}^+(Y)$ [H1, p. 92, Proposition 3.3] whence, again, $f_0^!\mathcal{G} \in \mathbf{D}_{\mathrm{c}}^+(X)$ [GD, p. 115, (5.3.13)].

Remarks 6.3.1. Let $\mathcal{X}$ be a locally noetherian formal scheme, and $\boldsymbol{j}\colon \mathbf{D}(\mathcal{A}_{\mathrm{t}}(\mathcal{X})) \to \mathbf{D}(\mathcal{X})$ the natural functor.

(i) The functor $\boldsymbol{\varGamma} := \boldsymbol{j}\mathbf{R}\varGamma'_{\mathcal{X}}\colon \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X})$ admits a natural map $\boldsymbol{\varGamma} \xrightarrow{\gamma} \mathbf{1}$, which induces a functorial isomorphism

$$\text{(A)} \qquad \operatorname{Hom}(\boldsymbol{\varGamma}\mathcal{E}, \boldsymbol{\varGamma}\mathcal{F}) \xrightarrow{\sim} \operatorname{Hom}(\boldsymbol{\varGamma}\mathcal{E}, \mathcal{F}) \qquad \big(\mathcal{E}, \mathcal{F} \in \mathbf{D}(\mathcal{X})\big),$$

see Proposition 5.2.1(c). Moreover $\boldsymbol{\varGamma}$ has a right adjoint, viz. $\boldsymbol{\Lambda} := \boldsymbol{\Lambda}_{\mathcal{X}}$ (see (6.1.4.1)).

The rest of (i) consists of (well-known) formal consequences of these properties.[22]

Since γ is functorial, it holds that $\gamma(\mathcal{F}) \circ \gamma(\boldsymbol{\varGamma}\mathcal{F}) = \gamma(\mathcal{F}) \circ \boldsymbol{\varGamma}(\gamma(\mathcal{F}))\colon \boldsymbol{\varGamma}\boldsymbol{\varGamma}\mathcal{F} \to \mathcal{F}$, so injectivity of the map in (a) (with $\mathcal{E} = \boldsymbol{\varGamma}\mathcal{F}$) yields $\gamma(\boldsymbol{\varGamma}\mathcal{F}) = \boldsymbol{\varGamma}(\gamma(\mathcal{F}))\colon \boldsymbol{\varGamma}\boldsymbol{\varGamma}\mathcal{F} \to \boldsymbol{\varGamma}\mathcal{F}$; and one finds after setting $\mathcal{F} = \boldsymbol{\varGamma}\mathcal{G}$ in (a) that this functorial map is an *isomorphism*

$$\text{(a)} \qquad \gamma(\boldsymbol{\varGamma}) = \boldsymbol{\varGamma}(\gamma)\colon \boldsymbol{\varGamma}\boldsymbol{\varGamma} \xrightarrow{\sim} \boldsymbol{\varGamma}.$$

Conversely, given (a) one can deduce that the map in (A) is an isomorphism, whose inverse takes $\alpha\colon \boldsymbol{\varGamma}\mathcal{E} \to \mathcal{F}$ to the composition $\boldsymbol{\varGamma}\mathcal{E} \xrightarrow{\sim} \boldsymbol{\varGamma}\boldsymbol{\varGamma}\mathcal{E} \xrightarrow{\boldsymbol{\varGamma}\alpha} \boldsymbol{\varGamma}\mathcal{F}$.

The functorial map $\mathbf{1} \xrightarrow{\lambda} \boldsymbol{\Lambda}$ "right-conjugate" to γ induces an isomorphism

$$\text{(B)} \qquad \operatorname{Hom}(\boldsymbol{\Lambda}\mathcal{E}, \boldsymbol{\Lambda}\mathcal{F}) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{E}, \boldsymbol{\Lambda}\mathcal{F}) \qquad \big(\mathcal{E}, \mathcal{F} \in \mathbf{D}(\mathcal{X})\big),$$

or equivalently (as above), λ induces an isomorphism

$$\text{(b)} \qquad \lambda(\boldsymbol{\Lambda}) = \boldsymbol{\Lambda}(\lambda)\colon \boldsymbol{\Lambda} \xrightarrow{\sim} \boldsymbol{\Lambda}\boldsymbol{\Lambda}.$$

Moreover, the isomorphism (A) transforms via adjointness to an isomorphism

$$\operatorname{Hom}(\mathcal{E}, \boldsymbol{\Lambda}\boldsymbol{\varGamma}\mathcal{F}) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{E}, \boldsymbol{\Lambda}\mathcal{F}) \qquad \big(\mathcal{E}, \mathcal{F} \in \mathbf{D}(\mathcal{X})\big),$$

whose meaning is that γ induces an isomorphism

$$\text{(c)} \qquad \boldsymbol{\Lambda}\boldsymbol{\varGamma} \xrightarrow{\sim} \boldsymbol{\Lambda}.$$

Similarly, (B) means that λ induces the conjugate isomorphism

$$\text{(d)} \qquad \boldsymbol{\varGamma}\boldsymbol{\Lambda} \xleftarrow{\sim} \boldsymbol{\varGamma}.$$

Similarly, that $\boldsymbol{\Lambda}(\lambda(\mathcal{F}))$—or $\gamma(\boldsymbol{\varGamma}(\mathcal{E}))$—is an isomorphism (respectively that $\lambda(\boldsymbol{\Lambda}(\mathcal{F}))$—or $\boldsymbol{\varGamma}(\gamma(\mathcal{E}))$—is an isomorphism) is equivalent to the first (respectively the second) of the following maps (induced by λ and γ respectively) being an isomorphism:

$$\text{(AB)} \qquad \operatorname{Hom}(\boldsymbol{\varGamma}\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \operatorname{Hom}(\boldsymbol{\varGamma}\mathcal{E}, \boldsymbol{\Lambda}\mathcal{F}) \xleftarrow{\sim} \operatorname{Hom}(\mathcal{E}, \boldsymbol{\Lambda}\mathcal{F}).$$

That (c) is an isomorphism also means that the "homology localization" functor $\boldsymbol{\Lambda}$ factors, via $\boldsymbol{\varGamma}$, through the essential image $\mathbf{D}_{\mathrm{t}}(\mathcal{X})$ of $\boldsymbol{\varGamma}$ (i.e., the full subcategory $\mathbf{D}_{\mathrm{t}}(\mathcal{X})$ whose objects are isomorphic to $\boldsymbol{\varGamma}\mathcal{E}$ for some $\mathcal{E}$); and similarly (d) being an isomorphism means that the "cohomology colocalization" functor $\boldsymbol{\varGamma}$ factors, via $\boldsymbol{\Lambda}$, through the essential image $\mathbf{D}\hat{}(\mathcal{X})$ of $\boldsymbol{\Lambda}$; and the isomorphisms $\boldsymbol{\varGamma}\boldsymbol{\Lambda}\boldsymbol{\varGamma} \cong \boldsymbol{\varGamma}$ and $\boldsymbol{\Lambda}\boldsymbol{\varGamma}\boldsymbol{\Lambda} \cong \boldsymbol{\Lambda}$ deduced from (a)–(d) signify that $\boldsymbol{\Lambda}$ and $\boldsymbol{\varGamma}$ induce quasi-inverse equivalences between the categories $\mathbf{D}_{\mathrm{t}}(\mathcal{X})$ and $\mathbf{D}\hat{}(\mathcal{X})$.

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[22] The *idempotence* of $\boldsymbol{\varGamma}$, expressed by (a), can be interpreted as follows. Set $\mathbf{D} := \mathbf{D}(\mathcal{X})$, $\mathbf{S} := \{\, \mathcal{E} \in \mathbf{D} \mid \boldsymbol{\varGamma}(\mathcal{E}) = 0 \,\}$, so that $\boldsymbol{\varGamma}$ factors uniquely as $\mathbf{D} \xrightarrow{q} \mathbf{D}/\mathbf{S} \xrightarrow{\boldsymbol{\varGamma}'} \mathbf{D}$ where q is the "Verdier quotient" functor. Then $\boldsymbol{\varGamma}'$ *is left-adjoint to* q, so that $\mathbf{S} \subset \mathbf{D}$ admits a "Bousfield colocalization." It follows from (c) and (d) below that $\mathbf{S} = \{\, \mathcal{E} \in \mathbf{D} \mid \boldsymbol{\Lambda}(\mathcal{E}) = 0 \,\}$, and (b) below means that *the functor* $\boldsymbol{\Lambda}'\colon \mathbf{D}/\mathbf{S} \to \mathbf{D}$ *defined by* $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}' \circ q$ *is left-adjoint to* q; thus $\mathbf{S} \subset \mathbf{D}$ also admits a "Bousfield localization." And $\mathbf{D}/\mathbf{S}$ is equivalent, via $\boldsymbol{\varGamma}'$ and $\boldsymbol{\Lambda}'$ respectively, to the categories $\mathbf{D}_{\mathrm{t}} \subset \mathbf{D}$ and $\mathbf{D}\hat{} \subset \mathbf{D}$ introduced below—categories denoted by $\mathbf{S}^{\perp}$ and ${}^{\perp}\mathbf{S}$ in [N2, Chapter 8].

(ii) If $\mathfrak{X}$ is properly algebraic, the natural functor $\boldsymbol{j}_{\vec{c}}\colon \mathbf{D}(\mathcal{A}_{\vec{c}}(\mathfrak{X})) \to \mathbf{D}_{\vec{c}}(\mathfrak{X})$ is an *equivalence,* and the inclusion $\mathbf{D}_{\vec{c}}(\mathfrak{X}) \hookrightarrow \mathbf{D}(\mathfrak{X})$ has a right adjoint $\mathbf{Q} := \boldsymbol{j}_{\vec{c}}\mathbf{R}Q_{\mathfrak{X}}$ (see Corollary 3.3.4.) Then (easy check, given Corollary 3.1.5 and Proposition 5.2.1) all of (i) holds with $\mathbf{D}$, $\mathbf{D}_{\mathrm{t}}$, and $\boldsymbol{\Lambda}$ replaced by $\mathbf{D}_{\vec{c}}$, $\mathbf{D}_{\mathrm{qct}}$, and $\boldsymbol{\Lambda}^{\vec{c}} := \mathbf{Q}\boldsymbol{\Lambda}$, respectively.

(iii) As in (i), $\boldsymbol{\Lambda}$ induces an equivalence from $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ to $\mathbf{D}_{\mathrm{\hat{qc}}}(\mathfrak{X})$, the essential image of $\boldsymbol{\Lambda}|_{\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})}$—or, since $\boldsymbol{\Lambda} \cong \boldsymbol{\Lambda}\boldsymbol{\varGamma}$, of $\boldsymbol{\Lambda}|_{\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})}$ (Proposition 5.2.1). So for any separated map $f\colon \mathfrak{X} \to \mathfrak{Y}$ of noetherian formal schemes, the functor

$$\boldsymbol{\Lambda}_{\mathfrak{Y}}\mathbf{R}f_{*}\mathbf{R}\varGamma'_{\mathfrak{X}}\colon \mathbf{D}_{\mathrm{\hat{qc}}}(\mathfrak{X}) \to \mathbf{D}_{\mathrm{\hat{qc}}}(\mathfrak{Y})$$

has the right adjoint $\boldsymbol{\Lambda}_{\mathfrak{X}}f_{\mathrm{t}}^{\times}\mathbf{R}\varGamma'_{\mathfrak{Y}} = \boldsymbol{\Lambda}_{\mathfrak{X}}f_{\mathrm{t}}^{\times} = f^{\#}$ (see Corollaries 6.1.5 and 6.1.4). Hence we get two "parallel" adjoint pseudofunctors [L4, (3.6.7)(d)] (where "3.6.6" = "3.6.2"):

$$(\mathbf{R}f_{*}, f_{\mathrm{t}}^{\times}) \text{ (on } \mathbf{D}_{\mathrm{qct}}) \quad \text{and} \quad (\boldsymbol{\Lambda}\mathbf{R}f_{*}\mathbf{R}\varGamma', f^{\#}) \text{ (on } \mathbf{D}_{\mathrm{\hat{qc}}}).$$

(Both of these correspond to the same adjoint pseudofunctor on the quotient $\mathbf{D}_{\mathrm{qc}}/\mathbf{S}_{\mathrm{qc}}$, see footnote under (i)).

(iv) Let $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(\mathfrak{X})$. Proposition 6.2.1 (for $\mathcal{E} := \mathcal{O}_{\mathfrak{X}}$) shows that $\mathcal{F} \cong \boldsymbol{\Lambda}\mathcal{F}$; and so $\mathbf{D}_{\mathrm{c}}(\mathfrak{X}) \subset \mathbf{D}_{\mathrm{\hat{qc}}}(\mathfrak{X})$. If $\mathfrak{X}$ is noetherian and $f\colon \mathfrak{X} \to \mathfrak{Y}$ is *proper* then $\mathbf{R}f_{*}\mathcal{F} \in \mathbf{D}_{\mathrm{c}}(\mathfrak{Y})$ (Proposition 3.5.1), and, by Propositions 5.2.7 and 6.2.1,

$$\boldsymbol{\Lambda}_{\mathfrak{Y}}\mathbf{R}f_{*}\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{F} \cong \boldsymbol{\Lambda}_{\mathfrak{Y}}\mathbf{R}\varGamma'_{\mathfrak{Y}}\mathbf{R}f_{*}\mathcal{F} \cong \boldsymbol{\Lambda}_{\mathfrak{Y}}\mathbf{R}f_{*}\mathcal{F} \cong \mathbf{R}f_{*}\mathcal{F}.$$

Moreover, under the hypotheses of Corollary 6.2.3, it holds that $f^{\#}(\mathbf{D}_{\mathrm{c}}^{+}(\mathfrak{Y})) \subset \mathbf{D}_{\mathrm{c}}^{+}(\mathfrak{X})$. In this case, $\mathfrak{X}$ being properly algebraic (Corollary 3.3.8), it follows from Corollary 6.2.2 that $f^{\#}$ coincides on $\mathbf{D}_{\mathrm{c}}^{+}(\mathfrak{Y})$ with the functor $f^{\times}$ of Corollary 4.1.1. So the second adjunction in (iii)—or the one from Corollary 4.1.1—restricts to an adjunction

$$\mathbf{D}_{\mathrm{c}}^{+}(\mathfrak{X}) \underset{f^{\#}=f^{\times}}{\overset{\mathbf{R}f_{*}}{\rightleftarrows}} \mathbf{D}_{\mathrm{c}}^{+}(\mathfrak{Y}).$$

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