Performance Evaluation 13 (1991) 181–204
North-Holland

Optimal decentralized flow control of Markovian queueing networks with multiple controllers

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Received September 1989
Revised February 1991

Abstract

Hsiao, M.-T.T. and A.A. Lazar, Optimal decentralized flow control of Markovian queueing networks with multiple controllers, Performance Evaluation 13 (1991) 181–204.

A Markovian queueing network model is used to derive decentralized flow control mechanisms in computer communication networks with multiple controllers. Under the network optimization criterion, finding the optimal decentralized flow control that maximizes the average network throughput under an average network delay bound becomes a team decision problem. It is shown that the network optimization problem depends on the parameters of the network only through the conditional estimates of the total arrival and departure rates. Using linear programming, the network optimal control is demonstrated to be a set of window-type mechanisms. Under the user optimization criterion, each individual user maximizes its average throughput subject to a constraint on its average time delay. Finding the optimal decentralized flow control under the individual user's performance results in a multiple objective optimization problem and leads to a game theoretic formulation. Structural results which simplify the problem are presented. It is shown that the user optimization problem depends on the parameters of the network and the action of the other users only through the conditional estimate of the user service rate. The Nash equilibrium solution under the game theoretic formulation is demonstrated to be a set of window-type mechanisms. Finally, the class of decentralized flow control problems with Nash equilibrium solutions is characterized.

Keywords: team decision, game theory, Nash equilibrium, BCMP networks, optimal flow control.

1. Introduction

Consider an arbitrary number of users sharing the communication facilities of a packet switching network. Each user adopts a decentralized flow control strategy by individually monitoring the available information with an acknowledgment protocol. The users are not aware of the presence of the others except through the time delay incurred during a session. Optimal strategies for all users are to be derived simultaneously. The above problem of decentralized flow control can be considered as a team decision problem [12] or a game theoretic problem [13]. These two classes of problems arise due to the fact that performance measures can be based on statistics for the entire network, or statistics for each individual user. If the network performance point of view is taken, the flow control problem consists of many decision makers (controllers) with a common objective (a team decision problem). From the alternative point of view, i.e., that of individual performance, the problem becomes a multiple objective optimization problem with noncooperative decision makers (a game theoretic formulation).

0166-5316/91/$03.50

In the control literature, the ideas of cooperative decision making appeared in team decision problems (e.g., [8,9]) while noncooperative game theory appeared in problems related to sequential strategies for dynamic systems with multiple decision makers and multiple objective functions (e.g., [5,18]). In team decision theory, the team players (decision makers) have access to decentralized information, and decide on individual controls based on a global objective. Hence, these problems can be treated in general as single objective optimization problems. For the game theoretic formulation (multiple objective optimization problems) the concept of a Stackelberg strategy (see e.g., [1]) is suitable for situations in which sequential actions are to be taken by the decision makers, i.e., there is a leader and a follower. These models are exemplified in pursuit-evasion type games. The concept of a Nash equilibrium strategy [28] is, on the other hand suitable for models where there is no natural distinction between the "players" to classify one as leader and others as followers.

A large body of related work has considered models of bargaining, competitive market and various other economic models and appeared in the economics literature (e.g., [6,7,26,27,32,34]). Here concepts of noncooperative and cooperative games form the basis of multiple decision making models [28–30].

The application of game theoretic models to resource sharing in computer networks has been scarce. Kurose [19] applied the concepts of Pareto optimality to the multiple access environment. Courcoubetis [3,4] treated the case of two processes sharing a resource in the Pareto optimal sense. A Pareto optimal strategy is efficient since no alternate distribution of resources exists which improves the performance of one set of users without degrading the performance of some other set of users.

In this article the optimal decentralized flow control of the Markovian queueing network model with multiple controllers described in Section 2 is investigated. The optimality criteria adopted are defined as follows: (i) the global (network) objective is to maximize the average network throughput subject to a bound on the average network time delay, i.e., the average is taken over all users of the network, and (ii) individual (user) objectives are to maximize each user's average throughput subject to the average user time delay constraint. In the second case there are more than one objective and constraint to be considered.

The basic contributions of this paper are as follows. The generalized Norton's equivalent results of [14] are employed to obtain a simple equivalent queueing model of the original Markovian system (Section 3).

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Structural results are first obtained for the network optimization criterion. A representation theorem is given which shows that the network optimization problem depends on the parameters of the network only through the conditional estimates of the total arrival rate and the total departure rates (Section 4.1). Using linear programming, the network optimal flow control is shown to be a window mechanism (Section 4.2). We then consider the user optimization problems under a game theoretic formulation (with constraints). Structural results are obtained for the user optimization criterion using the results of [14]. For the user optimization problem, we prove a separation theorem between flow control and estimation of the user service rate (Section 5.1). The Nash equilibrium solution of this formulation is demonstrated to be a collection of window mechanisms. The class of decentralized flow control problems for which Nash equilibrium solutions exist is characterized (Section 5.2). These results are summarized in Section 6.

2. Markovian queueing network models for decentralized flow control

In order to describe the behavior of large systems of various components or subcomponents, queueing network models are used. The stochastic nature of the input and output to and from a component is in general very complex. Only a very limited class of queueing network models are amenable to analytical solutions. Markovian queueing network models with the so-called "product form" solutions [2] are among those that can be analyzed exactly.

In computer communication networks, the arrivals and departures of packets to and from communication links and nodal processors constitute a stochastic input/output system. Naturally, queueing network models are employed to study the behavior of such systems. Indeed, such models have been used extensively since the early years when computer networks began to attract attention [17,31]. Through these extensive studies, it has been generally accepted that "product form" networks provide a reasonable approximation to the actual behavior of packet switched communication networks. In this article, we present a model for decentralized flow control in computer communication networks based on these Markovian "product form" queueing network models.

Since each user has to choose a flow control strategy, and the optimal choice for decentralized flow control of a user depends on the strategies of other users, we describe a model where all the users are under flow control and decentralized optimal flow control strategies can be obtained for all users simultaneously. This model also allows us to model the behavior of each individual user attempting to optimize on his/her performance objectives.

2.1. Network description

Consider a datagram network or a virtual-circuit packet switching network. By making the assumptions listed below, we model this network as a multi-class queueing network.

- The switching nodes have negligible nodal processing delays and there is no nodal blocking (i.e., there are ample buffers available).
- The nodes are connected by M uni-directional links. The routing of a packet, upon completion of service at a station, is determined by a fixed probability distribution. It can be routed to another node within the network or it can leave the network entirely with certain probabilities.
- Packets are acknowledged individually by an end-to-end protocol. Acknowledgments may be piggy-backed or stand-alone. It is assumed that a negligible delay is incurred in returning an acknowledgment, partly because it is much shorter than data messages, and partly because it may have higher priority. (This assumption, however, can easily be relaxed and incorporated into our model.)
- There are K classes of packets. Each class belongs to a particular source-destination pair, for which optimal flow control mechanisms are to be found. It is assumed that there is a maximum of N_k packets of class k, $k = 1, \dots, K$, where the N_k's are arbitrarily large numbers.
- Each source feeds into a controller which determines the rate of allowing packets into the network based on the number of unacknowledged packets of the corresponding source-destination pair.

2.2. Queueing model

The model for this network consists of M FCFS stations with exponentially distributed service time and class independent service rates μ^i, $1 \leqslant i \leqslant M$. K classes of packets access the network. Packets of class k, $k = 1, \ldots, K$, are allowed into the network through controller k, which is modeled by a station with exponentially distributed service time, at rate $\lambda^k_{j_k}$, where j_k is the number of outstanding packets of class k in the network. The routing is probabilistic. Class k packets enter the network through station i with probability $r^{k \cdot i}$. These packets move from station i to station j with probability r^{kij} and leave the network from station i with probability $r^{ki\cdot} = 1 - \sum_{j=1}^{M} r^{kij}$. Packets of class k, upon leaving the network, are fed back to controller k with probability 1. There are a total of N_k packets of class k in the kth closed subchain. Since there is a fixed number of each class of packets in the network, the feedback queues provide the decentralized information available to the controllers, i.e., the number of outstanding packets of each user. The queueing model of this network is shown in Fig. 1.

Let $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in_i})$ be the state of station i, where x_{ij} is the class of the packet at position j and n_i is the total number of packets at station i. Since the total number of each class of packets is fixed, and only class k packets enter controller queue k, the state of this system can be described by $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M)$. The steady-state probability for the network is given by [2]:

$$p(\boldsymbol{x}) = p(\boldsymbol{0}) \left[\prod_{l_1=0}^{\eta_1 - 1} \lambda^1_{l_1} \cdots \prod_{l_K=0}^{\eta_K - 1} \lambda^K_{l_K} \right] \prod_{i=1}^{M} \left[\left(\frac{\theta^{1i}}{\mu^i} \right)^{n_{1i}} \cdots \left(\frac{\theta^{Ki}}{\mu^i} \right)^{n_{Ki}} \right], \tag{1}$$

where n_{ki} is the number of class k packets at station i, and $\eta_k \triangleq \sum_{j=1}^{M} n_{kj}$, $k = 1, \ldots, K$, is the number of packets in the forward network. For each k, $k = 1, \ldots, K$, the visit ratios θ^{kj}, satisfy the linear equations

$$\theta^{kj} = \sum_{i=1}^{M} \theta^{ki} r^{kij} + r^{k \cdot j}, \tag{2}$$

for all j, $1 \leqslant j \leqslant M$. Note that the computation of $p(\boldsymbol{0})$, the normalization constant, is not necessary for the solution of our optimization problem.

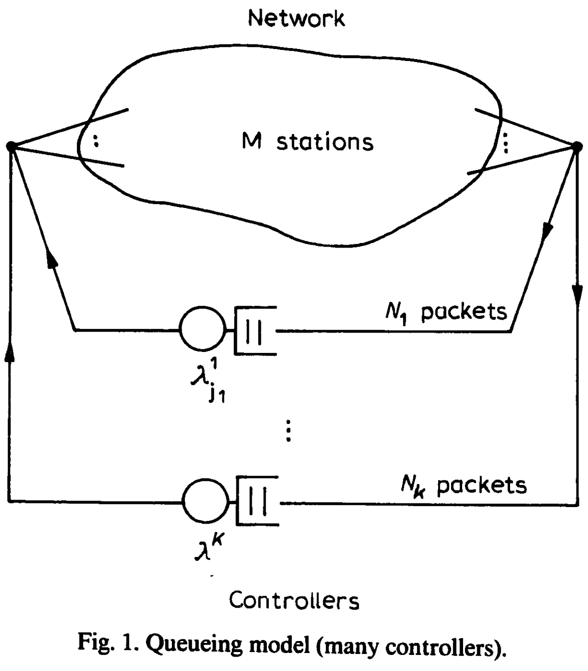

Fig. 1. Queueing model (many controllers).

Let $n_i = (n_{1i}, \dots, n_{Ki})$ be the aggregate state description of station i. By summing over appropriate states, the steady-state probability of state $n = (n_1, \dots, n_M)$ is given by:

$$p(n) = p(0) \left[\prod_{l_1=0}^{\eta_1 - 1} \lambda^1_{l_1} \cdots \prod_{l_K=0}^{\eta_K - 1} \lambda^K_{l_K} \right] \prod_{i=1}^{M} \left[\frac{(n_{1i} + \cdots + n_{Ki})!}{n_{1i}! \cdots ! n_{Ki}!} \left(\frac{\theta^{1i}}{\mu^i} \right)^{n_{1i}} \cdots \left(\frac{\theta^{Ki}}{\mu^i} \right)^{n_{Ki}} \right]. \tag{3}$$

From these probabilities, we can compute for the network model described above any first order statistics such as the average throughput, and the average time delay. In the next section, the optimization problem will be formalized.

2.3. Optimization criteria

Since the network consists of more than one class of packets, two optimization criteria are investigated. The objective is to maximize the average throughput subject to a constraint on the average time delay. For the multiple controller model, we are interested in the *network* performance as well as the performance of each *user* under flow control. The network performance defines the network optimization as a team decision problem. The users' performance define the user optimization problem and leads to a game theoretic formulation.

For each source destination pair, there is a maximum load into the system. This naturally gives rise to the definition of an admissible control. The following is a precise

Definition 1. The class of controls $\lambda^k = (\lambda^k_{jk})$, $0 \leqslant j_k \leqslant N_k - 1$, $k = 1, \dots, K$, satisfying the constraints

$$0 \leqslant \lambda^k_{j_k} \leqslant c^k,$$

for all j_k, $0 \leqslant j_k \leqslant N_k - 1$, where $c^k \in \mathbb{R}_+$, $k = 1, \dots, K$, is a constant, is called admissible.

Denote the average network throughput and average network time delay by $E\gamma_N$ and $E\tau_N$, respectively. Here, $N = (N_1, \dots, N_K)$. These averages can be computed based on the given steady-state probabilities of the network. The network optimization problem is defined as follows.

Definition 2. The control $\lambda = (\lambda^k)$, $1 \leqslant k \leqslant K$, is said to be network-optimal over the class of admissible controls for a given T, $T \in \mathbb{R}_+$, if the maximum

$$\max_{E\tau_N \leqslant T} E\gamma_N$$

is achieved.

On the other hand, since each of the K users has a different performance measure, they define a multiple objective optimization problem. We propose a game theoretic formulation to analyze this problem and define the so-called Nash equilibrium of a noncooperative game. Denote the average user k throughput and average user k time delay by $E\gamma^k_N$ and $E\tau^k_N$, respectively. The user optimization problem is defined as follows.

Definition 3. The control λ^k, of user k is said to be user-optimal over the class of admissible controls of user k, for a given T^k, $T^k \in \mathbb{R}_+$, and $(\lambda^1, \dots, \lambda^{k-1}, \lambda^{k+1}, \dots, \lambda^K)$, if the maximum

$$\max_{E\tau^k_N \leqslant T^k} E\gamma^k_N$$

is achieved.

For each user class k, a similar definition can be given for an user-optimal control for class k. Since each user has a different objective, this becomes a multiple-objective optimization problem. In order to describe the behavior of users who are only concerned with their own performance objectives, the Pareto optimal description is inadequate. In a totally individualistic environment, each user attempts to optimize an individual objective, without regards to those of the other users. In this case, the concept of a Nash equilibrium [28] strategy in a game theoretic setting is more appropriate. In order to analyze the decentralized flow control problem in this setting, it is necessary to introduce the concept of a strategy, i.e., a plan for playing a game. The static decentralized flow control problem can be thought of as a one step game in which each user (player) chooses a strategy in the beginning. This strategy corresponds to the choice of the controlled flow rate of packets. User k's strategy is $(\lambda^k_{j_k}, 1 \leqslant j_k \leqslant N_k)$, $k = 1,\ldots, K$. The constraints on the choice of these strategies is that they be chosen from the class of admissible controls as defined above. For a strategic game (see Appendix), the notion of a Nash equilibrium solution will be adopted. In accordance with intuitive ideas, the game theoretic model of a strategic game consists of K players who have to choose from a set of strategies in order to receive a certain payoff. The Nash equilibrium in a noncooperative game consists of a strategy K-tuple of the players where there is no incentive for any player to deviate from the current strategy, given that the other players do not change their strategies. The formal definition of a Nash equilibrium is given below.

Definition 4. The strategy K-tuple $\lambda^* = (\lambda^{1*},\ldots, \lambda^{K*})$ which satisfies

$$\varphi(\lambda^*) = \lambda^*,$$

where φ is the network reaction function, is called the Nash equilibrium (NE) strategy of the strategic game G.

Remark. The precise definition of the strategic game G and the function φ are given in the appendix.

In Section 4, we shall investigate the Nash equilibrium of this game theoretic model of the user optimization problem. We provide the structure of the equilibrium strategies as well as characterize the class of problems for which there exists Nash equilibrium solutions. For completeness, some background in game theory is given in the Appendix. (For a more detailed introduction to static and dynamic noncooperative game theory, the readers are referred to [1,24].)

3. Norton's equivalent formulation

The queueing model described in Section 2 for decentralized flow control with multiple controllers provides a detailed state description for the network. In this section we show that the average throughput and average time delay can be computed based on the Norton's equivalent [14] of the original network (the interpretation of the Norton's equivalent as a conditional estimate plays a fundamental role here). More specifically, suppose that $(\eta_1,\ldots, \eta_K)$, the number of each class of packets in the forward network, is observed. Then, it will be shown that the conditional departure rate estimates of the K classes of packets are sufficient for computing the average throughput and average time delay. For all k, $1 \leqslant k \leqslant K$, let

$$\mathscr{D}_k \triangleq \{x = (x_1, x_2,\ldots, x_n) \mid x_1 = k,\ n \geqslant 1\}.$$

Thus, $\mathscr{D}_k$ corresponds to the set of states (in the original network) where the first packet in a station is of class k, $1 \leqslant k \leqslant K$. Further, define the sets of states $\mathscr{S}(\eta_1,\ldots, \eta_K)$ and $\mathscr{N}(\eta_1,\ldots, \eta_K)$ by

$$\mathscr{S}(\eta_1,\ldots, \eta_K) \triangleq \{x \mid \#\text{ of class } k \text{ packets} = \eta_k,\ 1 \leqslant k \leqslant K\}$$

and

$$\mathscr{N}(\eta_1,\ldots, \eta_K) \triangleq \{\boldsymbol{n} \mid \#\text{ of class } k \text{ packets} = \eta_k,\ 1 \leqslant k \leqslant K\}.$$

These correspond to sets of states that have the property that the total number of class k packets is η_k, $1 \leqslant k \leqslant K$.

Now, by abuse of notation, the instantaneous departure rate of class k packets at time t, is

$$\mu_t^k = \sum_{i=1}^{M} \mu^i r^{ki\cdot} 1\left(X_t^i \in \mathscr{D}_k \right), \tag{4}$$

where X_t^i is the state of station i at time t and $1(\cdot)$ is the indicator function. To obtain expressions for the conditional estimates, denote the marginal probabilities by $p_{\eta_1 \cdots \eta_K} \triangleq \sum_{x \in \mathscr{S}(\eta_1,\ldots,\eta_K)} p(x)$. Then, the Norton's equivalent service rate for class k users is given by

$$\begin{aligned}
\nu_{\eta_1 \cdots \eta_K}^k &= E\left[\mu_t^k \mid \sum_{i=1}^{M} n_{ji} = \eta_j,\ j = 1,\ldots,K \right] \\
&= \sum_{x \in \mathscr{S}(\eta_1,\ldots,\eta_K)} \sum_{i=1}^{M} \mu^i r^{ki\cdot} 1\left(x^i \in \mathscr{D}_k \right) \frac{p(x)}{p_{\eta_1 \cdots \eta_K}} \\
&= \sum_{n \in \mathscr{N}(\eta_1,\ldots,\eta_K)} \sum_{i=1}^{M} \mu^i r^{ki\cdot} \left(\frac{n_{ki}}{n_{1i} + \cdots + n_{Ki}} \right) 1(n_{1i} + \cdots + n_{Ki} > 0) \frac{p(n)}{p_{\eta_1 \cdots \eta_K}} \\
&= \frac{\displaystyle\sum_{n \in \mathscr{N}(\eta_1,\ldots,\eta_K)} \sum_{i=1}^{M} \mu^i r^{ki\cdot} \left(\frac{n_{ki}}{n_i} \right) 1(n_i > 0) \prod_{j=1}^{M} \frac{(n_{1j} + \cdots + n_{Kj})!}{n_{1j}! \cdots n_{Kj}!} \left(\frac{\theta^{1j}}{\mu^j} \right)^{n_{1j}} \cdots \left(\frac{\theta^{Kj}}{\mu^j} \right)^{n_{Kj}}}{\displaystyle\sum_{n \in \mathscr{N}(\eta_1,\ldots,\eta_K)} \prod_{j=1}^{M} \frac{(n_{1j} + \cdots + n_{Kj})!}{n_{1j}! \cdots n_{Kj}!} \left(\frac{\theta^{1j}}{\mu^j} \right)^{n_{1j}} \cdots \left(\frac{\theta^{Kj}}{\mu^j} \right)^{n_{Kj}}},
\end{aligned}$$

where $n_i \triangleq \sum_{j=1}^{K} n_{ji}$. Note that the λ's do not appear in these estimates.

The equivalent network for the multiple controller model is shown in Fig. 2. The marginal probabilities satisfy a set of "detailed balance equations". This result is given in the following.

Lemma 1. *The marginal probabilities $p_{\eta_1 \cdots \eta_K}$ satisfy the equalities*

$$\lambda_{\eta_k}^k\, p_{\eta_1,\ldots,\eta_K} = \nu_{\eta_1,\ldots,\eta_k+1,\ldots,\eta_K}^k\, p_{\eta_1,\ldots,\eta_k+1,\ldots,\eta_K},$$

for all k, $1 \leqslant k \leqslant K$.

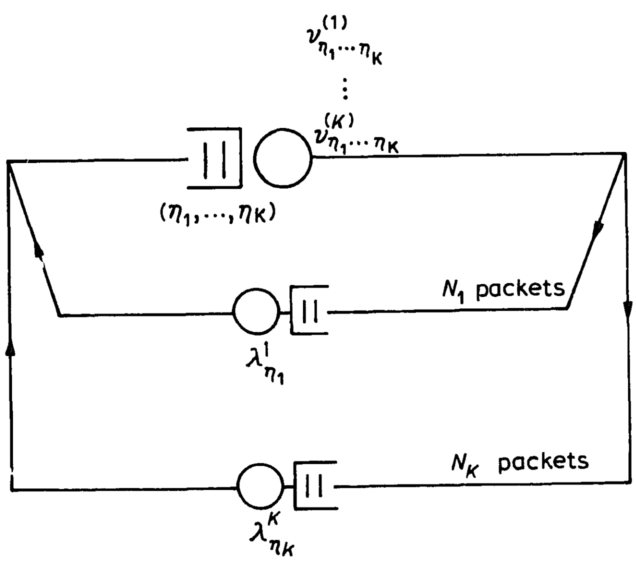

Fig. 2. Equivalent network of the many controllers model.

Proof. First note that by summing the traffic equations (2) over j, we have,

$$\sum_{j=1}^{M} \theta^{kj} = \sum_{j=1}^{M} \sum_{i=1}^{M} \theta^{ki} r^{kij} + 1$$

$$= \sum_{i=1}^{M} \theta^{ki}(1 - r^{ki\cdot}) + 1, \quad \text{for all } k, 1 \leqslant k \leqslant K.$$

Thus,

$$\sum_{i=1}^{M} \theta^{ki} r^{ki\cdot} = 1, \quad \text{for all } k, 1 \leqslant k \leqslant K.$$

Now, by substituting the expressions for $p_{\eta_1 \cdots \eta_K}$ and $\nu^k_{\eta_1 \cdots \eta_K}$,

$$\lambda^k_{\eta_k} p_{\eta_1 \cdots \eta_K} = \sum_{n \in \mathcal{N}(\eta_1, \ldots, \eta_K)} \sum_{i=1}^{M} \theta^{ki} r^{ki\cdot} \left[\prod_{l_1=0}^{\eta_1 - 1} \lambda^1_{l_1} \cdots \prod_{l_k=0}^{\eta_k} \lambda^k_{l_k} \cdots \prod_{l_K=0}^{\eta_K - 1} \lambda^K_{l_K} \right] \cdot$$

$$\prod_{j=1}^{M} \frac{(n_{1j} + \cdots + n_{Kj})!}{n_{1j}! \cdots n_{Kj}!} \left[\left(\frac{\theta^{1j}}{\mu^j} \right)^{n_{1j}} \cdots \left(\frac{\theta^{Kj}}{\mu^j} \right)^{n_{Kj}} \right] p(\mathbf{0})$$

$$= \sum_{n \in \mathcal{N}(\eta_1, \ldots, \eta_k + 1, \ldots, \eta_K)} \sum_{i=1}^{M} \mu^i r^{ki\cdot} \left(\frac{n_{ki}}{n_{1i} + \cdots + n_{Ki}} \right) 1(n_{1i} + \cdots + n_{Ki} > 0).$$

$$\left[\prod_{l_1=0}^{\eta_1 - 1} \lambda^1_{l_1} \cdots \prod_{l_K=0}^{\eta_K - 1} \lambda^K_{l_K} \right] \prod_{j=1}^{M} \frac{(n_{1j} + \cdots + n_{Kj})!}{n_{1j}! \cdots n_{Kj}!} \left(\frac{\theta^{1j}}{\mu^j} \right)^{n_{1j}} \cdots \left(\frac{\theta^{Kj}}{\mu^j} \right)^{n_{Kj}} p(\mathbf{0})$$

$$= \nu^k_{\eta_1, \ldots, \eta_k + 1, \ldots, \eta_K} p_{\eta_1, \ldots, \eta_k + 1, \ldots, \eta_K}. \qquad \square$$

The above conditional estimates can be used to compute the average throughput and time delay expressions, as shown in the next two sections.

4. Network optimization (global objective)

Using the conditional estimates in the previous section, the average network throughput is given by

$$E\gamma_N = \sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \left(\nu^1_{j_1 \cdots j_K} + \cdots + \nu^K_{j_1 \cdots j_K} \right) p_{j_1 \cdots j_K} \tag{6}$$

and the average network time delay is (via Little's formula [23])

$$E\tau_N = \frac{\displaystyle\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} (j_1 + \cdots + j_K) p_{j_1 \cdots j_K}}{\displaystyle\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \left(\nu^1_{j_1 \cdots j_K} + \cdots + \nu^K_{j_1 \cdots j_K} \right) p_{j_1 \cdots j_K}}. \tag{7}$$

Under the network optimization criterion, we seek to maximize a global objective (average network throughput) subject to a global constraint (average network time delay). The expressions for these global averages can be further simplified. We proceed by first giving the structural results for the network optimization problem.

4.1. Structural results for the network optimization problem

In this subsection, we derive the structure of the network optimal decentralized flow control with multiple controllers. The average network throughput and average network time delay can be computed based on π_i, the probability of there being a total of i packets in the network and $\hat{\nu}_i$, the total departure rate conditioned on a total of i packets in the network. From the expression for the average network throughput (6), we can write

$$E\gamma_N = \sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \left(\nu^1_{j_1 \cdots j_K} + \cdots + \nu^K_{j_1 \cdots j_K}\right) p_{j_1 \cdots j_K}$$

$$= \sum_{i=1}^{N_1 + \cdots + N_K} \sum_{j_1 + \cdots + j_K = i} \frac{\left(\nu^1_{j_1 \cdots j_K} + \cdots + \nu^K_{j_1 \cdots j_K}\right) p_{j_1 \cdots j_K}}{\pi_i} \pi_i = \sum_{i=1}^{N_1 + \cdots + N_K} \hat{\nu}_i \pi_i, \tag{8}$$

where $\hat{\nu}_i \triangleq \Sigma_{j_1 + \cdots + j_K = i} (\nu^1_{j_1 \cdots j_K} + \cdots + \nu^K_{j_1 \cdots j_K}) p_{j_1 \cdots j_K} / \pi_i$ and [1] $\pi_i \triangleq \Sigma_{j_1 + \cdots + j_K = i} p_{j_1 \cdots j_K}$.

The average network time delay (7) can be written as:

$$E\tau_N = \frac{\displaystyle\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} (j_1 + \cdots + j_K) p_{j_1 \cdots j_K}}{\displaystyle\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \left(\nu^1_{j_1 \cdots j_K} + \cdots + \nu^K_{j_1 \cdots j_K}\right) p_{j_1 \cdots j_K}} = \frac{\displaystyle\sum_{i=1}^{N_1 + \cdots + N_K} i \pi_i}{\displaystyle\sum_{i=1}^{N_1 + \cdots + N_K} \hat{\nu}_i \pi_i}. \tag{9}$$

Thus, the network flow control problem can be considered as a problem for a single class of packets entering a state-dependent Markovian service station. Intuitively, since the averages are computed over the entire network, and the service rates are class independent, the equivalent system can be characterized by estimates of the arrival and service rates conditioned on the total number of packets in the system. Such a result is given in the following Representation Theorem and is illustrated in Fig. 3.

Let Q^k_t be the number of class k, $1 \leqslant k \leqslant K$, packets in the network at time t.

Theorem 1. (*Representation Theorem*) *The network optimization problem depends on the parameters of the network only through the conditional estimates*

$$\hat{\nu}_{Q^1_t + \cdots + Q^K_t} \triangleq E\left[\nu^1_{Q^1_t \cdots Q^K_t} + \cdots + \nu^K_{Q^1_t \cdots Q^K_t} \mid Q^1_t + \cdots + Q^K_t\right] \tag{10}$$

and

$$\hat{\lambda}_{Q^1_t + \cdots + Q^K_t} \triangleq E\left[\lambda^1_{Q^1_t} + \cdots + \lambda^K_{Q^K_t} \mid Q^1_t + \cdots + Q^K_t\right]. \tag{11}$$

[1] The notation $\Sigma_{j_1 + \cdots + j_K = i}$ stands for the summation over all states $j_1, \ldots, j_K$, in the state space such that $j_1 + \cdots + j_K = i$.

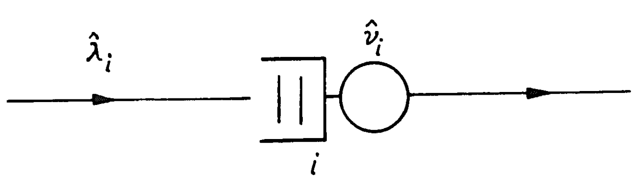

Fig. 3. Equivalent network for the network-optimization problem.

Proof. In view of eqs. (8) and (9), it is sufficient to show that π_i is also the equilibrium probability of a birth–death process with arrival rate $\hat{\lambda}_i$ and departure rate $\hat{\nu}_i$. To verify this, note that

$$\hat{\nu}_i = \frac{\sum_{j_1+\cdots+j_K=i} \left(\nu^1_{j_1\cdots j_K} + \cdots + \nu^K_{j_1\cdots j_K}\right) p_{j_1\cdots j_K}}{\sum_{j_1+\cdots+j_K=i} p_{j_1\cdots j_K}}$$

$$= \frac{\sum_{j_1+\cdots+j_K=i} \left(\lambda^1_{j_1-1} p_{j_1-1,j_2\cdots j_K} + \cdots + \lambda^K_{j_K-1} p_{j_1\cdots j_{K-1},j_K-1}\right)}{\sum_{j_1+\cdots+j_K=i} p_{j_1\cdots j_K}}$$

$$= \frac{\sum_{j_1+\cdots+j_K=i-1} \left(\lambda^1_{j_1} + \cdots + \lambda^K_{j_K}\right) p_{j_1\cdots j_K}}{\pi_{i-1}} \frac{\pi_{i-1}}{\pi_i}$$

$$= \hat{\lambda}_{i-1} \frac{\pi_{i-1}}{\pi_i}, \tag{12}$$

which is the balance equation of a simple birth–death process with rates $\hat{\lambda}_i$ and $\hat{\nu}_i$. Hence, the network optimization problem can be solved based on the conditional estimates (10) and (11). □

4.2. Optimal network flow control

The above theorem gives structural insight into the network optimization problem since the averages needed for the network optimization problem can be computed based on the equivalent network. The optimal network flow control can be found by analyzing the equivalent system. However, since the controls λ appear in the expressions for the conditional estimates $\hat{\lambda}_i$ and $\hat{\nu}_i$, it is more convenient to consider the problem in terms of the averages given by eqs. (6) and (7).

Denote the minimum and maximum average network time delay with any admissible control by T_{min} and T_{max}, respectively. Consider the case where $T_{min} \leqslant T \leqslant T_{max}$. To obtain the form of the optimal control, let us consider the problem as a K step optimization on $\lambda^1, \ldots, \lambda^K$, i.e.,

$$\max_{\lambda}\{\ \} = \max_{\lambda^1} \cdots \max_{\lambda^K}\{\ \}$$

For an arbitrary but fixed $\lambda^1, \ldots, \lambda^{K-1}$, the network optimization problem can be cast in the form of a linear program. This is formalized in the following

Lemma 2. *For fixed $\lambda^1, \ldots, \lambda^{K-1}$, the optimization problem on λ^K is equivalent to the following linear program (LPN):*

$$\max \sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \left(\nu^1_{j_1\cdots j_K} + \cdots + \nu^K_{j_1\cdots j_K}\right) p_{j_1\cdots j_K} \tag{13}$$

subject to:

$$\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \left[j_1 + \cdots + j_K - \left(\nu^1_{j_1\cdots j_K} + \cdots + \nu^K_{j_1\cdots j_K}\right)\right] p_{j_1\cdots j_K} + x = 0, \tag{14}$$

$$\lambda^1_{j_1} p_{j_1 \cdots j_K} = \nu^1_{j_1+1, j_2 \cdots j_K} p_{j_1+1, j_2 \cdots j_K} \qquad \begin{cases} 0 \leqslant j_k \leqslant N_k, \ k = 2, \ldots, K \\ 0 \leqslant j_1 \leqslant N_1 - 1, \end{cases} \tag{15}$$

$$\lambda^2_{j_2} p_{0 j_2 \cdots j_K} = \nu^2_{0, j_2+1, \cdots j_K} p_{0, j_2+1, \cdots j_K} \qquad \begin{cases} 0 \leqslant j_k \leqslant N_k, \ k = 3, \ldots, K \\ 0 \leqslant j_2 \leqslant N_2 - 1, \end{cases} \tag{16}$$

$$\vdots$$

$$\lambda^{K-1}_{j_{K-1}} p_{0 \cdots 0 j_{K-1} j_K} = \nu^{K-1}_{0 \cdots 0, j_{K-1}+1, j_K} p_{0 \cdots 0, j_{K-1}+1, j_K} \qquad \begin{cases} 0 \leqslant j_K \leqslant N_K \\ 0 \leqslant j_{K-1} \leqslant N_{K-1} - 1, \end{cases} \tag{17}$$

$$c^K p_{0 \cdots 0 j_K} = \nu^K_{0 \cdots 0, j_K+1} p_{0 \cdots 0, j_K+1} + y^K_{j_K} \qquad 0 \leqslant j_K \leqslant N_K - 1, \tag{18}$$

and

$$\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} p_{j_1 \cdots j_K} = 1, \tag{19}$$

where $p_{j_1 \cdots j_K}$, $y^K_{j_K}$, $x \geqslant 0$. *A control corresponding to the solution of (LPN) for fixed* $\lambda^1, \ldots, \lambda^{K-1}$, *is given by*:

$$\lambda^K_{j_K} = \begin{cases} 0 & \text{if } p_{j_1 \cdots j_K} = 0, \text{ all } \begin{matrix} 0 \leqslant j_k \leqslant N_k \\ k = 1, \ldots, K-1 \end{matrix}, \\ \dfrac{\nu^K_{j_1 \cdots j_{K-1}, j_K+1} p_{j_1 \cdots j_{K-1}, j_K+1}}{p_{j_1 \cdots j_K}} & \text{if } p_{j_1 \cdots j_K} > 0, \text{ some } j_1, \ldots, j_{K-1}. \end{cases}$$

Proof. Equation (13) is simply the average network throughput. The equality constraint (14) is derived from the time delay constraint, with slack variable x. Equations (18) are derived from the "detailed balance equations" and the admissibility constraint $0 \leqslant \lambda^K_{jK} \leqslant c^K$. Note that the equations form a set of linearly independent balance equations for solving the steady-state probabilities $p_{j_1 \cdots j_K}$. The slack variables $y^K_{j_K} = (c^K - \lambda^K_{j_K}) p_{j_1 \cdots j_K}$ are used to obtain the standard equality constraint formulation. For fixed $\lambda^1, \ldots, \lambda^{K-1}$, eqs. (15)–(17) are the remaining linearly independent balance equations for the steady state probabilities of the equivalent network. □

With the linear programming formulation, we can immediately obtain structural properties of the optimal solution, e.g., [22]. If the time delay constraint, T, is such that $T_{\min} \leqslant T \leqslant T_{\max}$, then a feasible solution exists. Since any continuous objective function achieves a maximum on a non-empty, compact and convex constraint set, there exists an optimal feasible solution. By the Fundamental Theorem of Linear Programming [25], there exists a basic optimal feasible solution. This result will be used to obtain the form of the optimal solution to (LPN) as in the following

Lemma 3. *Suppose that* $\lambda^1, \ldots, \lambda^{K-1}$ *are arbitrarily fixed and* $T_{\min} \leqslant T \leqslant T_{\max}$. *Then there exists an optimal network flow control of the form*:

$$\lambda^K_{j_K} = \begin{cases} 0 & L_K \leqslant j_K < N_K \\ 0 < \lambda^K_{m_K} \leqslant c^K & \text{at most one } m_K, \ 0 \leqslant m_K < L_K \\ c^K & 0 \leqslant j_K < L_K, \ j_K \neq m_K, \end{cases}$$

for some L_K, $0 \leqslant L_K < N_K$.

Proof. From the partial balance equations, it is clear that there exist integers $L_1, \ldots, L_K$ such that $0 \leqslant L_k \leqslant N_k$, $k = 1, \ldots, K$ and

$$p_{j_1 \cdots j_K} > 0 \quad 0 \leqslant j_k \leqslant L_k, \ k = 1, \ldots, K,$$
$$p_{j_1 \cdots j_K} = 0 \quad \text{otherwise.}$$

The constraint equations (14)–(19) thus become:

$$\sum_{j_1=0}^{L_1} \cdots \sum_{j_K=0}^{L_K} \left[j_1 + \cdots + j_K - \left(\nu^1_{j_1 \cdots j_K} + \cdots + \nu^K_{j_1 \cdots j_K} \right) \right] p_{j_1 \cdots j_K} + x = 0, \tag{20}$$

$$\lambda^1_{j_1} p_{j_1 \cdots j_K} = \nu^1_{j_1+1, j_2 \cdots j_K} p_{j_1+1, j_2 \cdots j_K} \quad \begin{cases} 0 \leqslant j_k \leqslant L_k, \; k = 2, \ldots, K \\ 0 \leqslant j_1 \leqslant L_1 - 1, \end{cases} \tag{21}$$

$$\lambda^2_{j_2} p_{0 j_2 \cdots j_K} = \nu^2_{0, j_2+1, \cdots j_K} p_{0, j_2+1, \cdots j_K} \quad \begin{cases} 0 \leqslant j_k \leqslant L_k, \; k = 3, \ldots, K \\ 0 \leqslant j_2 \leqslant L_2 - 1, \end{cases} \tag{22}$$

$$\vdots$$

$$\lambda^{K-1}_{j_{K-1}} p_{0 \cdots 0 j_{K-1} j_K} = \nu^{K-1}_{0 \cdots 0, j_{K-1}+1, j_K} p_{0 \cdots 0, j_{K-1}+1, j_K} \quad \begin{cases} 0 \leqslant j_K \leqslant L_K \\ 0 \leqslant j_{K-1} \leqslant L_{K-1} - 1, \end{cases} \tag{23}$$

$$c^K p_{0 \cdots 0 j_K} = \nu^K_{0 \cdots 0, j_K+1} p_{0 \cdots 0, j_K+1} + y^K_{j_K} \quad 0 \leqslant j_K \leqslant L_K - 1, \tag{24}$$

$$c^K p_{0 \cdots 0 L_K} = y^K_{L_K}, \tag{25}$$

$$\sum_{j_1=0}^{L_1} \cdots \sum_{j_K=0}^{L_K} p_{j_1 \cdots j_K} = 1. \tag{26}$$

The rank, R, of the constraint matrix is therefore $[(L_1+1)(L_2+1) \cdots (L_K+1)] + 2$, corresponding to the number of linearly independent constraint equations. The number of nonzero stationary probabilities variables in this case is $[(L_1+1)(L_2+1) \cdots (L_K+1)]$. In a basic solution at most R variables are positive. From eq. (25), $y^K_{L_K}$ is necessarily positive. Therefore, at most one of $y^K_0, y^K_1, \ldots, y^K_{L_K-1}, x$ is positive. The lemma then follows. □

The above results implies that one only needs to consider window policies for user K, irrespective of the policies of other users. By an inductive argument, the structure of the optimal solution for the network optimization problem can be obtained. Thus, for the general problem, we have proved the following theorem:

Theorem 2. *If $T_{\min} \leqslant T \leqslant T_{\max}$, then there exists a network optimal flow control of the form:*

$$\lambda^1_{j_1} = \begin{cases} 0 & L_1 \leqslant j_1 < N_1 \\ 0 < \lambda^1_{m_1} \leqslant c^1 & \text{at most one } m_1, 0 \leqslant m_1 < L_1 \\ c^1 & 0 \leqslant j_1 < L_1, \; j_1 \neq m_1, \end{cases}$$

$$\vdots$$

$$\lambda^K_{j_K} = \begin{cases} 0 & L_K \leqslant j_K < N_K \\ 0 < \lambda^K_{m_K} \leqslant c^K & \text{at most one } m_K, 0 \leqslant m_K < L_K \\ c^K & 0 \leqslant j_K < L_K, \; j_K \neq m_K, \end{cases}$$

for some $(L_1, \ldots, L_K)$, $0 \leqslant L_k < N_k$, $k = 1, \ldots, K$.

In other words, the optimal control is shown to be a collection of window flow control mechanisms. The optimal window sizes depend on the equivalent service rate ν^k, $k = 1, \ldots, K$, the maximum tolerated time delay T, and the maximum user packet generation rates c^k $k = 1, \ldots, K$. In order to demonstrate the use of the above theorem for obtaining optimal network flow control, we give an example of a bottleneck decentralized flow control in the next subsection.

4.3. Bottleneck modeling and decentralized flow control

The performance of a network can be greatly degraded by bottlenecks. Bottlenecks arise when the input flow exceeds the capacity of any processing or communication medium. One purpose of flow control is to make efficient use of network resources by limiting or rerouting traffic, thus avoiding bottlenecks [10,11,16].

In the following, the bottleneck is modeled as a station with exponentially distributed service time, mean $1/\mu$, and an FCFS service discipline. For simplicity, only two user classes are considered. Packets belonging to either class queue up in a common buffer for transmission via the bottleneck link. User class 1 packets are generated at the maximum rate of c^1 packets per second, while user class 2 packets are generated at the maximum rate of c^2 packets per second. The corresponding controlled packet generation rates are $\lambda^1_{j_1}$ and $\lambda^2_{j_2}$, respectively. This model is a particular case of the network described in the previous subsection and hence, the structure of the network optimal flow control can be obtained from Theorem 2.

In order to find the optimal window sizes for any particular set of user packet generation rates and time delay constraint, it is not necessary to run the corresponding linear program. The form of the optimal control provided by Theorem 2 gives a very explicit graphical way of computing the window sizes. As a numerical example, the service rate μ is normalized to 1. The maximum user packet generation rates are $c^1 = 0.3$, and $c^2 = 0.6$, respectively. The controls $\lambda^1_{j_1}$ and $\lambda^2_{j_2}$ are increased in the following way: if $\lambda^1_{j_1} = c^1$ for all $j_1 < l$, and $\lambda^1_{j_1} = 0$ for all $j_1 \geq l$, increase λ^1_l from 0 to c^1. This process is repeated for $0 \leq l < N_1$. The control $\lambda^2_{j_2}$ is increased in the same manner. Hence, we can plot the average network throughput and average network time delay as a function of the control. These are shown in Figs. 4 and 5, respectively.

In order to display the relationship between the maximum average time delay constraint and the maximum throughput, it is necessary to show their dependence on the control. To do so, first, the control of user 1 is parameterized and the control of user 2 is increased in the sense described above. Secondly, the control of user 2 is parameterized and that of user 1 is increased. The resulting throughput time delay tradeoffs are shown in Figs. 6 and 7, respectively.

Remark. The notation "$L_k = u$" means that $\lambda^k_{j_k} = c^k$, $0 \leq j_k \leq u$ and $\lambda^k_{j_k} = 0$, otherwise ($k = 1, 2$).

The optimal control is obtained by comparing these curves and choosing the control which corresponds to a maximum throughput for a given time delay constraint T. By superimposing the curves in Figs. 6 and 7, it can be seen that the optimal control allows near equal window sizes for the two users. If $c^1 < c^2$, one user 2 packet should enter the system first. If the time delay constraint has not yet been achieved, one user

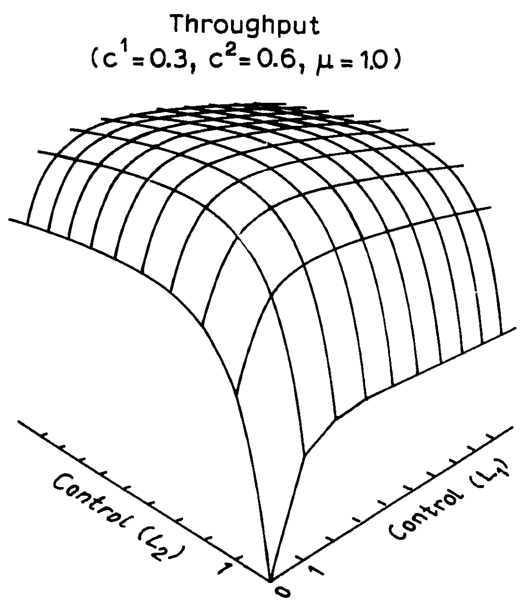

Fig. 4. Throughput versus control.

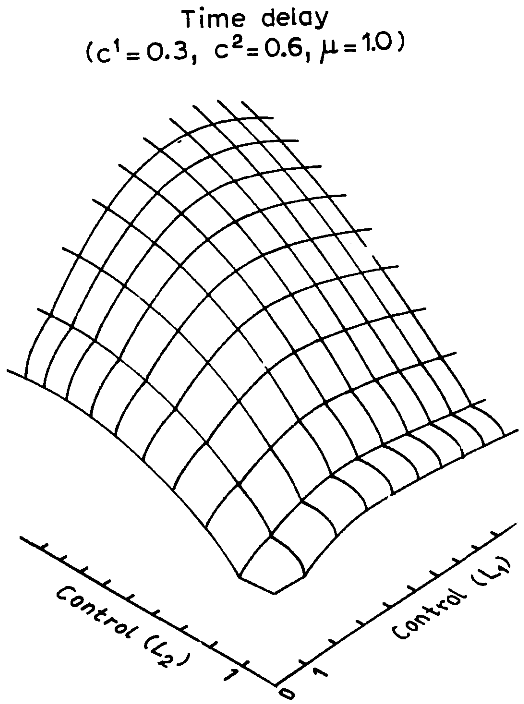

Fig. 5. Time delay versus control.

1 packet has to enter the system. This process is repeated alternating between user 2 and user 1 packets until the time delay constraint is achieved. It is important to note that only the order relation between c^1 and c^2 is needed in the determination of the optimal window sizes for the two users. This does not appear to be intuitive from a system point of view, since the resulting optimal strategy gives equal access to the system resources even though the users have different rates. From a fairness point of view, however, the result is intuitively pleasing since a global objective should not favor any particular type of users.

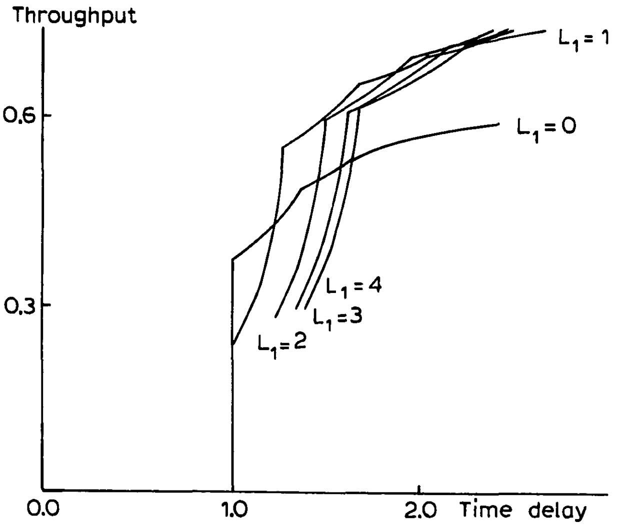

Fig. 6. Throughput versus time delay (λ^1 fixed).

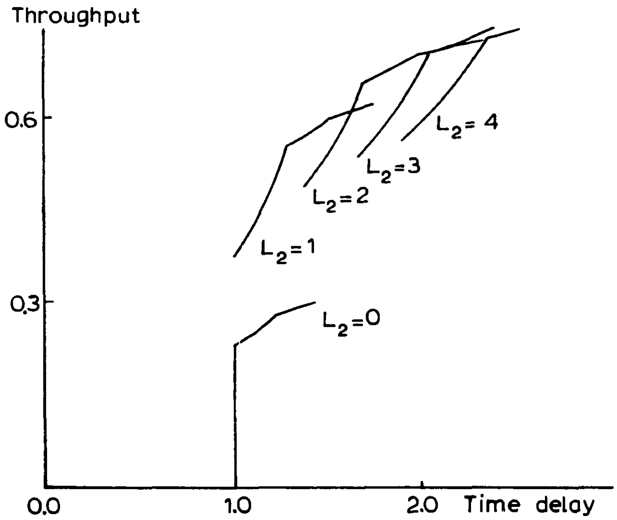

Fig. 7. Throughput versus time delay (λ^2 fixed).

5. User optimization (individual objectives)

The conditional estimates in the Norton's equivalent formulation in Section 3 can be used to compute the average user throughput and user time delay expressions as follows. For user k, the average throughput is given by

$$E\gamma_N^k = \sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \nu_{j_1 \cdots j_K}^k p_{j_1 \cdots j_K} \tag{27}$$

and the average time delay is (via Little's formula)

$$E\tau_N^k = \frac{\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} j_k p_{j_1 \cdots j_K}}{\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \nu_{j_1 \cdots j_K}^k p_{j_1 \cdots j_K}}. \tag{28}$$

Under the user optimization criterion, the users seek to maximize individual objectives (average user throughput) subject to individual constraints (average user time delay). Since there are K users, each with a different objective, the problem becomes a multi-objective optimization problem. In this section, we consider the decentralized flow control problem where each user has a different objective that he/she attempts to optimize in a decentralized and noncooperative manner. As in economic systems, this type of individual optimization leads almost always to conflicting situations.

For problems with more than one objective function, optimality has yet to be defined. One view is to treat the decentralized flow control problem as a noncooperative game in which each user is considered a player of a game under a set of rules [1]. The players would act individually in order to maximize their own payoff. Since the network users behave very much like payoff-maximizing players who act independently of one another, the game theoretic approach here appears to be a natural one.

The user optimal flow control as given by Definition 3 places a hard constraint on the average user time delay. Consider for example a network with a time-out retransmission protocol. If a packet is not acknowledged before the time-out expires, the packet is retransmitted. Hence, for any achieved user throughput, the utility to the user is zero unless the user time delay constraint is satisfied. This motivates the following definition of the utility function of a user.

Definition 5. The utility function of user k, $k = 1, \ldots, K$ is given by:

$$u^k(\lambda) = \begin{cases} E\gamma_N^k & E\tau_N^k \leqslant T^k \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda = (\lambda^1, \ldots, \lambda^K)$.

Specifically, the static game model for the decentralized user flow control problem consists of:
1) $\mathscr{K} = 1, \ldots, K$ (players $1, \ldots, K$),
2) $S^k = \{(\lambda^k_{j_k}, 0 \leqslant j_k \leqslant N_k) \mid 0 \leqslant \lambda^k_{j_k} \leqslant c^k, 0 \leqslant j_k \leqslant N_k\}$ $(k = 1, \ldots, K)$,
3) $h^k(\lambda) = u^k(\lambda)$, $k = 1, \ldots, K$.

Before providing the Nash equilibrium solution to the decentralized flow control problem, the structural properties of the user optimization problem will be presented.

5.1. Structural results for the user optimization problem

For users of class k, the optimization problem is to maximize the average user k throughput, $E\gamma_N^k$, subject to a constraint on the average user k time delay, $E\tau_N^k$. These averages can be expressed in terms of $p^k_{j_k}$ and $\hat{\mu}^k_{j_k}$, the marginal probability of there being j_k class k packets in the network, and the conditional estimate of the departure rate of class k packets given the number of such packets in the network, respectively. Specifically,

$$p^k_{j_k} \triangleq \sum_{j_1=0}^{N_1} \cdots \sum_{j_{k-1}=0}^{N_{k-1}} \sum_{j_{k+1}=0}^{N_{k+1}} \cdots \sum_{j_K=0}^{N_K} p_{j_1 \cdots j_K}$$

and

$$\hat{\mu}^k_{j_k} \triangleq \sum_{j_1=0}^{N_1} \cdots \sum_{j_{k-1}=0}^{N_{k-1}} \sum_{j_{k+1}=0}^{N_{k+1}} \cdots \sum_{j_K=0}^{N_K} \nu^k_{j_1 \cdots j_K} \frac{p_{j_1 \cdots j_K}}{p^k_{j_k}}.$$

To prove the assertion above, recall that the average user k throughput is given by

$$\begin{aligned} E\gamma_N^k &= \sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \nu^k_{j_1 \cdots j_K} p_{j_1 \cdots j_K} \\ &= \sum_{j_k=0}^{N_k} \sum_{j_1=0}^{N_1} \cdots \sum_{j_{k-1}=0}^{N_{k-1}} \sum_{j_{k+1}=0}^{N_{k+1}} \cdots \sum_{j_K=0}^{N_K} \nu^k_{j_1 \cdots j_K} \frac{p_{j_1 \cdots j_K}}{p^k_{j_k}} \cdot p^k_{j_k} \\ &= \sum_{j_k=0}^{N_k} \hat{\mu}^k_{j_k} p^k_{j_k}, \end{aligned} \tag{29}$$

and the average user k time delay is

$$E\tau_N^k = \frac{\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} j_k p_{j_1 \cdots j_K}}{\sum_{j_1=0}^{N_1} \cdots \sum_{j_K=0}^{N_K} \nu^k_{j_1 \cdots j_K} p_{j_1 \cdots j_K}} = \frac{\sum_{j_k=0}^{N_k} j_k p^k_{j_k}}{\sum_{j_k=0}^{N_k} \hat{\mu}^k_{j_k} p^k_{j_k}}. \tag{30}$$

The next theorem gives the structure of the optimization problem for user k. This result indicates that the optimization problem of user k can be reduced to the one shown in Fig. 3. Let Q_t^k, $k = 1, \ldots, K$ be the number of class k packets in the network at time t.

Theorem 3. (*Separation Theorem*) *The optimization problem of user k depends on the parameters of the network and the action of other users only through the conditional estimate of the class k packets departure rate*

$$\hat{\mu}^k_{Q^k_t} \triangleq E\left[\nu^k_{Q^1_t \cdots Q^K_t} \mid Q^k_t\right]. \tag{31}$$

Proof. By virtue of eqs. (29) and (30), it suffices to demonstrate that the marginal probabilities $p^k_{j_k}$, $0 \leqslant j_k \leqslant N_k$, satisfy the following balance equations of a simple birth–death queueing process with arrival rates $\lambda^k_{j_k}$ and departure rates $\hat{\mu}^k_{j_k}$:

$$\lambda^k_{j_k} p^k_{j_k} = \hat{\mu}^k_{j_k+1} p^k_{j_k+1}. \tag{32}$$

From the definition of the conditional estimates,

$$\begin{aligned}\hat{\mu}^k_{j_k} = E\left[\nu^k_{Q^1_t \cdots Q^K_t} \mid Q^k_t = j_k\right] &= \sum_{j_1=0}^{N_1} \cdots \sum_{j_{k-1}=0}^{N_{k-1}} \sum_{j_{k+1}=0}^{N_{k+1}} \cdots \sum_{j_K=0}^{N_K} \nu^k_{j_1 \cdots j_K} \frac{p_{j_1 \cdots j_K}}{p^k_{j_k}} \\ &= \sum_{j_1=0}^{N_1} \cdots \sum_{j_{k-1}=0}^{N_{k-1}} \sum_{j_{k+1}=0}^{N_{k+1}} \cdots \sum_{j_K=0}^{N_K} \lambda^k_{j_k-1} \frac{p_{j_1 \cdots, j_k-1, \ldots, j_K}}{p^k_{j_k}} \\ &= \lambda^k_{j_k-1} \frac{p^k_{j_k-1}}{p^k_{j_k}}.\end{aligned}$$

The theorem then follows. □

Remark. The estimates $\hat{\mu}^k_{j_k}$ are indeed independent of the arrival rates of user k packets, $\lambda^k_{j_k}$. To see this, note that $p^k_{j_k}$ can be written as

$$\prod_{l_k=0}^{j_k-1} \lambda^k_{l_k} \sum_{j_1=0}^{N_1} \cdots \sum_{j_{k-1}=0}^{N_{k-1}} \sum_{j_{k+1}=0}^{N_{k+1}} \cdots \sum_{j_K=0}^{N_K} \prod_{\substack{i=1 \\ i \neq k}}^{K} \left[\prod_{l_i=0}^{j_i-1} \lambda^i_{l_i}\right] \sigma(j_1, \ldots, j_K) p(\mathbf{0}),$$

where

$$\sigma(j_1, \ldots, j_K) \triangleq \sum_{\boldsymbol{n} \in \mathcal{N}(j_1, \ldots, j_K)} \prod_{i=1}^{M} \frac{(n_{1i} + \cdots + n_{Ki})!}{n_{1i}! \cdots n_{Ki}!} \left(\frac{\theta^{1i}}{\mu^i}\right)^{n_{1i}} \cdots \left(\frac{\theta^{Ki}}{\mu^i}\right)^{n_{Ki}}.$$

Hence,

$$\hat{\mu}^k_{j_k} = \frac{\displaystyle\sum_{j_1=0}^{N_1} \cdots \sum_{j_{k-1}=0}^{N_{k-1}} \sum_{j_{k+1}=0}^{N_{k+1}} \cdots \sum_{j_K=0}^{N_K} \prod_{\substack{i=1 \\ i \neq k}}^{K} \left[\prod_{l_i=0}^{j_i-1} \lambda^i_{l_i}\right] \sigma(j_1, \ldots, j_k - 1, \ldots, j_K)}{\displaystyle\sum_{j_1=0}^{N_1} \cdots \sum_{j_{k-1}=0}^{N_{k-1}} \sum_{j_{k+1}=0}^{N_{k+1}} \cdots \sum_{j_K=0}^{N_K} \prod_{\substack{i=1 \\ i \neq k}}^{K} \left[\prod_{l_i=0}^{j_i-1} \lambda^i_{l_i}\right] \sigma(j_1, \ldots, j_k, \ldots, j_K)}.$$

The structural results can be used to investigate the Nash equilibrium solution for the user optimization problems.

5.2. Optimal user flow control and the Nash equilibrium solution

As far as user k is concerned, the objective is to maximize the average user k throughput subject to an average user k time delay constraint. The separation theorem above provides a simple equivalent formulation for the optimization problem of user k. By virtue of eq. (32), the optimal control problem for user k can be formalized as a linear program on the variables $p^k_{j_k}$.

Lemma 4. *The optimization problem for user k is equivalent to the following linear program:*

$$\max \sum_{j_k=0}^{N_k} \hat{\mu}^k_{j_k} p^k_{j_k} \tag{33}$$

subject to

$$\sum_{j_k=0}^{N_k} \left(j_k - \hat{\mu}^k_{j_k} T^k \right) p^k_{j_k} + x = 0, \tag{34}$$

$$c^k p^k_{j_k} = \hat{\mu}^k_{j_k+1} p^k_{j_k+1} + y^k_{j_k},\ 0 \leqslant j_k \leqslant N_k - 1, \tag{35}$$

$$\sum_{j_k=0}^{N_k} p^k_{j_k} = 1, \tag{36}$$

where $p^k_{j_k}$, x, $y^k_{j_k} \geqslant 0$.

Proof. Equations (33) and (34) are the average user k throughput and time delay, respectively. To obtain the equality constraints (35), replace $\lambda^k_{j_k}$ in the balance equations (32) with c^k and introduce the slack variables $y^k_{j_k}$. The optimal control for user k can be obtained from the optimal solution of the linear program as

$$\lambda^k_{j_k} = \hat{\mu}^k_{j_k+1} \frac{p^k_{j_k+1}}{p^k_{j_k}}. \qquad \square$$

Using the linear programming formulation, the optimal control for user k can be shown to be a window-type mechanism, for any given fixed λ^i, $i \neq k$. (For other applications of this technique, see for example [15,22].) Suppose that for the given controls λ^i, $i \neq k$, the minimum and maximum average user k time delay (denoted by $T^k_{\min}$ and $T^k_{\max}$, respectively), that is achievable with any admissible control λ^k, are such that

$$T^k_{\min} \leqslant T^k \leqslant T^k_{\max}.$$

In other words, there exists a feasible solution to the constrained optimization problem of user k. An optimal feasible solution therefore exists by the continuity of the objective function and compactness of the constraint set. Then, the Fundamental Theorem of Linear Programming [25] implies that there exists a basic optimal feasible solution. The following Lemma makes use of this result to obtain the structure of the optimal solution.

Lemma 5. *For any given controls* λ^i, $i \neq k$, *suppose that* $T^k_{\min} \leqslant T^k \leqslant T^k_{\max}$. *Then, there exists an optimal control for the user k optimization problem of the form:*

$$\lambda^k_{j_k} = \begin{cases} 0 & L_k \leqslant j_k < N_k \\ 0 < \lambda^k_{m_k} \leqslant c^k & \text{at most one } m_k,\ 0 \leqslant m_k < L_k, \\ c^k & 0 \leqslant j_k < L_k,\ j_k \neq m_k \end{cases}$$

for some L_k, $0 \leqslant L_k < N_k$.

Proof. Note that from the balance equations (32), there exists an integer L_k such that $0 \leqslant L_k \leqslant N_k$, and

$$p^k_{j_k} > 0 \quad 0 \leqslant j_k \leqslant L_k,$$

$$p^k_{j_k} = 0 \quad \text{otherwise}.$$

The constraint equations (34)–(36) thus become:

$$\sum_{j_k=0}^{L_k} \left(j_k - \hat{\mu}^k_{j_k} T^k \right) p^k_{j_k} + x = 0, \tag{37}$$

$$c^k p^k_{j_k} = \hat{\mu}^k_{j_k+1} p^k_{j_k+1} + y^k_{j_k}, \ 0 \leqslant j_k \leqslant L_k - 1, \tag{38}$$

$$c^k p^k_{L_k} = y^k_{L_k}, \tag{39}$$

and

$$\sum_{j_k=0}^{L_k} p^k_{j_k} = 1. \tag{40}$$

The rank R, of the constraint matrix is therefore $L_k + 3$, corresponding to the number of linearly independent constraint equations. The number of nonzero stationary probabilities variables $p^k_{j_k}$ is $L_k + 1$. Now, there exists a basic optimal feasible solution with at most R positive variables. From eq. (39), $y^k_{L_k}$ is necessarily positive. Therefore, at most one of $y^k_0, y^k_1, \ldots, y^k_{L_k-1}, x$ is positive. This completes the proof. □

The above result implies that the optimal control for user k is to adopt a window-type policy for any control of the other users. The optimal window size, though, depends on the values of $\hat{\mu}^k_{j_k}$, which in turn depend on the control of the other users.

In order to obtain Nash equilibrium solutions to the decentralized user flow control problem, the reaction function of each user, ϕ^k as defined in the Appendix, has to be found. The above Lemma gives the structure of the reaction function of user k, given the controls λ^i, $i \neq k$. The Nash equilibrium strategy is a K-tuple of controls $(\lambda^{1*}, \ldots, \lambda^{K*})$ such that given the controls of other users i, λ^{i*}, $i \neq k$, the optimal strategy for user k is to adopt control λ^{k*}, for all k, $k = 1, \ldots, K$. That is, given the strategies (controls) of the other users, no user has an incentive to deviate from their present strategy. If such a strategy K-tuple exists, then there is a Nash equilibrium strategy of the window type, as shown in the following.

Theorem 4. *The Nash equilibrium solutions to the decentralized user flow control problem are of the form*:

$$\lambda^1_{j_1} = \begin{cases} 0 & L_1 \leqslant j_1 < N_1 \\ 0 < \lambda^1_{m_1} \leqslant c^1 & \text{at most one } m_1, \ 0 \leqslant m_1 < L_1 \\ c^1 & 0 \leqslant j_1 < L_1, \ j_1 \neq m_1, \end{cases}$$

$$\vdots$$

$$\lambda^K_{j_K} = \begin{cases} 0 & L_K \leqslant j_K < N_K \\ 0 < \lambda^K_{m_K} \leqslant c^K & \text{at most one } m_K, \ 0 \leqslant m_K < L_K \\ c^K & 0 \leqslant j_K < L_K, \ j_K \neq m_K, \end{cases}$$

for some $(L_1, \ldots, L_K)$, $0 \leqslant L_k < N_k$, $k = 1, \ldots, K$.

Proof. By Lemma 5, the reaction function of user k is given by a window type control. Therefore, we can restrict the range of the reaction function to the class of window-type controls. Since the Nash equilibrium is defined by the fixed point equation $\varphi(\lambda^*) = \lambda^*$, the theorem follows. □

It remains now to characterize the class of decentralized optimal flow control problems for which there exist Nash equilibrium solutions. The conditional estimates $\hat{\mu}^k_{j_k}$ provide a simplified formulation for the optimization problem of user k. These estimates can be considered as the conditional throughput of class k packets from the network given the number of class k packets in the network. For a large class of

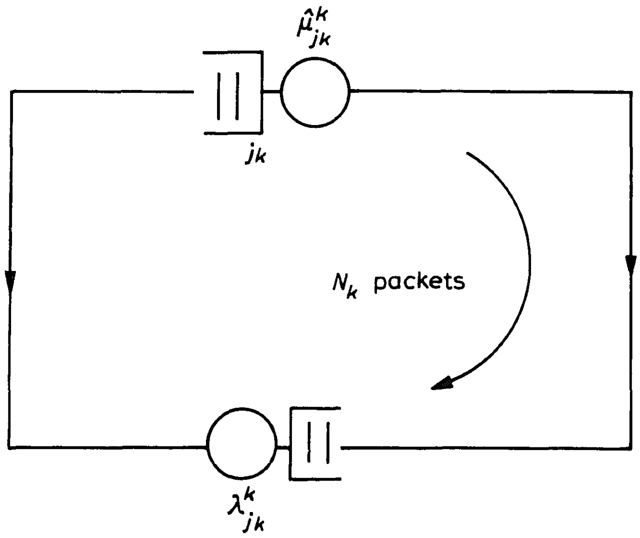

Fig. 8. Equivalent network for the optimization problem of user k.

networks, it has been shown that the throughput is a concave increasing function of the number of packets in the network [20,33]. Intuitively, since the network utilization is increasing with a diminishing rate in a completely shared network with fixed resources, the amount of resource available to a user (the throughput) is a concave increasing function of the number of the user's packets in the network. In the following discussion, we shall assume that the conditional estimates $\hat{\mu}^k_{j_k}$ are concave increasing in j_k, i.e.,

$$\hat{\mu}^k_l \leqslant \hat{\mu}^k_m \leqslant \hat{\mu}^k_n \tag{41}$$

and

$$\frac{\hat{\mu}^k_m - \hat{\mu}^k_l}{m-l} \geqslant \frac{\hat{\mu}^k_n - \hat{\mu}^k_l}{n-l}, \tag{42}$$

for all l, m and n such that $0 \leqslant l < m < n \leqslant N_k$.

The above property can be given a very natural interpretation: the average throughput and the average time delay of user k is increasing with the number of user k packets in the system. For networks with these properties, the optimal flow control problem of user k (as shown in Fig. 8) can be solved via a majorization argument and the optimal solution is explicitly given [21]. The optimal control is characterized by the set of increasing numbers [2], indexed by $L_k = 0, 1, 2, \dots, N_k$, and by abuse of notation,

$$T^{L_k}_{k,\max} = \frac{\sum_{j_k=0}^{L_k} j_k \rho^k_{j_k}}{\sum_{j_k=0}^{L_k} \hat{\mu}^k_{j_k} \rho^k_{j_k}} \leqslant \frac{L_k}{\hat{\mu}^k_{L_k}}, \tag{43}$$

where $\rho^k_{j_k} \triangleq \prod_{l_k=0}^{j_k-1} c^k / \hat{\mu}^k_{l_k+1}$ and the reaction function for user k is given by the following

Theorem 5. *With $\hat{\mu}^k_{j_k}$ determined by $\lambda^{[k]} \triangleq (\lambda^i, i \neq k)$, and given that $T^{L_k-1}_{k,\max} < T^k \leqslant T^{L_k}_{k,\max}$, for some L_k, $2 \leqslant L_k \leqslant N_k$, the reaction function of user k is given by*

$$\varphi^k(\lambda^{[k]}) = \lambda^k_{j_k} = \begin{cases} c^k & 0 \leqslant j_k \leqslant L_k - 2 \\ \lambda^k_{L_k-1} & j_k = L_k - 1 \\ 0 & L_k \leqslant j_k \leqslant N_k, \end{cases} \tag{44}$$

[2] These numbers may be interpreted as the maximum time delay for user k with a total of L_k packets in the network.

where

$$\lambda^k_{L_k-1} = \frac{\hat{\mu}^k_{L_k}}{\left(L_k - \hat{\mu}^k_{L_k}T^k\right)\rho^k_{L_k-1}} \sum_{l_k=0}^{L_k-1} \left(\hat{\mu}^k_{l_k}T^k - l_k\right)\rho^k_{l_k}. \tag{45}$$

Furthermore, the maximum user k throughput is given by

$$F^k(T^k) = \frac{\sum_{j_k=1}^{L_k-1} \left(L_k\hat{\mu}^k_{j_k} - j_k\hat{\mu}^k_{L_k}\right)\rho^k_{j_k}}{L_k - \hat{\mu}^k_{L_k}T^k + \sum_{j_k=1}^{L_k-1} \left[L_k - j_k - \left(\hat{\mu}^k_{L_k} - \hat{\mu}^k_{j_k}\right)T^k\right]\rho^k_{j_k}}. \tag{46}$$

Proof. See [21] or [15] for more general results. □

We are now ready to characterize the class of decentralized user optimal flow control problems for which there exist Nash equilibrium solutions. Let

$$\mathscr{T} \triangleq \left\{(T^1, T^2, \ldots, T^K) \in \mathbb{R}^K \mid E\tau^k_N = T^k,\ k = 1, 2, \ldots, K\right\},$$

where $E\tau^k_N$ are functions of $\lambda^1, \ldots, \lambda^K$ and each λ^k is a window mechanism of the type (44), $k = 1, \ldots, K$.

Theorem 6. *For any decentralized user flow control problems with time delay constraints such that $(T^1, T^2, \ldots, T^K) \in \mathscr{T}$, the Nash equilibrium exists.*

Proof. The average user k throughput, $E\gamma^k_N$, and time delay, $E\tau^k_N$ are increasing as λ^k, for $\hat{\mu}^k_{j_k}$ which satisfies (41) and (42). Hence, the time delay constraint is achieved for an optimal control for user k. Furthermore, this optimal control is unique. Suppose that $\lambda^{1*}, \ldots, \lambda^{K*}$ are window controls such that $E\tau^1_N = T^1, \ldots, E\tau^K_N = T^K$. Then, given the controls of users i, λ^{i*}, $i \neq k$, the optimal control for user k is λ^{k*}. Thus, $(\lambda^{1*}, \ldots, \lambda^{K*})$ is a Nash equilibrium solution for the decentralized user flow control problems with constraints $T^1, \ldots, T^K$. □

It only remains to characterize the properties of $\mathscr{T}$. It is easy to see that the set $\mathscr{T}$ can be generated by the set of all window controls $\lambda^1, \ldots, \lambda^K$ of the form (44). Hence, the elements of $\mathscr{T}$ can be generated by computing the time delays $E\tau^1_N, \ldots, E\tau^K_N$ for each possible selection of window control policies of the form (44). In the following, a scenario in which the concept of Nash equilibrium may be applied to computer networks is given.

5.3. A scenario

The concept of Nash equilibrium for decentralized optimal flow control introduced above can be applied to describe the behavior of a number of users logging onto a network. Consider the situation in which there are K users who are using the resources of a packet switched network. By the separation theorem, the optimal control policy for packets of class k can be found based on the estimates of the departure rates of class k packets from the network. In equilibrium, the users control the flow of their packets with a window mechanism and the estimates remain constant. When the estimates begin to change, as when a user logs off or a new user enters the network, a new control policy has to be computed based on the new estimates. The introduction of the new user to the network drives up the time delay of the other users and away from their original equilibrium. Therefore, the latter users have to reduce their window sizes in order to meet their time delay requirements. After the process of adjusting the window sizes in order to meet the time delay constraints, the network will operate at a new Nash equilibrium point. In order to achieve this, decentralized flow control algorithms need to be developed and their convergence properties investigated.

Although the result does not prescribe how a network of noncooperative users behave absolutely, it does specify the manner these users control their packets so as to achieve their individual objectives. In view of the separation theorem, each user only sees the network as a dedicated resource with variable service rates. In practice, these results should lead to efficient decentralized algorithms for the flow control of computer communication networks with many users. The application of the Nash equilibrium concepts in decentralized flow control with a throughput/time-delay criterion thus represents a novel approach for describing the behavior of a system of noncooperating users with decentralized information.

6. Conclusions

In this article, the decentralized flow control in computer communication networks with multiple controllers was studied. Under a global objective, the average network throughput is maximized subject to a constraint on the average network time delay. Representation results were given which provide insight into the network optimization problem. By using a linear programming formulation, the optimal control was shown to be a window-type mechanism. An example was given which demonstrates how the structural results can be used to obtain the network optimal control policy.

Under individual objectives, each user maximizes the corresponding average user throughput subject to the average user time delay constraint. The multi-objective optimization problem arising from individual users' performance measures was treated in a game theoretic setting. A separation result was proved which revealed that the optimization problem for user k can be solved based on estimates of the departure rates of class k packets from the network. The optimal control for an arbitrary user k was given explicitly as a function of the control of the other users. The solution using the Nash equilibrium concepts was presented. A scenario in which the Nash equilibrium concept was used to describe the action of different users sharing the resources of a packet switching network was also discussed.

Acknowledgement

The research reported here was supported in part by the Office of Naval Research under contract ONR-N00014-90-J-1289.

Appendix. Game theoretic preliminaries

The concept of a game in strategic (normal) form is defined in the following:

Definition 6. A game in strategic form (strategic game), G, consists of:

1) a set $\mathscr{K}$ (of players),
2) for each $k \in \mathscr{K}$ a set S^k (strategy set of k),
3) for each $k \in \mathscr{K}$ a function h^k (the payoff function of k):

$$h^k: \times_{k \in \mathscr{K}} S^k \to \mathbb{R}.$$

In other words, if $\mathscr{K} = \{1, 2, \dots, K\}$, the game consists of K players $P_1, \dots, P_K$. Player P_k, $k = 1, \dots, K$, chooses from the strategy set S^k and receives a payoff h^k equal to the utility corresponding to the chosen control strategy, a defined earlier. It is assumed that the players are noncooperative in the sense that the players do not have enough information or incentive to form a coalition which would increase the payoffs to all players simultaneously. Hence, the players act individualistically (selfishly in a sense) in order to maximize their individual payoffs.

Suppose that $P_2, \ldots, P_K$ have chosen strategies $\lambda^2 \in S^2, \ldots, \lambda^K \in S^K$. The *reaction* of P_1 is the strategy which maximizes P_1's own payoff, given that P_k has chosen strategy λ^k, $k = 2, \ldots, K$. This defines a mapping as follows:

Definition 7. The S^1-valued function $\varphi^1 : S^2 \times \cdots \times S^K \to S^1$, such that

$$\varphi^1(\lambda^2, \ldots, \lambda^K) = \arg\max_{\lambda^1 \in S^1} h^1(\lambda^1, \ldots, \lambda^K)$$

is called the reaction function of P_1.

The reaction functions of P_k, ϕ^k, $k = 2, \ldots, K$, are similarly defined.

Now, define the network reaction function φ by

$$\varphi(\lambda^1, \ldots, \lambda^K) = \left(\varphi^1(\lambda^{[1]}), \ldots, \phi^K(\lambda^{[K]})\right),$$

where $\lambda^{[k]} \triangleq (\lambda^1, \ldots, \lambda^{k-1}, \ \lambda^{k+1}, \ldots, \lambda^K)$. The Nash equilibrium strategy is then a fixed point of the function φ. The precise definition is repeated here for completeness.

Definition 4. The strategy K-tuple $\lambda^* = (\lambda^{1*}, \ldots, \lambda^{K*})$ which satisfies

$$\varphi(\lambda^*) = \lambda^*$$

is called the Nash equilibrium (NE) strategy of the strategic game G.

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