The Gaussian Multiple Access Diamond Channel

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Abstract—In this paper, we study the capacity of the diamond channel. We focus on the special case where the channel between the source node and the two relay nodes are two separate links of finite capacity and the link from the two relay nodes to the destination node is a Gaussian multiple access channel. We call this model the Gaussian multiple access diamond channel. We first propose an upper bound on the capacity. This upper bound is a single-letterization of the n-letter upper bound proposed by Traskov and Kramer, which is tighter than the cut-set bound. Next, we provide a lower bound based on sending correlated codes through the multiple access channel. Since the upper and lower bounds take on similar forms, it is expected that they coincide for certain channel parameters. To show this, we further focus on the symmetric case where the separate links to the relays are of the same capacity and the power constraints of the two relays are the same. For the symmetric case, we give necessary and sufficient conditions that the upper and lower bounds meet. Thus, for a Gaussian multiple access diamond channel that satisfies these conditions, we have found its capacity.

I. Introduction

The diamond channel was first introduced by Schein in 2001 [1]. It models the communication from a source node to a destination node with the help of two relay nodes. The channel between the source node and the two relays form a broadcast channel as the first stage and the channel between the two relays and the destination node form a multiple access channel as the second stage. The capacity of the diamond channel in its most general form is open. Achievability results were proposed in [1], while for the general diamond channel, the best known converse results is still the cut-set bound [2]. Capacity has been found for some special classes of diamond channels in [3], [4].

The problem of sending correlated codes through a multiple access channel was studied in [5]. This channel model can be regarded as a special case of the diamond channel where the broadcast channel between the source node and the two relay nodes are two separate links of finite capacity. We call this the multiple access diamond channel. Achievability results for the discrete multiple access diamond channel were proposed in [5], [6]. In [6], an uncomputable n-letter upper bound is also provided which is tighter than cut-set bound.

In this paper, we consider the multiple access diamond channel where the multiple access channel from the two relay nodes to the destination node is Gaussian, see Figure 1. We call this channel model the Gaussian multiple access diamond channel. We first obtain an upper bound on the capacity by single-letterizing the n-letter upper bound in [6]. The main technique we use in the upper bound derivation is the introduction of an imaginary random variable used to bound the correlation between the two relay signals. This technique has also been used in solving the Gaussian multiple description problem [7]. We then propose a lower bound on the capacity where the relays send correlated codewords into the channel. Comparing the upper and lower bounds, we find that they are of similar forms and therefore, when the channel parameters satisfy certain conditions, they would coincide, yielding the capacity. To illustrate this, we focus our attention on the symmetric case, where the power constraints of the relay nodes are the same and the links from the source node to the two relay nodes are of the same capacity. For the symmetric case, we give necessary and sufficient conditions that the upper and lower bounds meet. Thus, for a symmetric Gaussian multiple access diamond channel that satisfies these conditions, we have found its capacity.

II. System Model

We consider the Gaussian multiple access diamond channel, see Figure 1. The capacity of the link from the source node to Relay k is R_k, $k = 1, 2$. The received signal at the destination node is

$$Y = X_1 + X_2 + U$$

where X_1 and X_2 are the input signals from Relay 1 and Relay 2, respectively, and U is a zero-mean unit-variance Gaussian random variable. It is assumed that U is independent to (X_1, X_2).

Let W be a message that the source node would like to transmit to the destination node. Assume that W is uniformly distributed on $\{1, 2, \cdots, M\}$. An (M, n, ϵ_n) code consists of an encoding function at the source node

$$f^n : \{1, 2, \cdots, M\} \to \{1, 2, \cdots, 2^{nR_1}\} \times \{1, 2, \cdots, 2^{nR_2}\},$$

two encoding functions at the relays

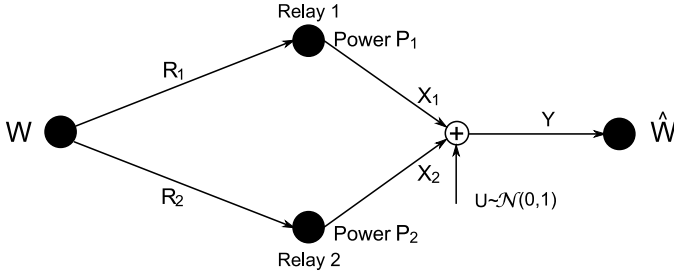$$f_k^n : \{1, 2, \cdots, 2^{nR_k}\} \to \mathbb{R}^n, \quad k = 1, 2$$

Fig. 1. The Gaussian multiple access diamond channel

which satisfy the average power constraint: for any x_k^n that Relay k input into the Gaussian multiple access channel, it satisfies

$$\frac{1}{n}\sum_{i=1}^{n} x_{ki}^2 \le P_k, \quad k = 1, 2$$

and one decoding function at the destination node

$$g^n : \mathbb{R}^n \to \{1, 2, \cdots, M\}$$

The probability of error is defined as

$$\epsilon_n = \sum_{w=1}^{M} \frac{1}{M}\Pr[g^n(Y^n) \ne w | W = w]$$

Rate R is said to be achievable if there exists a sequence of $\left(2^{nR}, n, \epsilon_n\right)$ codes such that $\epsilon_n \to 0$ as $n \to \infty$. The capacity of the Gaussian multiple access diamond channel is the supremum of all achievable rates.

We would like the characterize the capacity of the Gaussian multiple access diamond channel in terms of the channel parameters R_1, R_2, P_1 and P_2.

To simplify presentation, we define the following functions of $\rho \in [0, 1]$:

$$f_1(\rho) \triangleq R_1 + \frac{1}{2}\log[1 + (1 - \rho^2)P_2]$$

$$f_2(\rho) \triangleq R_2 + \frac{1}{2}\log[1 + (1 - \rho^2)P_1]$$

$$f_3(\rho) \triangleq R_1 + R_2 - \frac{1}{2}\log\left(\frac{1}{1 - \rho^2}\right)$$

$$f_4(\rho) \triangleq \frac{1}{2}\log\left(1 + P_1 + P_2 + 2\rho\sqrt{P_1 P_2}\right)$$

III. An Upper Bound

Theorem 1 *An upper bound on the capacity of the Gaussian multiple access diamond channel is*

$$\max(T_1, T_2)$$

where

$$T_1 = \max_{0 \le \rho \le \rho^*} \min\{f_1(\rho), f_2(\rho), f_3(\rho), f_4(\rho)\} \quad (1)$$

$$T_2 = \max_{\rho^* \le \rho \le 1} \min\{f_1(\rho), f_2(\rho), f_3(0), f_4(\rho)\}$$

and

$$\rho^* = \sqrt{1 + \frac{1}{4P_1 P_2}} - \frac{1}{2\sqrt{P_1 P_2}}$$

Remark: The cut-set bound for the Gaussian multiple access diamond channel is

$$\max_{0 \le \rho \le 1} \min\{f_1(\rho), f_2(\rho), f_3(0), f_4(\rho)\}$$

Hence, our upper bound is tighter than the cut-set bound.

Proof: From the cut-set bound, we always have

$$R \le R_1 + R_2 \quad (2)$$

We also have

$$
\begin{aligned}
nR &= H(W) \\
&= H(X_1^n, X_2^n) + H(W | X_1^n, X_2^n) \quad (3) \\
&\le H(X_1^n, X_2^n) + H(W | Y^n) \quad (4) \\
&\le H(X_1^n, X_2^n) + n\epsilon_n \quad (5) \\
&\le I(X_1^n, X_2^n; Y^n) + H(X_1^n, X_2^n | Y^n) + n\epsilon_n \\
&\le I(X_1^n, X_2^n; Y^n) + 2n\epsilon_n \quad (6) \\
&= h(Y^n) - h(Y^n | X_1^n, X_2^n) + 2n\epsilon_n \\
&= h(Y^n) - \frac{n}{2}\log(2\pi e) + 2n\epsilon_n \quad (7)
\end{aligned}
$$

where (3) is because without loss of generality, we may consider deterministic encoders, i.e., (X_1^n, X_2^n) is a deterministic function of W, (4) is because of Markov chain $W \to (X_1^n, X_2^n) \to Y^n$, and (5) and (6) both follow from Fano's inequality. We also have

$$
\begin{aligned}
nR &= H(W) \\
&\ge H(X_1^n, X_2^n) \quad (8) \\
&\ge I(X_1^n, X_2^n; Y^n) \\
&= h(Y^n) - \frac{n}{2}\log(2\pi e) \quad (9)
\end{aligned}
$$

(8) is follows from the same reasoning as (3). Now, we upper bound $h(Y^n)$ as

$$
\begin{aligned}
h(Y^n) &\le \sum_{i=1}^{n} h(Y_i) \\
&\le \sum_{i=1}^{n} \frac{1}{2}\log(2\pi e)\left(P_{1i} + P_{2i} + 2\rho_i\sqrt{P_{1i}P_{2i}} + 1\right)
\end{aligned}
$$
$$(10)$$

where in (10), we have defined $P_{ki} \triangleq E[X_{ki}^2], k = 1, 2$ and $\rho_i = \frac{E[X_{1i}X_{2i}]}{\sqrt{P_{1i}P_{2i}}}$, and used the fact that given power constraint, the Gaussian distribution maximizes the differential entropy. Another upper bound on R is

$$
\begin{aligned}
nR &\le H(X_1^n, X_2^n) + n\epsilon_n \quad (11) \\
&= H(X_1^n | X_2^n) + H(X_2^n) + n\epsilon_n \\
&\le H(X_1^n | X_2^n) + nR_2 + n\epsilon_n \\
&= I(X_1^n; Y^n | X_2^n) + H(X_1^n | Y^n, X_2^n) + nR_2 + n\epsilon_n \\
&\le I(X_1^n; Y^n | X_2^n) + nR_2 + 2n\epsilon_n
\end{aligned}
$$

$$\leq \sum_{i=1}^{n} I(X_{1i}; Y_i | X_{2i}) + nR_2 + 2n\epsilon_n$$

$$\leq \sum_{i=1}^{n} \frac{1}{2} \log \left[(1 - \rho_i^2) P_{1i} + 1 \right] + nR_2 + 2n\epsilon_n \quad (12)$$

where (11) is because of (5) and (12) follows from the same reasoning as (10). Similarly,

$$nR \leq \sum_{i=1}^{n} \frac{1}{2} \log \left[(1 - \rho_i^2) P_{2i} + 1 \right] + nR_1 + 2n\epsilon_n$$

Since the inputs from Relay 1 must satisfy the average power constraint P_1, we have

$$0 \leq \frac{1}{n} \sum_{i=1}^{n} \rho_i^2 P_{1i} \leq \frac{1}{n} \sum_{i=1}^{n} P_{1i} \leq P_1$$

Thus, there exists a $\rho_a \in [0, 1]$ such that $\rho_a^2 P_1 = \frac{1}{n} \sum_{i=1}^{n} \rho_i^2 P_{1i}$. Due to the concavity of the logarithm function, from (12), we have

$$R \leq \frac{1}{2} \log \left(\frac{1}{n} \sum_{i=1}^{n} \left[(1 - \rho_i^2) P_{1i} + 1 \right] \right) + R_2 + 2\epsilon_n$$

$$\leq \frac{1}{2} \log \left[(1 - \rho_a^2) P_1 + 1 \right] + R_2 + 2\epsilon_n \quad (13)$$

By a similar argument, from (10), we have

$$\frac{1}{n} h(Y^n)$$

$$\leq \frac{1}{2} \log(2\pi e) \left(\frac{1}{n} \sum_{i=1}^{n} \left[P_{1i} + P_{2i} + 2|\rho_i| \sqrt{P_{1i} P_{2i}} + 1 \right] \right)$$

$$\leq \frac{1}{2} \log(2\pi e) \left(P_1 + P_2 + \frac{1}{n} \sum_{i=1}^{n} 2\sqrt{\rho_i^2 P_{1i} P_{2i}} + 1 \right)$$

From Cauchy-Schwarz inequality, we have

$$\frac{1}{n} \sum_{i=1}^{n} \sqrt{\rho_i^2 P_{1i} P_{2i}} \leq \sqrt{\left(\frac{1}{n} \sum_{i=1}^{n} \rho_i^2 P_{1i} \right) \left(\frac{1}{n} \sum_{i=1}^{n} P_{2i} \right)}$$

$$\leq \sqrt{\rho_a^2 P_1 P_2}$$

Thus, we have

$$\frac{1}{n} h(Y^n) \leq \frac{1}{2} \log(2\pi e) \left(P_1 + P_2 + 2\rho_a \sqrt{P_1 P_2} + 1 \right) \quad (14)$$

Similarly, there exists a $\rho_b \in [0, 1]$ such that $\rho_b^2 P_2 = \frac{1}{n} \sum_{i=1}^{n} \rho_i^2 P_{2i}$, and we have

$$R \leq \frac{1}{2} \log \left[(1 - \rho_b^2) P_2 + 1 \right] + R_1 + 2\epsilon_n \quad (15)$$

$$\frac{1}{n} h(Y^n) \leq \frac{1}{2} \log(2\pi e) \left(P_1 + P_2 + 2\rho_b \sqrt{P_1 P_2} + 1 \right) \quad (16)$$

Define $\rho \in [0, 1]$, which is a function of $h(Y^n)$ as follows: If

$$\frac{1}{n} h(Y^n) \leq \frac{1}{2} \log(2\pi e)(1 + P_1 + P_2) \quad (17)$$

then $\rho = 0$; otherwise, ρ is such that

$$\frac{1}{n} h(Y^n) = \frac{1}{2} \log(2\pi e)(1 + P_1 + P_2 + 2\rho\sqrt{P_1 P_2}) \quad (18)$$

For the case when $\rho = 0$, from (13), (15), (7), (17) and (2), and letting $n \to \infty$, we have

$$R \leq \min \left(f_1(\rho), f_2(\rho), f_3(\rho), f_4(\rho) \right)$$

which means, for the case of $\rho = 0$, $R \leq T_1$.

For the case of $\rho > 0$, since $h(Y^n)$ must satisfy (14) and (16), we see that $\rho \leq \min(\rho_a, \rho_b)$. This means from (13), (15), (7) and (9), that we have

$$R \leq \min(f_1(\rho), f_2(\rho)) + 2\epsilon_n \quad (19)$$

$$f_4(\rho) \leq R \leq f_4(\rho) + 2\epsilon_n \quad (20)$$

If ρ further satisfy $0 < \rho \leq \rho^*$, which is equivalent to $\sqrt{P_1 P_2} \left(\frac{1}{\rho} - \rho \right) - 1 \geq 0$, we define additional random variables

$$Z_i = Y_i + U_i' \qquad i = 1, \ldots, n$$

where U'^n is an i.i.d. Gaussian sequence with mean zero and variance

$$N = \sqrt{P_1 P_2} \left(\frac{1}{\rho} - \rho \right) - 1 \quad (21)$$

and is independent to everything else. We have

$$2nR \leq 2H(X_1^n, X_2^n) + 2n\epsilon_n \quad (22)$$

$$= H(X_1^n, X_2^n) - I(X_1^n; X_2^n) + H(X_1^n) + H(X_2^n) + 2n\epsilon_n$$

$$\leq H(X_1^n, X_2^n) - I(X_1^n; X_2^n) + nR_1 + nR_2 + 2n\epsilon_n$$

$$\leq I(X_1^n, X_2^n; Y^n) - I(X_1^n; X_2^n) + nR_1 + nR_2 + 3n\epsilon_n \quad (23)$$

where (22) follows because of (5), and (23) follows from (6). Note that

$$I(X_1^n; X_2^n)$$

$$= I(X_1^n; Z^n) - I(X_1^n; Z^n | X_2^n) + I(X_1^n; X_2^n | Z^n)$$

$$\geq I(X_1^n; Z^n) - I(X_1^n; Z^n | X_2^n)$$

$$= I(X_1^n, X_2^n; Z^n) - I(X_2^n; Z^n | X_1^n) - I(X_1^n; Z^n | X_2^n) \quad (24)$$

We further have

$$I(X_1^n; Z^n | X_2^n) \leq \sum_{i=1}^{n} \frac{1}{2} \log \frac{(1 - \rho_i^2) P_{1i} + 1 + N}{1 + N} \quad (25)$$

$$\leq \frac{n}{2} \log \frac{(1 - \rho^2) P_1 + 1 + N}{1 + N} \quad (26)$$

where (25) follows by similar arguments as (12) and (26) follows by similar arguments as (13) and (19). Similarly, we have

$$I(X_2^n; Z^n | X_1^n) \leq \frac{n}{2} \log \frac{(1 - \rho^2) P_2 + 1 + N}{1 + N} \quad (27)$$

We also have

$$I(X_1^n, X_2^n; Z^n) = h(Z^n) - h(Z^n | X_1^n, X_2^n)$$
$$= h(Z^n) - \sum_{i=1}^{n} \frac{1}{2} \log(2\pi e)(1 + N)$$

From Entropy Power Inequality (EPI) [8, Lemma I], we have

$$h(Z^n) \geq \frac{n}{2} \log \left[2^{\left(\frac{2}{n} h(Y^n) \right)} + 2\pi e N \right]$$

Therefore,

$$h(Z^n) - h(Y^n)$$
$$\geq \frac{n}{2} \log \left[1 + \frac{2\pi e N}{2^{\left(\frac{2}{n} h(Y^n) \right)}} \right]$$
$$= \frac{n}{2} \log \left[1 + \frac{N}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1} \right] \quad (28)$$
$$= \frac{n}{2} \log \frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1 + N}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1}$$

where (28) follows from (18). Thus,

$$I(X_1^n, X_2^n; Y^n) - I(X_1^n, X_2^n; Z^n)$$
$$\leq \frac{n}{2} \log \frac{(N+1)(P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1)}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1 + N} \quad (29)$$

Using (23), (24), (26), (27) and (29), we have

$$2nR$$
$$\leq \frac{n}{2} \log(P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1)$$
$$- \frac{n}{2} \log \frac{(P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + 1 + N)(1 + N)}{((1 - \rho^2)P_1 + 1 + N)((1 - \rho^2)P_2 + 1 + N)}$$
$$+ nR_1 + nR_2 + 3n\epsilon_n$$

plugging in N in (21), we have

$$2R \leq f_4(\rho) + f_3(\rho) + 3\epsilon_n \quad (30)$$

Hence, for the case of $0 < \rho \leq \rho^*$, from (19), (20) and (30), and letting $n \to \infty$, we have proved $R \leq T_1$.

Finally, for the case where $\rho > \rho^*$, though we do not have (30), (19), (20) and (2) still hold, and by letting $n \to \infty$, we have proved $R \leq T_2$.

Hence, for all cases of $\rho \in [0, 1]$, we have proved that the achievable rate either satisfy $R \leq T_1$ or $R \leq T_2$, and thus, Theorem 1 is proved. ∎

IV. A LOWER BOUND

Theorem 2 *A lower bound of the capacity of the above Gaussian multiple access diamond channel is*

$$\max_{0 \leq \rho \leq \rho^\circ} \min \{ f_1(\rho), f_2(\rho), f_3(\rho), f_4(\rho) \} \quad (31)$$

where

$$\rho^\circ = \sqrt{1 - \exp(-2\min(R_1, R_2))}$$

Proof: Consider a pair of zero-mean jointly Gaussian random variables (X_1, X_2), such that the covariance of X_k

is P_k, $k = 1, 2$ and the correlation coefficient between X_1 and X_2 is ρ. Note that the condition $0 \leq \rho \leq \rho^o$ is equivalent to $\min(R_1, R_2) \geq \frac{1}{2} \log \frac{1}{1 - \rho^2}$.

Codebook generation: Randomly generate 2^{nR_1} independent codewords $x_1^n(i)$, $i = 1, \ldots, 2^{nR_1}$ according to $p(x_1)$ and randomly generate 2^{nR_2} independent codewords $x_2^n(i)$, $i = 1, \ldots, 2^{nR_2}$ according to $p(x_2)$. Then, with probability 1, for every codeword $x_1^n(i)$, $i = 1, \ldots, 2^{nR_1}$, there are $2^{n\left(R_2 - \frac{1}{2} \log \frac{1}{1 - \rho^2}\right)}$ x_2^n sequences jointly typical with $x_1^n(i)$ according to the given Gaussian distribution. Similarly, with probability 1, for every codeword $x_2^n(i), i = 1, \ldots, 2^{nR_2}$, there are $2^{n\left(R_1 - \frac{1}{2} \log \frac{1}{1 - \rho^2}\right)}$ x_1^n sequences jointly typical with $x_2^n(i)$. We collect all the jointly typical codeword pairs $(x_1^n(i), x_2^n(j))$ among all the possible (i, j) combinations and index them as $(x_1^n, x_2^n)(k)$, for $k = 1, \ldots, 2^{nR}$, where

$$R = R_1 + R_2 - \frac{1}{2} \log \frac{1}{1 - \rho^2} = f_3(\rho) \quad (32)$$

Encoding: When the message $W = w$, for $w = 1, \ldots, 2^{nR}$, the source nodes finds the pair (i, j) that corresponds to $(x_1^n, x_2^n)(w)$. It sends index $i \in \{1, 2, \cdots, 2^{nR_1}\}$ to Relay 1 and index $j \in \{1, 2, \cdots, 2^{nR_2}\}$ to Relay 2. Relay 1 upon receiving index i, sends $x_1^n(i)$ into the multiple access channel. Relay 2 upon receiving index j, sends $x_2^n(j)$ into the multiple access channel.

Decoding: Upon receiving Y^n, the receiver declares w is sent if $(x_1^n, x_2^n)(w)$ is jointly typical with the received codeword. If no such w exists, or if there is more than one such, an error is declared.

Probability of Error: By a similar argument as in [2, Sec. 14.3.1], we can show that the probability of error goes to zero if following conditions are satisfied

$$R_1 \leq I(X_1; Y, X_2), \quad R_2 \leq I(X_2; Y, X_1), \quad R \leq I(X_1, X_2; Y)$$

which means

$$R \leq \min \left(f_1(\rho), f_2(\rho), f_4(\rho) \right) \quad (33)$$

Thus, based on (32) and (33), Theorem 2 is proved. ∎

V. SYMMETRIC CASE AND CAPACITY

Comparing the upper and lower bounds proposed in Theorems 1 and 2, we see that they take on similar forms, more specifically, the four functions after the minimum in (1) is exactly the same as that in (31). Thus, if the parameters of the Gaussian multiple access diamond channel, R_1, R_2, P_1 and P_2, is such that $\rho^o \geq \rho^*$ and $T_1 \geq T_2$, the upper and lower bounds meet providing us with the exact capacity of the channel.

To show that there indeed exist channels such that the upper and lower bounds meet, in this section, we focus on the symmetric case, i.e., $P_1 = P_2 = P$ and $R_1 = R_2 = R_0$.

If the channel is such that $R_0 \geq \frac{1}{2} \log (1 + 4P)$, it is clear that the multiple access channel in the second stage is the bottleneck of the whole network, and thus, the capacity is equal to $\frac{1}{2} \log (1 + 4P)$. On the other hand, if the channel

is such that $\frac{1}{2}\log(1+2P) \geq 2R_0$, it is clear that the two separate links in the first stage is the bottleneck of the whole network, and the capacity is equal to $2R_0$. Thus, we only need to focus on the nontrivial cases where

$$\frac{1}{4}\log(1+2P) < R_0 < \frac{1}{2}\log(1+4P)$$

Define the following functions of $\rho \in [0,1]$:

$$f_1^s(\rho) \triangleq R_0 + \frac{1}{2}\log[1+(1-\rho^2)P]$$

$$f_3^s(\rho) \triangleq 2R_0 - \frac{1}{2}\log\frac{1}{1-\rho^2}$$

$$f_4^s(\rho) \triangleq \frac{1}{2}\log[1+2(1+\rho)P]$$

Theorem 3 *For the symmetric Gaussian multiple access diamond channel, if the channel parameters satisfy*

$$\rho^o \geq \bar{\rho}_3, \quad \rho^* \geq \bar{\rho}_1, \quad f_1^s(\rho^*) \leq f_3^s(\bar{\rho}_3) \qquad (34)$$

then its capacity is $f_3^s(\bar{\rho}_3)$. Here, $\bar{\rho}_1$ and $\bar{\rho}_3$ are the positive roots of the second order equations $f_1^s(\rho) = f_4^s(\rho)$ and $f_3^s(\rho) = f_4^s(\rho)$, respectively.

Proof: : The proof of Theorem 3 is omitted due to limited space. ∎

To show that there indeed exist symmetric Gaussian multiple access diamond channels that satisfy (34), take for example, $P = 3$ and $R_0 = 1.2$, we then have

$$\rho^o = 0.9003, \rho^* = 0.8471, \bar{\rho}_1 = 0.7734, \bar{\rho}_3 = 0.7643,$$

$$f_1^s(\rho^*) = 1.6426, f_3^s(\bar{\rho}_3) = 1.7671$$

Thus, for $P = 3$ and $R_0 = 1.2$, (34) is satisfied and the upper and lower bounds coincide, yielding the capacity, which is 1.7671.

To illustrate further, we plot the upper and lower bounds in Theorems 1 and 2 and depict them in Figure 2 and Figure 3 for the cases of $P = 3$ and $P = 30$, respectively. The cut-set bound is also plotted to show the improvement of our upper bound over the cut-set bound, which is the best-known upper bound on the capacity. As can be seen, for small R_0, the proposed upper bound is much smaller than the cut-set bound. Also, the gap between our lower and upper bounds is rather small, especially when R_0 is relatively small and/or P is relatively large.

VI. CONCLUSIONS

We have studied the Gaussian multiple access diamond channel and provided upper and lower bounds on the capacity. Focusing on the symmetric case, we gave necessary and sufficient conditions that the upper and lower bounds meet. Thus, for a symmetric Gaussian multiple access diamond channels that satisfies these conditions, we have found its capacity.

REFERENCES

[1] B. E. Schein. *Distributed Coordination in Network Information Theory*. PhD thesis, Massachusetts Institute of Technology, 2001.
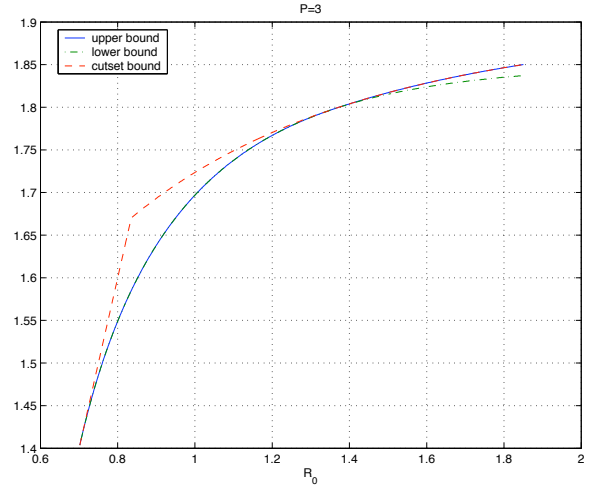
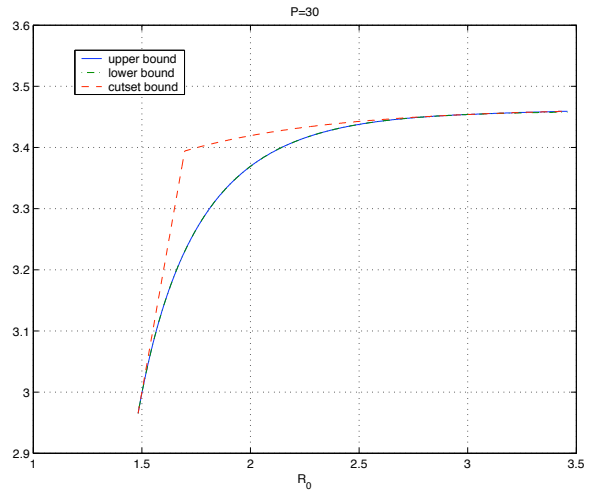Fig. 2. Comparison of upper and lower bounds with $P = 3$

Fig. 3. Comparison of upper and lower bounds with $P = 30$

[2] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. John Wiley and Sons, 1991.
[3] W. Kang and S. Ulukus. Capacity of a class of diamond channels. In *46th Annual Allerton Conference on Communications, Control and Computing*, pages 1426–1431, Monticello, IL, September 2008.
[4] R. Tandon and S. Ulukus. Diamond channel with partially separated relays. In *Proc. IEEE International Symp. on Information Theory (ISIT)*, pages 644–648, Austin, TX, June 2010.
[5] R. Ahlswede and T. S. Han. On source coding with side information via a multiple-access channel and related problems in multi-user information theory. *IEEE Trans. Inform. Theory*, 29(3):396–412, 1983.
[6] D. Traskov and G. Kramer. Reliable communication in networks with multi-access interference. In *Proc. Conf. IEEE Information Theory Workshop (ITW)*, Lake Tahoe, CA, Sep. 2007.
[7] L. Ozarow. On a source-coding problem with two channels and three receivers. *Bell Syst. Tech. J.*, 59:1909–1921, December 1980.
[8] P. Bergmans. A simple converse for broadcast channels with additive white Gaussian noise. *IEEE Trans. on Information Theory*, 20:279 –280, March 1974.