Ruin Probabilities with Dependent Rates of Interest

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Abstract

In this paper, we study ruin probabilities in two generalized risk models. The effects of timing of payments and interest on the ruin probabilities in the models are considered. The rates of interest are assumed to have a dependent autoregressive structure. Generalized Lundberg inequalities for the ruin probabilities are derived by a renewal recursive technique. An illustrative application is given to the compound binomial risk process.

Keywords: Ruin probability; rate of interest; Lundberg inequality; NWUC; compound binomial risk process.

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1 Introduction

Let $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ be two sequences of independent and identically distributed (i.i.d.) nonnegative random variables. Define

$$\psi(u) = \Pr\left\{\bigcup_{k=1}^{\infty}(U_k < 0)\right\}, \tag{1.1}$$

where

$$U_k = u + \sum_{t=1}^{k}(X_t - Y_t),\ k = 1, 2, \cdots, \tag{1.2}$$

or equivalently, the stochastic process $\{U_n,\ n = 1, 2, \cdots\}$ satisfies

$$U_n = U_{n-1} + X_n - Y_n,\ n = 1, 2, \cdots, \tag{1.3}$$

with $U_0 = u \geq 0$.

The quantity $\psi(u)$ arises in many applied probability models and has been extensively studied in risk theory and queueing theory as the ruin probability in different risk models and the tail probability of the equilibrium waiting time in the G/G/1 queue, respectively. See, for example, Grandell (1991), Rolski *et al* (1999), Ross (1996), and Willmot and Lin (2001), and references therein.

In a risk theoretic setting, if Y_n denotes the total claims during the nth period, i.e. from time $n-1$ to time n, and X_n represents the total premiums during the nth period, then $\psi(u)$ is the ultimate ruin probability with the initial surplus u in the classical risk model (1.2). An important point to note about the classical risk model is that the surplus level at time n does not depend on the timing of payment of premiums or claims because there is no interest component in the model. Such issues are, however, important if we include such an element in the model.

In this paper, we consider two generalizations of the classical risk model, in which the effects of timing of payments and interest on the surplus process and on the ruin probability can be included. In doing so, let $\{I_n,\ n = 1, 2, \cdots\}$ be another sequence of nonnegative random variables, and define two generalized processes by

$$U_n = (U_{n-1} + X_n)(1 + I_n) - Y_n,\ n = 1, 2, \cdots, \tag{1.4}$$

and

$$U_n = U_{n-1}(1+I_n) + X_n - Y_n, \; n = 1, 2, \cdots, \tag{1.5}$$

respectively.

It is not hard to check that (1.4) implies

$$U_n = u\prod_{k=1}^{n}(1+I_k) + \sum_{k=1}^{n}\left((X_k(1+I_k) - Y_k)\prod_{t=k+1}^{n}(1+I_t)\right), \; n = 1, 2, \cdots, \tag{1.6}$$

while (1.5) is equivalent to

$$U_n = u\prod_{k=1}^{n}(1+I_k) + \sum_{k=1}^{n}\left((X_k - Y_k)\prod_{t=k+1}^{n}(1+I_t)\right), \; n = 1, 2, \cdots, \tag{1.7}$$

where $\prod_{t=a}^{b}(1+I_t) = 1$ if $a > b$.

Mathematically, (1.4) and (1.5) are the generalizations of the classical risk model (1.3). In fact, (1.3) is a special case of (1.4) and (1.5) when $I_n = 0$ for $n = 1, 2, \cdots$. Furthermore, models (1.4) and (1.5) can be interpreted in a risk theoretic setting. To see that, assume that an insurer would receive interest on its surplus each period. Let I_n denote the rate of interest during the nth period, i.e. from time $n-1$ to time n. Assume that Y_n is the amount of claims during the nth period, and is paid at the end of the nth period, i.e. at time n. If X_n is the amount of premiums during the nth period, and is received at the beginning of the nth period, i.e. at time $n-1$, then, the surplus of the insurer at time n denoted by U_n with the initial surplus u satisfies (1.4). On the other hand, if X_n is received at the end of the nth period, then U_n satisfies (1.5). Therefore, (1.4) and (1.5) allow us to adopt the effects of timing of payment and interest on the surplus, and the ruin probabilities in the models.

The effects of interest on ruin probabilities have been discussed in several references. Sundt and Teugels (1995, 1997) have studied the effects of a constant rate on the ruin probability in the compound Poisson risk model. For more topics in the continuous time risk models with rates of interest, see Asmussen (2000), Rolski *et al* (1999), and references therein. Yang (1999) has considered a special case of (1.7) when $\{I_n, n = 1, 2, \cdots\}$ are identical constants. Further, Cai (2002) has discussed models (1.6) and (1.7) when $\{I_n, n = 1, 2, \cdots\}$ are independent and identically distributed (i.i.d.) rates of interest.

However, the assumption of constant and i.i.d. rates are not particularly realistic since rates of interest are usually statistically dependent over time.

In this paper, we consider a dependent model for $\{I_n,\ n = 1, 2, \cdots\}$, in which $\{I_n,\ n = 1, 2, \cdots\}$ are assumed to have a dependent autoregressive structure of order one, i.e. I_n satisfies,

$$I_n = \alpha I_{n-1} + W_n,\ n = 1, 2, \cdots, \tag{1.8}$$

where $0 \leq \alpha < 1$ and $I_0 = i_0 \geq 0$ are constants, and $\{W_n, n = 1, 2, \cdots\}$ is a sequence of i.i.d. nonnegative random variables. Furthermore, we assume that $\{Y_n, n = 1, 2, \cdots\}$, $\{X_n, n = 1, 2, \cdots\}$, and $\{W_n, n = 1, 2, \cdots\}$ are independent, and they have common distribution functions $F(y) = \Pr\{Y_1 \leq y\}$, $H(x) = \Pr\{X_1 \leq x\}$, and $G(w) = \Pr\{W_1 \leq w\}$, respectively, with $F(0) = 0$.

We note that if $\alpha = 0$, (1.8) gives the stochastic model of i.i.d. rates of interest; if $\alpha = 0$ and $W_n = i$, a constant, for all $n = 1, 2, \cdots$, (1.8) yields the model of a constant rate. (1.8) is a dependent model for rates of interest, see, for example, Kellison (1991). In addition, (1.8) is equivalent to

$$I_n = \alpha^n i_0 + \alpha^{n-1} W_1 + \cdots + \alpha W_{n-1} + W_n,\ n = 1, 2, \cdots, \tag{1.9}$$

which implies that the rates of interest depend heavily on the recent rates. Furthermore, we point out that the mean $EI_n = EW_1/(1-\alpha) + (i_0 - EW_1/(1-\alpha))\alpha^n$ has a constant limit of $EW_1/(1-\alpha)$ as $n \to \infty$, is increasing in n when $EW_1 \geq i_0(1-\alpha)$ and decreasing when $EW_1 \leq i_0(1-\alpha)$. But I_n itself is neither increasing nor decreasing in n. The model can also be interpreted as the average business increasing or decreasing over time.

We denote the ultimate ruin probability in model (1.6) with a rate model (1.8), the initial surplus u, and the initial rate i_0 by

$$\phi(u, i_0) = \Pr\{\bigcup_{k=1}^{\infty} (U_k < 0)\}$$

where U_k is given in (1.6), and define the ultimate ruin probability in model (1.7) with a rate model (1.8), the initial surplus u, and the initial rate i_0 by

$$\varphi(u, i_0) = \Pr\{\bigcup_{k=1}^{\infty} (U_k < 0)\}$$

where U_k is given in (1.7).

To calculate the ruin probability $\psi(u)$ is not easy but the calculations of the ruin probabilities $\phi(u, i_0)$ and $\varphi(u, i_0)$ are even harder due to the complication of models (1.6) and (1.7). One of probabilistic methods concerned commonly in risk theory is to derive probability inequalities for the ruin probabilities. An celebrated result for the ruin probability $\psi(u)$ in the classical risk model is the Lundberg inequality, which states if $EX_1 > EY_1$ (we still assume that this condition holds in this paper) and there is a constant $R > 0$ satisfying

$$Ee^{-R(X_1-Y_1)} = 1\,, \tag{1.10}$$

then

$$\psi(u) \;\leq\; e^{-Ru}\,, \quad u \geq 0\,, \tag{1.11}$$

see, for example, Rolski *et al* (1999).

It is not difficult to see that

$$\phi(u, i_0) \;\leq\; \varphi(u, i_0) \;\leq\; \psi(u),\; u \geq 0. \tag{1.12}$$

Thus, the ruin probability $\psi(u)$ in the classical risk model is reduced by adding the interest income to the surplus. Also, the timing of payments has effects on the ruin probabilities $\phi(u, i_0)$ and $\varphi(u, i_0)$. Indeed, it is intuitive to see (1.12) since the premiums are paid earlier in model (1.4) than in model (1.5) and there is no interest income in (1.2). On the other hand, considering (1.12), any interesting upper bounds for $\phi(u, i_0)$ and $\varphi(u, i_0)$, saying

$$\phi(u, i_0) \;\leq\; \Delta(u, i_0) \;\text{ and }\; \varphi(u, i_0) \;\leq\; \Lambda(u, i_0),\; u \geq 0,$$

should satisfy

$$\Delta(u, i_0) \;\leq\; \Lambda(u, i_0) \;\leq\; e^{-Ru},\; u \geq 0. \tag{1.13}$$

In this paper, we will derive probability inequalities for $\phi(u, i_0)$ and $\varphi(u, i_0)$, which both are generalizations of the Lundberg upper bound and satisfy (1.13). In Section 2, we first derive integral equations for $\phi(u, i_0)$ and $\varphi(u, i_0)$, and then give probability inequalities for $\phi(u,\ i_0)$ and $\varphi(u,\ i_0)$ in Section 3. Finally, we give an illustrative application to the ruin probability in the compound binomial risk process under the dependent rates of interest in Section 4.

2 Integral equations for ruin probabilities

Define the finite time ruin probability in model (1.6) with a rate model (1.8), the initial surplus u, and the initial rate i_0 by

$$\begin{aligned}\phi_n(u, i_0) &= \Pr\{\bigcup_{k=1}^{n}(U_k < 0)\} \\ &= \Pr\left\{\bigcup_{k=1}^{n}\left(u\prod_{t=1}^{k}(1+I_t) + \sum_{j=1}^{k}(X_j(1+I_j) - Y_j)\prod_{t=j+1}^{k}(1+I_t) < 0\right)\right\}.\end{aligned}$$

Then

$$\lim_{n\to\infty}\phi_n(u, i_0) = \phi(u, i_0).$$

We first give the following integral equations for $\phi_n(u, i_0)$ and $\phi(u, i_0)$.

Lemma 2.1 For $n = 1, 2, \cdots$,

$$\begin{aligned}&\phi_{n+1}(u,\ i_0) \\ = &\int_0^\infty\int_0^\infty \bar{F}((u+x)(1+\alpha i_0+w))dH(x)dG(w) \\ + &\int_0^\infty\int_0^\infty\int_0^{(u+x)(1+\alpha i_0+w)} \phi_n((u+x)(1+\alpha i_0+w)-y,\ \alpha i_0+w)dF(y)dH(x)dG(w)\end{aligned}$$

and

$$\begin{aligned}&\phi(u,\ i_0) \\ = &\int_0^\infty\int_0^\infty \bar{F}((u+x)(1+\alpha i_0+w))dH(x)dG(w) \\ + &\int_0^\infty\int_0^\infty\int_0^{(u+x)(1+\alpha i_0+w)} \phi((u+x)(1+\alpha i_0+w)-y,\ \alpha i_0+w)dF(y)dH(x)dG(w).\end{aligned}$$

Proof. From (1.6) and (1.8), we have $U_1 = (u + X_1)(1 + I_1) - Y_1 = (u + X_1)(1 + \alpha i_0 + W_1) - Y_1$. Given $Y_1 = y, X_1 = x$, and $W_1 = w$, let $h = \alpha i_0 + w$. Thus, if $y > (u+x)(1+\alpha i_0+w) = (u+x)(1+h)$, then

$$\Pr\{U_1 < 0 |\, Y_1 = y, X_1 = x, W_1 = w\} = 1,$$

which implies

$$\Pr\left\{\bigcup_{k=1}^{n+1}(U_k < 0)\,|\, Y_1 = y, X_1 = x, W_1 = w\right\} = 1.$$

Let $\{\tilde{Y}_n,\, n=1,2,\cdots\}, \{\tilde{X}_n,\, n=1,2,\cdots\}$, and $\{\tilde{W}_n,\, n=1,2,\cdots\}$ are independent copies of $\{Y_n,\, n=1,2,\cdots\}, \{X_n,\, n=1,2,\cdots\}$, and $\{W_n,\, n=1,2,\cdots\}$, respectively. Then, given $W_1=w$, by (1.9), we know that

$$\begin{aligned}
I_t &= \alpha^t i_0 + \alpha^{t-1}W_1 + \alpha^{t-2}W_2 + \cdots + \alpha W_{t-1} + W_t \\
&= \alpha^{t-1}(\alpha i_0 + w) + \alpha^{t-2}W_2 + \cdots + \alpha W_{t-1} + W_t \\
&= \alpha^{t-1}h + \alpha^{t-2}W_2 + \cdots + \alpha W_{t-1} + W_t
\end{aligned}$$

has the same distribution as that of $\tilde{I}_{t-1} = \alpha^{t-1}h + \alpha^{t-2}\tilde{W}_1 + \cdots + \alpha\tilde{W}_{t-2} + \tilde{W}_{t-1}$, where $\{\tilde{I}_n,\ n=1,2,\cdots\}$ has a similar autoregressive structure as that of $\{I_n,\ n=1,2,\cdots\}$, namely,

$$\tilde{I}_n = \alpha\tilde{I}_{n-1} + \tilde{W}_n,\ n=1,2,\cdots,$$

but with a different initial rate $\tilde{I}_0 = \tilde{i}_0 = h = \alpha i_0 + w$.

Hence, if $0 \le y \le (u+x)(1+\alpha i_0 + w) = (u+x)(1+h)$, then

$$\Pr\{U_1 < 0 |\, Y_1 = y, X_1 = x, W_1 = w\} = 0,$$

which implies by (1.4) that for $0 \le y \le (u+x)(1+h)$,

$$\begin{aligned}
&\Pr\left\{\bigcup_{k=1}^{n+1}(U_k<0)\,|\,Y_1=y, X_1=x, W_1=w\right\} \\
=\;& \Pr\left\{\bigcup_{k=2}^{n+1}(U_k<0)\,|\,Y_1=y, X_1=x, W_1=w\right\} \\
=\;& \Pr\left\{\bigcup_{k=2}^{n+1}\left(((u+x)(1+h)-y)\prod_{t=2}^{k}(1+I_t) + \sum_{j=2}^{k}(X_j(1+I_j)-Y_j)\prod_{t=j+1}^{k}(1+I_t)<0\right)\right\} \\
=\;& \Pr\left\{\bigcup_{k=1}^{n}\left(((u+x)(1+h)-y)\prod_{t=1}^{k}(1+\tilde{I}_t) + \sum_{j=1}^{k}(\tilde{X}_j(1+\tilde{I}_j)-\tilde{Y}_j)\prod_{t=j+1}^{k}(1+\tilde{I}_t)<0\right)\right\} \\
=\;& \phi_n((u+x)(1+h)-y,\ \tilde{i}_0) \\
=\;& \phi_n((u+x)(1+h)-y,\ h).
\end{aligned}$$

Therefore, by conditioning on Y_1, X_1, and W_1, we get

$$\phi_{n+1}(u,i_0) = \Pr\left\{\bigcup_{k=1}^{n+1}(U_k<0)\right\}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \int_0^\infty \Pr\left\{\bigcup_{k=1}^{n+1}(U_k<0)\,|\,Y_1=y, X_1=x, W_1=w\right\} dF(y)dH(x)dG(w) \\
&= \int_0^\infty \int_0^\infty \int_{(u+x)(1+h)}^\infty dF(y)dH(x)dG(w) \\
&+ \int_0^\infty \int_0^\infty \int_0^{(u+x)(1+h)} \phi_n((u+x)(1+h)-y,\ h)dF(y)dH(x)dG(w) \\
&= \int_0^\infty \int_0^\infty \bar{F}((u+x)(1+h))dH(x)dG(w) \\
&+ \int_0^\infty \int_0^\infty \int_0^{(u+x)(1+h)} \phi_n((u+x)(1+h)-y,\ h)dF(y)dH(x)dG(w). \qquad (2.1)
\end{aligned}
$$

Thus, the integral equation for $\phi(u, i_0)$ in Lemma 2.1 follows from letting $n \to \infty$ in (2.1), $\lim_{n\to\infty} \phi_n(u, i_0) = \phi(u, i_0)$, and the Lebesgue dominated convergence theorem. □

Similarly, by denoting the finite time ruin probability in model (1.7) with a rate model (1.8), the initial surplus u, and the initial rate i_0 by

$$
\begin{aligned}
\varphi_n(u, i_0) &= \Pr\{\bigcup_{k=1}^{n}(U_k<0)\} \\
&= \Pr\left\{\bigcup_{k=1}^{n}\left(u\prod_{t=1}^{k}(1+I_t) + \sum_{j=1}^{k}(X_j - Y_j)\prod_{t=j+1}^{k}(1+I_t) < 0\right)\right\},
\end{aligned}
$$

we then get the following integral equations for $\varphi_n(u, i_0)$ and $\varphi(u, i_0)$.

Lemma 2.2 For $n = 1, 2, \cdots$,

$$
\begin{aligned}
&\varphi_{n+1}(u, i_0) \\
&= \int_0^\infty \int_0^\infty \bar{F}(u(1+\alpha i_0 + w) + x)dH(x)dG(w) \\
&+ \int_0^\infty \int_0^\infty \int_0^{u(1+\alpha i_0+w)+x} \varphi_n(u(1+\alpha i_0 + w) + x - y,\ \alpha i_0 + w)dF(y)dH(x)dG(w)
\end{aligned}
$$

and

$$
\begin{aligned}
&\varphi(u, i_0) \\
&= \int_0^\infty \int_0^\infty \bar{F}(u(1+\alpha i_0 + w) + x)dH(x)dG(w) \\
&+ \int_0^\infty \int_0^\infty \int_0^{u(1+\alpha i_0+w)+x} \varphi(u(1+\alpha i_0 + w) + x - y,\ \alpha i_0 + w)dF(y)dH(x)dG(w).
\end{aligned}
$$

Proof. In this case,

$$U_k = u \prod_{t=1}^{k}(1+I_t) + \sum_{j=1}^{k}(X_j - Y_j) \prod_{t=j+1}^{k}(1+I_t).$$

The rest of the proof of Lemma 2.2 is similar to that of Lemma 2.1. □

3 Probability inequalities for ruin probabilities

Using the integral equations derived in Section 2 for $\phi(u,\ i_0)$ and $\varphi(u,\ i_0)$, we can derive probability inequalities for $\phi(u,\ i_0)$ and $\varphi(u,\ i_0)$. We first get a probability inequality for $\phi(u,\ i_0)$.

Theorem 3.1 Suppose that $R_1 > 0$ is a constant satisfying

$$Ee^{-R_1(X_1(1+W_1)-Y_1)} = 1\,. \tag{3.1}$$

Then,

$$\phi(u,\ i_0) \ \le\ \beta_1\, Ee^{R_1 Y_1}\ Ee^{-R_1(u+X_1)(1+\alpha i_0+W_1)}, \quad u \ge 0 \tag{3.2}$$

where

$$(\beta_1)^{-1} = \inf_{t\ge 0} \frac{\int_t^\infty e^{R_1 y} dF(y)}{e^{R_1 t}\bar{F}(t)}\,. \tag{3.3}$$

Proof. For any $x \ge 0$, we have

$$\begin{aligned} \bar{F}(x) &= \left(\frac{\int_x^\infty e^{R_1 y} dF(y)}{e^{R_1 x}\bar{F}(x)}\right)^{-1} e^{-R_1 x}\int_x^\infty e^{R_1 y} dF(y) \\ &\le \beta_1 e^{-R_1 x}\int_x^\infty e^{R_1 y} dF(y) \qquad (3.4) \\ &\le \beta_1 e^{-R_1 x}\, Ee^{R_1 Y_1}. \qquad (3.5) \end{aligned}$$

Then, for any $u \ge 0$ and any $i_0 \ge 0$,

$$\begin{aligned} \phi_1(u,\ i_0) &= \Pr\{Y_1 > (u+X_1)(1+I_1)\} = \Pr\{Y_1 > (u+X_1)(1+\alpha i_0 + W_1)\} \\ &= \int_0^\infty\int_0^\infty \bar{F}((u+x)(1+\alpha i_0+w))dH(x)dG(w), \end{aligned}$$

which implies by (3.5) that

$$\begin{aligned}\phi_1(u,\ i_0) &\leq \beta_1\, Ee^{R_1 Y_1} \int_0^\infty \int_0^\infty e^{-R_1(u+x)(1+\alpha i_0+w)} dH(x)dG(w)\\ &= \beta_1\, Ee^{R_1 Y_1}\, Ee^{-R_1(u+X_1)(1+\alpha i_0+W_1)}\,.\end{aligned}$$

Under an inductive hypothesis, we assume for any $u \geq 0$ and any $i_0 \geq 0$,

$$\phi_n(u,\ i_0) \ \leq\ \beta_1\, Ee^{R_1 Y_1}\, Ee^{-R_1(u+X_1)(1+\alpha i_0+W_1)}. \tag{3.6}$$

Thus, for $0 \leq y \leq (u+x)(1+\alpha i_0+w)$, by (3.6), $\alpha i_0 \geq 0$, and $W_1 \geq 0$, we have

$$\begin{aligned}&\phi_n((u+x)(1+\alpha i_0+w)-y,\ \alpha i_0+w)\\ \leq\ & \beta_1\, Ee^{R_1 Y_1}\, Ee^{-R_1((u+x)(1+\alpha i_0+w)-y+X_1)(1+\alpha(\alpha i_0+w)+W_1)}\\ \leq\ & \beta_1\, Ee^{R_1 Y_1}\, Ee^{-R_1((u+x)(1+\alpha i_0+w)-y+X_1)(1+W_1)}\\ \leq\ & \beta_1\, Ee^{R_1 Y_1}\, Ee^{-R_1((u+x)(1+\alpha i_0+w)-y)-R_1X_1(1+W_1)}\\ =\ & \beta_1\, Ee^{R_1 Y_1}\, Ee^{-R_1X_1(1+W_1)}\ e^{-R_1((u+x)(1+\alpha i_0+w)-y)}\\ =\ & \beta_1\, e^{-R_1((u+x)(1+\alpha i_0+w)-y)}.\end{aligned} \tag{3.7}$$

Thus, by Lemma 2.1, (3.4), and (3.7), we get

$$\begin{aligned}&\phi_{n+1}(u,\ i_0)\\ \leq\ & \beta_1 \int_0^\infty \int_0^\infty e^{-R_1(u+x)(1+\alpha i_0+w)} \int_{(u+x)(1+\alpha i_0+w)}^\infty e^{R_1 y} dF(y)dH(x)dG(w)\\ +\ & \beta_1 \int_0^\infty \int_0^\infty e^{-R_1(u+x)(1+\alpha i_0+w)} \int_0^{(u+x)(1+\alpha i_0+w)} e^{R_1 y} dF(y)dH(x)dG(w)\\ =\ & \beta_1 \int_0^\infty \int_0^\infty e^{-R_1(u+x)(1+\alpha i_0+w)} \int_0^\infty e^{R_1 y} dF(y)dH(x)dG(w)\\ =\ & \beta_1 Ee^{R_1 Y_1} Ee^{-R_1(u+X_1)(1+\alpha i_0+W_1)}\,.\end{aligned}$$

Hence, for any $n = 1, 2, \cdots$, (3.6) holds. Therefore, (3.2) follows from letting $n \to \infty$ in (3.6) and $\lim_{n\to\infty} \phi_n(u, i_0) = \phi(u, i_0)$. □

An improved upper bound in Theorem 3.1 can be obtained when F is new worse than used in convex ordering (NWUC), see Cao and Wang (1991) for the definition and properties of NWUC.

Corollary 3.1 Under the conditions of Theorem 3.1, if F is new worse than used in convex ordering (NWUC), then,

$$\phi(u,\ i_0) \ \leq\ Ee^{-R_1(u+X_1)(1+\alpha i_0+W_1)},\ u \geq 0\,. \tag{3.8}$$

Proof. By Proposition 6.1.1 of Willmot and Lin (2001), we know that if F is NWUC then $\beta_1 = (Ee^{R_1 Y_1})^{-1}$. Thus (3.8) follows from (3.2). □

We can show that the upper bound in Theorem 3.1 is less than the Lundberg upper bound. In doing so, we have the following result about R_1 and R.

Proposition 3.1 If $EX_1 > EY_1$ and $R_1 > 0$ in (3.1) and $R > 0$ in (1.10) exist, then $R_1 \geq R$, in particular, if both X_1 and W_1 are not degenerate at 0, then $R_1 > R$.

Proof. By considering the following functions:

$$f(r) = Ee^{-r(X_1(1+W_1)-Y_1)} - 1$$

and

$$g(r) = Ee^{-r(\,X_1-Y_1)} - 1\,,$$

we have,

$$f''(r) = E\left[(X_1(1+W_1)-Y_1)^2\, e^{-r(X_1(1+W_1)-Y_1)}\right] \geq 0\,,$$

which implies that $f(r)$ is a convex function. In addition, $f(0) = 0$ and $f'(0) = EY_1 - EX_1(1+W_1) \ \leq\ EY_1 - EX_1 < 0$. Similarly, $g(r)$ is also a convex function with $g(0) = 0$ and $g'(0) = E(Y_1 - X_1) < 0$. Therefore, if $R_1 > 0$ and $R > 0$ exist, then they are unique positive roots of $f(r)$ and $g(r)$ respectively on $(0, \infty)$. Further, if $r > 0$ such that $g(r) \geq 0$, then $r \geq R$. However,

$$e^{-R_1(X_1(1+W_1)-Y_1)} \ \leq\ e^{-R_1(X_1-Y_1)}.$$

Thus,

$$1 = Ee^{-R_1(X_1(1+W_1)-Y_1)} \ \leq\ Ee^{-R_1(X_1-Y_1)},$$

or

$$g(R_1) = Ee^{-R_1(X_1-Y_1)} - 1 \ \geq \ 0,$$

which implies that $R_1 > R$, in particular, if both X_1 and W_1 are not degenerate at 0, then

$$1 = Ee^{-R_1(X_1(1+W_1)-Y_1)} \ < \ Ee^{-R_1(X_1-Y_1)},$$

or

$$g(R_1) = Ee^{-R_1(X_1-Y_1)} - 1 \ > \ 0,$$

which implies that $R_1 > R$. □

Denote the upper bound in Theorem 3.1 by $A(u, i_0)$, i.e.

$$A(u, i_0) = \beta_1 Ee^{R_1 Y_1} Ee^{-R_1(u+X_1)(1+\alpha i_0+W_1)},$$

we have the following result.

Proposition 3.2 For any $u \geq 0$,

$$A(u, i_0) \ \leq \ e^{-Ru}.$$

Proof. By $W_1 \geq 0$, $\alpha i_0 \geq 0$, (3.1), and Proposition 3.1, we have for $u \geq 0$,

$$\begin{aligned}
& A(u, i_0) \\
= \ & \beta_1 Ee^{R_1 Y_1} Ee^{-R_1 u(1+\alpha i_0+W_1)-R_1 X_1(1+\alpha i_0+W_1)} \\
\leq \ & \beta_1 Ee^{R_1 Y_1} Ee^{-R_1 u(1+\alpha i_0)-R_1 X_1(1+W_1)} \\
= \ & \beta_1 Ee^{R_1 Y_1} Ee^{-R_1 X_1(1+W_1)} e^{-R_1 u(1+\alpha i_0)} \\
= \ & \beta_1 e^{-R_1 u(1+\alpha i_0)} \\
\leq \ & e^{-R_1 u(1+\alpha i_0)} \ \leq \ e^{-Ru}.
\end{aligned}$$

□

Proposition 3.2 means that the upper bound in Theorem 3.1 is less than the Lundberg upper bound. Next, we obtain the following probability inequality for $\varphi(u, \ i_0)$.

Theorem 3.2 Let $R > 0$ be a constant satisfying (1.10). Then,

$$\varphi(u,\ i_0)\ \le\ \beta\, Ee^{-Ru(1+\alpha i_0+W_1)},\ u \ge 0 \tag{3.9}$$

where

$$(\beta)^{-1} = \inf_{t\ge 0} \frac{\int_t^\infty e^{Ry} dF(y)}{e^{Rt}\bar{F}(t)}\,. \tag{3.10}$$

Proof. Similarly to (3.4) and (3.5), we have for any $x \ge 0$,

$$\begin{aligned}\bar{F}(x) &\le \beta e^{-Rx} \int_x^\infty e^{Ry} dF(y) \\ &\le \beta e^{-Rx}\, Ee^{RY_1},\end{aligned}$$

which implies that for any $u \ge 0$ and any $i_0 \ge 0$,

$$\begin{aligned}\varphi_1(u,\ i_0) &= \Pr\{Y_1 > u(1+I_1) + X_1\} \\ &= \int_0^\infty \int_0^\infty \bar{F}(u(1+\alpha i_0 + w) + x) dH(x) dG(w) \\ &\le \beta\, Ee^{RY_1} \int_0^\infty \int_0^\infty e^{-R(u(1+\alpha i_0+w)+x)} dH(x) dG(w) \\ &= \beta\, Ee^{RY_1}\, Ee^{-R(u(1+\alpha i_0+W_1)+X_1)} \\ &= \beta\, Ee^{-Ru(1+\alpha i_0+W_1)}\,.\end{aligned}$$

Using the similar inductive arguments as those used in the proof of Theorem 3.1, we can prove that for any $n = 1, 2, \cdots$, $u \ge 0$, and $i_0 \ge 0$,

$$\varphi_n(u,\ i_0)\ \le\ \beta\, Ee^{-Ru(1+\alpha i_0+W_1)}\,. \tag{3.11}$$

Therefore, (3.9) follows from letting $n \to \infty$ in (3.11) and $\lim_{n\to\infty} \varphi_n(u, i_0) = \varphi(u, i_0)$. □

Also, a refinement of the upper bound in Theorem 3.2 is available when F is NWUC.

Corollary 3.2 Under the conditions of Theorem 3.2, if F is NWUC, then

$$\varphi(u,\ i_0)\ \le\ (Ee^{RY_1})^{-1} Ee^{-Ru(1+\alpha i_0+W_1)},\ u \ge 0\,. \tag{3.12}$$

Proof. The proof of Corollary 3.2 is similar to that of Corollary 3.1. □

Denote the upper bound in Theorem 3.2 by $B(u, i_0)$, i.e.

$$B(u, i_0) = \beta E e^{-Ru(1+\alpha i_0+W_1)},$$

we can show that (1.13) holds for $A(u, i_0)$ and $B(u, i_0)$.

Proposition 3.3 For any $u \geq 0$

$$A(u, i_0) \ \leq \ B(u, i_0) \ \leq \ e^{-Ru}. \tag{3.13}$$

Proof. The proof of proposition 3.3 is similar to that of Proposition 3.2. □

Proposition 3.3 means that the upper bound in Theorem 3.1 for $\phi(u, i_0)$ is less than the upper bound in Theorem 3.2 for $\varphi(u, i_0)$, and (1.13) is satisfied by the upper bounds derived in this paper.

Further, we point out that the upper bounds in Theorems 3.1 and 3.2 are optimal in the sense that if $\{X_n = cT_n, n = 1, 2, \cdots\}$, $c > 0$, $\alpha = W_1 = 0$, and T_1 and Y_1 are exponential random variables, then the upper bounds in Theorems 3.1 and 3.2 equal the exact value of the ruin probability in the compound Poisson risk process with exponential claim sizes. See, for example, Grandell (1991).

In addition, if $\alpha = 0$ and $W_n = 0$, then $R_1 = R$ and $\beta_1 = \beta$. Thus, the upper bounds in Theorems 3.1 and 3.2 are reduced to βe^{-Ru}, which yields an improvement on the Lundberg upper bound since $0 \leq \beta \leq 1$. For further refinements of the Lundberg upper bound in different applied probability models, see Grandell (1997), Willmot (1996), Willmot and Lin (2001), and references therein.

4 Ruin in the compound binomial risk model with interest

In this section, we consider a discrete-time risk model and assume that a discrete time stochastic process $\{U_t, t = 1, 2, \cdots\}$ satisfies

$$U_t = u + t - \sum_{i=1}^{N_t} P_i, \ t = 0, 1, 2, \cdots, \tag{4.1}$$

where $U_0 = u \geq 0$, $\{N_t, t = 1, 2, \cdots\}$ is a binomial process with $EN_t = tq$ and $0 < q < 1$, and $\{P_t, t = 1, 2, \cdots,\}$ is a sequence of i.i.d. positive random variables and is independent of the binomial process $\{N_t, t = 1, 2, \cdots\}$. This model is called a compound binomial risk model. It is a discrete time analogue of the continuous time compound Poisson risk model and can be used as an approximation to the continuous time compound Poisson risk model.

It is assumed in the compound binomial risk model that in each time period $(t, t+1], t = 0, 1, 2, \cdots$, the probability of a claim is q and the probability of no clam is $1-q$. The occurrences of a claim in different time periods are independent events, and the premiums for each time period are one. The amount of the ith claim is P_i. Thus, the surplus of an insurance company at time t with the initial surplus u is given by (4.1). In addition, the positive loading condition holds, i.e. $qEP_1 < 1$.

The ruin probability in the compound binomial risk model has been studied by Cheng *et al* (2000), DeVylder and Marceau (1996), and references therein. However, these studies have not considered the effects of interest and the timing of the payment on the surplus process $\{U_t, t = 1, 2, \cdots\}$. In this section, we modify the compound binomial risk model to adopt such effects and give the upper bounds for the ruin probability in the modified compound binomial risk model as the applications of the results in Section 3.

Let $\{\varepsilon_t, t = 1, 2, \cdots\}$ be a sequence of i.i.d. Bernoulli random variables with $\Pr\{\varepsilon_1 = 1\} = 1 - \Pr\{\varepsilon_1 = 0\} = q$, and assume that $\{\varepsilon_t, t = 1, 2, \cdots\}$ are independent of $\{P_t, t = 1, 2, \cdots\}$. Thus, the surplus process $\{U_t, t = 0, 1, 2, \cdots\}$ in the compound binomial risk model has the same distribution as the process $\{U_t^*, t = 0, 1, 2, \cdots\}$ defined by

$$U_t^* = u + \sum_{k=1}^{t}(1 - Y_k), \quad t = 0, 1, 2, \cdots, \tag{4.2}$$

where $U_0^* = u$ and $Y_k = \varepsilon_k P_k, k = 1, 2, \cdots$. Therefore, the ruin probability in the surplus process $\{U_t, t = 0, 1, 2, \cdots\}$ equals the ruin probability in the surplus process $\{U_t^*, t = 0, 1, 2, \cdots\}$, i.e.

$$\Pr\{U_t < 0, \text{ for some}\, t = 1, 2, \cdots\} = \Pr\{U_t^* < 0, \text{ for some}\, t = 1, 2, \cdots\}, \tag{4.3}$$

which is the ruin probability in the compound binomial risk model and has been considered by the above references.

Assume that the insurer would receive interest on its surplus during each time period and the rate of interest during the nth time period is I_n satisfying (1.8). Suppose that the premiums are paid at the beginning of each period and the claims are paid at the end of each period. Denote the ruin probability in the compound binomial risk model with a rate model (1.8), the initial surplus u, the initial rate i_0 by $\psi(u, i_0)$. Thus, R_1 in (3.1) satisfies

$$\begin{aligned}1 &= Ee^{-R_1(1+W_1-\varepsilon_1 P_1)} = e^{-R_1}Ee^{-R_1W_1}E(e^{R_1\varepsilon_1P_1}) \\ &= e^{-R_1}Ee^{-R_1W_1}E(E(e^{R_1\varepsilon_1P_1}|\varepsilon_1)) \\ &= e^{-R_1}Ee^{-R_1W_1}(qEe^{R_1P_1}+1-q).\end{aligned}$$

Therefore, Theorems 3.1 gives that if $\kappa_1 > 0$ be a constant satisfying

$$qEe^{\kappa_1P_1}+1-q = \frac{e^{\kappa_1}}{Ee^{-\kappa_1W_1}},$$

then

$$\psi(u, i_0) \le (qEe^{\kappa_1P_1}+1-q)Ee^{-\kappa_1(u+1)(1+\alpha i_0+W_1)}, \ u \ge 0. \tag{4.4}$$

Similarly, using Theorem 3.2, we can obtain the upper bound for the ruin probability in the compound binomial risk model when the premiums are paid at the end of each period with a dependent rates of interest in (1.8).

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