CERTAIN NIL CLEAN CONDITIONS ON ZERO-DIVISORS

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Abstract: An element of a ring is (very) nil clean if it is the sum of an (a very) idempotent and a nilpotent element. In this paper we investigate the uniqueness of (very) nil cleanness, especially, on zero-divisors. A ring R is very (D-very) nil clean if every element (zero-divisor) can be uniquely written as the sum of a (very) idempotent and a nilpotent element. The structure of these rings are determined. For instance, we prove that a ring R is very nil clean if and only if R is abelian; $J(R)$ is nil and $R/J(R)$ is isomorphic to $\mathbb{Z}_3$, a Boolean ring, or $\mathbb{Z}_3 \bigoplus B$ where B is a Boolean. A periodic ring R is D-very nil clean if and only if R is abelian and $R/J(R)$ is isomorphic to a field F, $\mathbb{Z}_3 \bigoplus \mathbb{Z}_3$, $\mathbb{Z}_3 \bigoplus B$ where B is Boolean, or a Boolean ring. In particular, the structure of uniquely (D-uniquely) nil clean rings are also studied.

Key words: Zero-divisor; Very Nil Clean Ring; D-Very Nil Clean Ring; Uniquely Nil Clean Ring; D-Uniquely Nil Clean Ring.

MR(2010) Subject Classification: 16E50, 16U99.

1. INTRODUCTION

Let R be a ring with an identity. We say that $a \in R$ is a left (right) zero-divisor if there exists a nonzero $b \in R$ such that $ab = 0(ba = 0)$. An element that is a left and a right zero-divisor is simply called a zero-divisor. Zero-divisors occur in many classes of rings. An element a in a ring R is (uniquely) nil clean if it is the sum of (unique) an idempotent $e \in R$ and a nilpotent. A ring R is (uniquely) nil clean provided that every element in R is (unique) nil clean. An element a in R is a very idempotent provided that a or $-a$ is an idempotent. An element $a \in R$ is called very nil clean provided that there exists a very idempotent $e \in R$ such that $a - e \in N(R)$, and that $a - f \in N(R)$ with a very idempotent $f \in R$ implies that $e^2 = f^2$. A ring R is very nil clean if every element in R is very nil clean. The motivation of this paper is to explore structure of rings with these nil clean conditions on zero-divisors.

A ring R is called D-very nil clean provided that every zero-divisor in R is very nil clean. A ring R is called D-uniquely nil clean provided that every zero-divisor in R is uniquely nil clean. In Section 2, we prove that a ring R is very nil clean if and only if R is abelian; $J(R)$

is nil and $R/J(R)$ is isomorphic to $\mathbb{Z}_3$, a Boolean ring, or $\mathbb{Z}_3 \bigoplus B$ where B is a Boolean. A ring R is called a periodic ring if for any $a \in R$ there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$. In Section 3, we shall determine the structure of D-very nil clean rings in periodic case. We show that a periodic ring R is D-very nil clean if and only if R is abelian and $R/J(R)$ is isomorphic to a field F, $\mathbb{Z}_3 \bigoplus \mathbb{Z}_3$, $\mathbb{Z}_3 \bigoplus B$ where B is Boolean, or a Boolean ring. Furthermore, in Section 4, we are concern several special cases. We prove that a ring R is an abelian ring in which every zero-divisor in R is a very idempotent or a nilpotent element if and only if R is isomorphic to one of the following: a D-ring, a Boolean ring, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, and $\mathbb{Z}_3 \bigoplus B$ where B is a Boolean. In Section 5, we characterize uniquely nil clean rings. We prove that a ring R is uniquely nil clean if and only if $2 \in R$ is nilpotent and R is very nil clean. From this, we show that RG is uniquely nil clean if and only if R is uniquely nil clean and $I(R, G)$ is nil. Finally, in the last section, we explore the structure of D-uniquely nil clean rings. We show that a ring R is a D-uniquely nil clean ring if and only if R is a D-ring or R is uniquely nil clean.

Throughout, all rings are associative with an identity. We use $Id(R), N(R)$ and $J(R)$ to denote the sets of all idempotents, all nilpotent elements and the Jacobson radical of a ring R. $Z(R)$ and $NZ(R)$ stand for the sets of all zero-divisors and non zero-divisors of a ring R.

2. VERY NIL CLEAN RINGS

The aim of this is to investigate the structure of very nil clean rings. The necessary and sufficient conditions under which a group ring is very nil clean are also obtained.

Lemma 2.1 [2, Theorem 2.28]. *Let R be a ring. Then every element in R is a very idempotent if and only if R is isomorphic to one of the following:*

(a) $\mathbb{Z}_3$,

(b) *a Boolean ring, or*

(c) $\mathbb{Z}_3 \bigoplus B$ *where B is a Boolean.*

Theorem 2.2. *Let R be a ring. Then R is very nil clean if and only if*

(1) *R is abelian;*

(2) *$J(R)$ is nil;*

(3) *$R/J(R)$ is isomorphic to one of the following:*

  (a) $\mathbb{Z}_3$,

  (b) *a Boolean ring, or*

  (c) $\mathbb{Z}_3 \bigoplus B$ *where B is a Boolean.*

Proof. Suppose that R is very nil clean. For any idempotent $e \in R$ and any $a \in R$, $e + ea(1-e) \in R$ is an idempotent. Since $\big(e + ea(1-e)\big) + 0 = e + ea(1-e)$, by the uniqueness, $ea(1-e) = 0$; hence, $ea = eae$. Likewise, $ae = eae$, and so $ea = ae$. Thus, R is abelian. Let $x \in J(R)$. Then we have a very idempotent $e \in R$ such that $w := x - e \in N(R)$. Clearly, $e \in R$ is central; hence, $wx = w(w+e) = (w+e)w = xw$. This implies that

$1-e=(1-x)+w=(1-x)\big(1+(1-x)^{-1}w\big)\in U(R)$, and so $1-e=1$. Thus, $e=0$, and then $x=w\in N(R)$. That is, $J(R)$ is nil. Let $a\in R$. Then there exists a central very idempotent $e\in R$ such that $w:=a-e\in N(R)$. If $e^2=e$, then $a-a^2=w-2ew-w^2\in N(R)$. If $e^2=-e$, then $a+a^2=w+2ew+w^2\in N(R)$. In any case, we can find some $n\in\mathbb{N}$ such that $a^n=a^{n+1}f(a)$ where $f(t)\in R[t]$. In view of Herstein's Theorem, R is periodic, and then $N(R)$ forms an ideal of R. Therefore, $J(R)=N(R)$, and so every element in $R/J(R)$ is a very idempotent. In light of Lemma 2.1, (3) is satisfied.

Conversely, assume that $(1)-(3)$ hold. Let $a\in R$. Then $\overline{a}$ is a very idempotent, in terms of Lemma 2.1. As $J(R)$ is nil, every idempotent lifts modulo $J(R)$, and so every very idempotent lifts modulo $J(R)$. Thus, we can find a very idempotent $e\in R$ such that $\overline{a}=\overline{e}$. Hence, $v:=a-e\in J(R)\subseteq N(R)$. If there exists a very idempotent $f\in R$ such that $w:=a-f\in N(R)$, then $e^2-f^2=(a-v)^2-(a-w)^2=(-av-va+v^2)+(aw+wa-w^2)$. As $v\in J(R)$, we see that $-av-va+v^2\in J(R)$. Furthermore, $aw+wa-w^2\in N(R)$ since $aw=wa$. This implies that $1-(e^2-f^2)=-(-av-va+v^2)+\big(1-(aw+wa-w^2)\big)\in U(R)$. As $e^2,f^2\in R$ are idempotents, we have $(e^2-f^2)^3=e^2-f^2$, and so $(e^2-f^2)\big(1-(e^2-f^2)\big)=0$. Accordingly, $e^2=f^2$, as asserted. □

Corollary 2.3. *Let R be a local ring. Then R is very nil clean if and only if*

(1) *$J(R)$ is nil;*

(2) *$R/J(R)$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. Suppose that R is very nil clean. Then $J(R)$ is nil. Since every idempotent in R is 0 or 1, $R/J(R)$ is isomorphic to $\mathbb{Z}_3$, or a Boolean ring. If $R/J(R)$ is isomorphic to a Boolean ring, then $R/J(R)\cong\mathbb{Z}_2$, as desired.

The converse is clear from Theorem 2.2, as R is abelian. □

Corollary 2.4. *Let R be a commutative ring. Then every element in R is the sum of a very idempotent and a nilpotent element if and only if*

(1) *$J(R)$ is nil;*

(2) *$R/J(R)$ is isomorphic to one of the following:*

  (a) *$\mathbb{Z}_3$,*

  (b) *a Boolean ring, or*

  (c) *$\mathbb{Z}_3\bigoplus B$ where B is a Boolean.*

Proof. This is obvious by Theorem 2.2 □

Let $R=\mathbb{Z}_{2^m\times 3^n}(m,n\in\mathbb{N})$. Then $R\cong\mathbb{Z}_{2^m}\oplus\mathbb{Z}_{3^n}$. But $J\big(\mathbb{Z}_{2^m}\oplus\mathbb{Z}_{3^n}\big)=2\mathbb{Z}_{2^m}\oplus 3\mathbb{Z}_{3^n}$ is nil, and so $R/J(R)\cong\mathbb{Z}_2\oplus\mathbb{Z}_3$. Therefore every element in R is the sum of a very idempotent and a nilpotent element, in terms of Corollary 2.4.

Corollary 2.5. *Let R be a ring, and let $\sigma: R\to R$ be an endomorphism. Then $R[[x,\sigma]]$ is very nil clean if and only if*

(1) *R is very nil clean;*

(2) *$\sigma(e)=e$ for all idempotents $e\in R$.*

Proof. Suppose that $R[[x,\sigma]]$ is very nil clean. Then $R[[x,\sigma]]$ is abelian, by Theorem 2.2. For any idempotent $e\in R$, we see that $ex=xe=\sigma(e)x$, and so $e=\sigma(e)$. Let $a\in R$ be a zero-divisor. Then $a\in R[[x,\sigma]]$ is a zero-divisor. By hypothesis, there exists a unique idempotent $e(x)\in R[[x,\sigma]]$ such that $a-e(x)\in N\big(R[[x,\sigma]]\big)$. Hence, we have a unique idempotent $e(0)\in R$ such that $a-e(0)\in N(R)$. Therefore, R is very nil clean.

Conversely, assume that (1) and (2) hold. Let $e(x)=\sum_{i=0}^{\infty}e_ix^i\in R[[x,\sigma]]$ be an idempotent. Then

$$\begin{gathered} e_0^2=e_0;\\ e_0e_1+e_1\sigma(e_0)=e_1;\\ e_0e_2+e_1\sigma(e_1)+e_2\sigma^2(e_0)=e_1;\\ \vdots \end{gathered}$$

In view of Theorem 2.2, R is abelian. Hence, $(2e_0-1)e_1=0$. As $(2e_0-1)^2=1$, we get $e_1=0$. Likewise, $e_2=\cdots=0$. Hence, $e(x)=e_0\in R$ is an idempotent. This implies that $R[[x,\sigma]]$ is abelian. By using Theorem 2.2 again, $R/J(R)$ is one of the following:

(a) $\mathbb{Z}_3$,

(b) *a Boolean ring, or*

(c) $\mathbb{Z}_3\bigoplus B$ *where* B *is a Boolean.*

Obviously,

$$R[[x,\sigma]]/J(R[[x,\sigma]])\cong R/J(R).$$

Therefore $R[[x,\sigma]]/J(R[[x,\sigma]])$ is one of the preceding. Accordingly, we complete the proof, in terms of Theorem 2.2. □

Proposition 2.6. *Let R be a ring, suppose that*

(1) $N(R)$ *is commutative;*

(2) *Every element in R is the sum of a very idempotent and a nilpotent element.*

Then $J(R)$ is nil and $R/J(R)$ is isomorphic to one of the following:

(a) $\mathbb{Z}_3$,

(b) *a Boolean ring, or*

(c) $\mathbb{Z}_3\bigoplus B$ *where* B *is a Boolean.*

Proof. Let $a\in J(R)$. Then there exists an idempotent $e\in R$ such that $a+e\in N(R)$ or $a-e\in N(R)$. If $v:=a+e\in N(R)$, then $1-e=(1+a)-v\in U(R)$, and so $e=0$. If $w:=a-e\in N(R)$, then $1-e=(1-a)+w\in U(R)$, and so $e=0$. In any case, we have $a\in N(R)$. Therefore, $J(R)$ is nil.

Let $e\in R$ be an idempotent, and let $x\in N(R)$. Then $(1-e)xe\in N(R)$, and so $x(1-e)xe=(1-e)xex$. Hence, $x^2e-xexe=xex-exex$, and so $ex^2e-exexe=0$. Thus, $ex^2e=(exe)^2$. Write $x^m=0(m\geq 1)$. By induction, we get $(exe)^{2^m}=ex^{2^m}e=0$, and then $exe\in N(R)$. This implies that $xe=exe+(1-e)xe\in N(R)$. Likewise, $ex\in N(R)$. Let $r\in R$. Then we have an idempotent $e\in R$ such that $r+e\in N(R)$ or $r-e\in N(R)$. Since $N(R)$ is commutative, we see that $rx,xr\in N(R)$. We infers that $N(R)$ is an ideal of R. Hence, $N(R)=J(R)$. It follows that every element in $R/J(R)$ a very idempotent. Therefore the result follows, by Lemma 2.1. □

Let $P(R)$ be the intersection of all prime ideals of R, i.e., $P(R)$ is the prime radical of R. As is well known, $P(R)$ is the intersection of all minimal prime ideals of R.

Theorem 2.7. *Let R be a ring. Then R is very nil clean if and only if*

(1) *R is abelian;*

(2) *$R/P(R)$ is very nil clean.*

Proof. Suppose that R is very nil clean. Then R is abelian. In view of Theorem 2.2, R is clean, and so it is an exchange ring. Thus, $R/P(R)$ is abelian. Obviously, $J\big(R/P(R)\big) = J(R)/P(R)$ is nil. Further, $\big(R/P(R)\big)/J\big(R/P(R)\big) \cong R/J(R)$. By Theorem 2.2 again, $R/P(R)$ is very nil clean.

Conversely, assume that (1) and (2) hold. For any $x \in J(R)$, we see that $\overline{x} \in J\big(R/P(R)\big)$ is nilpotent. Since $P(R)$ is nil, we see that $x \in R$ is an nilpotent; hence that $J(R)$ is nil. As $R/J(R) \cong \big(R/P(R)\big)/J\big(R/P(R)\big)$, it follows from Theorem 2.2 that R is very clean, as asserted. □

Let R be a ring, and let G be a group. The augmentation ideal $I(R,G)$ of the group ring RG is the kernel of the homomorphism from RG to R induced by collapsing G to 1. That is, $I(R,G) = ker(\omega)$, where $\omega = \{\sum\limits_{g\in G} r_g g \mid \sum\limits_{g\in G} r_g = 0\}$.

Lemma 2.8. *Let R be a ring, and let G be a group. If RG is very nil clean, then so is R.*

Proof. Let $a \in R$. Then we have a very idempotent $e \in RG$ such that $a - e \in N(RG)$ and that such representation is unique. Hence, $a - \omega(e) = \omega(a - e) \in N(R)$. Obviously, $\omega(e) \in R$ is a very idempotent. If $a - f \in N(R)$ for a very idempotent $f \in R$, then $e^2 = f^2$, as desired. □

Theorem 2.9. *Let R be a ring, and let G be a group. If $I(R,G)$ is nil, then RG is very nil clean if and only if so is R.*

Proof. One direction is obvious by Lemma 2.8. Conversely, assume that R is very nil clean. Let $x \in RG$. Then $x = \omega(x) + \big(x - \omega(x)\big)$. By hypothesis, there exists a very idempotent $e \in R$ such that $w := \omega(x) - e \in N(R)$. Hence, $x = e + \big(w + (x - \omega(x))\big)$. Since $ker(\omega)$ is nil, we see that $v := w + (x - \omega(x)) \in N(R)$. Assume that $x = f + w$ where $f \in RG$ is an very idempotent and $w \in N(RG)$. Then $f - \omega(f) \in ker(\omega)$ is nil. As R is very nil clean, R is abelian. Hence, $\big(f - \omega(f)\big)\big(1 - (f - \omega(f))^2\big) = 0$, and so $f = \omega(f) \in R$. It is easy to verify that $vw = (x - e)(x - f) = (x - f)(x - e) = wv$, and then $e - f = w - v \in N(R)$. It follows from $(e - f)\big(1 - (e - f)^2\big) = 0$ that $e = f$, as needed. □

Corollary 2.10. *Let R be a ring with a prime $p \in J(R)$, and let G be a locally finite p-group. Then RG is very nil clean if and only if R is very nil clean.*

Proof. One direction is obvious. Conversely, assume that R is very nil clean. Then $J(R)$ is nil by Theorem 2.2. We first suppose G is finite and prove the claim by induction on $|G|$. As the center of a nontrivial finite p-group contains more than one element, we may take $x \in G$ be an element in the center with the order p. Let (x) be the subgroup of G generated by x. Then $\overline{G} = G/(x)$ has smaller order. By induction hypothesis, $ker\big(\overline{\omega}\big)$ is nil, where $\overline{\omega} : R\overline{G} \to R, \sum\limits_{\overline{g}\in\overline{G}} r_{\overline{g}}\overline{g}$. Let $\varphi : RG \to R\overline{G}, \sum\limits_{g} r_g g \to \sum\limits_{g} r_g \overline{g}$. Then $ker(\varphi) = (1-x)RG$. Since $x^p = 1$, we see that $(1-x)^p \in pRG$; hence, $1-x \in RG$ is nilpotent. But $\varphi\big(ker(\omega)\big) = ker\big(\overline{\omega}\big)$

is nil. For any $z \in ker(\omega)$, we have some $m \in \mathbb{N}$ such that $z^m \in ker(\varphi)$ is nilpotent. Thus, $z \in RG$ is nilpotent. We conclude that $ker(\omega)$ is nil, and therefore RG is very nil clean, in terms of Theorem 2.2. □

3. FACTORIZATION OF ZERO-DIVISORS

A ring R is D-very nil clean provided that every zero-divisor of R is very nil clean. This section is concern on such rings. We begin with the relation of these rings with very nil clean ones.

Theorem 3.1. *Let R be a ring. Then R is very nil clean if and only if*

(1) *R is periodic;*

(2) *R is D-very nil clean;*

(3) $U(R) = \{x \pm 1 \mid x \in N(R)\}$.

Proof. Suppose that R is very nil clean. As in the proof of Theorem 2.2, R is periodic. (2) is obvious. Let $x \in U(R)$. Then we have a very idempotent $e \in R$ such that $w := x - e \in N(R)$. As R is abelian, we see that $e = x - w$ and $ew = we$, and so $e = \pm 1$. Therefore $x = w \pm 1$, as desired.

Conversely, assume that $(1) - (3)$ hold. Let $a \in R$. Then we have distinct $m, n \in \mathbb{N}(m > n)$ such that $a^m = a^n$. If a is a zero-divisor, then a is very nil clean. If a is a non zero-divisor, $a^{m-n} = 1$. By (3), we see that a is very nil clean. This completes the proof. □

Lemma 3.2. *Every D-very nil clean ring is abelian.*

Proof. Let $e \in R$ be an idempotent, and let $x \in R$. Then $e + ex(1-e) \in R$ is an idempotent. If $e = 1$, then $ex = exe$. If $1 - e = ex(1-e)$, then $ex = exe$. If $e \neq 1$ and $1 - e \neq ex(1-e)$, then $e + ex(1-e) \in R$ is a zero-divisor, as

$$(1-e)\big(e + ex(1-e)\big) = 0 = \big(e + ex(1-e)\big)\big(1 - e - ex(1-e)\big).$$

Since $e + ex(1-e) = e + ex(1-e) + 0$, by hypothesis, $e^2 = \big(e + ex(1-e)\big)^2$, and then $ex(1-e) = 0$. That is, $ex = exe$. Likewise, $xe = exe$. Thus, $ex = xe$. This completes the proof. □

We say that a ring R is a D-ring if every zero-divisor in R is nilpotent. For instance, $\mathbb{Z}_{p^k}(p \text{ is prime}, k \geq 1)$. This concept coincides with that introduced by Abu-Khuzam and Yaqub [1] for a commutative ring.

Theorem 3.3. *Every D-very nil clean ring is a D-ring or the product of two very nil clean rings.*

Proof. Let R be a D-very nil clean ring. In view of Lemma 3.2, R is abelian.

Case I. R is indecomposable. Then every zero-divisor is nilpotent or invertible. The later is imposable, and so R is a D-ring.

Case II. R is decomposable. Write $R = A \oplus B$. Let $a \in A$. Then $(a, 0) \in R$ is a zero-divisor. By hypothesis, there exists a very idempotent $(e, e') \in R$ such that $(a, 0) - (e, e') \in N(R)$, and that $(a, 0) - (f, f') \in N(R)$ with a very idempotent $(f, f') \in R$ implies that

$(e,e')^2=(f,f')^2$. Thus, $a-e\in N(R)$. If there exists a very idempotent $g\in A$ such that $a-g\in N(A)$. Then $(a,0)-(g,0)\in N(R)$. This implies that $(g,0)^2=(e,e')^2$, and so $g^2=e^2$. Therefore A is very nil clean. Similarly, B is very nil clean, as asserted. □

Example 3.4. *$\mathbb{Z}_3$ is very nil clean, but $\mathbb{Z}_3\oplus\mathbb{Z}_3$ is not very nil clean.*

Proof. Clearly, $\mathbb{Z}_3$ is very nil clean. One easily checks that $(1,2)\in\mathbb{Z}_3\oplus\mathbb{Z}_3$ is not very nil clean, and we are through. □

Lemma 3.5. *Let R be a ring. Then every zero-divisor in R is a very idempotent if and only if R is isomorphic to one of the following:*

(1) *a domain,*

(2) $\mathbb{Z}_3\bigoplus\mathbb{Z}_3$*,*

(3) $\mathbb{Z}_3\bigoplus B$ *where B is a Boolean, or*

(4) *a Boolean ring.*

Proof. Suppose that every zero-divisor in R is a very idempotent. By Lemma 3.2, R is abelian.

Case I. R is indecomposable. Then $Id(R)=\{0,1\}$ and $-Id(R)=\{0,-1\}$. Thus, the only zero-divisor is zero. Hence, R is a domain.

Case II. R is decomposable. Then we have $S,T\neq 0$ such that $R=S\oplus T$. For any $t\in T$, $(0,t)\in R$ is a zero-divisor. By hypothesis, $(0,t)$ or $-(0,t)$ is an idempotent; hence that t or $-t$ is an idempotent in T. Therefore every element in T is a very idempotent. In light of Lemma 2.1, T is isomorphic to one of the following:

(a) $\mathbb{Z}_3$*,*

(b) *a Boolean ring, or*

(c) $\mathbb{Z}_3\bigoplus B$ *where B is a Boolean.*

Likewise, S is isomorphic to one of the preceding. Thus, R is isomorphic to one of the following: R is isomorphic to one of the following:

(i) $\mathbb{Z}_3\bigoplus\mathbb{Z}_3$*,*

(ii) *a Boolean ring, or*

(iii) $\mathbb{Z}_3\bigoplus B$ *where B is a Boolean.*

(iv) $\mathbb{Z}_3\bigoplus\mathbb{Z}_3\bigoplus B$ *where B is a Boolean.*

But in Case (iv), $(1,2,0)\in\mathbb{Z}_3\bigoplus\mathbb{Z}_3\bigoplus B$ is a zero-divisor, while it is not a very idempotent. Therefore Case (iv) will not appear, as desired.

Conversely, if R is a domain, then every zero-divisor is zero. If $R=\mathbb{Z}_3\bigoplus\mathbb{Z}_3$, then $NZ(R)=\{(1,1),(1,2),(2,1)\}, Id(R)=\{(0,0),(0,1),(1,0),(1,1)\}$ and $-Id(R)=\{(0,0),$ $(0,2),$ $(2,0),$ $(2,2)\}$. Therefore $R=NZ(R)\bigcup Id(R)\bigcup -Id(R)$. If $R=\mathbb{Z}_3\bigoplus B$ where B is a Boolean, then $Id(R)=\{(0,b),(1,b)\mid b\in B\}$ and $-Id(R)=\{(0,b),(2,b)\mid b\in B\}$. Therefore $R=Id(R)\bigcup -Id(R)$. If R is a Boolean ring, then every element in R is an idempotent. In any case, every element in R is a very idempotent, and we are done. □

We come now to the main result of the section.

Theorem 3.6. *Let R be a periodic ring. Then R is D-very nil clean if and only if*

(1) *R is abelian;*

(2) *$R/J(R)$ is isomorphic to one of the following:*

   (a) *a field F,*

   (b) $\mathbb{Z}_3 \bigoplus \mathbb{Z}_3$,

   (c) $\mathbb{Z}_3 \bigoplus B$ *where B is Boolean, or*

   (d) *a Boolean ring.*

Proof. Suppose that R is D-very nil clean. Then R is abelian. In view of [4, Theorem], $N(R)$ is an ideal of R. As R is periodic, $J(R)$ is nil; hence, $J(R) = N(R)$. As every idempotent lifts modulo $N(R)$, we see that $R/J(R)$ is abelian. Let $\overline{a} \in R/N(R)$ be a zero-divisor. If $a \in R$ is not a zero-divisor, then $a \in U(R)$, and so $\overline{a} \in U\big(R/N(R)\big)$, a contradiction. Thus, $a \in R$ is a zero-divisor. By hypothesis, a is the sum of a very idempotent and a nilpotent. Hence, $\overline{a}$ is a very idempotent. That is, every zero-divisor in $R/J(R)$ is a very idempotent. In light of Lemma 3.5, $R/J(R)$ is isomorphic to one of the following:

(i) *a domain F,*

(ii) $\mathbb{Z}_3 \bigoplus \mathbb{Z}_3$,

(iii) $\mathbb{Z}_3 \bigoplus B$ *where B is Boolean, or*

(iv) *a Boolean ring.*

If $R = F$ is a domain, then for any $a \in R$, $a = 0$ or $a^m = 1$ for some $m \in \mathbb{N}$. This shows that R is a field, as required.

Conversely, assume that (1) and (2) hold. In view of [4, Theorem], $N(R)$ forms an ideal of R. Let $a \in R$ be a zero divisor. Then $\overline{a} \in R/J(R)$ is a zero-divisor; otherwise, $\overline{a} \in R/J(R)$ is invertible, and so $a \in R$ is invertible, a contradiction. According to Lemma 3.5, $\overline{a}$ is a very idempotent in $R/J(R)$. As R is periodic, $J(R)$ is nilpotent, and so every idempotent modulo $J(R)$. This implies that $v := a - e \in N(R)$ for some very idempotent $e \in R$. Let $f \in R$ be a very idempotent such that $w := a - f \in N(R)$. Then $e^2 - f^2 = (a - v)^2 - (a - w)^2 = -av - va + v^2 + aw + wa - w^2 \in N(R)$. As $e, f \in R$ are very clean, we se that $e^2, f^2 \in R$ are idempotents. It is easy to verify that $(e^2 - f^2)\big(1 - (e^2 - f^2)^2\big) = 0$, and so $e^2 = f^2$. Therefore we complete the proof. □

Let R be a ring, and let $\sigma : R \to R$ be an endomorphism. $T(R, \sigma) = \{\begin{pmatrix} a & b \\ & a \end{pmatrix} \mid a, b \in R\}$. Here,

$$\begin{pmatrix} a & b \\ & a \end{pmatrix} + \begin{pmatrix} a' & b' \\ & a' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ & a + a' \end{pmatrix};$$
$$\begin{pmatrix} a & b \\ & a \end{pmatrix} \begin{pmatrix} a' & b' \\ & a' \end{pmatrix} = \begin{pmatrix} aa' & ab' + b\sigma(a') \\ & aa' \end{pmatrix}.$$

Corollary 3.7. *Let R be a periodic ring, and let $\sigma : R \to R$ be an endomorphism. Then $T(R, \sigma)$ is D-very nil clean if and only if*

(1) *R is D-very nil clean;*

(2) *$\sigma(e) = e$ for all idempotents $e \in R$.*

Proof. Obviously, $T(R,\sigma)$ is periodic. Suppose that $T(R,\sigma)$ is D-very nil clean. Then $T(R,\sigma)$ is abelian, by Theorem 3.6. Let $e \in R$ be an idempotent, and let $x \in R$. Then

$$\begin{pmatrix} e & \\ & e \end{pmatrix}\begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}\begin{pmatrix} e & \\ & e \end{pmatrix},$$

and so $e = \sigma(e)$. Let $a \in R$ be a zero divisor. Then $\begin{pmatrix} a & \\ & a \end{pmatrix} \in T(R,\sigma)$ is a zero-divisor. By hypothesis, there exists a very idempotent $\begin{pmatrix} e & g \\ & e \end{pmatrix} \in T(R,\sigma)$ such that

$$\begin{pmatrix} a & \\ & a \end{pmatrix} - \begin{pmatrix} e & g \\ & e \end{pmatrix} \in N\big(T(R,\sigma)\big).$$

It follows that we have a very idempotent $e \in R$ such that $a - e \in N(R)$. If $a - f \in N(R)$ with a very idempotent $f \in R$, then we have a very idempotent $\begin{pmatrix} f & 0 \\ & f \end{pmatrix} \in T(R,\sigma)$ such that

$$\begin{pmatrix} a & \\ & a \end{pmatrix} - \begin{pmatrix} f & 0 \\ & f \end{pmatrix} \in N\big(T(R,\sigma)\big).$$

Thus, we get $\begin{pmatrix} e & g \\ & e \end{pmatrix}^2 = \begin{pmatrix} f & 0 \\ & f \end{pmatrix}^2$. It follows that $e^2 = f^2$, and therefore R is D-very nil clean.

Conversely, assume that (1) and (2) hold. Then $T(R,\sigma)$ is abelian, as in the preceding discussion. By using Theorem 3.6, $R/J(R)$ is isomorphic to one of the following:

(a) *a field* F,

(b) $\mathbb{Z}_3 \bigoplus \mathbb{Z}_3$,

(c) $\mathbb{Z}_3 \bigoplus B$ *where* B *is Boolean, or*

(d) *a Boolean ring.*

But $T(R,\sigma)/J\big(T(R,\sigma)\big) \cong R/J(R)$, we see that $T(R,\sigma)/J\big(T(R,\sigma)\big)$ is isomorphic to one of the preceding. Consequently, $T(R,\sigma)$ is D-very nil clean, in terms of Theorem 3.6. □

4. SPECIAL CASES

The purpose of this section is to explore the structure of rings in which every zero-divisor is a very idempotent or a nilpotent element. These form a subset of all D-very nil clean rings.

Lemma 4.1. *Every ring in which every element is a very idempotent or a nilpotent element is abelian.*

Proof. Let $e \in R$ be an idempotent, and let $x \in R$. Then $1 - ex(1-e) \in U(R)$. If $\big(1 - ex(1-e)\big)^2 = 1 - ex(1-e)$, then $ex(1-e) = 0$, and so $ex = exe$. If $\big(1 - ex(1-e)\big)^2 = -\big(1 - ex(1-e)\big)$, then $ex(1-e) = 2$. and so $ex(1-e) = 2e(1-e) = 0$. Hence, $ex = exe$.

If $1 - ex(1-e) \in N(R)$, this will be a contradiction. Thus, $ex = exe$. Likewise, $xe = exe$. Therefore $ex = xe$, hence the result. $\square$

Lemma 4.2. *Let R be a ring. Then $R = N(R) \bigcup Id(R) \bigcup -Id(R)$ if and only if R is isomorphic to one of the following:*

(a) $\mathbb{Z}_3$, $\mathbb{Z}_4$,

(b) *a Boolean ring, or*

(c) $\mathbb{Z}_3 \bigoplus B$ *where B is a Boolean.*

Proof. Suppose that $R = N(R) \bigcup Id(R) \bigcup -Id(R)$. In view of Lemma 4.1, R is abelian.

Case I. R is indecomposable. Then $R = N(R) \bigcup \{0, 1, -1\}$. Let $a \in N(R)$. Then $1 - a \in U(R)$. If $(1-a)^2 = 1 - a$, then $a = 0$. If $(1-a)^2 = -(1-a)$, then $a = 2$. If $1 - a \in N(R)$, then we get a contradiction. Therefore $R = \{0, 1, -1, 2$, and so $R \cong \mathbb{Z}_2, \mathbb{Z}_3$ or $\mathbb{Z}_4$.

Case II. R is decomposable. Write $R = S \oplus T$. For any $t \in T$, $(1, t) \in R$ is not nilpotent. Then $(1, t) \in R$ is a very idempotent. This implies that $t \in T$ is a very idempotent. Hence, every element in T is a very idempotent. Likewise, every element in S is a very idempotent. In view of Lemma 2.1, S and T are isomorphic to one of the following:

(i) $\mathbb{Z}_3$,

(ii) *a Boolean ring, or*

(iii) $\mathbb{Z}_3 \bigoplus B$ *where B is a Boolean.*

One easily checks that $(1, 2) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $(1, 2, 0) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B$ where B is Boolean are neither a very idempotent nor a nilpotent. Therefore R is isomorphic to one of $(a) - (c)$. The converse is obvious. $\square$

Theorem 4.3. *Let R be a ring. Then R is an abelian ring in which every zero-divisor in R is a very idempotent or a nilpotent element if and only if R is isomorphic to one of the following:*

(a) *a D-ring,*

(b) *a Boolean ring,*

(c) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$,

(d) $\mathbb{Z}_3 \bigoplus B$ *where B is a Boolean.*

Proof. Suppose that R is an abelian ring in which every zero-divisor in R is a very idempotent or a nilpotent element.

Case I. R is indecomposable. Then every very idempotent is $0, 1$ or -1. Hence, every zero-divisor in R is nilpotent. Hence, R is a D-ring.

Case II. R is decomposable. Write $R = S \oplus T$. For any $t \in T$, $(0, t)$ is a very idempotent or a nilpotent element. We infer that every element in T is a very idempotent or a nilpotent element. Similarly, every element in S is a very idempotent or a nilpotent element. By virtue of Lemma 4.2, S and T are both isomorphic to one of the following:

(a) $\mathbb{Z}_3$, $\mathbb{Z}_4$,

(b) *a Boolean ring, or*

(c) $\mathbb{Z}_3 \bigoplus B$ *where* B *is a Boolean.*

But one easily checks that $Z(R) \neq Id(R) \bigcup -Id(R) \bigcup N(R)$ for any of those types

(1) $\mathbb{Z}_3 \oplus \mathbb{Z}_4$,

(2) $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus B$ *where* B *is a Boolean ring,*

(3) $\mathbb{Z}_4 \bigoplus \mathbb{Z}_4$,

(4) $\mathbb{Z}_4 \oplus B$ *where* B *is a Boolean ring, and*

(5) $\mathbb{Z}_3 \bigoplus \mathbb{Z}_4 \oplus B$ *where* B *is a Boolean ring.*

Therefore R is isomorphic to one of $(a) - (d)$.

Conversely, R is abelian, as every D-ring is connected. One easily checks that any of these four types of rings satisfy the desired condition. □

Corollary 4.4. *Let* R *be a ring. Then the following are equivalent:*

(1) *R is an abelian ring in which every zero-divisor in R is an idempotent or a nilpotent element.*

(2) *R is a D-ring or a Boolean ring.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 4.3, R is isomorphic to one of the following:

(a) *a D-ring,*

(b) *a Boolean ring,*

(c) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$,

(d) $\mathbb{Z}_3 \bigoplus B$ *where* B *is a Boolean.*

But in the case $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $(0, 2) \notin Id(R) \bigcup N(R)$. In the case $\mathbb{Z}_3 \bigoplus B$, $(2, 0) \notin Id(R) \bigcup N(R)$. Therefore proving (2).

(2) $\Rightarrow$ (1) This is obvious. □

The following example shows that the abelian condition in Corollary 4.4 is necessary.

Example 4.5. Every zero-divisor in $T_2\big(\mathbb{Z}_2\big)$ is an idempotent or a nilpotent element. But $T_2\big(\mathbb{Z}_2\big)$ is neither a Boolean ring nor a D-ring. In this case, R is not abelian.

Proposition 4.6. *Let R be a ring in which every element in R is a very idempotent or a nilpotent element. Then every zero-divisor in R which is not 2 is a very idempotent.*

Proof. Let $2 \neq a \in R$. Assume that $a \in R$ is nilpotent. Then $1-a \in U(R)$. If $(1-a)^2 = 1-a$, then $a = 0$. If $(1-a)^2 = -(1-a)$, then $a = 2$, a contradiction. If $1 - a \in N(R)$, then $1 - a = 0$, an absurd. Therefore $a \in R$ is a very idempotent, as asserted. □

5. UNIQUELY NIL CLEAN RINGS

A ring R is said to be uniquely nil clean provided that for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a-e \in R$ is nilpotent. Recently, this type of nil clean rings was studied in [9]. We shall give the connection of uniquely nil clean rings and very nil clean rings, and then obtain the conditions under which a group ring is uniquely nil clean. We add several new characterizations of such rings.

Theorem 5.1. *Let R be a ring. Then R is uniquely nil clean if and only if*

(1) *R is abelian.*

(2) *$R/J(R)$ is Boolean and $J(R)$ is nil.*

Proof. Assume that R is uniquely nil clean. As in the proof of Theorem 2.2, R is abelian, and so R is strongly nil clean. For any $a \in R$, it follows that $a-a^2 \in N(R)$. Write $(a-a^2)^m = 0$ for some $m \in \mathbb{N}$. Then $a^m = a^{m+1}b$ for a $b \in R$, and so R is strongly π-regular. If $x \in J(R)$, we have some $n \in \mathbb{N}$ such that $x^n = x^{n+1}y$ for a $y \in R$; hence, $x^n(1-xy) = 0$. Clearly, $1-xy \in U(R)$, and so $x^n = 0$. This implies that $J(R)$ is nil. As R is abelian, it follows by [4, Theorem] that $N(R)$ forms an ideal of R. Thus, $N(R) \subseteq J(R)$, and therefore $R/J(R)$ is Boolean.

Assume that (1) and (2) hold. For any $a \in R$, $a-a^2 \in J(R)$, and so we have an idempotent $e \in R$ such that $a-e \in J(R)$, as $J(R)$ is nil. Write $a = e+v$. Then $v \in J(R) \subseteq N(R)$. If there exists an idempotent $f \in R$ and a $w \in N(R)$ such that $a = f+w$, then $e-f = (a-v)-(a-w) = w-v$. Clearly, $wv = (a-f)(a-e) = (a-e)(a-f) = vw$, and so $e-f \in N(R)$. Since $(e-f)^3 = e-f$, we see that $e-f = 0$, and then $e = f$. Therefore R is uniquely nil clean. □

Corollary 5.2. *Let R be a ring. Then R is uniquely nil clean if and only if R is uniquely clean, and $J(R)$ is nil.*

Proof. This is obvious from Theorem 5.1. □

Corollary 5.3. *A ring R is uniquely nil clean if and only if*

(1) *R is π-regular;*

(2) *Every idempotent in R is central;*

(3) *$R/J(R)$ is Boolean.*

Proof. Suppose that R is uniquely nil clean. Then R is strongly nil clean, and so R is π-regular. Furthermore, proving (2) and (3) by Theorem 5.1.

Conversely, assume that $(1),(2)$ and (3) hold. Clearly, R is an exchange ring, and so every idempotent lifts modulo $J(R)$. Let $x \in J(R)$. Since R is π-regular, we have $n \in \mathbb{N}$ and $y \in R$ such that $x^n = x^nyx^n$; hence, $x^n(1-yx^n) = 0$. This implies that $x^n = 0$. That is, $J(R)$ is nil. Therefore we complete the proof, thanks to Theorem 5.1. □

We note that $\mathbb{Z}/4\mathbb{Z}$ is uniquely nil clean and $\mathbb{Z}/6\mathbb{Z}$ is not uniquely nil clean, though they are both π-regular rings with all idempotents central.

Theorem 5.4. *Let R be a ring. Then R is uniquely nil clean if and only if*

(1) *R is abelian.*

(2) *R is strongly nil clean.*

Proof. One direction is obvious by Theorem 5.1.

Conversely, assume that (1) and (2) hold. For any $a \in R$, there exists an idempotent $e \in R$ and a $w \in N(R)$ such that $a = e + w$. Write $a = f + v, f = f^2 \in R, v \in N(R)$. In light of Theorem 5.1, $N(R)$ forms an ideal of R, and so $e - f = (a - w) - (a - v) = v - w \in N(R)$. As R is abelian, $(e - f)^3 = e - f$; hence, $e = f$, as desired. □

Corollary 5.5. *Every homomorphic image of a uniquely nil clean ring is uniquely nil clean.*

Proof. Let R be uniquely nil clean, and let I be an ideal of R. In light of Theorem 5.4, R is abelian strongly nil clean. This shows that R/I is strongly nil clean. Clearly, R is strongly clean. We infer that every idempotent lifts modulo I, and then R/I is abelian. By Theorem 5.2 again, R/I is uniquely nil clean, as required. □

Corollary 5.6. *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely nil clean.*

(2) *$T = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ is uniquely nil clean.*

(3) *$R[x]/(x^n)$ is uniquely nil clean for all $n \geq 2$.*

Proof. This is obvious by Theorem 5.4. □

Theorem 5.7. *Let R be a ring. Then R is uniquely nil clean if and only if*

(1) *$2 \in R$ is nilpotent;*

(2) *R is very nil clean.*

Proof. Suppose that R is uniquely nil clean. In view of Theorem 5.1, $\overline{2}^2 = \overline{2}$ in $R/J(R)$, and so $2 \in J(R)$ is nilpotent. By Theorem 5.1 and Theorem 2.2, we observe that every uniquely nil clean ring is very nil clean.

Conversely, assume that (1) and (2) hold. As $2 \in \mathbb{Z}_3$ is not nilpotent. In view of Theorem 2.2, we see that R is abelian, $J(R)$ is nil, and that $R/J(R)$ is Boolean. The result follows by Theorem 5.1. □

Corollary 5.8. *Let R be a ring. Then R is uniquely nil clean if and only if*

(1) *R is abelian;*

(2) *$R/P(R)$ is uniquely nil clean.*

Proof. One direction is obvious, by Theorem 2.7 and Corollary 5.7.

Conversely, assume that (1) and (2) hold. By virtue of Theorem 5.7, $\overline{2} \in R/P(R)$ is nilpotent. We infer that $2 \in R$ is nilpotent. Furthermore, $R/P(R)$ is very nil clean. According to Theorem 2.7, R is very nil clean. By using Theorem 5.7 again, R is uniquely nil clean. □

Corollary 5.9. *Let R be a ring, and G be a group. Then RG is uniquely nil clean if and only if R is uniquely nil clean and $I(R,G)$ is nil.*

Proof. Suppose RG is uniquely nil clean. Then RG is very nil clean and $2 \in N(RG)$, by Theorem 5.7. Hence, R is very nil clean and $2 \in N(R)$. By using Theorem 5.7 again, R is uniquely nil clean. On the other hand, RG is uniquely clean, thanks to Corollary 5.2. Hence, $RG/J(RG)$ is Boolean. For any $g \in G$, we see that $(1 - g) - (1 - g)^2 \in J(RG)$; hence, $1 - g \in J(RG)$. This implies that $ker(\omega) \subseteq J(RG)$ is nil, as desired.

Conversely, assume that R is uniquely nil clean and $ker(\omega)$ is nil. In light of Theorem 5.7 and Theorem 2.9, $2 \in N(R)$ and RG is very nil clean. By using Theorem 5.7 again, RG is uniquely nil clean, as asserted. □

Example 5.10. Let G be a 3-group. Then $\mathbb{Z}_3G$ is is not uniquely nil clean, while it is very nil clean.

6. D-UNIQUELY NIL RINGS

A ring R is said to be D-uniquely nil clean provided that for any zero-divisor $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in R$ is nilpotent. The aim of this is to give the connection of D-uniquely nil clean rings and uniquely nil clean rings, and then characterize the structure of D-uniquely nil clean periodic rings.

Lemma 6.1. *Every D-uniquely nil clean ring is abelian.*

Proof. This is similar to that in Lemma 3.2. □

Theorem 6.2. *Let R be a ring. Then R is a D-uniquely nil clean ring if and only if R is a D-ring or R is uniquely nil clean.*

Proof. Suppose R is a D-uniquely nil clean ring. Then R is abelian by Lemma 6.1.

Case I. R is indecomposable. Let $a \in R$ be a zero-divisor. Then $a \in R$ is nilpotent or $a \in U(R)$. This shows that every zero-divisor is nilpotent, i.e., R is a D-ring.

Case II. R is decomposable. Write $R = A \oplus B$. For any $x \in A$, $(x, 0) \in R$ is a zero-divisor. Hence, we can find a unique idempotent $(e, f) \in R$ such that $(x, 0) - (e, f) \in N(R)$. Thus, $x - e \in N(R)$ for an idempotent $e \in R$. If there exists an idempotent $g \in R$ such that $x - g \in N(R)$. Then $(x, 0) - (g, f) \in N(R)$. By the uniqueness, we get $g = e$, and therefore A is uniquely nil clean. Similarly, B is uniquely nil clean, and then R is uniquely nil clean.

Conversely, if R is a D-ring, then R is a D-uniquely nil clean ring. So we assume that R is uniquely nil clean, and therefore R is a D-uniquely nil clean ring. □

Corollary 6.3. *Let R be D-uniquely nil clean. Then the ring* $T(R,R) = \{\begin{pmatrix} a & b \\ & a \end{pmatrix} \mid a, b \in R\}$ *is D-uniquely nil clean.*

Proof. By virtue of Theorem 6.2, R is a D-ring or R is uniquely nil clean.

Case I. R is a D-ring. Let $\begin{pmatrix} a & x \\ & a \end{pmatrix} \in Z\big(T(R,R)\big)$. Then $0 \neq \begin{pmatrix} b & y \\ & b \end{pmatrix}, \begin{pmatrix} c & z \\ & c \end{pmatrix}$ $\in T(R,R)$ such that

$$\begin{pmatrix} a & x \\ & a \end{pmatrix}\begin{pmatrix} b & y \\ & b \end{pmatrix} = \begin{pmatrix} c & z \\ & c \end{pmatrix}\begin{pmatrix} a & x \\ & a \end{pmatrix} = 0.$$

If $b, c \neq 0$, it follows from $ab = 0 = ca$ that $a \in R$ is a zero-divisor. If $b = c = 0$, then $y, z \neq 0$. It follows from $ay = 0 = za$ that $a \in R$ is a zero-divisor. If $b = 0, c \neq 0$ or $b \neq 0, c = 0$, then $ca = 0 = ay$ or $ab = 0 = za$. Then $a \in R$ is a zero-divisor. Hence, $a \in R$ is nilpotent. Therefore we conclude that $\begin{pmatrix} a & x \\ & a \end{pmatrix}$ is nilpotent, and so $T(R,R)$ is a D-ring.

Case II. R is uniquely nil clean. In light of Corollary 5.6, $T(R,R)$ is uniquely nil clean.

Therefore $T(R,R)$ is D-uniquely nil clean, in terms of Theorem 6.2. □

Proposition 6.4. *A ring R is D-uniquely nil clean if and only if for any zero-divisor $a \in R$ there exists a central idempotent $e \in R$ such that $a - e \in N(R)$.*

Proof. One direction is obvious from Lemma 6.1. Conversely, letting $e \in R$ be an idempotent, we have a central idempotent $f \in R$ such that $w := e - f \in N(R)$. Thus, $(e-f)^3 = e - f$, and so $(e-f)\big(1-(e-f)^2\big) = 0$. This implies that $e = f$, and then R is abelian. Let $a \in R$ be a zero-divisor. Then there exists a central $e \in R$ such that $a - e \in N(R)$. If there exists an idempotent $f \in R$ such that $a - f \in N(R)$, then $e - f = (a-f) - (a-e) \in N(R)$. It follows from $(e-f)^3 = e - f$ that $e = f$, which completes the proof. □

Theorem 6.5. *Let R be a periodic ring. Then R is D-uniquely nil clean if and only if*

(1) *R is abelian;*

(2) *R is local or $R/J(R)$ is Boolean.*

Proof. Suppose that R is D-uniquely nil clean. Then R is abelian by Lemma 6.1.

In view of [4, Theorem], $N(R)$ forms an ideal of R. Hence, $J(R) = N(R)$. Let $\overline{a} \in R/J(R)$ is a zero-divisor. Then $a \in R$ is a divisor; otherwise, $a \in U(R)$ as R is periodic, a contradiction. Hence, a is the sum of an idempotent and a nilpotent element. This shows that $\overline{a}$ is an idempotent. Therefore, every zero-divisor in $R/J(R)$ is an idempotent.

Set $S = R/J(R)$. Suppose that S has a nonzero zero-divisor. Then we have some $x, y \in R$ such that $xy = 0, x, y \neq 0$. Hence, $(yx)^2 = 0$. If $yx \neq 0$, then $yx \in R$ is a zero-divisor. So $yx \in R$ is an idempotent. Thus, $yx = (yx)^2 = 0$. This implies that $x \in R$ is a zero-divisor, and so $x = x^2$. It follows that $1 - x \in R$ is a zero-divisor; hence that $1 - x = (1-x)^2$. Therefore $x^2 = x$.

Let $a \in R$. Then $\big(xa(1-x)\big)^2 = 0$. Hence, $xa(1-x) = 0$; otherwise, $xa(1-x) \in R$ is an idempotent, and so $xa(1-x) = 0$, a contradiction. Thus, $xa(1-x) = 0$, hence, $xa = xax$. Likewise, $ax = xax$. Thus, $xa = ax$. If $xa = 0$, then $a \in R$ is a zero-divisor, and so it is an idempotent. If $xa \neq 0$, it follows from $xa(1-x) = 0$ that $xa \in R$ is a zero-divisor, and so $xa = (xa)^2$. Hence, $xa(1-a) = 0$. This implies that $1 - a \in R$ is a zero-divisor, and then $1 - a = (1-a)^2$. Thus, $a = a^2$. Therefore $a \in R$ is an idempotent. Consequently, $R/J(R)$ is Boolean or $R/J(R)$ is a domain. If $R/J(R)$ is a domain, the periodic property implies that R is a field. Thus, R is local or $R/J(R)$ is Boolean.

Conversely, assume that (1) and (2) hold. Let $a \in R$ be a zero-divisor. If R is local, then $a \in J(R)$. As R is local, we see that $J(R)$ is nil; hence, $a = 0 + a$ is the sum of an idempotent and a nilpotent. If $a = e + w$ with an idempotent $e \in R$ and a nilpotent $w \in R$, then $e = a - w$ with $aw = (e+w)w = w(e+w) = wa$, as R is abelian. This shows that $e \in R$ is nilpotent. Hence, $e = 0$. Thus, there exists a unique idempotent $e \in R$ such that $a - e \in N(R)$. If $R/J(R)$ is Boolean, we can find an idempotent $e \in R$ such that $a - e \in N(R)$, as $J(R)$ is nil. Since R is abelian, we see that such idempotent e is unique. Therefore R is D-uniquely nil clean. □

Corollary 6.6. *Let R be a periodic ring, and let $\sigma : R \to R$ be an endomorphism. Then $T(R,\sigma)$ is D-uniquely nil clean if and only if*

(1) *R is D-uniquely nil clean;*

(2) *$\sigma(e) = e$ for all idempotents $e \in R$.*

Proof. As R is a periodic ring, then so is $T(R,\sigma)$. Suppose that $T(R,\sigma)$ is D-uniquely nil clean. Then $T(R,\sigma)$ is abelian, by Lemma 6.1. Thus, $e=\sigma(e)$ for any idempotent $e\in R$, as in the proof of Corollary 3.7. Let $a\in R$ be a zero divisor. Then $\begin{pmatrix} a & \\ & a \end{pmatrix}\in T(R,\sigma)$ is a zero-divisor. By hypothesis, there exists a unique idempotent $\begin{pmatrix} e & f \\ & e \end{pmatrix}\in T(R,\sigma)$ such that

$$\begin{pmatrix} a & \\ & a \end{pmatrix}-\begin{pmatrix} e & f \\ & e \end{pmatrix}\in N\big(T(R,\sigma)\big).$$

It follows that we have a unique idempotent $e\in R$ such that $a-e\in N(R)$. Therefore, R is D-uniquely nil clean.

Conversely, assume that (1) and (2) hold. Let $\begin{pmatrix} e & f \\ & e \end{pmatrix}\in T(R,\sigma)$ be an idempotent. Then $e=e^2$ and $ef+f\sigma(e)=f$. As R is D-uniquely nil clean, we see that R is abelain. Hence, $(2e-1)f=0$. It follows from $(2e-1)^2=1$ that $f=0$. This implies that $T(R,\sigma)$ is abelian. In view of Theorem 6.5, $R/J(R)$ is a division ring or a Boolean ring. Since $T(R,\sigma)/J\big(T(R,\sigma)\big)\cong R/J(R)$, we see that $T(R,\sigma)$ is local, or $T(R,\sigma)/J\big(T(R,\sigma)\big)$ is Boolean. Therefore $T(R,\sigma)$ is D-uniquely nil clean, in terms of Theorem 6.5. □

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