Rings over which all modules are Gorenstein (resp., strongly Gorenstein) projective

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Abstract. One of the main results of this paper is the characterization of the rings over which all modules are strongly Gorenstein projective. We show that these kinds of rings are very particular cases of the well-known quasi-Frobenius rings. We give examples of rings over which all modules are Gorenstein projective but not necessarily strongly Gorenstein projective.

Key Words. (Strongly) Gorenstein projective, injective, and flat modules; quasi-Frobenius rings.

1 Introduction

Throughout this paper all rings are commutative with identity element and all modules are unital.
It is convenient to use "m-local" to refer to (not necessarily Noetherian) rings with a unique maximal ideal m.

It is by now a well-established fact that even if a ring R is non-Noetherian, there exist Gorenstein projective and injective R-modules. We assume that the reader is familiar with the Gorenstein dimensions theory. Some references are [3, 5, 6, 7].

Recall the definitions of the Gorenstein projective and injective modules:

Definitions 1.1 ([6])

1. *An R-module M is said to be Gorenstein projective (G-projective for short), if there exists an exact sequence of projective modules*

$$\mathbf{P} = \ \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ *and such that* $\mathrm{Hom}_R(-, Q)$ *leaves the sequence* $\mathbf{P}$ *exact whenever Q is a projective R-module.*
The exact sequence $\mathbf{P}$ *is called a complete projective resolution.*

2. *The Gorenstein injective (G-injective for short) modules are defined dually.*

Recently in [3], the authors studied a simple particular case of Gorenstein projective and injective modules, which are defined, respectively, as follows:

Definitions 1.2 ([3]) 1. *An R-module M is said to be strongly Gorenstein projective (SG-projective for short), if there exists an exact sequence of projective modules*

$$\mathbf{P} = \ \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Im}(f)$ *and such that* $\mathrm{Hom}_R(-, Q)$ *leaves the sequence* $\mathbf{P}$ *exact whenever Q is a projective module.*
The exact sequence $\mathbf{P}$ *is called a strongly complete projective resolution.*

2. *The strongly Gorenstein injective (SG-injective for short) modules are defined dually.*

The principal role of the strongly Gorenstein projective and injective modules is to give a simple characterization of Gorenstein projective and injective modules, respectively, as follows:

Theorem 1.3 ([3], Theorem 2.7) *A module is Gorenstein projective (resp., injective) if, and only if, it is a direct summand of a strongly Gorenstein projective (resp., injective) module.*

The important of this last result manifests in showing that the strongly Gorenstein projective and injective modules are simpler characterizations than their Gorenstein correspondent modules. For instance:

Proposition 1.4 ([3], Proposition 2.9) *A module M is strongly Gorenstein projective if, and only if, there exists a short exact sequence*

$$0 \to M \to P \to M \to 0,$$

where P is a projective module, and $\mathrm{Ext}(M, Q) = 0$ *for any projective module Q.*

The aim of this paper is to investigate the two following classes of rings:

1. The rings over which all modules are Gorenstein projective (resp., injective), which are called **G-semisimple** rings.
2. The rings over which all modules are strongly Gorenstein projective (resp., injective), which are called **SG-semisimple** rings.

In Section 2, we show that the G-semisimple ring is just the well-known quasi-Frobenius ring; i.e., the Noetherian and self-injective ring.
The SG-semisimple ring is then a particular case of the quasi-Frobenius ring. In Section 3, we characterize the SG-semisimple rings. Namely, we show that the local SG-semisimple ring is precisely the ring which has a unique ideal (see Theorem 3.5).

Before starting, we need to recall some useful results about quasi-Frobenius rings (for more details about these kinds of rings see for example [1, 8]).

The quotient ring R/I, where R is a principal ideal domain and I is any nonzero ideal of R, is a classical example of a quasi-Frobenius ring [9, Exercise 9.24]. The quasi-Frobenius ring has several characterizations. Here, we need the following:

Theorem 1.5 ([8], Theorems 1.50, 7.55, and 7.56) *For a ring R, the following are equivalent:*

1. *R is quasi-Frobenius;*
2. *R is Artinian and self-injective;*
3. *Every projective R-module is injective;*
4. *Every injective R-module is projective;*
5. *R is Noetherian and, for every ideal I, $Ann(Ann(I)) = I$, where $Ann(I)$ denotes the annihilator of I.*

The quasi-Frobenius rings are closely related with the perfect rings; i.e., the rings over which all flat modules are projective. These kinds of rings are introduced by Bass in [2]. They have the following characterization (needed later):

Theorem 1.6 ([2], Theorem P and Examples (6)) *For a ring R, the following are equivalent:*

1. *R is perfect;*

2. *A direct limit of projective R-modules is projective;*

3. *R is a finite direct product of local rings, each with T-nilpotent maximal ideal (i.e., if we pick a sequence $a_1, a_2, ...$ of elements in the maximal ideal, then for some index m, $a_1a_2...a_m = 0$).*

From Theorems 1.5 and 1.6 above and [8, Lemma 5.64], we may give the following structural characterization of quasi-Frobenius rings, which will be used later:

Proposition 1.7 *A ring R is quasi-Frobenius if, and only if, $R = R_1 \times ... \times R_n$ where each R_i is a local quasi-Frobenius rings.*

2 G-semisimple rings

In this section we investigate the rings over which all modules are Gorenstein projective (resp., injective).

First, observe the following:

Proposition 2.1 *Let R be a ring. The following are equivalent:*

1. *Every R-module is Gorenstein projective;*

2. *Every R-module is Gorenstein injective.*

Proof. It suffices to prove the implication (1) ⇒ (2), and the proof of the converse implication is analogous.
Assume that every module is Gorenstein projective. Then, any injective module is projective (since, as a Gorenstein projective module, it embeds in a projective module). This is equivalent, by Theorem 1.5, to say that every projective module is injective. Thus, easily we may show that a complete projective resolution is also a complete injective resolution. This completes the proof. ■

From Proposition 2.1 and the fact that a Gorenstein projective (resp., injective) module is projective (resp., injective) if, and only if, it has finite projective (resp., injective) dimension [7, Proposition 2.27], a semisimple ring is a particular case of a ring satisfying either of the equivalent statements of Proposition 2.1. Namely, the semisimple ring is precisely the ring of finite global dimension over which all modules are Gorenstein projective (resp., injective).
By this reason and according to the terminology of the classical ring theory, we set the following definition:

Definition 2.2 *A ring which satisfies either of the equivalent conditions of Proposition 2.1 will be called G-semisimple.*

Note that Proposition 2.1 is already existed over Noetherian rings (see for example [6, Theorem 12.3.1]), such that each of the statement (1) and (2) is equivalent to say that the ring is quasi-Frobenius. Here, from Proposition 2.1 and its proof, the G-semisimple ring is the same quasi-Frobenius. Thus we have a new characterization of quasi-Frobenius rings which is completed as follows:

Theorem 2.3 *For any ring R, the following are equivalent:*

1. *R is G-semisimple;*
2. *Every Gorenstein injective R-module is Gorenstein projective;*
3. *Every strongly Gorenstein injective R-module is strongly Gorenstein projective;*
4. *Every Gorenstein projective R-module is Gorenstein injective;*
5. *Every strongly Gorenstein injective R-module is strongly Gorenstein projective;*
6. *R is quasi-Frobenius.*

Proof. (1) $\Leftrightarrow$ (6). From Proposition 2.1 and its proof.
The proof of each other equivalences is similar to that of the implication (1) $\Rightarrow$ (2) of Proposition 2.1. ■

As mentioned after Proposition 2.1, we have the following relationship between the semisimple ring and the G-semisimple ring.

Proposition 2.4 *A G-semisimple ring is semisimple if, and only if, it has finite global dimension.*

This result is well-known by: a quasi-Frobenius ring is semisimple if, and only if, it has finite global dimension (see for example [9, Exercise 9.2]).
Finally, it is important to say that there exist several examples of a G-semisimple ring which is not semisimple. For instance, $\mathbb{Z}/4\mathbb{Z}$.

3 SG-semisimple rings

We investigate, in this section, the rings over which all modules are strongly Gorenstein projective (resp., injective).
First, we set the following result:

Proposition 3.1 *Let R be a ring. The following are equivalent:*

1. *Every R-module is strongly Gorenstein projective;*
2. *Every R-module is strongly Gorenstein injective.*

Proof. It suffices to prove the implication (1) $\Rightarrow$ (2), and the proof of the converse implication is analogous.
Assume that every module is strongly Gorenstein projective. Then, by Theorem 2.3, R is G-semisimple (i.e., quasi-Frobenius). Thus, easily we can show that a strongly complete projective resolution is also a strongly complete injective resolution. This completes the proof. ■

From the reason above and according to the terminology used in the previous section, we set the following definition:

Definition 3.2 *A ring which satisfies either of the equivalent statements of Proposition 3.1 above is called strongly G-semisimple (SG-semisimple for short).*

Naturally, an SG-semisimple ring is G-semisimple (i.e., quasi-Frobenius). Later, we give examples of SG-semisimple rings and other examples of G-semisimple rings which are not SG-semisimple (see Corollaries 3.9 and 3.10). Before that, we give a characterization of SG-semisimple rings. We begin by the following structure theorem.

Theorem 3.3 *An SG-semisimple ring is a finite direct product of local SG-semisimple rings.*

Proof. The result is a simple consequence of Proposition 1.7 and the following characterization of the (strongly) Gorenstein projective modules over a finite direct product of rings. ■

Lemma 3.4 *Let $R = R_1 \times ... \times R_n$ be a finite direct product of rings R_i. An R-module M is (strongly) Gorenstein projective if, and only if, $M = M_1 \oplus ... \oplus M_n$, where each M_i is a (strongly) Gorenstein projective R_i-module.*

Proof. This follows from the structure of (projective) modules and homomorphisms over a finite direct product of rings (see for example [4, Subsection 2.6]). ■

Theorem 3.3 leads us to restrict the study of the SG-semisimple rings to the local SG-semisimple rings.
The main result, in this section, is the following characterization of local SG-semisimple rings.

Theorem 3.5 *Let R be an m-local ring. The following are equivalent:*

1. *R is SG-semisimple;*
2. *R/m is a strongly Gorenstein projective R-module;*
3. *R has a unique ideal (which is necessarily m).*

To prove this theorem, we need the following results:

Lemma 3.6 *Let R be an m-local ring and let $x \neq 0$ be a zero-divisor element of R. If the ideal xR is strongly Gorenstein projective, then $Ann(Ann(xR)) = Ann(xR)$. Particularly, if $xR = m$, we get $Ann(m) = m$.*

Proof. Since R is a local ring, there exists, from [3, Proposition 2.12], a short exact sequence of R-modules:

$$(\star) \qquad 0 \to xR \to R^n \to xR \to 0,$$

where n is a positive integer.
Consider also the following canonical short exact sequence of R-modules:

$$0 \to Ann(xR) \to R \to xR \to 0.$$

From Schanuel's lemma [9, Theorem 3.62], we have

$$Ann(xR) \oplus R^n \cong R \oplus (xR).$$

Thus, since R is a local ring and $Ann(xR) \neq 0$, n must equal to 1, and so the sequence $(\star)$ becomes:

$$0 \to xR \to R \xrightarrow{f} xR \to 0.$$

Now, let $\alpha \in R$ with $f(1) = \alpha x$. Since f is surjective, there exists $\beta \in R$ such that $f(\beta) = x$. So, $x = \beta\alpha x$, and then $(1 - \beta\alpha)x = 0$, which means that $(1 - \beta\alpha) \in Ann(xR) \subseteq m$. Then, $\beta\alpha$ is invertible and so is α. This implies that:

$$\mathrm{Ker}\, f = \{y \in R \,|\, 0 = yf(1) = y\alpha x\} = Ann(xR).$$

Consequently, $xR \cong \mathrm{Ker}\, f = Ann(xR)$. Therefore, $Ann(Ann(xR)) = Ann(xR)$, as desired.
Now, suppose that $m = xR$. We have $m = xR \subseteq Ann(Ann(xR)) = Ann(xR) \subseteq m$. This implies the desired result. ■

Lemma 3.7 *Let R be an m-local ring and let I be a nonzero proper ideal of R. If R/I is a strongly Gorenstein projective R-module, then I is a cyclic strongly Gorenstein projective ideal, generated by a zero-divisor element of R.*

Proof. Since R is a local ring, there exists, from [3, Proposition 2.12], a short exact sequence of R-modules:

$$(*) \qquad 0 \to R/I \to R^n \to R/I \to 0,$$

where n is a positive integer.
Consider also the following short exact sequence of R-modules:

$$(**) \qquad 0 \to I \to R \to R/I \to 0.$$

From Schanuel's lemma [9, Theorem 3.62], we have

$$I \oplus R^n \cong R \oplus R/I.$$

Thus, since R is a local ring, $n = 1$ and $I = xR$ for some element x of R. Here, x must be a zero-divisor element of R (by the above isomorphism).
Now, to show that I is strongly Gorenstein projective, note at first that it is Gorenstein projective (by the sequence $(**)$ and from [7, Theorem 2.5]). Then, $\mathrm{Ext}(I, P) = 0$ for every projective R-module P (by [7, Proposition 2.3]). On the other hand, the two sequences $(*)$ and $(**)$ with Horseshoe's lemma [9, Lemma 6.20] give the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I & \to & Q & \to & I & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R & \to & R \oplus R & \to & R & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R/I & \to & R & \to & R/I & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that Q is a projective R-module. Therefore, by the top horizontal sequence and Proposition 1.4, I is a strongly Gorenstein projective ideal, as desired ■

Lemma 3.8 *If R is a local G-semisimple ring, then every R-module M is of the form: $M = R^{(I)} \oplus N$, where I is an index set and N an R-module with $Ann(x) \neq 0$ for every element $x \in N$.*

Proof. We may assume that M admits an element x such that $rx \neq 0$ for all $0 \neq r \in R$. Consider the set E of all free submodules of M. The set E is not empty, since xR is a free submodule of M. On the other hand, since R is a local G-semisimple ring and from Theorem 1.6, a direct limit of free R-modules is a free R-module. Then, for every subchain E_i of E, $\cup E_i$ is a free submodule of M. Then, by Zorn's lemma, E admits a maximal element F. We may set $F = R^{(I)}$ which is injective (since R is G-semisimple). Then, F is a direct summand of M and so $M = F \oplus N$ for some R-module N. If there exists $x \in N$ such that $rx \neq 0$ for all $r \in R$, then $xR \cong R$ is injective and then a direct summand of N. But, the free submodule $F \oplus xR$ contradicts the maximality of F. This completes the proof. ■

Proof of Theorem 3.5. The implication (1) ⇒ (2) is obvious.
We claim (2) ⇒ (3). From Lemma 3.7, $m = xR$ is a cyclic strongly Gorenstein projective ideal and x is zero-divisor. Then, by Lemma 3.6, $m^2 = 0$. Therefore, the standard argument shows that m is the unique ideal of R.
(3) ⇒ (1). We may assume that R is not a field. Clearly $m = xR$ (for some $0 \neq x \in R$) and $m^2 = 0$. Then, from Theorem 1.5, R is G-semisimple (i.e., quasi-Frobenius). Then, by the short exact sequence

$$0 \to Ann(m) = m \to R \to m \to 0,$$

m is a strongly Gorenstein projective R-module.
Now, consider any R-module M. By Lemma 3.8, there exists an index set I such that $M \cong R^{(I)} \oplus N$, where N is an R-module with $Ann(y) \neq 0$ for every nonzero element $y \in N$. Then, necessarily $xN = 0$, and so $N \cong (R/m)^{(J)}$ for some index set J. Since $R/m \cong m$ is a strongly Gorenstein projective R-module and, by [3, Proposition 2.2], N is a strongly Gorenstein projective R-module. Therefore, M is a strongly Gorenstein projective R-module. This completes the proof. ■

We end by some examples of G-semisimple and SG-semisimple rings.
The following gives examples of local SG-semisimple rings.

Corollary 3.9 *For every principal ideal domain R and every prime ideal p of R, the ring R/p^2 is a local SG-semisimple ring.*

Proof. Trivial. ■

As an example of G-semisimple rings which are not SG-semisimple, we set the following result.

Corollary 3.10 *For every principal ideal domain R and every nonzero ideal I of R, which is neither prime nor a square of a prime ideal, the ring R/I is a local G-semisimple, but it is not SG-semisimple.*

Proof. Trivial. ■

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