SMOOTH ERGODIC THEORY AND NONUNIFORMLY HYPERBOLIC DYNAMICS

LUIS BARREIRA AND YAKOV PESIN

Contents

Introduction

The goal of this chapter is to describe the contemporary status of nonuniform hyperbolicity theory. We present the core notions and results of the theory as well as discuss recent developments and some open problems. We also describe essentially all known examples of nonuniformly hyperbolic systems. Following the principles of the Handbook we include informal discussions of many results and sometimes outline their proofs.

Originated in the works of Lyapunov [170] and Perron [193, 194] the nonuniform hyperbolicity theory has emerged as an independent discipline in the works of Oseledets [191] and Pesin [197]. Since then it has become one of the major parts of the general dynamical systems theory and one of the main tools in studying highly sophisticated behavior associated with "deterministic chaos". We refer the reader to the article [105] by Hasselblatt and Katok in Volume 1A of the Handbook for a discussion on the role of nonuniform hyperbolicity theory, its relations to and interactions with other areas of dynamics. See also the article [104] by Hasselblatt in the same volume for a brief account of nonuniform hyperbolicity theory in view of the general hyperbolicity theory, and the book by Barreira and Pesin [24] for a detailed presentation of the core of the nonuniform hyperbolicity theory.

Nonuniform hyperbolicity conditions can be expressed in terms of the Lyapunov exponents. Namely, a dynamical system is nonuniformly hyperbolic if it admits an invariant measure with nonzero Lyapunov exponents almost everywhere. This provides an efficient tool in verifying the nonuniform hyperbolicity conditions and determines the importance of the nonuniform hyperbolicity theory in applications.

We emphasize that the nonuniform hyperbolicity conditions are *weak enough* not to interfere with the topology of the phase space so that any compact smooth manifold of dimension ≥ 2 admits a volume-preserving C^∞ diffeomorphism which is nonuniformly hyperbolic. On the other hand, these conditions are *strong enough* to ensure that any $C^{1+\alpha}$ nonuniformly hyperbolic diffeomorphism has positive entropy with respect to any invariant *physical* measure (by physical measure we mean either a smooth measure or a Sinai–Ruelle–Bowen (SRB) measure). In addition, any ergodic component has positive measure and up to a cyclic permutation the restriction of the map to this component is Bernoulli. Similar results hold for systems with continuous time.

It is conjectured that dynamical systems of class $C^{1+\alpha}$ with nonzero Lyapunov exponents preserving a given smooth measure are typical in some sense. This remains one of the major open problems in the field and its affirmative solution would greatly benefit and boost the applications of the nonuniform hyperbolicity theory. We stress that the systems under consideration should be of class $C^{1+\alpha}$ for some $\alpha > 0$: not only the nonuniform hyperbolicity theory for C^1 systems is substantially less interesting but one should also expect a "typical" C^1 map to have some zero Lyapunov exponents (unless the map is Anosov).

In this chapter we give a detailed account of the topics mentioned above as well as many others. Among them are: 1) stable manifold theory (including the construction of local and global stable and unstable manifolds and their absolute continuity); 2) local ergodicity problem (i.e., finding conditions which guarantee that every ergodic component of positive measure is open (mod 0)); 3) description of the topological properties of systems with nonzero Lyapunov exponents (including the density of periodic orbits, the closing and shadowing properties, and the approximation by horseshoes); and 4) computation of the dimension and the entropy of arbitrary hyperbolic measures. We also describe some methods which allow one to establish that a given system has nonzero Lyapunov exponents (for example, the cone techniques) or to construct a hyperbolic measure with "good" ergodic properties (for example, the Markov extension approach). Finally, we outline a version of nonuniform hyperbolicity theory for systems with singularities (including billiards).

The nonuniform hyperbolicity theory covers an enormous area of dynamics and despite the scope of this survey there are several topics not covered or barely mentioned. Among them are nonuniformly hyperbolic one-dimensional transformations, random dynamical systems with nonzero Lyapunov exponents, billiards and related systems (for example systems of hard balls), and numerical computation of Lyapunov exponents. For more information on these topics we refer the reader to the articles in the Handbook [90, 93, 124, 143, 147]. Here the reader finds the ergodic theory of random transformations [143, 93] (including a version of Pesin's entropy formula in [143]), nonuniform one-dimensional dynamics [168, 124], ergodic properties and decay of correlations for nonuniformly expanding maps [168], the dynamics of geodesic flows on compact manifolds of nonpositive curvature [147], homoclinic bifurcations and dominated splitting [209] and dynamics of partially hyperbolic systems with nonzero Lyapunov exponents [106]. Last but no least, we would like to mention the article [90] on the Teichmüller geodesic flows showing in particular, that the Kontsevich–Zorich cocycle over the Teichmüller flow is nonuniformly hyperbolic [89]. Although we included comments of historical nature concerning some main notions and basic results, the chapter is not meant to present a complete historical account of the field.

Acknowledgments. First of all we would like to thank O. Sarig who wrote the appendix containing some background information on decay of correlations. We are in debt to D. Dolgopyat, A. Katok, and the referees for valuable comments. L. Barreira was partially supported by the Center for Mathematical Analysis, Geometry, and Dynamical Systems, through FCT by Program POCTI/FEDER. Ya. Pesin was partially supported by the National Science Foundation. Some parts of the chapter were written when Ya. Pesin visited the Research Institute for Mathematical Science (RIMS) at Kyoto University. Ya. Pesin wishes to thank RIMS for hospitality.

1. Lyapunov exponents of dynamical systems

Let $f^t\colon M \to M$ be a dynamical system with discrete time, $t \in \mathbb{Z}$, or continuous time, $t \in \mathbb{R}$, of a smooth Riemannian manifold M. Given a point $x \in M$, consider the family of linear maps $\{d_x f^t\}$ which is called *the system in variations* along the trajectory $f^t(x)$. It turns out that for a "typical" trajectory one can obtain a sufficiently complete information on stability of the trajectory based on the information on the asymptotic stability of the "zero solution" of the system in variations.

In order to characterize the asymptotic stability of the "zero solution", given a vector $v \in T_xM$, define the *Lyapunov exponent* of v at x by

$$\chi^+(x,v) = \varlimsup_{t\to+\infty} \frac{1}{t} \log \|d_x f^t v\|.$$

For every $\varepsilon > 0$ there exists $C = C(v,\varepsilon) > 0$ such that if $t \geq 0$ then

$$\|d_x f^t v\| \leq C e^{(\chi^+(x,v)+\varepsilon)t}\|v\|.$$

The Lyapunov exponent possesses the following basic properties:

1. $\chi^+(x,\alpha v) = \chi^+(x,v)$ for each $v \in V$ and $\alpha \in \mathbb{R}\setminus\{0\}$;
2. $\chi^+(x,v+w) \leq \max\{\chi^+(x,v),\chi^+(x,w)\}$ for each v, $w \in V$;
3. $\chi^+(x,0) = -\infty$.

The study of the Lyapunov exponents can be carried out to a certain extent using only these three basic properties. This is the subject of the abstract theory of Lyapunov exponents (see [24]). As a simple consequence of the basic properties we obtain that the function $\chi^+(x,\cdot)$ attains only finitely many values on $T_xM \setminus \{0\}$. Let $p^+(x)$ be the number of distinct values and

$$\chi_1^+(x) < \cdots < \chi_{p^+(x)}^+(x),$$

the values themselves. The Lyapunov exponent $\chi^+(x,\cdot)$ generates the filtration $\mathcal{V}_x^+$ of the tangent space T_xM

$$\{0\} = V_0^+(x) \subsetneqq V_1^+(x) \subsetneqq \cdots \subsetneqq V_{p^+(x)}^+(x) = T_xM,$$

where $V_i^+(x) = \{v \in T_xM : \chi^+(x,v) \le \chi_i^+(x)\}$. The number

$$k_i^+(x) = \dim V_i^+(x) - \dim V_{i-1}^+(x)$$

is the *multiplicity* of the value $\chi_i^+(x)$. We have

$$\sum_{i=1}^{p^+(x)} k_i^+(x) = \dim M.$$

The collection of pairs

$$\operatorname{Sp}\chi^+(x) = \{(\chi_i^+(x), k_i^+(x)) : 1 \le i \le p^+(x)\}$$

is called the *Lyapunov spectrum* of the exponent $\chi^+(x,\cdot)$.

The functions $\chi_i^+(x)$, $p^+(x)$, and $k_i^+(x)$ are *invariant* under f and (Borel) *measurable* (but not necessarily continuous).

One can obtain another Lyapunov exponent for f by reversing the time. Namely, for every $x \in M$ and $v \in T_xM$ let

$$\chi^-(x,v) = \overline{\lim_{t\to-\infty}} \frac{1}{|t|} \log\|d_xf^tv\|.$$

The function $\chi^-(x,\cdot)$ possesses the same basic properties as $\chi^+(x,\cdot)$ and hence, takes on finitely many values on $T_xM \setminus \{0\}$:

$$\chi_1^-(x) > \cdots > \chi_{p^-(x)}^-(x),$$

where $p^-(x) \le \dim M$. Denote by $\mathcal{V}_x^-$ the filtration of T_xM associated with $\chi^-(x,\cdot)$:

$$T_xM = V_1^-(x) \supsetneqq \cdots \supsetneqq V_{p^-(x)}^-(x) \supsetneqq V_{p^-(x)+1}^-(x) = \{0\},$$

where $V_i^-(x) = \{v \in T_xM : \chi^-(x,v) \le \chi_i^-(x)\}$. The number

$$k_i^-(x) = \dim V_i^-(x) - \dim V_{i+1}^-(x)$$

is the *multiplicity* of the value $\chi_i^-(x)$. The collection of pairs

$$\operatorname{Sp}\chi^-(x) = \{(\chi_i^-(x), k_i^-(x)) \colon i = 1, \dots, p^-(x)\}$$

is called the *Lyapunov spectrum* of the exponent $\chi^-(x,\cdot)$.

We now introduce the crucial concept of Lyapunov regularity. Roughly speaking it asserts that the forward and backward behavior of the system along a "typical" trajectory comply in a quite strong way.

A point x is called *Lyapunov forward regular point* if

$$\lim_{t\to+\infty} \frac{1}{t} \log|\det d_xf^t| = \sum_{i=1}^{p^+(x)} k_i^+(x)\chi_i^+(x).$$

Similarly, a point x is called *Lyapunov backward regular* if

$$\lim_{t\to-\infty} \frac{1}{|t|} \log|\det d_xf^t| = \sum_{i=1}^{p^-(x)} k_i^-(x)\chi_i^-(x).$$

If a point x is forward (backward) regular then so is any point along its trajectory and we can say that the whole trajectory is forward (backward) regular. Note that there may be trajectories which are neither forward nor backward regular and that forward (backward) regularity does not necessarily imply backward (forward) regularity (an example is a flow in $\mathbb{R}^3$ that progressively approaches zero and infinity when time goes to $+\infty$, oscillating between the two, but which tends to a given point when time goes to $-\infty$).

Given $x \in M$, we say that the filtrations $\mathcal{V}_x^+$ and $\mathcal{V}_x^-$ comply if:

1. $p^+(x) = p^-(x) \stackrel{\text{def}}{=} p(x)$;
2. the subspaces $E_i(x) = V_i^+(x) \cap V_i^-(x)$, $i = 1, \ldots, p(x)$ form a splitting of the tangent space
$$T_xM = \bigoplus_{i=1}^{p(x)} E_i(x).$$

We say that a point x is *Lyapunov regular* or simply *regular* if:

1. the filtrations $\mathcal{V}_x^+$ and $\mathcal{V}_x^-$ comply;
2. for $i = 1, \ldots, p(x)$ and $v \in E_i(x) \setminus \{0\}$ we have
$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|d_x f^t v\| = \chi_i^+(x) = -\chi_i^-(x) \stackrel{\text{def}}{=} \chi_i(x)$$
with uniform convergence on $\{v \in E_i(x) : \|v\| = 1\}$;
3. $$\lim_{t\to\pm\infty} \frac{1}{t} \log|\det d_x f^t| = \sum_{i=1}^{p(x)} \chi_i(x) \dim E_i(x).$$

Note that if x is regular then so is the point $f^t(x)$ for any t and thus, one can speak of the whole trajectory as being regular.

In order to simplify our notations in what follows, we will drop the superscript $+$ from the notation of the Lyapunov exponents and the associated quantities if it does not cause any confusion.

The following criterion for regularity is quite useful in applications. Denote by $V(v_1, \ldots, v_k)$ the k-volume of the parallelepiped defined by the vectors $v_1, \ldots, v_k$.

Theorem 1.1 (see [58])**.** *If x is Lyapunov regular then the following statements hold:*

1. *for any vectors $v_1, \ldots, v_k \in T_xM$ there exists the limit*
$$\lim_{t\to\pm\infty} \frac{1}{t} \log V(d_x f^t v_1, \ldots, d_x f^t v_k);$$
if, in addition, $v_1, \ldots, v_k \in E_i(x)$ and $V(v_1, \ldots, v_k) \neq 0$ then
$$\lim_{t\to\pm\infty} \frac{1}{t} \log V(d_x f^t v_1, \ldots, d_x f^t v_k) = \chi_i(x)k;$$
2. *if $v \in E_i(x) \setminus \{0\}$ and $w \in E_j(x) \setminus \{0\}$ with $i \neq j$ then*
$$\lim_{t\to\pm\infty} \frac{1}{t} \log|\sin \angle(d_x f^t v, d_x f^t w)| = 0.$$

Furthermore, if these properties hold then x is Lyapunov regular.

Forward and backward regularity of a trajectory does not automatically yields that the filtrations comply and hence, forward and backward regularity do not, in general, imply Lyapunov regularity. Roughly speaking the forward behavior of a trajectory may not depend on its backward behavior while Lyapunov regularity requires some compatibility between the forward and backward behavior expressed in terms of the filtrations $\mathcal{V}_{\chi^+}$ and $\mathcal{V}_{\chi^-}$. However, if a trajectory $\{f^n(x)\}$ returns infinitely often to an arbitrary small neighborhood of x as $n \to \pm\infty$ one may expect the forward and backward behavior to comply in a certain sense.

The following celebrated result of Oseledets [191] gives a rigorous mathematical description of this phenomenon and shows that regularity is "typical" from the measure-theoretical point of view.

Theorem 1.2 (Multiplicative Ergodic Theorem)**.** *If f is a diffeomorphism of a smooth Riemannian manifold M, then the set of Lyapunov regular points has full measure with respect to any f-invariant Borel probability measure on M.*

This theorem is a particular case of a more general statement (see Section 5.3).

The notion of Lyapunov exponent was introduced by Lyapunov [170], with background and motivation coming from his study of differential equations. A comprehensive but somewhat outdated reference for the theory of Lyapunov exponents as well as its applications to the theory of differential equations is the book of Bylov, Vinograd, Grobman, and Nemyckii [58] which is available only in Russian. A part of this theory is presented in modern language in [24].

The notion of forward regularity originated in the work of Lyapunov [170] and Perron [193, 194] in connection with the study of the stability properties of solutions of linear ordinary differential equations with nonconstant coefficients (see [24] for a detailed discussion).

2. Examples of systems with nonzero exponents

2.1. **Hyperbolic invariant measures.** Smooth ergodic theory studies topological and ergodic properties of smooth dynamical systems with nonzero Lyapunov exponents. Let f be a diffeomorphism of a complete (not necessarily compact) smooth Riemannian manifold M. The map f should be of class at least $C^{1+\alpha}$, $\alpha > 0$. We assume that there exists an f-invariant set Λ with the property that for every $x \in \Lambda$ the values of the Lyapunov exponent at x are nonzero. More precisely, there exists a number $s = s(x)$, $1 \le s < p(x)$ such that

$$\chi_1(x) < \cdots < \chi_s(x) < 0 < \chi_{s+1}(x) < \cdots < \chi_{p(x)}(x). \tag{2.1}$$

We say that f has nonzero exponents on Λ. Let us stress that according to our definition there should always be at least one negative value and at least one positive value of the exponent.

Assume now that f preserves a Borel probability measure ν on M. We call ν *hyperbolic* if (2.1) holds for almost every $x \in M$. It is not known whether *a diffeomorphism f which has nonzero exponents on a set Λ possesses a hyperbolic measure ν with $\nu(\Lambda) = 1$.*

In the case ν is ergodic the values of the Lyapunov exponent are constant almost everywhere, i.e., $k_i(x) = k_i^\nu$ and $\chi_i(x) = \chi_i^\nu$ for $i = 1, \dots, p(x) = p^\nu$. The collection of pairs

$$\operatorname{Sp}\chi^\nu = \{(\chi_i^\nu, k_i^\nu) \colon i = 1, \dots, p^\nu\}$$

is called the *Lyapunov spectrum of the measure.* The measure ν is hyperbolic if none of the numbers χ_i^ν in its spectrum is zero.

We now discuss the case of dynamical systems with continuous time. Let f^t be a smooth flow on a smooth Riemannian manifold M. It is generated by a vector field X on M such that $X(x) = \frac{df^t(x)}{dt}|_{t=0}$. Clearly, $\chi(x, v) = 0$ for every v in the direction of $X(x)$, i.e., for $v = \alpha X(x)$ with some $\alpha \in \mathbb{R}$.

We say that *the flow f^t has nonzero exponents on an invariant set* Λ if for every $x \in \Lambda$ all the Lyapunov exponents, but the one in the direction of the flow, are nonzero, at least one of them is negative and at least one of them is positive. More precisely, there exists a number $s = s(x)$, $1 \le s < p(x) - 1$ such that

$$\chi_1(x) < \cdots < \chi_s(x) < \chi_{s+1}(x) = 0 < \chi_{s+2}(x) < \cdots < \chi_{p(x)}(x), \tag{2.2}$$

where $\chi_{s+1}(x)$ is the value of the exponent in the direction of $X(x)$.

Assume now that a flow f^t preserves a Borel probability measure ν on M. We call ν *hyperbolic* if (2.2) holds for almost every $x \in M$.

There are two classes of hyperbolic invariant measures on compact manifolds for which one can obtain a sufficiently complete description of its ergodic properties. They are:

1. *smooth measures*, i.e., measures which are equivalent to the Riemannian volume with the Radon–Nikodim derivative bounded from above and bounded away from zero (see Section 11);
2. *Sinai–Ruelle–Bowen measures* (see Section 14).

Dolgopyat and Pesin [78] proved that any compact manifold of dimension ≥ 2 admits a volume-preserving diffeomorphism with nonzero Lyapunov exponents, and Hu, Pesin and Talitskaya [121] showed that any compact manifold of dimension ≥ 3 admits a volume-preserving flow with nonzero Lyapunov exponents; see Section 13.1 for precise statements and further discussion. However, there are few particular examples of volume-preserving systems with nonzero Lyapunov exponents. In the following subsections we present some basic examples of such systems to illustrate some interesting phenomena associated with nonuniform hyperbolicity.

2.2. **Diffeomorphisms with nonzero exponents on the 2-torus.** The first example of a diffeomorphism with nonzero Lyapunov exponent, which is not an Anosov map, was constructed by Katok [130]. This is an area-preserving ergodic (indeed, Bernoulli) diffeomorphism $G_{\mathbb{T}^2}$ of the two-dimensional torus $\mathbb{T}^2$ which is obtained by a "surgery" of an area-preserving hyperbolic toral automorphism A with two eigenvalues $\lambda > 1$ and $\lambda^{-1} < 1$. The main idea of Katok's construction is to destroy the uniform hyperbolic structure associated with A by slowing down trajectories in a small neighborhood U of the origin (which is a fixed hyperbolic point for A). This means that the time, a trajectory of a "perturbed" map $G_{\mathbb{T}^2}$ stays in U, gets larger and larger the closer the trajectory passes by the origin, while the map is unchanged outside U. In particular, it can be arranged that the trajectories, starting on the stable and unstable separatrices of the origin, have zero exponents and thus, $G_{\mathbb{T}^2}$ is a not an Anosov map. Although a "typical" trajectory may spend arbitrarily long periods of time in U, the average time it stays in U is proportional to the measure of U and hence, is small. This alone does not automatically guarantee that a "typical" trajectory has nonzero exponents. Indeed, one should make sure that between the time the trajectory enters and exits U a vector in small cone around the unstable direction of A does not turn into a vector in a small cone around the stable direction of A. If this occurs the vector may contract, while travelling outside U, so one may loose control over its length.

The construction depends upon a real-valued function ψ which is defined on the unit interval $[0, 1]$ and has the following properties:

1. ψ is a C^∞ function except at the origin;
2. $\psi(0)=0$ and $\psi(u)=1$ for $u \geq r_0$ where $0<r_0<1$;
3. $\psi'(u)>0$ for every $0<u<r_0$;
4. the following integral converges:

$$\int_0^1 \frac{du}{\psi(u)} < \infty.$$

Consider the disk D_r centered at 0 of radius r and a coordinate system (s_1, s_2) in D_r formed by the eigendirections of A such that

$$D_r = \{(s_1, s_2) : {s_1}^2 + {s_2}^2 \leq r^2\}.$$

Observe that A is the time-one map of the flow generated by the following system of differential equations:

$$\dot{s}_1 = s_1 \log \lambda, \quad \dot{s}_2 = -s_2 \log \lambda.$$

Fix a sufficiently small number $r_1 > r_0$ and consider the time-one map g generated by the following system of differential equations in D_{r_1}:

$$\dot{s}_1 = s_1 \psi({s_1}^2 + {s_2}^2) \log \lambda, \quad \dot{s}_2 = -s_2 \psi({s_1}^2 + {s_2}^2) \log \lambda. \tag{2.3}$$

Our choice of the function ψ guarantees that $g(D_{r_2}) \subset D_{r_1}$ for some $r_2 < r_1$, and that g is of class C^∞ in $D_{r_1} \setminus \{0\}$ and coincides with A in some neighborhood of the boundary ∂D_{r_1}. Therefore, the map

$$G(x) = \begin{cases} A(x) & \text{if } x \in \mathbb{T}^2 \setminus D_{r_1} \\ g(x) & \text{if } x \in D_{r_1} \end{cases}$$

defines a homeomorphism of the torus $\mathbb{T}^2$ which is a C^∞ diffeomorphism everywhere except at the origin. The map $G(x)$ is a slowdown of the automorphism A at 0.

Denote by W^u and W^s the projections of the eigenlines in $\mathbb{R}^2$ to $\mathbb{T}^2$ corresponding to the eigenvalues λ and λ^{-1}. Set $W = W^u \cup W^s$ and $X = \mathbb{T}^2 \setminus W$. Note that the set W is everywhere dense in $\mathbb{T}^2$.

Let $x = (0, s_2) \in D_{r_1} \cap W^s$. For a vertical vector $v \in T_x\mathbb{T}^2$,

$$\begin{aligned} \chi(x, v) &= \overline{\lim_{t\to+\infty}} \frac{\log|s_2(t)|}{t} \\ &= \overline{\lim_{t\to+\infty}} (\log|s_2(t)|)' = \overline{\lim_{t\to+\infty}} (-\psi(s_2(t)^2) \log \lambda), \end{aligned}$$

where $s_2(t)$ is the solution of (2.3) with the initial condition $s_2(0) = s_2$. In view of the choice of the function ψ, we obtain that $\chi(x, v) = 0$. Similarly, $\chi(x, v) = 0$ whenever x, $v \in W^u$. In particular, G is not an Anosov diffeomorphism.

Choose $x \in X \setminus D_{r_1}$ and define the stable and unstable cones in $T_x\mathbb{T}^2 = \mathbb{R}^2$ by

$$\begin{aligned} C^s(x) &= \{(v_1, v_2) \in \mathbb{R}^2 : |v_1| \leq \alpha|v_2|\}, \\ C^u(x) &= \{(v_1, v_2) \in \mathbb{R}^2 : |v_2| \leq \alpha|v_1|\}, \end{aligned}$$

where $v_1 \in W^u$, $v_2 \in W^s$ and $0 < \alpha < 1/4$. The formulae

$$E^s(x) = \bigcap_{j=0}^{\infty} dG^{-j} C^s(G^j(x)), \quad E^u(x) = \bigcap_{j=0}^{\infty} dG^j C^u(G^{-j}(x))$$

define one-dimensional subspaces at x such that $\chi(x,v) < 0$ for $v \in E^s(x)$ and $\chi(x,v) > 0$ for $v \in E^u(x)$. The map G is uniformly hyperbolic on $X \setminus D_{r_1}$: there is a number $\mu > 1$ such that for every $x \in X \setminus D_{r_1}$,

$$\|dG|E^s(x)\| \le \frac{1}{\mu}, \quad \|dG^{-1}|E^u(x)\| \le \frac{1}{\mu}.$$

One can show that the stable and unstable subspaces can be extended to $W \setminus \{0\}$ to form two one-dimensional continuous distributions on $\mathbb{T}^2 \setminus \{0\}$.

The map G preserves the probability measure $d\nu = \kappa_0^{-1}\kappa\, dm$ where m is area and the density κ is a positive C^∞ function that is infinite at 0. It is defined by the formula

$$\kappa(s_1,s_2) = \begin{cases} (\psi({s_1}^2 + {s_2}^2))^{-1} & \text{if } (s_1,s_2) \in D_{r_1} \\ 1 & \text{otherwise} \end{cases},$$

and

$$\kappa_0 = \int_{\mathbb{T}^2} \kappa\, dm.$$

Consider the map φ of the torus given by

$$\varphi(s_1,s_2) = \frac{1}{\sqrt{\kappa_0({s_1}^2 + {s_2}^2)}} \left(\int_0^{{s_1}^2+{s_2}^2} \frac{du}{\psi(u)} \right)^{1/2} (s_1,s_2) \tag{2.4}$$

in D_{r_1} and $\varphi = \mathrm{Id}$ in $\mathbb{T}^2 \setminus D_{r_1}$. It is a homeomorphism and is a C^∞ diffeomorphism except at the origin. It also commutes with the involution $I(t_1,t_2) = (1-t_1, 1-t_2)$. The map $G_{\mathbb{T}^2} = \varphi \circ G \circ \varphi^{-1}$ is of class C^∞, area-preserving and has nonzero Lyapunov exponents almost everywhere. One can show that $G_{\mathbb{T}^2}$ is ergodic and is a Bernoulli diffeomorphism.

2.3. **Diffeomorphisms with nonzero exponents on the 2-sphere.** Using the diffeomorphism $G_{\mathbb{T}^2}$ Katok [130] constructed a diffeomorphisms with nonzero exponents on the 2-sphere S^2. Consider a toral automorphism A of the torus $\mathbb{T}^2$ given by the matrix $A = \left(\begin{smallmatrix} 5 & 8 \\ 8 & 13 \end{smallmatrix}\right)$. It has four fixed points $x_1 = (0,0)$, $x_2 = (1/2, 0)$, $x_3 = (0, 1/2)$ and $x_4 = (1/2, 1/2)$.

For $i = 1, 2, 3, 4$ consider the disk D_r^i centered at x_i of radius r. Repeating the construction from the previous section we obtain a diffeomorphism g_i which coincides with A outside $D_{r_1}^i$. Therefore, the map

$$G_1(x) = \begin{cases} A(x) & \text{if } x \in \mathbb{T}^2 \setminus D \\ g_i(x) & \text{if } x \in D_{r_1}^i \end{cases}$$

defines a homeomorphism of the torus $\mathbb{T}^2$ which is a C^∞ diffeomorphism everywhere expect at the points x_i. Here $D = \bigcup_{i=1}^4 D_{r_1}^i$. The Lyapunov exponents of G_1 are nonzero almost everywhere with respect to the area.

Consider the map

$$\varphi(x) = \begin{cases} \varphi_i(x) & \text{if } x \in D_{r_1}^i \\ x & \text{otherwise,} \end{cases}$$

where φ_i are given by (2.4) in each disk $D_{r_2}^i$. It is a homeomorphism of $\mathbb{T}^2$ which is a C^∞ diffeomorphism everywhere expect at the points x_i. The map $G_2 = \varphi \circ G_1 \circ \varphi^{-1}$ is of class C^∞, area-preserving and has nonzero Lyapunov exponents almost everywhere.

Consider the map $\zeta\colon \mathbb{T}^2 \to S^2$ defined by

$$\zeta(s_1, s_2) = \left(\frac{{s_1}^2 - {s_2}^2}{\sqrt{{s_1}^2 + {s_2}^2}}, \frac{2 s_1 s_2}{\sqrt{{s_1}^2 + {s_2}^2}} \right).$$

This map is a double branched covering and is C^∞ everywhere except at the points x_i, $i = 1, 2, 3, 4$ where it branches. It commutes with the involution I and preserves the area. Consider the map $G_{S^2} = \zeta \circ G_2 \circ \zeta^{-1}$. One can show that it is a C^∞ diffeomorphism which preserves the area and has nonzero Lyapunov exponents almost everywhere. Furthermore, one can show that G_{S^2} is ergodic and indeed, is a Bernoulli diffeomorphism.

2.4. **Analytic diffeomorphisms with nonzero exponents.** We describe an example due to Katok and Lewis [134] of a volume-preserving analytic diffeomorphism of a compact smooth Riemannian manifold. It is a version of the well-known blow-up procedure from algebraic geometry.

Setting $X = \{x \in \mathbb{R}^n \colon \|x\| > 1\}$ consider the map $\varphi\colon \mathbb{R}^n \setminus \{0\} \to X$ given by

$$\varphi(x) = \frac{(\|x\|^n + 1)^{1/n}}{\|x\|} x.$$

It is easy to see that φ has Jacobian 1 with respect to the standard coordinates on $\mathbb{R}^n$.

Let A be a linear hyperbolic transformation of $\mathbb{R}^n$. The diffeomorphism $f = \varphi \circ A \circ \varphi^{-1}$ extends analytically to a neighborhood of the boundary which depends on A. This follows from the formula

$$\varphi \circ A \circ \varphi^{-1}(x) = \left(\frac{\|x\|^n - 1}{\|x\|} + \frac{1}{\|Ax\|^n} \right)^{1/n} Ax.$$

Let (r, θ) be the standard polar coordinates on X so that $X = \{(r, \theta) \colon r > 1, \theta \in S^{n-1}\}$. Introducing new coordinates (s, θ), where $s = r^n - 1$, observe that these coordinates extend analytically across the boundary and have the property that the standard volume form is proportional to $ds \wedge d\theta$. Let B be the quotient of $\bar{X}$ under the identification of antipodal points on the boundary. The map f induces a map F of B which preserves the volume form $ds \wedge d\theta$, has nonzero Lyapunov exponents and is analytic.

2.5. **Pseudo-Anosov maps.** Pseudo-Anosov maps were singled out by Thurston in connection with the problem of classifying diffeomorphisms of a compact C^∞ surface M up to isotopy (see [240, 85]). According to Thurston's classification, a diffeomorphism f of M is isotopic to a homeomorphism g satisfying one of the following properties (see [85, Exposé 9]):

1. g is of finite order and is an isometry with respect to a Riemannian metric of constant curvature on M;
2. g is a "reducible" diffeomorphism, that is, a diffeomorphism leaving invariant a closed curve;
3. g is a *pseudo-Anosov map*.

Pseudo-Anosov maps are surface homeomorphisms that are differentiable except at most at finitely many points called *singularities*. These maps minimize both the number of periodic points (of any given period) and the topological entropy in their isotopy classes. A pseudo-Anosov map is Bernoulli with respect to an absolutely continuous invariant measure with C^∞ density which is positive except at the singularities (see [85, Exposé 10]).

We proceed with a formal description. Let $\{x_1, \dots, x_m\}$ be a finite set of points and ν a Borel measure on M. Write $\mathcal{D}_a = \{z \in \mathbb{C} : |z| < a\}$.

We say that $(\mathcal{F}, \nu)$ is a *measured foliation* of M with *singular points* $x_1, \dots, x_m$ if $\mathcal{F}$ is a partition of M for which the following properties hold:

1. there is a collection of C^∞ charts $\varphi_k \colon U_k \to \mathbb{C}$ for $k = 1, \dots, \ell$ and some $\ell \geq m$ with $\bigcup_{k=1}^{\ell} U_k = M$;
2. for each $k = 1, \dots, m$ there is a number $p = p(k) \geq 3$ of elements of $\mathcal{F}$ meeting at x_k such that:
   (a) $\varphi_k(x_k) = 0$ and $\varphi_k(U_k) = \mathcal{D}_{a_k}$ for some $a_k > 0$;
   (b) if C is an element of $\mathcal{F}$ then $C \cap U_k$ is mapped by φ_k to a set
   $$\{z : \mathrm{Im}(z^{p/2}) = \text{constant}\} \cap \varphi_k(U_k);$$
   (c) the measure $\nu|U_k$ is the pullback under φ_k of
   $$|\mathrm{Im}(dz^{p/2})| = |\mathrm{Im}(z^{(p-2)/2}dz)|;$$
3. for each $k > m$ we have:
   (a) $\varphi_k(U_k) = (0, b_k) \times (0, c_k) \subset \mathbb{R}^2 \equiv \mathbb{C}$ for some b_k, $c_k > 0$;
   (b) if C is an element of $\mathcal{F}$ then $C \cap U_k$ is mapped by φ_k to a segment
   $$\{(x, y) : y = \text{constant}\} \cap \varphi_k(U_k);$$
   (c) the measure $\nu|U_k$ is given by the pullback of $|dy|$ under φ_k.

The elements of $\mathcal{F}$ are called *leaves* of the foliation, and ν a *transverse* measure. For $k = 1, \dots, m$, each point x_k is called a *$p(k)$-prong singularity* of $\mathcal{F}$ and each of the leaves of $\mathcal{F}$ meeting at x_k is called a *prong* of x_k. If, in addition, we allow single leaves of $\mathcal{F}$ to terminate in a point (called a *spine*, in which case we set $p = 1$ above), then $(\mathcal{F}, \nu)$ is called a *measured foliation with spines*.

The transverse measure is consistently defined on chart overlaps, because whenever $U_j \cap U_k \neq \varnothing$, the transition functions $\varphi_k \circ {\varphi_j}^{-1}$ are of the form
$$(\varphi_k \circ {\varphi_j}^{-1})(x, y) = (h_{jk}(x, y), c_{jk} \pm y),$$
where h_{jk} is a function, and c_{jk} is a constant.

A surface homeomorphism f is called *pseudo-Anosov* if it satisfies the following properties:

1. f is differentiable except at a finite number of points $x_1, \dots, x_m$;
2. there are two measured foliations $(\mathcal{F}^s, \nu^s)$ and $(\mathcal{F}^u, \nu^u)$ with the same singularities $x_1, \dots, x_m$ and the same number of prongs $p = p(k)$ at each point x_k, for $k = 1, \dots, m$;
3. the leaves of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ are transversal at nonsingular points;
4. there are C^∞ charts $\varphi_k \colon U_k \to \mathbb{C}$ for $k = 1, \dots, \ell$ and some $\ell \geq m$, such that for each k we have:
   (a) $\varphi_k(x_k) = 0$ and $\varphi_k(U_k) = \mathcal{D}_{a_k}$ for some $a_k > 0$;
   (b) leaves of $\mathcal{F}^s$ are mapped by φ_i to components of the sets
   $$\{z : \mathrm{Re}\, z^{p/2} = \text{constant}\} \cap \mathcal{D}_{a_k};$$
   (c) leaves of $\mathcal{F}^u$ are mapped by φ_i to components of the sets
   $$\{z : \mathrm{Im}(z^{p/2}) = \text{constant}\} \cap \mathcal{D}_{a_k};$$

(d) there exists a constant $\lambda > 1$ such that

$$f(\mathcal{F}^s, \nu^s) = (\mathcal{F}^s, \nu^s/\lambda) \quad \text{and} \quad f(\mathcal{F}^u, \nu^u) = (\mathcal{F}^u, \lambda\nu^u).$$

If, in addition, $(\mathcal{F}^s, \nu^s)$ and $(\mathcal{F}^u, \nu^u)$ are measured foliations with spines (with $p = p(k) = 1$ when there is only one prong at x_k), then f is called a *generalized pseudo-Anosov homeomorphism*.

We call $\mathcal{F}^s$ and $\mathcal{F}^u$ the *stable* and *unstable foliations*, respectively. At each singular point x_k, with $p = p(k)$, the *stable* and *unstable prongs* are, respectively, given by

$$P^s_{kj} = {\varphi_k}^{-1}\left\{\rho e^{i\tau} : 0 \le \rho < a_k, \tau = \frac{2j+1}{p}\pi\right\},$$

$$P^u_{kj} = {\varphi_k}^{-1}\left\{\rho e^{i\tau} : 0 \le \rho < a_k, \tau = \frac{2j}{p}\pi\right\},$$

for $j = 0, 1, \ldots, p-1$. We define the *stable* and *unstable sectors* at x_k by

$$S^s_{kj} = {\varphi_k}^{-1}\left\{\rho e^{i\tau} : 0 \le \rho < a_k, \frac{2j-1}{p}\pi \le \tau \le \frac{2j+1}{p}\pi\right\},$$

$$S^u_{kj} = {\varphi_k}^{-1}\left\{\rho e^{i\tau} : 0 \le \rho < a_k, \frac{2j}{p}\pi \le \tau \le \frac{2j+2}{p}\pi\right\},$$

respectively, for $j = 0, 1, \ldots, p-1$.

Since f is a homeomorphism, $f(x_k) = x_{\sigma_k}$ for $k = 1, \ldots, m$, where σ is a permutation of $\{1, \ldots, m\}$ such that $p(k) = p(\sigma_k)$ and f maps the stable prongs at x_k into the stable prongs at x_{σ_k} (provided the numbers a_k are chosen such that $a_k/\lambda^{2/p} \le a_{\sigma_k}$). Hence, we may assume that σ is the identity permutation, and

$$f(P^s_{kj}) \subset P^s_{kj} \quad \text{and} \quad f^{-1}(P^u_{kj}) \subset P^u_{kj}$$

for $k = 1, \ldots, m$ and $j = 0, \ldots, p-1$. Consider the map

$$\Phi_{kj} \colon \varphi_k(S^s_{kj}) \to \{z : \operatorname{Re} z \ge 0\},$$

given by

$$\Phi_{kj}(z) = 2z^{p/2}/p,$$

where $p = p(k)$. Write $\Phi_{kj}(z) = s_1 + is_2$ and $z = t_1 + it_2$, where s_1, s_2, t_1, t_2 are real numbers. Define a measure ν on each stable sector by

$$d\nu|S^s_{kj} = \varphi^*_k \Phi^*_{kj}(ds_1 ds_2)$$

if $k = 1, \ldots, m$, $j = 0, \ldots, p(i)-1$, and on each "nonsingular" neighborhood by

$$d\nu|U_k = \varphi^*_k(dt_1 dt_2)$$

if $k > m$. The measure ν can be extended to an f-invariant measure with the following properties:

1. ν is equivalent to the Lebesgue measure on M; moreover, ν has a density which is smooth everywhere except at the singular points x_k, where it vanishes if $p(k) \ge 3$, and goes to infinity if $p(k) = 1$;
2. f is Bernoulli with respect to ν (see [85, Section 10]).

One can show that the periodic points of any pseudo-Anosov map are dense.

If M is a torus, then any pseudo-Anosov map is an Anosov diffeomorphism (see [85, Exposé 1]). However, if M has genus greater than 1, a pseudo-Anosov map cannot be made a diffeomorphism by a coordinate change which is smooth outside the singularities or even outside a sufficiently small neighborhood of the singularities (see [97]). Thus, in order to find smooth models of pseudo-Anosov maps one may have to apply some nontrivial construction which is global in nature. In [97], Gerber and Katok constructed, for every pseudo-Anosov map g, a C^∞ diffeomorphism which is topologically conjugate to f through a homeomorphism isotopic to the identity and which is Bernoulli with respect to a smooth measure (that is, a measure whose density is C^∞ and positive everywhere).

In [96], Gerber proved the existence of real analytic Bernoulli models of pseudo-Anosov maps as an application of a conditional stability result for the smooth models constructed in [97]. The proofs rely on the use of Markov partitions. The same results were obtained by Lewowicz and Lima de Sá [162] using a different approach.

2.6. **Flows with nonzero exponents.** The first example of a volume-preserving ergodic flow with nonzero Lyapunov exponents, which is not an Anosov flow, was constructed by Pesin in [195]. The construction is a "surgery" of an Anosov flow and is based on slowing down trajectories near a given trajectory of the Anosov flow.

Let φ_t be an Anosov flow on a compact three-dimensional manifold M given by a vector field X and preserving a smooth ergodic measure μ. Fix a point $p_0 \in M$. There is a coordinate system x, y, z in a ball $B(p_0, d)$ (for some $d > 0$) such that p_0 is the origin (i.e., $p_0 = 0$) and $X = \partial/\partial z$.

For each $\varepsilon > 0$, let $T_\varepsilon = S^1 \times D_\varepsilon \subset B(0, d)$ be the solid torus obtained by rotating the disk

$$D_\varepsilon = \{(x, y, z) \in B(0, d) : x = 0 \text{ and } (y - d/2)^2 + z^2 \le (\varepsilon d)^2\}$$

around the z-axis. Every point on the solid torus can be represented as (θ, y, z) with $\theta \in S^1$ and $(y, z) \in D_\varepsilon$.

For every $0 \le \alpha \le 2\pi$, consider the cross-section of the solid torus $\Pi_\alpha = \{(\theta, y, z) : \theta = \alpha\}$. We construct a new vector field $\widetilde{X}$ on $M \setminus T_\varepsilon$. We describe the construction of $\widetilde{X}$ on the cross-section Π_0 and we obtain the desired vector field $\widetilde{X}|\Pi_\alpha$ on an arbitrary cross-section by rotating it around the z-axis.

Consider the Hamiltonian flow given by the Hamiltonian $H(y, z) = y(\varepsilon^2 - y^2 - z^2)$. In the annulus $\varepsilon^2 \le y^2 + z^2 \le 4\varepsilon^2$ the flow is topologically conjugated to the one shown on Figure 1. However, the Hamiltonian vector field $(-2yz, 3y^2 + z^2 - \varepsilon^2)$ is not everywhere vertical on the circle $y^2 + z^2 = 4\varepsilon^2$. To correct this consider a C^∞ function $p\colon [\varepsilon, \infty) \to [0, 1]$ such that $p|[\varepsilon, 3\varepsilon/2] = 1$, $p|[2\varepsilon, \infty) = 0$, and p is strictly decreasing in $(3\varepsilon/2, 2\varepsilon)$. The flow defined by the system of differential equations

$$\begin{cases} y' = -2yzp(\sqrt{y^2 + z^2}) \\ z' = (3y^2 + z^2 - \varepsilon^2)p(\sqrt{y^2 + z^2}) + 1 - p(\sqrt{y^2 + z^2}) \end{cases}$$

has now the behavior shown on Figure 1. Denote by $\overline{\varphi}_t$ and $\overline{X}$ the corresponding flow and vector field in coordinates x, y, z.

By changing the time one can obtain a flow $\overline{\varphi}_t$ in the annulus $\varepsilon^2 \le y^2 + z^2 \le 4\varepsilon^2$ so that the new flow $\widetilde{\varphi}_t$ preserves the measure μ. As a result we have a smooth

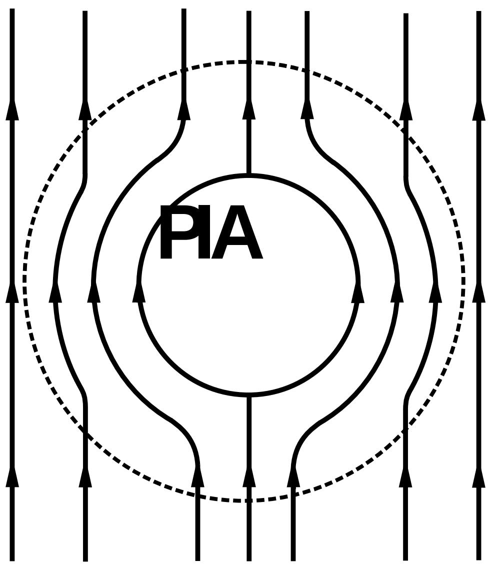

FIGURE 1. A cross-section Π_α and the flow $\widetilde{\varphi}_t$

vector field $\widetilde{X}$ on $M \setminus T_\varepsilon$ such that the flow $\widetilde{\varphi}_t$ generated by $\widetilde{X}$ has the following properties:

1. $\widetilde{X}|(M \setminus T_{2\varepsilon}) = X|(M \setminus T_{2\varepsilon})$;
2. for any $0 \le \alpha, \beta \le 2\pi$, the vector field $\widetilde{X}|\Pi_\beta$ is the image of the vector field $\widetilde{X}|\Pi_\alpha$ under the rotation around the z-axis that moves Π_α onto Π_β;
3. for every $0 \le \alpha \le 2\pi$, the unique two fixed points of the flow $\widetilde{\varphi}_t|\Pi_\alpha$ are those in the intersection of Π_α with the hyperplanes $z = \pm\varepsilon d$;
4. for every $0 \le \alpha \le 2\pi$ and $(y, z) \in D_{2\varepsilon} \setminus \operatorname{int} D_\varepsilon$, the trajectory of the flow $\widetilde{\varphi}_t|\Pi_\alpha$ passing through the point (y, z) is invariant under the symmetry $(\alpha, y, z) \mapsto (\alpha, y, -z)$;
5. the flow $\widetilde{\varphi}_t|\Pi_\alpha$ preserves the conditional measure induced by the measure μ on the set Π_α.

The orbits of the flows φ_t and $\widetilde{\varphi}_t$ coincide on $M \setminus T_{2\varepsilon}$, the flow $\widetilde{\varphi}_t$ preserves the measure μ and the only fixed points of this flow are those on the circles $\{(\theta, y, z) : z = -\varepsilon d\}$ and $\{(\theta, y, z) : z = \varepsilon d\}$.

On $T_{2\varepsilon} \setminus \operatorname{int} T_\varepsilon$ consider the new coordinates θ_1, θ_2, r with $0 \le \theta_1, \theta_2 < 2\pi$ and $\varepsilon d \le r \le 2\varepsilon d$ such that the set of fixed points of $\widetilde{\varphi}_t$ consists of those for which $r = \varepsilon d$, and $\theta_1 = 0$ or $\theta_1 = \pi$.

Define the flow on $T_{2\varepsilon} \setminus \operatorname{int} T_\varepsilon$ by

$$(\theta_1, \theta_2, r, t) \mapsto (\theta_1, \theta_2 + [2 - r/(\varepsilon d)]^4 t \cos\theta_1, r),$$

and let $\hat{X}$ be the corresponding vector field. Consider the flow ψ_t on $M \setminus \operatorname{int} T_\varepsilon$ generated by the vector field Y on $M \setminus \operatorname{int} T_\varepsilon$

$$Y(x) = \begin{cases} X(x), & x \in M \setminus \operatorname{int} T_{2\varepsilon} \\ \widetilde{X}(x) + \hat{X}(x), & x \in \operatorname{int} T_{2\varepsilon} \setminus \operatorname{int} T_\varepsilon \end{cases}.$$

The flow ψ_t has no fixed points, preserves the measure μ and for μ-almost every $x \in M \setminus T_{2\varepsilon}$,

$$\chi(x, v) < 0 \text{ if } v \in E^s(x), \text{ and } \chi(x, v) > 0 \text{ if } v \in E^u(x),$$

where $E^u(x)$ and $E^s(x)$ are respectively stable and unstable subspaces of the Anosov flow φ_t at x.

Set $M_1 = M \setminus T_\varepsilon$ and consider a copy $(\widetilde{M}_1, \widetilde{\psi}_t)$ of the flow (M_1, ψ_t). Gluing the manifolds M_1 and $\widetilde{M}_1$ along their boundaries ∂T_ε one obtains a three-dimensional smooth Riemannian manifold D without boundary. We define a flow F_t on D by

$$F_t x = \begin{cases} \psi_t x, & x \in M_1 \\ \widetilde{\psi}_t x, & x \in \widetilde{M}_1 \end{cases}.$$

It is clear that the flow F_t is smooth and preserves a smooth hyperbolic measure.

2.7. **Geodesic flows.** Our next example is the geodesic flow on a compact smooth Riemannian manifold of nonpositive curvature. Let M be a compact smooth p-dimensional Riemannian manifold with a Riemannian metric of class C^3.

The *geodesic flow* g_t acts on the tangent bundle TM by the formula

$$g_t(v) = \dot{\gamma}_v(t),$$

where $\dot{\gamma}_v(t)$ is the unit tangent vector to the geodesic $\gamma_v(t)$ defined by the vector v (i.e., $\dot{\gamma}_v(0) = v$; this geodesic is uniquely defined). The geodesic flow generates a vector field V on TM given by

$$V(v) = \frac{d(g_t(v))}{dt}\Big|_{t=0}.$$

Since M is compact the flow g_t is well-defined for all $t \in \mathbb{R}$ and is a smooth flow.

We recall some basic notions from Riemannian geometry of nonpositively curved manifolds (see [82, 81] for a detailed exposition). We endow the second tangent space $T(TM)$ with a special Riemannian metric. Let $\pi\colon TM \to M$ be the natural projection (i.e., $\pi(x, v) = x$ for each $x \in M$ and each $v \in T_xM$) and $K\colon T(TM) \to TM$ the linear (connection) operator defined by $K\xi = (\nabla Z)(t)|_{t=0}$, where $Z(t)$ is any curve in TM such that $Z(0) = d\pi\xi$, $\frac{d}{dt}Z(t)|_{t=0} = \xi$ and ∇ is the covariant derivative. The *canonical metric* on $T(TM)$ is given by

$$\langle \xi, \eta \rangle_v = \langle d_v\pi\xi, d_v\pi\eta \rangle_{\pi v} + \langle K\xi, K\eta \rangle_{\pi v}.$$

The set $SM \subset TM$ of the unit vectors is invariant with respect to the geodesic flow, and is a compact manifold of dimension $2p - 1$. In what follows we consider the geodesic flow restricted to SM.

The study of hyperbolic properties of the geodesic flow is based upon the description of solutions of the variational equation for the flow. This equation along a given trajectory $g_t(v)$ of the flow is the *Jacobi equation* along the geodesic $\gamma_v(t)$:

$$Y''(t) + R_{XY}X(t) = 0. \tag{2.5}$$

Here $Y(t)$ is a vector field along $\gamma_v(t), X(t) = \dot{\gamma}(t)$, and R_{XY} is the curvature operator. More precisely, the relation between the variational equations and the Jacobi equation (2.5) can be described as follows. Fix $v \in SM$ and $\xi \in T_vSM$. Let $Y_\xi(t)$ be the unique solution of (2.5) satisfying the initial conditions $Y_\xi(0) = d_v\pi\xi$ and $Y'_\xi(0) = K\xi$. One can show that the map $\xi \mapsto Y_\xi(t)$ is an isomorphism for which $d_{g_tv}\pi d_vg_t\xi = Y_\xi(t)$ and $Kd_vg_t\xi = Y'_\xi(t)$. This map establishes the identification between solutions of the variational equation and solutions of the Jacobi equation (2.5).

Recall that the Fermi coordinates $\{e_i(t)\}$, for $i = 1, \ldots, p$, along the geodesic $\gamma_v(t)$ are obtained by the time t parallel translation along $\gamma_v(t)$ of an orthonormal basis $\{e_i(0)\}$ in $T_{\gamma_v(0)}M$ where $e_1(t) = \dot\gamma(t)$. Using these coordinates we can rewrite Equation (2.5) in the matrix form

$$\frac{d^2}{dt^2}A(t) + K(t)A(t) = 0, \tag{2.6}$$

where $A(t) = (a_{ij}(t))$ and $K(t) = (k_{ij}(t))$ are matrix functions with entries $k_{ij}(t) = K_{\gamma_v(t)}(e_i(t), e_j(t))$.

Two points $x = \gamma(t_1)$ and $y = \gamma(t_2)$ on the geodesic γ are called *conjugate* if there exists a nonidentically zero Jacobi field Y along γ such that $Y(t_1) = Y(t_2) = 0$. Two points $x = \gamma(t_1)$ and $y = \gamma(t_2)$ are called *focal* if there exists a Jacobi field Y along γ such that $Y(t_1) = 0$, $Y'(t_1) \neq 0$ and $\frac{d}{dt}\|Y(t)\|^2|_{t=t_2} = 0$.

We say that the manifold M has:

1. *no conjugate points* if on each geodesic no two points are conjugate;
2. *no focal points* if on each geodesic no two points are focal;
3. *nonpositive curvature* if for any $x \in M$ and any two vectors v_1, $v_2 \in T_xM$ the sectional curvature $K_x(v_1, v_2)$ satisfies
$$K_x(v_1, v_2) \leq 0. \tag{2.7}$$

If the manifold has no focal points then it has no conjugate points and if it has nonpositive curvature then it has no focal points.

From now on we consider only manifolds with no conjugate points. The boundary value problem for Equation (2.6) has a unique solution, i.e., for any numbers s_1, s_2 and any matrices A_1, A_2 there exists a unique solution $A(t)$ of (2.6) satisfying $A(s_1) = A_1$ and $A(s_2) = A_2$.

Proposition 2.1. *Given $s \in \mathbb{R}$, let $A_s(t)$ be the unique solution of Equation* (2.6) *satisfying the boundary conditions: $A_s(0) = \mathrm{Id}$ (where Id is the identity matrix) and $A_s(s) = 0$. Then there exists the limit*

$$\lim_{s\to\infty} \frac{d}{dt}A_s(t)\Big|_{t=0} = A^+.$$

[Eberlein, [80]] We define the *positive limit solution* $A^+(t)$ of (2.6) as the solution that satisfies the initial conditions:

$$A^+(0) = \mathrm{Id} \quad \text{and} \quad \frac{d}{dt}A^+(t)\Big|_{t=0} = A^+.$$

This solution is nondegenerate (i.e., $\det A^+(t) \neq 0$ for every $t \in \mathbb{R}$) and $A^+(t) = \lim_{s\to+\infty} A_s(t)$.

Similarly, letting $s \to -\infty$, define the *negative limit solution* $A^-(t)$ of Equation (2.6).

For every $v \in SM$ set

$$E^+(v) = \{\xi \in T_vSM : \langle \xi, V(v)\rangle = 0 \text{ and } Y_\xi(t) = A^+(t)d_v\pi\xi\}, \tag{2.8}$$

$$E^-(v) = \{\xi \in T_vSM : \langle \xi, V(v)\rangle = 0 \text{ and } Y_\xi(t) = A^-(t)d_v\pi\xi\}, \tag{2.9}$$

where V is the vector field generated by the geodesic flow.

Proposition 2.2 (Eberlein, [80])**.** *The following properties hold:*

1. *the sets* $E^-(v)$ *and* $E^+(v)$ *are linear subspaces of* T_vSM*;*
2. $\dim E^-(v) = \dim E^+(v) = p-1$*;*
3. $d_v\pi E^-(v) = d_v\pi E^+(v) = \{w \in T_{\pi v}M : w \text{ is orthogonal to } v\}$*;*
4. *the subspaces* $E^-(v)$ *and* $E^+(v)$ *are invariant under the differential* d_vg_t*, i.e.,* $d_vg_tE^-(v) = E^-(g_tv)$ *and* $d_vg_tE^+(v) = E^+(g_tv)$*;*
5. *if* $\tau\colon SM \to SM$ *is the involution defined by* $\tau v = -v$*, then*
$$E^+(-v) = d_v\tau E^-(v) \quad \text{and} \quad E^-(-v) = d_v\tau E^+(v);$$
6. *if* $K_x(v_1,v_2) \ge -a^2$ *for some* $a>0$ *and all* $x\in M$*, then* $\|K\xi\| \le a\|d_v\pi\xi\|$ *for every* $\xi \in E^+(v)$ *and* $\xi \in E^-(v)$*;*
7. *if* $\xi\in E^+(v)$ *or* $\xi \in E^-(v)$*, then* $Y_\xi(t) \ne 0$ *for every* $t\in\mathbb{R}$*;*
8. $\xi \in E^+(v)$ *(respectively,* $\xi\in E^-(v)$*) if and only if*
$$\langle \xi, V(v)\rangle = 0 \quad \text{and} \quad \|d_{g_tv}\pi d_vg_t\xi\| \le c$$
for every $t>0$ *(respectively,* $t<0$*) and some* $c>0$*;*
9. *if the manifold has no focal points then for any* $\xi\in E^+(v)$ *(respectively,* $\xi \in E^-(v)$*) the function* $t\mapsto \|Y_\xi(t)\|$ *is nonincreasing (respectively, nondecreasing).*

In view of Properties 6 and 8, we have $\xi\in E^+(v)$ (respectively, $\xi\in E^-(v)$) if and only if $\langle\xi, V(v)\rangle = 0$ and $\|d_vg_t\xi\|\le c$ for $t>0$ (respectively, $t<0$), for some constant $c>0$. This observation and Property 4 justify to call $E^+(v)$ and $E^-(v)$ the *stable* and *unstable subspaces*.

In general, the subspaces $E^-(v)$ and $E^+(v)$ do not span the whole second tangent space T_vSM. Eberlein (see [80]) has shown that if they do span T_vSM for every $v\in SM$, then the geodesic flow is Anosov. This is the case when the curvature is strictly negative. For a general manifold without conjugate points consider the set

$$\Delta = \left\{ v\in SM : \overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t K_{\pi(g_sv)}(g_sv, g_sw)\,ds < 0 \right.$$
$$\left. \text{for every } w\in SM \text{ orthogonal to } v \right\}. \quad (2.10)$$

It is easy to see that Δ is measurable and invariant under g_t. The following result shows that the Lyapunov exponents are nonzero on the set Δ.

Theorem 2.3 (Pesin [197])**.** *Assume that the Riemannian manifold* M *has no conjugate points. Then for every* $v\in\Delta$ *we have* $\chi(v,\xi)<0$ *if* $\xi\in E^+(v)$ *and* $\chi(v,\xi)>0$ *if* $\xi\in E^-(v)$*.*

The geodesic flow preserves the Liouville measure μ on the tangent bundle. Denote by m the Lebesgue measure on M. It follows from Theorem 2.3 that if the set Δ has positive Liouville measure then the geodesic flow $g_t|\Delta$ has nonzero Lyapunov exponents almost everywhere. It is, therefore, crucial to find conditions which would guarantee that Δ has positive Liouville measure.

We first consider the two-dimensional case.

Theorem 2.4 (Pesin [197])**.** *Let* M *be a smooth compact surface of nonpositive curvature* $K(x)$ *and genus greater than* 1*. Then* $\mu(\Delta)>0$*.*

In the multidimensional case one can establish the following criterion for positivity of the Liouville measure of the set Δ.

Theorem 2.5 (Pesin [197])**.** *Let M be a smooth compact Riemannian manifold of nonpositive curvature. Assume that there exist $x \in M$ and a vector $v \in S_xM$ such that*

$$K_x(v, w) < 0$$

for any vector $w \in S_xM$ which is orthogonal to v. Then $\mu(\Delta) > 0$.

One can show that if $\mu(\Delta) > 0$ then the set Δ is open $\pmod 0$ and is everywhere dense (see Theorem 17.7 below).

3. Lyapunov exponents associated with sequences of matrices

In studying the stability of trajectories of a dynamical system f one introduces the system of variations $\{d_xf^m, m \in \mathbb{Z}\}$ and uses the Lyapunov exponents for this systems (see Section 1). Consider a family of trivializations τ_x of M, i.e., linear isomorphisms $\tau_x \colon T_xM \to \mathbb{R}^n$ where $n = \dim M$. The sequence of matrices

$$A_m = \tau_{f^{m+1}(x)} \circ d_{f^m(x)}f \circ \tau^{-1}_{f^m(x)} \colon \mathbb{R}^n \to \mathbb{R}^n$$

can also be used to study the linear stability along the trajectory $f^m(x)$.

In this section we extend our study of Lyapunov exponents for sequences of matrices generated by smooth dynamical systems to arbitrary sequences of matrices. This will also serve as an important intermediate step in studying Lyapunov exponents for the even more general case of cocycles over dynamical systems.

3.1. **Definition of the Lyapunov exponent.** Let $\mathcal{A}^+ = \{A_m\}_{m\geq 0} \subset GL(n, \mathbb{R})$ be a one-sided sequence of matrices. Set $\mathcal{A}_m = A_{m-1} \cdots A_1A_0$ and consider the function $\chi^+ \colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ given by

$$\chi^+(v) = \chi^+(v, \mathcal{A}^+) = \varlimsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}_mv\|. \tag{3.1}$$

We make the convention $\log 0 = -\infty$, so that $\chi^+(0) = -\infty$.

The function $\chi^+(v)$ is called the *forward Lyapunov exponent of v (with respect to the sequence $\mathcal{A}^+$)*. It has the following basic properties:

1. $\chi^+(\alpha v) = \chi^+(v)$ for each $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R} \setminus \{0\}$;
2. $\chi^+(v + w) \leq \max\{\chi^+(v), \chi^+(w)\}$ for each v, $w \in \mathbb{R}^n$;
3. $\chi^+(0) = -\infty$.

As an immediate consequence of the basic properties we obtain that there exist a positive integer p^+, $1 \leq p^+ \leq n$, a collection of numbers $\chi_1 < \chi_2 < \cdots < \chi_{p^+}$, and linear subspaces

$$\{0\} = V_0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \cdots \subsetneqq V_{p^+} = \mathbb{R}^n$$

such that $V_i = \{v \in \mathbb{R}^n : \chi^+(v) \leq \chi_i\}$, and if $v \in V_i \setminus V_{i-1}$, then $\chi^+(v) = \chi_i$ for each $i = 1, \ldots, p^+$. The spaces V_i form the *filtration* $\mathcal{V}_{\chi^+}$ of $\mathbb{R}^n$ associated with χ^+. The number

$$k_i = \dim V_i - \dim V_{i-1}$$

is called the *multiplicity* of the value χ_i, and the collection of pairs

$$\operatorname{Sp}\chi^+ = \{(\chi_i, k_i) \colon i = 1, \ldots, p^+\}$$

the *Lyapunov spectrum* of χ^+. We also set

$$n_i = \dim V_i = \sum_{j=1}^{i} k_j.$$

In a similar way, given a sequence of matrices $\mathcal{A}^- = \{A_m\}_{m<0}$, define the *backward Lyapunov exponent (with respect to the sequence* $\mathcal{A}^-$*)* $\chi^- \colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ by

$$\chi^-(v) = \chi^-(v, \mathcal{A}^-) = \overline{\lim_{m\to-\infty}} \frac{1}{|m|} \log\|\mathcal{A}_m v\|, \tag{3.2}$$

where $\mathcal{A}_m = (A_m)^{-1} \cdots (A_{-2})^{-1}(A_{-1})^{-1}$ for each $m < 0$. Let $\chi_1^- > \cdots > \chi_{p^-}^-$ be the *values* of χ^-, for some integer $1 \le p^- \le n$. The subspaces

$$\mathbb{R}^n = V_1^- \supsetneqq \cdots \supsetneqq V_{p^-}^- \supsetneqq V_{p^-+1}^- = \{0\},$$

where $V_i^- = \{v \in \mathbb{R}^n : \chi^-(v) \le \chi_i^-\}$, form the *filtration* $\mathcal{V}_{\chi^-}$ of $\mathbb{R}^n$ associated with χ^-. The number

$$k_i^- = \dim V_i^- - \dim V_{i+1}^-$$

is the *multiplicity* of the value χ_i^-, and the collection of pairs

$$\operatorname{Sp}\chi^- = \{(\chi_i^-, k_i^-) \colon i = 1, \ldots, p^-\}$$

is the *Lyapunov spectrum* of χ^-.

In the case when the sequence of matrices is obtained by iterating a given matrix A, i.e., $\mathcal{A}_m = A^m$ the Lyapunov spectrum is calculated as follows. Take all the eigenvalues with absolute value r. Then $\log r$ is a value of the Lyapunov exponent and the multiplicity is equal to the sum of the multiplicities of the exponents with this absolute value.

Equality (3.1) implies that for every $\varepsilon > 0$ there exists $C_+ = C_+(v, \varepsilon) > 0$ such that if $m \ge 0$ then

$$\|\mathcal{A}_m v\| \le C_+ e^{(\chi^+(v)+\varepsilon)m}\|v\|. \tag{3.3}$$

Similarly, (3.2) implies that for every $\varepsilon > 0$ there exists $C_- = C_-(v, \varepsilon) > 0$ such that if $m \ge 0$ then

$$\|\mathcal{A}_{-m} v\| \le C_- e^{(\chi^-(v)+\varepsilon)m}\|v\|. \tag{3.4}$$

Given vectors $v_1, \ldots, v_k \in \mathbb{R}^n$, we denote by $V(v_1, \ldots, v_k)$ the volume of the k-parallelepiped formed by $v_1, \ldots, v_k$. The *forward* and *backward k-dimensional Lyapunov exponents of the vectors* $v_1, \ldots, v_k$ are defined, respectively, by

$$\begin{aligned}\chi^+(v_1, \ldots, v_k) &= \chi^+(v_1, \ldots, v_k, \mathcal{A}^+) \\ &= \overline{\lim_{m\to+\infty}} \frac{1}{m} \log V(\mathcal{A}_m v_1, \ldots, \mathcal{A}_m v_k), \\ \chi^-(v_1, \ldots, v_k) &= \chi^-(v_1, \ldots, v_k, \mathcal{A}^-) \\ &= \overline{\lim_{m\to-\infty}} \frac{1}{|m|} \log V(\mathcal{A}_m v_1, \ldots, \mathcal{A}_m v_k).\end{aligned}$$

These exponents depend only on the linear space generated by the vectors $v_1, \ldots, v_k$. Since $V(v_1, \ldots, v_k) \le \prod_{i=1}^k \|v_i\|$ we obtain

$$\chi^+(v_1, \ldots, v_k) \le \sum_{i=1}^k \chi^+(v_i). \tag{3.5}$$

A similar inequality holds for the backward Lyapunov exponent.

The inequality (3.5) can be strict. Indeed, consider the sequence of matrices $A_m = \left(\begin{smallmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix}\right)$ and the vectors $v_1 = (1, 0)$, $v_2 = (1, 1)$. We have $\chi^+(v_1) = \chi^+(v_2) = \log 2$. On the other hand, since $\det A_m = 1$, we have $\chi^+(v_1, v_2) = 0 < \chi^+(v_1) + \chi^+(v_2)$.

3.2. **Forward and backward regularity.** We say that a sequence of matrices $\mathcal{A}^+$ is *forward regular* if

$$\lim_{m\to+\infty}\frac{1}{m}\log|\det\mathcal{A}_m| = \sum_{i=1}^{n}\chi_i', \tag{3.6}$$

where χ_1', …, χ_n' are the finite values of the exponent χ^+ counted with their multiplicities. By (3.5), this is equivalent to

$$\varliminf_{m\to+\infty}\frac{1}{m}\log|\det\mathcal{A}_m| \ge \sum_{i=1}^{n}\chi_i'.$$

The forward regularity is equivalent to the statement that there exists a positive definite symmetric matrix Λ such that

$$\lim_{m\to\infty}\|\mathcal{A}_m\Lambda^{-m}\| = 0, \quad \lim_{m\to\infty}\|\Lambda^m\mathcal{A}_m^{-1}\| = 0. \tag{3.7}$$

Let $\mathcal{A}^+ = \{A_m\}_{m\ge0}$ and $\mathcal{B}^+ = \{B_m\}_{m\ge0}$ be two sequences of matrices. They are called *equivalent* if there is a nondegenerate matrix C such that $A_m = C^{-1}B_mC$ for every $m \ge 0$. Any sequence of linear transformations, which is equivalent to a forward regular sequence, is itself forward regular.

The Lyapunov exponent χ^+ is said to be

1. *exact with respect to the collection of vectors* $\{v_1, \dots, v_k\} \subset \mathbb{R}^n$ if

$$\chi^+(v_1, \dots, v_k) = \lim_{m\to+\infty}\frac{1}{m}\log V(\mathcal{A}_mv_1, \dots, \mathcal{A}_mv_k);$$

2. *exact* if for any $1 \le k \le n$, the exponent χ^+ is exact with respect to every collection of vectors $\{v_1, \dots, v_k\} \subset \mathbb{R}^n$.

If the Lyapunov exponent is exact then in particular for $v \in \mathbb{R}^n$ one has

$$\chi^+(v) = \lim_{m\to+\infty}\frac{1}{m}\log\|\mathcal{A}_mv\|;$$

equivalently (compare with (3.3) and (3.4)): for every $\varepsilon > 0$ there exists $C = C(v,\varepsilon) > 0$ such that if $m \ge 0$ then

$$C^{-1}e^{(\chi^+(v)-\varepsilon)m} \le \|\mathcal{A}_mv\| \le Ce^{(\chi^+(v)+\varepsilon)m}.$$

Theorem 3.1 (Lyapunov [170])**.** *If the sequence of matrices $\mathcal{A}^+$ is forward regular, then the Lyapunov exponent χ^+ is exact.*

The following simple example demonstrates that the Lyapunov exponent χ^+ may be exact even for a sequence of matrices which is not forward regular. In other words, the existence of the limit in (3.1) does *not* guarantee that the Lyapunov exponent χ^+ is forward regular.

Example 3.2. *Let $\mathcal{A}^+ = \{A_m\}_{m\ge0}$ be the sequence of matrices where $A_0 = \left(\begin{smallmatrix}1&0\\2&4\end{smallmatrix}\right)$ and $A_m = \left(\begin{smallmatrix}1&0\\-2^{m+1}&4\end{smallmatrix}\right)$ for each $m \ge 1$ so that $\mathcal{A}_m = \left(\begin{smallmatrix}1&0\\2^m&4^m\end{smallmatrix}\right)$ for every $m \ge 1$. Given a vector $v = (a,b) \ne (0,0)$ we have $\chi^+(v) = \log 2$ if $b = 0$, and $\chi^+(v) = \log 4$ if $b \ne 0$. This implies that χ^+ is exact with respect to every vector v. Let $v_1 = (1,0)$ and $v_2 = (0,1)$. Then $\chi^+(v_1) = \log 2$ and $\chi^+(v_2) = \log 4$. Since $\det\mathcal{A}_m = 4^m$ we obtain $\chi^+(v_1,v_2) = \log 4$. Therefore, χ^+ is exact with respect to $\{v_1,v_2\}$, and hence with respect to every collection of two vectors. On the other hand,*

$$\chi^+(v_1,v_2) = \log 4 < \log 2 + \log 4 = \chi^+(v_1) + \chi^+(v_2)$$

and the sequence of matrices $\mathcal{A}^+$ is not forward regular.

In the one-dimensional case the situation is different.

Proposition 3.3. *A sequence of numbers $\mathcal{A}^+ \subset GL(1,\mathbb{R}) = \mathbb{R} \setminus \{0\}$ is forward regular if and only if the Lyapunov exponent χ^+ is exact.*

We now present an important characteristic property of forward regularity which is very useful in applications. We say that a basis $\mathbf{v} = (v_1, \ldots, v_n)$ of $\mathbb{R}^n$ is *normal* with respect to the filtration $\mathcal{V} = \{V_i : i = 0, \ldots, p^+\}$ if for every $1 \le i \le p^+$ there exists a basis of V_i composed of vectors from $\{v_1, \ldots, v_n\}$.

Theorem 3.4 (see [58])**.** *A sequence of matrices $\mathcal{A}^+$ is forward regular if and only if for any normal basis $\mathbf{v}$ of $\mathbb{R}^n$ with respect to the filtration $\mathcal{V}_{\chi^+}$ and any subset $K \subset \{1, \ldots, n\}$, we have:*

1.
$$\chi^+(\{v_i\}_{i\in K}) = \lim_{m\to+\infty} \frac{1}{m} \log V(\{\mathcal{A}_m v_i\}_{i\in K}) = \sum_{i\in K} \chi^+(v_i);$$
2. *if σ_m is the angle between the subspaces* $\operatorname{span}\{\mathcal{A}_m v_i : i \in K\}$ *and* $\operatorname{span}\{\mathcal{A}_m v_i : i \notin K\}$*, then*
$$\lim_{m\to+\infty} \frac{1}{m} \log|\sin\sigma_m| = 0.$$

A sequence of matrices $\mathcal{A}^- = \{A_m\}_{m<0}$ is called *backward regular* if

$$\lim_{m\to-\infty} \frac{1}{|m|} \log|\det \mathcal{A}_m| = \sum_{i=1}^{n} \widetilde{\chi}'_i,$$

where $\widetilde{\chi}'_1, \ldots, \widetilde{\chi}'_n$ are the finite values of χ^- counted with their multiplicities.

Given a sequence $\mathcal{A}^- = \{A_m\}_{m<0}$, we construct a new sequence $\mathcal{B}^+ = \{B_m\}_{m\ge0}$ by setting $B_m = (A_{-m-1})^{-1}$. The backward regularity of $\mathcal{A}^-$ is equivalent to the forward regularity of $\mathcal{B}^+$. This reduction allows one to translate any fact about forward regularity into a corresponding fact about backward regularity.

For example, the backward regularity of the sequence of matrices $\mathcal{A}^-$ implies that the Lyapunov exponent χ^- is exact. Moreover, if the Lyapunov exponent χ^- is exact (in particular, if it is backward regular) then

$$\chi^-(v) = \lim_{m\to-\infty} \frac{1}{|m|} \log\|\mathcal{A}_m v\|$$

for every $v \in \mathbb{R}^n$. This is equivalent to the following: for every $\varepsilon > 0$ there exists $C = C(v, \varepsilon) > 0$ such that if $m \ge 0$ then

$$C^{-1} e^{-(\chi^-(v)-\varepsilon)m} \le \|\mathcal{A}_{-m} v\| \le C e^{-(\chi^-(v)+\varepsilon)m}.$$

3.3. **A criterion for forward regularity of triangular matrices.** Let $\mathcal{A}^+ = \{A_m\}_{m\ge0}$ be a sequence of matrices. One can write each A_m in the form $A_m = R_m T_m$, where R_m is orthogonal, and T_m is lower triangular. In general, the diagonal entries of T_m alone do *not* determine the values of the Lyapunov exponent associated with $\mathcal{A}^+$. Indeed, let $\mathcal{A}^+ = \{A_m\}_{m\ge0}$ be a sequence of matrices where $A_m = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}$ for each $m > 0$. We have $A_m = R_m T_m$, where $R_m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is orthogonal and $T_m = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ is diagonal. Since ${A_m}^2 = -\mathrm{Id}$, we obtain $\chi^+(v, \mathcal{A}^+) = 0$ for every $v \ne 0$ whereas the values of the Lyapunov exponent for $\mathcal{T} = \{T_m\}_{m\ge0}$ are equal to $\pm\log 2$.

However, in certain situations one can reduce the study of sequences of arbitrary matrices to the study of sequences of lower triangular matrices (see Section 5.3). Therefore, we shall consider sequences of lower triangular matrices and present a useful criterion of regularity of the Lyapunov exponent. This criterion is used in the proof of the Multiplicative Ergodic Theorem 5.5, which is one of the central results in smooth ergodic theory. We write $\log^+ a = \max\{\log a, 0\}$ for a positive number a.

Theorem 3.5 (see [58]). *Let $\mathcal{A}^+ = \{(a_{ij}^m)\}_{m\geq 0} \subset GL(n,\mathbb{R})$ be a sequence of lower triangular matrices such that:*

1. *for each $i = 1, \ldots, n$, the following limit exists and is finite:*
$$\lim_{m\to+\infty} \frac{1}{m}\sum_{k=0}^{m} \log|a_{ii}^k| \stackrel{\text{def}}{=} \lambda_i;$$
2. *for any i, $j = 1, \ldots, n$, we have*
$$\overline{\lim_{m\to+\infty}} \frac{1}{m} \log^+|a_{ij}^m| = 0.$$

Then the sequence $\mathcal{A}^+$ is forward regular, and the numbers λ_i are the values of the Lyapunov exponent χ^+ (counted with their multiplicities but possibly not ordered).

Let us comment on the proof of this theorem. If we count each exponent according to its multiplicity we have exactly n exponents. To verify (3.6) we will produce a basis $v_1, \ldots, v_n$ which is normal with respect to the standard filtration (i.e., related with the standard basis by an upper triangular coordinate change) such that $\chi^+(v_i) = \lambda_i$.

If the exponents are ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then the standard basis is in fact normal. To see this notice that while multiplying lower triangular matrices one obtains a matrix whose off-diagonal entries contain a polynomially growing number of terms each of which can be estimated by the growth of the product of diagonal terms below.

However, if the exponents are not ordered that way then an element e_i of the standard basis will grow according to the maximal of the exponents λ_j for $j \geq i$. In order to produce the right growth one has to compensate the growth caused by off-diagonal terms by subtracting from the vector e_i a certain linear combination of vectors e_j for which $\lambda_j > \lambda_i$. This can be done in a unique fashion. The detailed proof proceeds by induction.

A similar criterion of forward regularity holds for sequences of upper triangular matrices.

Using the correspondence between forward and backward sequences of matrices we immediately obtain the corresponding criterion for backward regularity.

3.4. **Lyapunov regularity.** Let $\mathcal{A} = \{A_m\}_{m\in\mathbb{Z}}$ be a sequence of matrices in $GL(n,\mathbb{R})$. Set $\mathcal{A}^+ = \{A_m\}_{m\geq 0}$ and $\mathcal{A}^- = \{A_m\}_{m<0}$. Consider the forward and backward Lyapunov exponents χ^+ and χ^- specified by the sequence $\mathcal{A}$, i.e., by the sequences $\mathcal{A}^+$ and $\mathcal{A}^-$ respectively; see (3.1) and (3.2). Denote by
$$\mathcal{V}_{\chi^+} = \{V_i^+ : i = 1, \ldots, p^+\} \quad \text{and} \quad \mathcal{V}_{\chi^-} = \{V_i^- : i = 1, \ldots, p^-\}$$
the filtrations of $\mathbb{R}^n$ associated with the Lyapunov exponents χ^+ and χ^-.

We say that the filtrations $\mathcal{V}_{\chi^+}$ and $\mathcal{V}_{\chi^-}$ *comply* if the following properties hold:

1. $p^+ = p^- \stackrel{\text{def}}{=} p$;
2. there exists a decomposition
$$\mathbb{R}^n = \bigoplus_{i=1}^{p} E_i$$
into subspaces E_i such that if $i = 1, \ldots, p$ then
$$V_i^+ = \bigoplus_{j=1}^{i} E_j \quad \text{and} \quad V_i^- = \bigoplus_{j=i}^{p} E_j$$
(note that necessarily $E_i = V_i^+ \cap V_i^-$ for $i = 1, \ldots, p$);
3. if $v \in E_i \setminus \{0\}$ then
$$\lim_{m\to\pm\infty} \frac{1}{m} \log\|\mathcal{A}_m v\| = \chi_i,$$
with uniform convergence on $\{v \in E_i : \|v\| = 1\}$.

We say that the sequence $\mathcal{A}$ is *Lyapunov regular* or simply *regular* if:

1. $\mathcal{A}$ is simultaneously forward and backward regular (i.e., $\mathcal{A}^+$ is forward regular and $\mathcal{A}^-$ is backward regular);
2. the filtrations $\mathcal{V}_{\chi^+}$ and $\mathcal{V}_{\chi^-}$ comply.

Notice that the constant cocycle generated by a single matrix A (see Section 3.1) is Lyapunov regular since
$$\sum_{i=1}^{p} \chi_i \dim E_i = \log|\det A|.$$

Proposition 3.6. *If $\mathcal{A}$ is regular then:*

1. *the exponents χ^+ and χ^- are exact;*
2. *$\chi_i^- = -\chi_i$, $\dim E_i = k_i$, and*
$$\lim_{m\to\pm\infty} \frac{1}{m} \log|\det(\mathcal{A}_m|E_i)| = \chi_i k_i.$$

Simultaneous forward and backward regularity of a sequence of matrices $\mathcal{A}$ is not sufficient for Lyapunov regularity. Forward (respectively, backward) regularity does not depend on the backward (respectively, forward) behavior of $\mathcal{A}$, i.e., for $m \le 0$ (respectively, $m \ge 0$). On the other hand, Lyapunov regularity requires some compatibility between the forward and backward behavior which is expressed in terms of the filtrations $\mathcal{V}_{\chi^+}$ and $\mathcal{V}_{\chi^-}$.

Example 3.7. *Let*
$$A_m = \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} & \text{if } m \ge 0 \\ \begin{pmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{pmatrix} & \text{if } m < 0 \end{cases}.$$
Note that
$$\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} = R^{-1} \begin{pmatrix} 5/4 & -3/4 \\ -3/4 & 5/4 \end{pmatrix} R,$$
where R is the rotation by $\pi/4$ around 0. We have $\chi^+(1,0) = \chi^-(1,1) = \log 2$ and $\chi^+(0,1) = \chi^-(1,-1) = -\log 2$. Hence, $V_1^+ \ne V_1^-$, and thus, $\mathcal{A}$ is not regular. On the other hand, since $\det A_m = 1$*, we have*
$$\chi^+(v_1, v_2) = \chi^-(v_1, v_2) = \log 2 - \log 2 = 0,$$

and the exponents $\chi^+(v_1, v_2)$ *and* $\chi^-(v_1, v_2)$ *are exact for any linearly independent vectors* $v_1,\ v_2 \in \mathbb{R}^2$. *Therefore, the sequence* $\mathcal{A}$ *is simultaneously forward and backward regular.*

4. Cocycles and Lyapunov exponents

4.1. **Cocycles and linear extensions.** In what follows we assume that X is a measure space which is endowed with a σ-algebra of measurable subsets and that $f\colon X \to X$ is an invertible measurable transformation. For most substantive statements we will assume that f preserves a finite measure.

A function $\mathcal{A}\colon X \times \mathbb{Z} \to GL(n, \mathbb{R})$ is called a *linear multiplicative cocycle over* f or simply a *cocycle* if the following properties hold:

1. for every $x \in X$ we have $\mathcal{A}(x, 0) = \mathrm{Id}$ and if $m,\ k \in \mathbb{Z}$ then
$$\mathcal{A}(x, m+k) = \mathcal{A}(f^k(x), m)\mathcal{A}(x, k); \tag{4.1}$$
2. for every $m \in \mathbb{Z}$ the function $\mathcal{A}(\cdot, m)\colon X \to GL(n, \mathbb{R})$ is measurable.

If $\mathcal{A}$ is a cocycle, then $\mathcal{A}(f^{-m}(x), m)^{-1} = \mathcal{A}(x, -m)$ for every $x \in X$ and $m \in \mathbb{Z}$.

Given a measurable function $A\colon X \to GL(n, \mathbb{R})$ and $x \in X$, define the cocycle
$$\mathcal{A}(x, m) = \begin{cases} A(f^{m-1}(x)) \cdots A(f(x))A(x) & \text{if } m > 0 \\ \mathrm{Id} & \text{if } m = 0 \\ A(f^m(x))^{-1} \cdots A(f^{-2}(x))^{-1} A(f^{-1}(x))^{-1} & \text{if } m < 0 \end{cases}.$$
The map A is called the *generator* of the cocycle $\mathcal{A}$. One also says that the cocycle $\mathcal{A}$ is *generated* by the function A. Each cocycle $\mathcal{A}$ is generated by the function $A(\cdot) = \mathcal{A}(\cdot, 1)$.

The sequences of matrices that we discussed in the previous section are cocycles over the shift map $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$.

A cocycle $\mathcal{A}$ over f induces a *linear extension* $F\colon X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ of f to $X \times \mathbb{R}^n$, or a *linear skew product*, defined by
$$F(x, v) = (f(x), A(x)v).$$
In other words, the action of F on the fiber over x to the fiber over $f(x)$ is given by the linear map $A(x)$. If $\pi\colon X \times \mathbb{R}^n \to X$ is the projection, $\pi(x, v) = x$, then the following diagram
$$\begin{array}{ccc} X \times \mathbb{R}^n & \xrightarrow{\ F\ } & X \times \mathbb{R}^n \\ \pi \big\downarrow & & \big\downarrow \pi \\ X & \xrightarrow{\ f\ } & X \end{array}$$
is commutative. Notice that for each $m \in \mathbb{Z}$,
$$F^m(x, v) = (f^m(x), \mathcal{A}(x, m)v).$$
Linear extensions are particular cases of bundle maps of measurable vector bundles which we now consider. Let E and X be measure spaces and $\pi\colon E \to X$ a measurable map. One says that E is a *measurable vector bundle* over X if for every $x \in X$ there exists a measurable subset $Y_x \subset X$ containing x such that there exists a measurable map with measurable inverse $\pi^{-1}(Y_x) \to Y_x \times \mathbb{R}^n$. A bundle map

$F\colon E \to E$ over a measurable map $f\colon X \to X$ is a measurable map which makes the following diagram commutative:

$$\begin{array}{ccc} E & \xrightarrow{F} & E \\ \pi\big\downarrow & & \big\downarrow\pi \\ X & \xrightarrow{f} & X \end{array} .$$

The following proposition shows that from the measure theory point of view every vector bundle over a compact metric space is trivial, and hence, without loss of generality, one may always assume that $E = X \times \mathbb{R}^n$. In other words every bundle map of E is essentially a linear extension provided that the base space X is a compact metric space.

Proposition 4.1. *If E is a measurable vector bundle over a compact metric space (X, ν), then there is a subset $Y \subset X$ such that $\nu(Y) = 1$ and $\pi^{-1}(Y)$ is (isomorphic to) a trivial vector bundle.*

4.2. **Cohomology and tempered equivalence.** Let $A\colon X \to GL(n, \mathbb{R})$ be the generator of a cocycle $\mathcal{A}$ over the invertible measurable transformation $f\colon X \to X$. The cocycle $\mathcal{A}$ acts on the linear coordinate v_x on the fiber $\{x\} \times \mathbb{R}^n$ of $X \times \mathbb{R}^n$ by $v_{f(x)} = A(x)v_x$. Let $L(x) \in GL(n, \mathbb{R})$ be a linear coordinate change in each fiber, given by $u_x = L(x)v_x$ for each $x \in X$. We assume that the function $L\colon X \to GL(n, \mathbb{R})$ is measurable. Consider the function $B\colon X \to GL(n, \mathbb{R})$ for which $u_{f(x)} = B(x)u_x$. One can easily verify that

$$A(x) = L(f(x))^{-1}B(x)L(x),$$

and that B generates a new cocycle $\mathcal{B}$ over f. One can naturally think of the cocycles $\mathcal{A}$ and $\mathcal{B}$ as equivalent. However, since the function L is in general only measurable, without any additional assumption on L the measure-theoretical properties of the cocycles $\mathcal{A}$ and $\mathcal{B}$ can be very different. We now introduce a sufficiently general class of coordinate changes which make the notion of equivalence productive.

Let $Y \subset X$ be an f-invariant nonempty measurable set. A measurable function $L\colon X \to GL(n, \mathbb{R})$ is said to be *tempered on Y with respect to f* or simply *tempered on Y* if for every $x \in Y$ we have

$$\lim_{m \to \pm\infty} \frac{1}{m} \log\|L(f^m(x))\| = \lim_{m \to \pm\infty} \frac{1}{m} \log\|L(f^m(x))^{-1}\| = 0.$$

A cocycle over f is said to be *tempered on Y* if its generator is tempered on Y. If the real functions $x \mapsto \|L(x)\|$, $\|L(x)^{-1}\|$ are bounded or, more generally, have finite essential supremum, then the function L is tempered with respect to any invertible transformation $f\colon X \to X$ on any f-invariant nonempty measurable subset $Y \subset X$. The following statement provides a more general criterion for a function L to be tempered.

Proposition 4.2. *Let $f\colon X \to X$ be an invertible transformation preserving a probability measure ν, and $L\colon X \to GL(n, \mathbb{R})$ a measurable function. If*

$$\log\|L\|,\ \log\|L^{-1}\| \in L^1(X, \nu),$$

then L is tempered on some set of full ν-measure.

Let A, $B\colon X \to GL(n,\mathbb{R})$ be the generators, respectively, of two cocycles $\mathcal{A}$ and $\mathcal{B}$ over an invertible measurable transformation f, and $Y \subset X$ a measurable subset. The cocycles $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent on* Y or *cohomologous on* Y, if there exists a measurable function $L\colon X \to GL(n,\mathbb{R})$ which is tempered on Y such that for every $x \in Y$, we have

$$A(x) = L(f(x))^{-1}B(x)L(x). \tag{4.2}$$

This is clearly an equivalence relation and if two cocycles $\mathcal{A}$ and $\mathcal{B}$ are equivalent, we write $\mathcal{A} \sim_Y \mathcal{B}$. Equation (4.2) is called *cohomology equation.*

It follows from (4.2) that for any $x \in Y$ and $m \in \mathbb{Z}$,

$$\mathcal{A}(x,m) = L(f^m(x))^{-1}\mathcal{B}(x,m)L(x). \tag{4.3}$$

Proposition 4.2 immediately implies the following.

Corollary 4.3. *If* $L\colon X \to \mathbb{R}$ *is a measurable function such that* $\log\|L\|$, $\log\|L^{-1}\| \in L^1(X,\nu)$ *then any two cocycles* $\mathcal{A}$ *and* $\mathcal{B}$ *satisfying* (4.3) *are equivalent cocycles.*

We now consider the notion of equivalence for cocycles over different transformations. Let $f\colon X \to X$ and $g\colon Y \to Y$ be invertible measurable transformations. Assume that f and g are *measurably conjugated*, i.e., that $h \circ f = g \circ h$ for some invertible measurable transformation $h\colon X \to Y$. Let $\mathcal{A}$ be a cocycle over f and $\mathcal{B}$ a cocycle over g. The cocycles $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent* if there exists a measurable function $L\colon X \to GL(n,\mathbb{R})$ which is tempered on Y with respect to g, such that for every $x \in Y$, we have

$$A(h^{-1}(x)) = L(g(x))^{-1}B(x)L(x).$$

4.3. **Examples and basic constructions with cocycles.** We describe various examples of measurable cocycles over dynamical systems. Perhaps the simplest example is provided by the rigid cocycles generated by a single matrix. Starting from a given cocycle one can build other cocycles using some basic constructions in ergodic theory and algebra. Thus one obtains power cocycles, induced cocycles, and exterior power cocycles.

Let $\mathcal{A}$ be a measurable cocycle over a measurable transformation f of a Lebesgue space X. We will call a cocycle $\mathcal{A}$ *rigid* if it is equivalent to a cocycle whose generator A is a constant map. Rigid cocycles naturally arise in the classical Floquet Theory (where the dynamical system in the base is a periodic flow), and among smooth cocycles over translations on the torus with rotation vector satisfying a Diophantine condition (see [83, 151] and the references therein). In the setting of actions of groups other than $\mathbb{Z}$ and $\mathbb{R}$ rigid cocycles appear in the measurable setting for actions of higher rank semisimple Lie groups and lattices in such groups (see [261]), and in the smooth setting for hyperbolic actions of higher rank Abelian groups (see [136, 137]).

Given $m \geq 1$, consider the transformation $f^m\colon X \to X$ and the measurable cocycle $\mathcal{A}^m$ over f^m with the generator

$$A^m(x) \stackrel{\text{def}}{=} A(f^{m-1}(x)) \cdots A(x).$$

The cocycle $\mathcal{A}^m$ is called the *m-th power cocycle* of $\mathcal{A}$.

Assume that f preserves a measure ν and let $Y \subset X$ be a measurable subset of positive ν-measure. By Poincaré's Recurrence Theorem the set $Z \subset Y$ of points

$x \in Y$ such that $f^n(x) \in Y$ for infinitely many positive integers n, has measure $\nu(Z) = \nu(Y)$. We define the transformation $f_Y : Y \to Y \pmod 0$ as follows

$$f_Y(x) = f^{k_Y(x)}(x) \text{ where } k_Y(x) = \min\{k \geq 1 : f^k(x) \in Y\}.$$

The function k_Y and the map f_Y are measurable on Z. We call k_Y the *(first) return time* to Y and f_Y the *(first) return map* or *induced transformation* on Y.

Proposition 4.4 (see, for example, [67])**.** *The measure ν is invariant under f_Y, the function $k_Y \in L^1(X,\nu)$ and $\int_Y k_Y\, d\nu = \nu(\bigcup_{n\geq 0} f^n Y)$.*

Since $k_Y \in L^1(X,\nu)$, it follows from Birkhoff's Ergodic Theorem that the function

$$\tau_Y(x) = \lim_{k\to+\infty} \frac{1}{k} \sum_{i=0}^{k-1} k_Y(f_Y^i(x))$$

is well-defined for ν-almost all $x \in Y$ and that $\tau_Y \in L^1(X,\nu)$.

If $\mathcal{A}$ is a measurable linear cocycle over f with generator A, we define the *induced cocycle* $\mathcal{A}_Y$ over f_Y to be the cocycle with the generator

$$A_Y(x) = A^{k_Y(x)}(x).$$

Finally, given a cocycle $\mathcal{A}$ we define the cocycle $\mathcal{A}^{\wedge k} : X \times \mathbb{Z} \to (GL(n,\mathbb{R}))^{\wedge k}$ by $\mathcal{A}^{\wedge k}(x,m) = \mathcal{A}(x,m)^{\wedge k}$ (see Section 5.1 for the definition of exterior power). We call $\mathcal{A}^{\wedge k}$ the *k-fold exterior power cocycle* of $\mathcal{A}$.

4.4. **Hyperbolicity of cocycles.** The crucial notion of nonuniformly hyperbolic diffeomorphisms was introduced by Pesin in [196, 197]. In terms of cocycles this is the special case of derivative cocycles (see Section 6.1). Pesin's approach can readily be extended to general cocycles.

Consider a family of inner products $\langle\cdot,\cdot\rangle = \{\langle\cdot,\cdot\rangle_x : x \in X\}$ on $\mathbb{R}^n$. Given $x \in X$ we denote by $\|\cdot\|_x$ the norm and by $\sphericalangle(\cdot,\cdot)_x$ the angle induced by the inner product $\langle\cdot,\cdot\rangle_x$. In order to simplify the notation we often write $\|\cdot\|$ and $\sphericalangle(\cdot,\cdot)$ omitting the reference point x.

Let $Y \subset X$ be an f-invariant nonempty measurable subset. Let also λ, $\mu : Y \to (0,+\infty)$ and $\varepsilon : Y \to [0,\varepsilon_0]$, $\varepsilon_0 > 0$, be f-invariant measurable functions such that $\lambda(x) < \mu(x)$ for every $x \in Y$.

We say that a cocycle $\mathcal{A} : X \times \mathbb{Z} \to GL(n,\mathbb{R})$ over f is *nonuniformly partially hyperbolic in the broad sense* if there exist measurable functions C, $K : Y \to (0,+\infty)$ such that

1. for every $x \in Y$,
$$\text{either } \lambda(x)e^{\varepsilon(x)} < 1 \text{ or } \mu(x)e^{-\varepsilon(x)} > 1; \tag{4.4}$$
2. there exists a decomposition $\mathbb{R}^n = E_1(x) \oplus E_2(x)$, depending measurably on $x \in Y$, such that $A(x)E_1(x) = E_1(f(x))$ and $A(x)E_2(x) = E_2(f(x))$;
3. (a) for $v \in E_1(x)$ and $m > 0$,
$$\|\mathcal{A}(x,m)v\| \leq C(x)\lambda(x)^m e^{\varepsilon(x)m}\|v\|;$$
   (b) for $v \in E_2(x)$ and $m < 0$,
$$\|\mathcal{A}(x,m)v\| \leq C(x)\mu(x)^m e^{\varepsilon(x)|m|}\|v\|;$$
   (c) $\sphericalangle(E_1(f^m(x)), E_2(f^m(x))) \geq K(f^m(x))$ for every $m \in \mathbb{Z}$;

(d) for $m \in \mathbb{Z}$,

$$C(f^m(x)) \le C(x)e^{|m|\varepsilon(x)}, \quad K(f^m(x)) \ge K(x)e^{-|m|\varepsilon(x)}.$$

We say that a cocycle $\mathcal{A}\colon X \times \mathbb{Z} \to GL(n, \mathbb{R})$ over f is *nonuniformly (completely) hyperbolic* if the requirement (4.4) is replaced by the following stronger one: for every $x \in Y$,

$$\lambda(x)e^{\varepsilon(x)} < 1 < \mu(x)e^{-\varepsilon(x)}.$$

Proposition 4.5. *If a cocycle $\mathcal{A}$ over f is partially hyperbolic in the broad sense, then for every $x \in Y$,*

1. $\mathcal{A}(x, m)E_1(x) = E_1(f^m(x))$ *and* $\mathcal{A}(x, m)E_2(x) = E_2(f^m(x))$*;*
2. *for $v \in E_1(x)$ and $m < 0$,*
$$\|\mathcal{A}(x, m)v\| \ge C(f^m(x))^{-1}\lambda(x)^m e^{-\varepsilon(x)|m|}\|v\|;$$
3. *for $v \in E_2(x)$ and $m > 0$,*
$$\|\mathcal{A}(x, m)v\| \ge C(f^m(x))^{-1}\mu(x)^m e^{-\varepsilon(x)m}\|v\|.$$

The set Y is nested by the invariant sets $Y_{\lambda\mu\varepsilon}$ for which $\lambda(x) \le \lambda$, $\mu(x) \le \mu$ and $\varepsilon(x) \le \varepsilon$, i.e., $Y = \bigcup Y_{\lambda\mu\varepsilon}$ and $Y_{\lambda'\mu'\varepsilon'} \subset Y_{\lambda''\mu''\varepsilon''}$ provided $\lambda' \le \lambda''$, $\mu' \le \mu''$ and $\varepsilon' \le \varepsilon''$. On each of these sets the above estimates hold with $\lambda(x)$, $\mu(x)$ and $\varepsilon(x)$ replaced by λ, μ and ε respectively.

Even when a cocycle is continuous or smooth one should expect the functions λ, μ, ε, C, and K to be only measurable, the function C to be unbounded and K to have values arbitrarily close to zero.

If these functions turn to be continuous we arrive to the special case of uniformly hyperbolic cocycles. More precisely, we say that the cocycle $\mathcal{A}\colon X \times \mathbb{Z} \to GL(n, \mathbb{R})$ over f is *uniformly partially hyperbolic in the broad sense* if there exist $0 < \lambda < \mu < \infty$, $\lambda < 1$, and constants $c > 0$ and $\gamma > 0$ such that the following conditions hold

1. there exists a decomposition $\mathbb{R}^n = E_1(x) \oplus E_2(x)$, depending continuously on $x \in Y$, such that $A(x)E_1(x) = E_1(f(x))$ and $A(x)E_2(x) = E_2(f(x))$;
2. (a) for $v \in E_1(x)$ and $m > 0$,
$$\|\mathcal{A}(x, m)v\| \le c\lambda^m\|v\|;$$
   (b) for $v \in E_2(x)$ and $m < 0$,
$$\|\mathcal{A}(x, m)v\| \le c\mu^m\|v\|;$$
   (c) $\sphericalangle(E_1(f^m(x)), E_2(f^m(x))) \ge \gamma$ for every $m \in \mathbb{Z}$.

The principal example of uniformly hyperbolic cocycles are cocycles generated by Anosov diffeomorphisms and more generally Axiom A diffeomorphisms. The principal examples of nonuniformly hyperbolic cocycles are cocycles with nonzero Lyapunov exponents.

We will see below that a nonuniformly hyperbolic cocycle on a set Y of full measure (with respect to an invariant measure) is in fact, uniformly hyperbolic on a set $Y_\delta \subset Y$ of measure at least $1-\delta$ for arbitrarily small $\delta > 0$. This observation is crucial in studying topological and measure-theoretical properties of such cocycles. However, the "parameters" of uniform hyperbolicity, i.e., the numbers c and γ may vary with δ approaching ∞ and 0, respectively. We stress that this can only occur with a sub-exponential rate. We proceed with the formal description.

4.5. **Regular sets of hyperbolic cocycles.** Nonuniformly hyperbolic cocycles turn out to be uniformly hyperbolic on some compact but noninvariant subsets, called *regular sets*. They are nested and exhaust the whole space. Nonuniform hyperbolic structure appears then as a result of deterioration of the hyperbolic structure when a trajectory travels from one of these subsets to another. We first introduce regular sets for arbitrary cocycles and then establish their existence for nonuniformly hyperbolic cocycles.

Let $\mathcal{A}$ be a cocycle over X and $\varepsilon\colon X \to [0,+\infty)$ an f-invariant measurable function. Given $0 < \lambda < \mu < \infty$, $\lambda < 1$, and $\ell \geq 1$, denote by $\Lambda^\ell = \Lambda^\ell_{\lambda\mu}$ the set of points $x \in X$ for which there exists a decomposition $\mathbb{R}^n = E_{1x} \oplus E_{2x}$ such that for every $k \in \mathbb{Z}$ and $m > 0$,

1. if $v \in \mathcal{A}(x,k)E_{1x}$ then
$$\|\mathcal{A}(f^k(x),m)v\| \leq \ell\lambda^m e^{\varepsilon(x)(m+|k|)}\|v\|$$
and
$$\|\mathcal{A}(f^k(x),-m)v\| \geq \ell^{-1}\lambda^{-m} e^{-\varepsilon(x)(|k-m|+m)}\|v\|;$$
2. if $v \in \mathcal{A}(x,k)E_{2x}$ then
$$\|\mathcal{A}(f^k(x),-m)v\| \leq \ell\mu^{-m} e^{\varepsilon(x)(m+|k|)}\|v\|$$
and
$$\|\mathcal{A}(f^k(x),m)v\| \geq \ell^{-1}\mu^{m} e^{-\varepsilon(x)(|k+m|+m)}\|v\|;$$
3. $$\sphericalangle(E_{1f^k(x)}, E_{2f^k(x)}) \geq \ell^{-1} e^{-\varepsilon(x)|k|}.$$

The set Λ^ℓ is called a *regular set* (or a *Pesin set*).

It is easy to see that these sets have the following properties:

1. $\Lambda^\ell \subset \Lambda^{\ell+1}$;
2. for $m \in \mathbb{Z}$ we have $f^m(\Lambda^\ell) \subset \Lambda^{\ell'}$, where
$$\ell' = \ell \exp\left(|m| \sup\{\varepsilon(x) : x \in \Lambda^\ell\}\right);$$
3. the set $\Lambda = \Lambda_{\lambda\mu} \stackrel{\text{def}}{=} \bigcup_{\ell\geq 1} \Lambda^\ell$ is f-invariant;
4. if X is a topological space and $\mathcal{A}$ and ε are continuous then the sets Λ^ℓ are closed and the subspaces E_{1x} and E_{2x} vary continuously with $x \in \Lambda^\ell$ (with respect to the Grassmannian distance).

Every cocycle $\mathcal{A}$ over f, which is nonuniformly hyperbolic on a set $Y \subset X$, admits a nonempty regular set. Indeed, for each $0 < \lambda < \mu < \infty$, $\lambda < 1$, and each integer $\ell \geq 1$ let $Y^\ell \subset X$ be the set of points for which
$$\lambda(x) \leq \lambda < \mu \leq \mu(x),\ C(x) \leq \ell, \text{ and } K(x) \geq \ell^{-1}.$$
We have $Y^\ell \subset Y^{\ell+1}$, $Y^\ell \subset \Lambda^\ell$ and $E_{1x} = E_1(x)$, $E_{2x} = E_2(x)$ for every $x \in Y$.

4.6. **Lyapunov exponents for cocycles.** We extend the notion of Lyapunov exponent to cocycles over dynamical systems.

Let $\mathcal{A}$ be a cocycle over an invertible measurable transformation f of a measure space X with generator $A\colon X \to GL(n,\mathbb{R})$. Note that for each $x \in X$ the cocycle $\mathcal{A}$ generates a sequence of matrices $\{A_m\}_{m\in\mathbb{Z}} = \{A(f^m(x))\}_{m\in\mathbb{Z}}$. Therefore, every cocycle can be viewed as a collection of sequences of matrices which are indexed by the trajectories of f. One can associate to each of these sequences of matrices a Lyapunov exponent.

However, one should carefully examine the dependence of the Lyapunov exponent when one moves from a sequence of matrices to another one (see Proposition 4.6 below). This is what constitutes a substantial difference in studying cocycles over dynamical systems and sequences of matrices (see Sections 5.1 and 5.3 below). We now proceed with the formal definition of the Lyapunov exponent for cocycles.

Consider the generator $A\colon X \to GL(n,\mathbb{R})$ of the cocycle $\mathcal{A}$. Given a point $(x,v) \in X \times \mathbb{R}^n$, we define the *forward Lyapunov exponent of* (x,v) *(with respect to the cocycle* $\mathcal{A}$*)* by

$$\chi^+(x,v) = \chi^+(x,v,\mathcal{A}) = \varlimsup_{m\to+\infty} \frac{1}{m}\log\|\mathcal{A}(x,m)v\|.$$

Note that the number $\chi^+(x,v)$ does not depend on the norm $\|\cdot\|$ induced by an inner product on $\mathbb{R}^n$. With the convention $\log 0 = -\infty$, we obtain $\chi^+(x,0) = -\infty$ for every $x \in X$.

There exist a positive integer $p^+(x) \le n$, a collection of numbers

$$\chi_1^+(x) < \chi_2^+(x) < \cdots < \chi_{p^+(x)}^+(x),$$

and a filtration $\mathcal{V}_x^+$ of linear subspaces

$$\{0\} = V_0^+(x) \subsetneqq V_1^+(x) \subsetneqq \cdots \subsetneqq V_{p^+(x)}^+(x) = \mathbb{R}^n,$$

such that:

1. $V_i^+(x) = \{v \in \mathbb{R}^n : \chi^+(x,v) \le \chi_i^+(x)\}$;
2. if $v \in V_i^+(x) \setminus V_{i-1}^+(x)$, then $\chi^+(x,v) = \chi_i^+(x)$.

The numbers $\chi_i^+(x)$ are called the *values* of the Lyapunov exponent χ^+ at x. The number

$$k_i^+(x) = \dim V_i^+(x) - \dim V_{i-1}^+(x)$$

is called the *multiplicity* of the value $\chi_i^+(x)$. We also write

$$n_i^+(x) \overset{\text{def}}{=} \dim V_i^+(x) = \sum_{j=1}^{i} k_j^+(x).$$

The *Lyapunov spectrum* of χ^+ at x is the collection of pairs

$$\mathrm{Sp}_x^+\mathcal{A} = \{(\chi_i^+(x), k_i^+(x))\colon i = 1,\ \ldots,\ p^+(x)\}.$$

Observe that $k_i^+(f(x)) = k_i^+(x)$ and hence, $\mathrm{Sp}_{f(x)}^+\mathcal{A} = \mathrm{Sp}_x^+\mathcal{A}$.

Proposition 4.6. *The following properties hold:*

1. *the functions* χ^+ *and* p^+ *are measurable;*
2. $\chi^+ \circ f = \chi^+$ *and* $p^+ \circ f = p^+$*;*
3. $A(x)V_i^+(x) = V_i^+(f(x))$ *and* $\chi_i^+(f(x)) = \chi_i^+(x)$.

For every $(x,v) \in X \times \mathbb{R}^n$, we set

$$\chi^-(x,v) = \chi^-(x,v,\mathcal{A}) = \varlimsup_{m\to-\infty} \frac{1}{|m|}\log\|\mathcal{A}(x,m)v\|.$$

We call $\chi^-(x,v)$ the *backward Lyapunov exponent of* (x,v) *(with respect to the cocycle* $\mathcal{A}$*).* One can show that for every $x \in X$ there exist a positive integer $p^-(x) \le n$, the *values*

$$\chi_1^-(x) > \chi_2^-(x) > \cdots > \chi_{p^-(x)}^-(x),$$

and the *filtration* $\mathcal{V}_x^-$ of $\mathbb{R}^n$ associated with χ^- at x,

$$\mathbb{R}^n = V_1^-(x) \supsetneqq \cdots \supsetneqq V_{p^-(x)}^-(x) \supsetneqq V_{p^-(x)+1}^-(x) = \{0\},$$

such that $V_i^-(x) = \{v \in \mathbb{R}^n : \chi^-(x,v) \le \chi_i^-(x)\}$. The number

$$k_i^-(x) = \dim V_i^-(x) - \dim V_{i+1}^-(x)$$

is called the *multiplicity* of the value $\chi_i^-(x)$. We define the *Lyapunov spectrum* of χ^- at x by

$$\mathrm{Sp}_x^- \mathcal{A} = \{(\chi_i^-(x), k_i^-(x)) \colon i = 1, \ldots, p^-(x)\}.$$

Any nonuniformly (partially or completely) hyperbolic cocycle has nonzero Lyapunov exponents. More precisely,

1. If $\mathcal{A}$ is a nonuniformly partially hyperbolic cocycle (in the broad sense) on Y, then
$$Y \subset \left\{x \in X : \chi^+(x,v) \neq 0 \text{ for some } v \in \mathbb{R}^n \setminus \{0\}\right\}.$$
2. If $\mathcal{A}$ is a nonuniformly hyperbolic cocycle on Y, then
$$Y \subset \left\{x \in X : \chi^+(x,v) \neq 0 \text{ for all } v \in \mathbb{R}^n\right\}.$$

The converse statement is also true but is much more difficult. It is a manifestation of the Multiplicative Ergodic Theorem 5.5 of Oseledets. Namely, a cocycle whose Lyapunov exponents do not vanish almost everywhere is nonuniformly hyperbolic on a set of full measure (see Theorem 5.11).

Lyapunov exponents of a cocycle are invariants of a coordinate change which satisfies the tempering property as the following statement shows.

Proposition 4.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be equivalent cocycles on Y over a measurable transformation $f \colon X \to X$, and $L \colon X \to GL(n,\mathbb{R})$ a measurable function satisfying* (4.2) *that is tempered on Y. If $x \in Y$ then:*

1. *the forward and backward Lyapunov spectra coincide at x, i.e.,*
$$\mathrm{Sp}_x^+ \mathcal{A} = \mathrm{Sp}_x^+ \mathcal{B} \quad \textit{and} \quad \mathrm{Sp}_x^- \mathcal{A} = \mathrm{Sp}_x^- \mathcal{B};$$
2. *$L(x)$ preserves the forward and backward filtrations of $\mathcal{A}$ and $\mathcal{B}$, i.e.,*
$$L(x) V_i^+(x,\mathcal{A}) = V_i^+(x,\mathcal{B}), \quad i = 1, \ldots, p^+(x),$$
and
$$L(x) V_i^-(x,\mathcal{A}) = V_i^-(x,\mathcal{B}), \quad i = 1, \ldots, p^-(x).$$

5. Regularity and Multiplicative Ergodic Theorem

5.1. **Lyapunov regularity.** We extend the concept of regularity to cocycles over dynamical systems. Let $\mathcal{A}$ be a cocycle over an invertible measurable transformation f of a measure space X. As we saw, given $x \in X$, the cocycle $\mathcal{A}$ generates the sequence of matrices $\{A_m\}_{m\in\mathbb{Z}} = \{A(f^m(x))\}_{m\in\mathbb{Z}}$.

We say that x is *forward (*respectively*, backward) regular* for $\mathcal{A}$ if the sequence of matrices $\{A(f^m(x))\}_{m\in\mathbb{Z}}$ is forward (respectively, backward) regular.

Clearly, if x is a forward (respectively, backward) regular point for $\mathcal{A}$ then so is the point $f^m(x)$ for every $m \in \mathbb{Z}$. Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are equivalent cocycles on Y then the point $y \in Y$ is forward (respectively, backward) regular for $\mathcal{A}$ if and only if it is forward (respectively, backward) regular for $\mathcal{B}$.

Consider the filtrations $\mathcal{V}^+ = \{\mathcal{V}_x^+\}_{x\in X}$ and $\mathcal{V}^- = \{\mathcal{V}_x^-\}_{x\in X}$ of $\mathbb{R}^n$ associated with the Lyapunov exponents χ^+ and χ^- specified by the cocycle $\mathcal{A}$. For each

$x \in X$ these filtrations determine filtrations $\mathcal{V}_x^+$ and $\mathcal{V}_x^-$ of the Lyapunov exponents $\chi^+(x,\cdot)$ and $\chi^-(x,\cdot)$ for the sequence of matrices $\{A_m\}_{m\in\mathbb{Z}} = \{A(f^m(x))\}_{m\in\mathbb{Z}}$.

We say that the filtrations $\mathcal{V}^+$ and $\mathcal{V}^-$ comply at a point $x \in X$ if the filtrations $\mathcal{V}_x^+$ and $\mathcal{V}_x^-$ comply with respect to the sequence of matrices $\{A(f^m(x))\}_{m\in\mathbb{Z}}$. In other words, the filtrations $\mathcal{V}^+$ and $\mathcal{V}^-$ comply at $x \in X$ if the following properties hold:

1. $p^+(x) = p^-(x) \stackrel{\text{def}}{=} p(x)$;
2. there exists a decomposition
$$\mathbb{R}^n = \bigoplus_{i=1}^{p(x)} E_i(x) \tag{5.1}$$
into subspaces $E_i(x)$ such that $A(x)E_i(x) = E_i(f(x))$ and for $i = 1, \ldots, p(x)$,
$$V_i^+(x) = \bigoplus_{j=1}^{i} E_j(x) \quad \text{and} \quad V_i^-(x) = \bigoplus_{j=i}^{p(x)} E_j(x);$$
3. if $v \in E_i(x) \setminus \{0\}$ then
$$\lim_{m\to\pm\infty} \frac{1}{m} \log\|\mathcal{A}(x,m)v\| = \chi_i^+(x) = -\chi_i^-(x) \stackrel{\text{def}}{=} \chi_i(x),$$
with uniform convergence on $\{v \in E_i(x) : \|v\| = 1\}$.

We call the decomposition (5.1) the *Oseledets' decomposition* at the point x.

Property 2 requires some degree of compatibility between forward and backward regularity and is equivalent to the following: for $i = 1, \ldots, p(x)$ the spaces
$$E_i(x) = V_i^+(x) \cap V_i^-(x) \tag{5.2}$$
satisfy (5.1).

A point $x \in X$ is said to be *Lyapunov regular* or simply *regular* for $\mathcal{A}$ if the following conditions hold:

1. x is simultaneously forward and backward regular for $\mathcal{A}$;
2. the filtrations $\mathcal{V}^+$ and $\mathcal{V}^-$ comply at x.

The set of regular points is f-invariant. If $\mathcal{A}$ and $\mathcal{B}$ are equivalent cocycles on Y then $y \in Y$ is regular for $\mathcal{A}$ if and only if it is regular for $\mathcal{B}$. Under fairly general assumptions the set of regular points has full measure with respect to any invariant measure (see Theorem 5.5).

For each integer k, $1 \le k \le n$, let $(\mathbb{R}^n)^{\wedge k}$ be the space of alternating k-linear forms on $\mathbb{R}^n$. For any linear transformation A of $\mathbb{R}^n$, the *k-fold exterior power* $A^{\wedge k}$ of A is the unique linear transformation $A^{\wedge k}$ of $(\mathbb{R}^n)^{\wedge k}$ such that
$$A^{\wedge k}(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$$
for any $v_1, \ldots, v_k \in (\mathbb{R}^n)^{\wedge 1} \equiv \mathbb{R}^n$. One can define an inner product in $(\mathbb{R}^n)^{\wedge k}$ by requiring that for any $v_1 \wedge \cdots \wedge v_k$, $w_1 \wedge \cdots \wedge w_k \in (\mathbb{R}^n)^{\wedge k}$,
$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det B,$$
where $B = (b_{ij})$ is the $k \times k$ matrix with entries $b_{ij} = \langle v_i, w_j \rangle$ for each i and j. The induced norm satisfies the following properties:

1. $\|v_1 \wedge \cdots \wedge v_k\| \le \|v_1 \wedge \cdots \wedge v_\ell\| \cdot \|v_{\ell+1} \wedge \cdots \wedge v_k\|$ for any $\ell < k$;
2. for every linear transformations A, B of $\mathbb{R}^n$ and $1 \le k$, $\ell \le n$ the induced operator norm in $(\mathbb{R}^n)^{\wedge k}$ satisfies:

(a) $\|(AB)^{\wedge k}\| \le \|A^{\wedge k}\| \cdot \|B^{\wedge k}\|$;
(b) $\|A^{\wedge(k+\ell)}\| \le \|A^{\wedge k}\| \cdot \|A^{\wedge \ell}\| \le \|A\|^{k+\ell}$;
(c) $\|A^{\wedge k}\| = \prod_{j=1}^{k} d_j$, where $d_1 \ge d_2 \ge \cdots \ge d_n \ge 0$ are the eigenvalues of $(A^*A)^{1/2}$.

Proposition 5.1. *Let $x \in X$ be a regular point for $\mathcal{A}$. The following statements hold:*

1. *the exponents $\chi^+(x,\cdot)$ and $\chi^-(x,\cdot)$ are exact;*
2. *for $i = 1, \ldots, p(x)$,*
   (a) $\dim E_i(x) = k_i^+(x) = k_i^-(x) \stackrel{\text{def}}{=} k_i(x)$*;*
   (b) *for any vectors $v_1, \ldots, v_{k_i(x)} \in E_i(x)$ with $V(v_1, \ldots, v_{k_i(x)}) \neq 0$,*
$$\lim_{m\to\pm\infty} \frac{1}{m} \log V(\mathcal{A}(x,m)v_1, \ldots, \mathcal{A}(x,m)v_{k_i(x)}) = \chi_i(x)k_i(x).$$
3. *for $k = 1, \ldots, n$,*
$$\lim_{m\to\pm\infty} \frac{1}{m} \log\|\mathcal{A}(x,m)^{\wedge k}\| = \sum_{j=1}^{k} \chi'_{n-j+1}(x).$$

Identifying the space $E_i(f^m(x))$ with $\mathbb{R}^{k_i(x)}$ one can rewrite Property 2b in the following way: for $i = 1, \ldots, p(x)$,
$$\lim_{m\to\pm\infty} \frac{1}{m} \log|\det(\mathcal{A}(x,m)|E_i(x))| = \chi_i(x)k_i(x).$$

Furthermore, for every regular point $x \in X$, $1 \le i, j \le p(x)$ with $i \neq j$, and every distinct vectors $v, w \in H_i(x)$,
$$\lim_{m\to\pm\infty} \frac{1}{m} \log|\sin\sphericalangle(E_i(f^m(x)), E_j(f^m(x)))| = 0,$$
i.e., the angles between any two spaces $E_i(x)$ and $E_j(x)$ can grow at most sub-exponentially along the orbit of x, and
$$\lim_{m\to\pm\infty} \frac{1}{m} \log|\sin\sphericalangle(d_xf^m v, d_xf^m w)| = 0.$$

5.2. **Lyapunov exponents and basic constructions with cocycles.**

Proposition 5.2. *For every $x \in X$ and every $v \in \mathbb{R}^n$, if the exponent $\chi^+(x,v,\mathcal{A})$ is exact, then $\chi^+(x,v,\mathcal{A}^m)$ is exact and*
$$\chi^+(x,v,\mathcal{A}^m) = m\chi^+(x,v,\mathcal{A}).$$

It follows that if $x \in X$ is Lyapunov regular with respect to the cocycle $\mathcal{A}$ then so it is with respect to the cocycle $\mathcal{A}^m$. Moreover, the Oseledets' decomposition at a regular point $x \in X$ for the cocycle $\mathcal{A}$ provides the Oseledets' decomposition at x for the cocycle $\mathcal{A}^m$.

Proposition 5.3. *Let $\mathcal{A}$ be a measurable cocycle over f and $Y \subset X$ a measurable subset of positive ν-measure. For ν-almost every $x \in Y$ and every $v \in \mathbb{R}^n$, if $\chi^+(x,v,\mathcal{A})$ is exact then $\chi^+(x,v,\mathcal{A}_Y)$ is exact and*
$$\chi^+(x,v,\mathcal{A}_Y) = \tau_Y(x)\chi^+(x,v,\mathcal{A}).$$

It follows that ν-almost every $x \in Y$ is regular with respect to the cocycle $\mathcal{A}_Y$ if and only if it is regular with respect to the cocycle $\mathcal{A}$. Moreover, the Oseledets' decomposition at x for $\mathcal{A}$ provides the Oseledets' decomposition at x for $\mathcal{A}_Y$.

Proposition 5.4. *For every $x \in X$, $k = 1, \ldots, n$, and $v_1 \wedge \cdots \wedge v_k \in (\mathbb{R}^n)^{\wedge k}$, if the exponent $\chi^+(x, v_i, \mathcal{A})$ is exact for $i = 1, \ldots, k$, then $\chi^+(x, v_1 \wedge \cdots \wedge v_k, \mathcal{A}^{\wedge k})$ is exact and*

$$\chi^+(x, v_1 \wedge \cdots \wedge v_k, \mathcal{A}^{\wedge k}) = \sum_{i=1}^{k} \chi^+(x, v_i, \mathcal{A}).$$

It follows that if $x \in X$ is Lyapunov regular with respect to the cocycle $\mathcal{A}$ then so it is with respect to the cocycle $\mathcal{A}^{\wedge k}$. Moreover, from the Oseledets' decomposition $\bigoplus_{i=1}^{s(x)} E_i(x)$ at a regular point $x \in X$ for the cocycle $\mathcal{A}$ we obtain the Oseledets' decomposition

$$\bigoplus_{i_1, \ldots, i_k} E_{i_1}(x)^{\wedge 1} \wedge \cdots \wedge E_{i_k}(x)^{\wedge 1}$$

of $(\mathbb{R}^n)^{\wedge k}$ at x for the cocycle $\mathcal{A}^{\wedge k}$.

5.3. **Multiplicative Ergodic Theorem I: Oseledets' approach.** Lyapunov regularity is a strong condition which imposes certain requirements on the forward and backward behavior of trajectories. It is also not easy to verify this condition. Nevertheless, it turns out that Lyapunov regularity is "typical" in the measure-theoretical sense.

Theorem 5.5 (Multiplicative Ergodic Theorem, Oseledets [191]; see also [24] and [174])**.** *Let f be an invertible measure preserving transformation of a Lebesgue space (X, ν) and $\mathcal{A}$ a measurable cocycle over f whose generator satisfies the following integrability condition*

$$\log^+\|A\|,\ \log^+\|A^{-1}\| \in L^1(X, \nu), \tag{5.3}$$

where $\log^+ a = \max\{\log a, 0\}$. Then the set of regular points for $\mathcal{A}$ has full ν-measure.

Let us notice that Property (5.3) holds for any cocycle $A\colon X \to GL(n, \mathbb{R})$ for which there is a positive constant c such that $\|A(x)^{\pm 1}\| \le c$ for ν-almost all $x \in X$.

For one-dimensional cocycles, i.e., cocycles with values in $GL(1, \mathbb{R})$, the Multiplicative Ergodic Theorem amounts to Birkhoff's Ergodic Theorem since

$$\log|\mathcal{A}(x, m)| = \sum_{j=0}^{m-1} \log|A(f^j(x))|.$$

The main idea of Oseledets in proving the Multiplicative Ergodic Theorem is to reduce the general case to the case of triangular cocycles and then use a version of Theorem 3.5 to establish regularity.

The reduction to triangular cocycles goes as follows. First one constructs an extension of the transformation f,

$$F\colon X \times SO(n, \mathbb{R}) \to X \times SO(n, \mathbb{R}),$$

where $SO(n, \mathbb{R})$ is the group of orthogonal $n \times n$ matrices. Given $(x, U) \in X \times SO(n, \mathbb{R})$ one can apply the Gram–Schmidt orthogonalization procedure to the columns of the matrix $A(x)U$ and write

$$A(x)U = R(x, U)T(x, U), \tag{5.4}$$

where $R(x,U)$ is orthogonal and $T(x,U)$ is lower triangular (with positive entries on the diagonal). The two matrices $R(x,U)$ and $T(x,U)$ are uniquely defined, and their entries are linear combinations of the entries of U. Set

$$F(x,U) = (f(x), R(x,U)).$$

Consider the projection $\pi\colon (x,U) \mapsto U$. By (5.4), we obtain

$$T(x,U) = ((\pi \circ F)(x,U))^{-1} A(x) \pi(x,U). \tag{5.5}$$

Let $\widetilde{\mathcal{A}}$ and $\mathcal{T}$ be two cocycles over F defined respectively by $\widetilde{\mathcal{A}}(x,U) = A(x)$ and $\mathcal{T}(x,U) = T(x,U)$. Since $\|U\| = 1$ for every $U \in SO(n,\mathbb{R})$, it follows from (5.5) that the cocycles $\widetilde{\mathcal{A}}$ and $\mathcal{T}$ are equivalent on $X \times SO(n,\mathbb{R})$. Therefore a point $(x,U) \in X \times SO(n,\mathbb{R})$ is regular for $\widetilde{\mathcal{A}}$ if and only if it is regular for $\mathcal{T}$.

By the Representation Theorem for Lebesgue spaces we may assume that X is a compact metric space and that $f\colon X \to X$ is Borel measurable. Let $\mathcal{M}$ be the set of all Borel probability measures $\widetilde{\nu}$ on $X \times SO(n,\mathbb{R})$ which satisfy

$$\widetilde{\nu}(B \times SO(n,\mathbb{R})) = \nu(B) \tag{5.6}$$

for all measurable sets $B \subset X$. Then $\mathcal{M}$ is a compact convex subset of a locally convex topological vector space. The map $F_*\colon \mathcal{M} \to \mathcal{M}$ defined by

$$(F_*\widetilde{\nu})(B) = \widetilde{\nu}(F^{-1}B)$$

is a bounded linear operator. By the Tychonoff Fixed Point Theorem, there exists a fixed point $\widetilde{\nu}_0 \in \mathcal{M}$ for the operator F_*, i.e., a measure $\widetilde{\nu}_0$ such that $\widetilde{\nu}_0(F^{-1}B) = \widetilde{\nu}_0(B)$ for every measurable set $B \subset X \times SO(n,\mathbb{R})$. By (5.6), we conclude that the set of regular points for $\mathcal{A}$ has full ν-measure if and only if the set of regular points for $\widetilde{\mathcal{A}}$ has full $\widetilde{\nu}_0$-measure, and hence, if and only if the set of regular points for $\mathcal{T}$ has full $\widetilde{\nu}_0$-measure.

We may now assume that $A(x) = (a_{ij}(x))$ is a lower triangular matrix (i.e., $a_{ij}(x) = 0$ if $i < j$). Write $A(x)^{-1} = (b_{ij}(x))$ and note that $b_{ii}(x) = 1/a_{ii}(x)$ for each i. By (5.3), $\log^+|a_{ij}|$, $\log^+|b_{ij}| \in L^1(X,\nu)$. It follows from Birkhoff's Ergodic Theorem that for ν-almost every $x \in X$,

$$\lim_{m\to+\infty} \frac{1}{m} \log^+|a_{ij}(f^m(x))| = \lim_{m\to-\infty} \frac{1}{m} \log^+|b_{ij}(f^m(x))| = 0. \tag{5.7}$$

Note that

$$|\log|a_{ii}|| = \log^+|a_{ii}| + \log^-|a_{ii}| = \log^+|a_{ii}| + \log^+|b_{ii}|. \tag{5.8}$$

By (5.3) and (5.8), we obtain $\log|a_{ii}| = -\log|b_{ii}| \in L^1(X,\nu)$. Birkhoff's Ergodic Theorem guarantees the existence of measurable functions $\lambda_i \in L^1(X,\nu)$, $i = 1, \ldots, n$, such that for ν-almost every $x \in X$,

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{k=0}^{m-1} \log|a_{ii}(f^k(x))| = \lim_{m\to-\infty} \frac{1}{m} \sum_{k=m}^{-1} \log|b_{ii}(f^k(x))| = \lambda_i(x). \tag{5.9}$$

Let $Y \subset X$ be the set of points for which (5.7) and (5.9) hold. It is a set of full ν-measure. The proof is concluded by showing that Y consists of regular points for $\mathcal{A}$. Indeed, by Theorem 3.5, the sequence $\{A(f^m(x))\}_{m\in\mathbb{Z}}$ is simultaneously forward and backward regular for every $x \in Y$. Moreover, the numbers $\lambda_i(x)$ are the forward Lyapunov exponents counted with their multiplicities (but possibly not ordered), and are the symmetric of the backward Lyapunov exponents counted with their multiplicities (but possibly not ordered either). We conclude that $p^+(x) =$

$p^-(x) \stackrel{\text{def}}{=} p(x)$ and $\chi_i^-(x) = -\chi_i^+(x)$ for $i = 1, \ldots, p(x)$. The hardest and more technical part of the proof is to show that the spaces $E_1(x), \ldots, E_{p(x)}(x)$, defined by (5.2), satisfy (5.1).

Consider the set N of points which are *not* Lyapunov regular. This set has zero measure with respect to *any* invariant Borel probability measure but in general is not empty. For example, for the derivative cocycle (see below Section 6.1) generated by a volume-preserving Anosov diffeomorphism the set of nonregular points has positive Hausdorff dimension provided that the Riemannian volume is *not* the measure of maximal entropy (see [26]).

On another end, Herman [109] (see also Section 7.3.1) and Walters [245] constructed examples of continuous cocycles with values in $SL(2,\mathbb{R})$ over *uniquely ergodic* homeomorphisms of compact metric spaces for which the set of nonregular points is not empty.

Furman [92] found additional conditions on the cocycle over a uniquely ergodic homeomorphism which guarantee that *every* point is Lyapunov regular. Namely, the generator of the cocycle should be either 1) *continuously diagonalizable*, i.e., continuously equivalent to a diagonal matrix, or 2) *one-point Lyapunov spectrum*, or 3) continuously equivalent to an *eventually positive function*, i.e., for some $n \geq 0$ all the entries of $A(x, n)$ are positive.

5.4. **Multiplicative Ergodic Theorem II: Raghunathan's approach.** We describe another approach to the proof of the Multiplicative Ergodic Theorem due to Raghunathan [211]. It exploits the Sub-Additive Ergodic Theorem. The work of Raghunathan also contains an extension to local fields (such as the field $\mathbb{Q}_p$ of p-adic numbers).

Let $f\colon X \to X$ be a measurable transformation. A measurable function $\mathcal{B}\colon X \times \mathbb{Z} \to \mathbb{R} \setminus \{0\}$ is called a *sub-additive cocycle over f* if for every $x \in X$ the following properties hold:

1. $\mathcal{B}(x, 0) = 1$;
2. if m, $k \in \mathbb{Z}$ then

$$\mathcal{B}(x, m + k) \leq \mathcal{B}(f^k(x), m) + \mathcal{B}(x, k).$$

If $\mathcal{A}\colon X \times \mathbb{Z} \to GL(n, \mathbb{R})$ is a multiplicative cocycle over f (see (4.1)), then $\mathcal{B} = \log\|\mathcal{A}\|$ is a sub-additive cocycle. Indeed, by (4.1),

$$\log\|\mathcal{A}(x, m + k)\| \leq \log\|\mathcal{A}(f^k(x), m)\| + \log\|\mathcal{A}(x, k)\|.$$

The following statement is an immediate consequence of Kingman's Sub-additive Ergodic Theorem (see [215]).

Theorem 5.6. *Let f be an invertible measure preserving transformation of a Lebesgue space (X, ν), and $\mathcal{A}$ a measurable multiplicative cocycle over f whose generator satisfies* (5.3)*. Then there exist f-invariant measurable functions $\varphi_+\colon X \to \mathbb{R}$ and $\varphi_-\colon X \to \mathbb{R}$ such that for ν-almost every $x \in X$,*

$$\varphi_+(x) = \lim_{m \to +\infty} \frac{1}{m} \log\|\mathcal{A}(x, m)\| = -\lim_{m \to -\infty} \frac{1}{m} \log\|\mathcal{A}(x, m)^{-1}\|,$$

$$\varphi_-(x) = \lim_{m \to -\infty} \frac{1}{m} \log\|\mathcal{A}(x, m)\| = -\lim_{m \to +\infty} \frac{1}{m} \log\|\mathcal{A}(x, m)^{-1}\|.$$

Moreover φ_+, $\varphi_- \in L^1(X,\nu)$ *and*

$$\int_X \varphi_+ \, d\nu = \lim_{m\to+\infty} \frac{1}{m} \int_X \log\|\mathcal{A}(x,m)\| \, d\nu(x)$$
$$= -\lim_{m\to-\infty} \frac{1}{m} \int_X \log\|\mathcal{A}(x,m)^{-1}\| \, d\nu(x),$$
$$\int_X \varphi_- \, d\nu = \lim_{m\to-\infty} \frac{1}{m} \int_X \log\|\mathcal{A}(x,m)\| \, d\nu(x)$$
$$= -\lim_{m\to+\infty} \frac{1}{m} \int_X \log\|\mathcal{A}(x,m)^{-1}\| \, d\nu(x).$$

As an immediate corollary we obtain that the values of the Lyapunov exponents $\chi_i^+(x)$ and $\chi_i^-(x)$ are integrable functions provided that (5.3) holds.

Let $\mathcal{A}$ be a measurable multiplicative cocycle over a transformation f. For each $i = 1, \ldots, n$ the function $\log\|\mathcal{A}^{\wedge i}\|$ is a sub-additive cocycle.

We present now Raghunathan's version of the Multiplicative Ergodic Theorem 5.5. Let us stress that Raghunathan considered the case of noninvertible transformations but his methods can be adapted to invertible transformations and we state the corresponding result here; we refer the reader to Section 5.7 where we consider the case of noninvertible transformations.

Theorem 5.7 (Raghunathan [211])**.** *Let f be an invertible measure preserving transformation of a Lebesgue space (X,ν), and $\mathcal{A}$ a measurable multiplicative cocycle over f whose generator satisfies* (5.3)*. Then there exists a set $Y \subset X$ of full ν-measure such that if $x \in Y$ then:*

1. *x is a regular point for $\mathcal{A}$;*
2. *there exist matrices A_x^+ and A_x^- such that*
$$\lim_{m\to\pm\infty} (\mathcal{A}(x,m)^*\mathcal{A}(x,m))^{1/(2|m|)} = A_x^\pm;$$
3. *the distinct eigenvalues of A_x^+ are the numbers $e^{\chi_1(x)}, \ldots, e^{\chi_{s(x)}(x)}$;*
4. *the distinct eigenvalues of A_x^- are the numbers $e^{\chi_1(x)}, \ldots, e^{\chi_{s(x)}(x)}$.*

5.5. **Tempering kernels and the Reduction Theorem.** The results in the previous sections allow one to obtain a "normal form" of a general measurable cocycle associated with its Lyapunov exponent. Let us begin with the simple particular case of a rigid cocycle $\mathcal{A}$, i.e., a cocycle whose generator is a constant map A (see Section 4.6). It is easy to see that the cocycle $\mathcal{A}$ is equivalent to the rigid cocycle $\mathcal{B}$ whose generator is the Jordan block form of the matrix A. We consider $\mathcal{B}$ as the "normal form" of $\mathcal{A}$, and say that $\mathcal{A}$ is reduced to $\mathcal{B}$.

A general measurable cocycle $\mathcal{A}$ satisfying the integrability condition (5.3) is so to speak "weakly" rigid, i.e., it can be reduced to a constant cocycle up to an arbitrarily small error. We consider this constant cocycle as a "normal form" of $\mathcal{A}$. More precisely, by the Oseledets–Pesin Reduction Theorem 5.10 below given $\varepsilon > 0$, there exists a cocycle $\mathcal{A}_\varepsilon$ which is equivalent to $\mathcal{A}$ and has block form, such that the generator A_ε^i of each block satisfies

$$e^{\chi_i(x)-\varepsilon}\|v\| \le \|A_\varepsilon^i(x)v\| \le e^{\chi_i(x)+\varepsilon}\|v\|$$

for each regular point x and each $v \in E_i(x)$, where $\{E_i(x) : i = 1, \ldots, p(x)\}$ is the Oseledets' decomposition at x (see (5.1)). We say that $\mathcal{A}_\varepsilon$ is the *reduced form* of $\mathcal{A}$.

To proceed with the description of normal forms we first introduce a family of inner products $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_x$ on $\mathbb{R}^n$ for $x \in X$. We start with the following auxiliary result.

Proposition 5.8. *For each $\varepsilon > 0$ and each regular point $x \in X$ for $\mathcal{A}$, the formula*

$$\langle u, v\rangle'_{x,i} = \sum_{m\in\mathbb{Z}} \langle \mathcal{A}(x,m)u, \mathcal{A}(x,m)v\rangle e^{-2\chi_i(x)m-2\varepsilon|m|} \tag{5.10}$$

determines a scalar product on $E_i(x)$.

For a fixed $\varepsilon > 0$ we introduce a new inner product on $\mathbb{R}^n$ by

$$\langle u, v\rangle'_x = \sum_{i=1}^{p(x)} \langle u_i, v_i\rangle'_{x,i},$$

where u_i and v_i are the projections of the vectors u and v over $E_i(x)$ along $\bigoplus_{j\neq i} E_j(x)$. We call $\langle\cdot,\cdot\rangle'_x$ a *Lyapunov inner product at x*, and the corresponding norm $\|\cdot\|'_x$ a *Lyapunov norm at x*. The sequence of weights $\{e^{-2\chi_i(x)m-2\varepsilon|m|}\}_{m\in\mathbb{Z}}$ in (5.10) is called a *Pesin Tempering Kernel*. The value of $\langle u, v\rangle'_x$ depends on the number ε. The Lyapunov inner product has the following properties.

Proposition 5.9. *The following properties hold:*

1. *The inner product $\langle\cdot,\cdot\rangle'_x$ depends measurably on the regular point x.*
2. *For every regular point $x \in X$ and $i \neq j$, the spaces $E_i(x)$ and $E_j(x)$ are orthogonal with respect to the Lyapunov inner product.*

A coordinate change $C_\varepsilon\colon X \to GL(n,\mathbb{R})$ is called a *Lyapunov change of coordinates* if for each regular point $x \in X$ and $u, v \in \mathbb{R}^n$ it satisfies:

$$\langle u, v\rangle_x = \langle C_\varepsilon(x)u, C_\varepsilon(x)v\rangle'_x. \tag{5.11}$$

Note that the identity (5.11) does not determine the function $C_\varepsilon(x)$ uniquely.

The following result known as Oseledets–Pesin Reduction Theorem provides a complete description of normal forms for cocycles.

Theorem 5.10 (see [135]). *Let $f\colon X \to X$ be an invertible measure preserving transformation of the Lebesgue space (X,ν), and $\mathcal{A}$ a measurable cocycle over f. Given $\varepsilon > 0$ and a regular point x,*

1. *there exists a Lyapunov change of coordinates C_ε which sends the orthogonal decomposition $\bigoplus_{i=1}^{p(x)} \mathbb{R}^{k_i(x)}$ to the decomposition $\bigoplus_{i=1}^{p(x)} E_i(x)$ of $\mathbb{R}^n$;*
2. *the cocycle $A_\varepsilon(x) = C_\varepsilon(f(x))^{-1}A(x)C_\varepsilon(x)$ has the block form*

$$A_\varepsilon(x) = \begin{pmatrix} A_\varepsilon^1(x) & & \\ & \ddots & \\ & & A_\varepsilon^{s(x)}(x) \end{pmatrix}, \tag{5.12}$$

   where each block $A_\varepsilon^i(x)$ is a $k_i(x) \times k_i(x)$ matrix, and the entries are zero above and below the matrices $A_\varepsilon^i(x)$;
3. *each block $A_\varepsilon^i(x)$ satisfies*

$$e^{\chi_i(x)-\varepsilon} \le \|A_\varepsilon^i(x)^{-1}\|^{-1} \le \|A_\varepsilon^i(x)\| \le e^{\chi_i(x)+\varepsilon};$$

4. *if the integrability condition (5.3) holds then the map C_ε is tempered ν-almost everywhere, and the spectra of $\mathcal{A}$ and $\mathcal{A}_\varepsilon$ coincide ν-almost everywhere.*

In the particular case of cocycles with values in $GL(2,\mathbb{R})$ Thieullen [239] showed that if the two Lyapunov exponents are equal the cocycle is conjugate to one of the following: a rotation cocycle, an upper triangular cocycle, or a diagonal cocycle modulo a rotation by $\pi/2$.

An important manifestation of the Oseledets–Pesin Reduction Theorem is a criterion of nonuniform hyperbolicity (partial or complete) of measurable cocycles via the values of their Lyapunov exponents.

Theorem 5.11. *Let $f\colon X \to X$ be an invertible measure preserving transformation of the Lebesgue space (X,ν), and $\mathcal{A}$ a measurable cocycle over f whose generator satisfies* (5.3)*. Then the following properties hold:*

1. *if the set*
$$Z_{ph} = \{x \in X : \chi^{+}(x,v) \neq 0 \text{ for some } v \in \mathbb{R}^n \setminus \{0\}\}$$
has measure $\nu(Z_{ph}) > 0$ then $\mathcal{A}$ is nonuniformly partially hyperbolic in the broad sense on some set $W_{ph} \subset Z_{ph}$ with $\nu(W_{ph}) = \nu(Z_{ph})$;
2. *if the set*
$$Z_{h} = \{x \in X : \chi^{+}(x,v) \neq 0 \text{ for all } v \in \mathbb{R}^n \setminus \{0\}\}$$
has measure $\nu(Z_{h}) > 0$ then $\mathcal{A}$ is nonuniformly hyperbolic on some set $W_{h} \subset Z_{h}$ with $\nu(W_{h}) = \nu(Z_{h})$.

This theorem was first proved by Pesin in [197] for the special case of derivative cocycles (see the definition of the derivative cocycle in the next section) but the argument can readily be extended to the case of general cocycle.

The proof of this crucial statement is based upon the following observation. Given a regular point x and a small ε there exists a number $m(x,\varepsilon)$ such that for $m \geq m(x,\varepsilon)$,
$$\chi_i - \varepsilon \leq \frac{1}{n}\log\|\mathcal{A}_{ix}^{m}\| \leq \chi_i + \varepsilon, \quad -\chi_i - \varepsilon \leq \frac{1}{n}\log\|\mathcal{A}_{ix}^{-m}\| \leq -\chi_i + \varepsilon,$$
and
$$-\chi_i - \varepsilon \leq \frac{1}{n}\log\|\mathcal{B}_{ix}^{m}\| \leq -\chi_i + \varepsilon, \quad \chi_i - \varepsilon \leq \frac{1}{n}\log\|\mathcal{B}_{ix}^{-m}\| \leq \chi_i + \varepsilon,$$
where $\mathcal{A}_{ix}^{m} = \mathcal{A}(x,nm)|E_i(x)$ and $\mathcal{B}_{ix}^{m} = \mathcal{B}(x,m)|E_i^{*}(x)$ with $E_i^{*}(x)$ the dual space to $E_i(x)$. Here
$$\mathcal{B}(x,m) = \begin{cases} (A(x)^{*})^{-1}(A(f(x))^{*})^{-1}\cdots(A(f^{m-1}(x))^{*})^{-1} & \text{if } m > 0 \\ \mathrm{Id} & \text{if } m = 0\ . \\ A(f^{-1}(x))^{*}A(f^{-2}(x))^{*}\cdots A(f^{m}(x))^{*} & \text{if } m < 0 \end{cases}$$
Set
$$D_1^{\pm}(x,\varepsilon) = \min_{1\leq i\leq s}\min_{0\leq j\leq m(x,\varepsilon)}\left\{1, \|\mathcal{A}_{ix}^{j}\|e^{(-\chi_i\pm\varepsilon)j}, \|\mathcal{B}_{ix}^{j}\|e^{(\chi_i\pm\varepsilon)j}\right\},$$
$$D_2^{\pm}(x,\varepsilon) = \max_{1\leq i\leq s}\max_{0\leq j\leq m(x,\varepsilon)}\left\{1, \|\mathcal{A}_{ix}^{j}\|e^{(-\chi_i\pm\varepsilon)j}, \|\mathcal{B}_{ix}^{j}\|e^{(\chi_i\pm\varepsilon)j}\right\}$$
and
$$D_1(x,\varepsilon) = \min\{D_1^{+}(x,\varepsilon), D_1^{-}(x,\varepsilon)\},$$
$$D_2(x,\varepsilon) = \max\{D_2^{+}(x,\varepsilon), D_2^{-}(x,\varepsilon)\},$$
$$D(x,\varepsilon) = \max\{D_1(x,\varepsilon)^{-1}, D_2(x,\varepsilon)\}.$$

The function $D(x,\varepsilon)$ is measurable, and if $m \ge 0$ and $1 \le i \le p$ then

$$\begin{aligned} D(x,\varepsilon)^{-1} e^{(\pm\chi_i-\varepsilon)m} &\le \|\mathcal{A}_{ix}^{\pm m}\| \le D(x,\varepsilon) e^{(\pm\chi_i+\varepsilon)m}, \\ D(x,\varepsilon)^{-1} e^{(\pm\chi_i-\varepsilon)m} &\le \|\mathcal{B}_{ix}^{\pm m}\| \le D(x,\varepsilon) e^{(\pm\chi_i+\varepsilon)m}. \end{aligned} \tag{5.13}$$

Moreover, if $d \ge 1$ is a number for which the inequalities (5.13) hold for all $m \ge 0$ and $1 \le i \le p$ with $D(x,\varepsilon)$ replaced by d then $d \ge D(x,\varepsilon)$. Therefore,

$$\begin{aligned} D(x,\varepsilon) = \inf\{d \ge 1 : &\text{ the inequalities (5.13) hold for all } n \ge 0 \\ &\text{ and } 1 \le i \le p \text{ with } D(x,\varepsilon) \text{ replaced by } d\}. \end{aligned} \tag{5.14}$$

We wish to compare the values of the function $D(x,\varepsilon)$ at the points x and $f^j(x)$. We introduce the identification map $\tau_x \colon (\mathbb{R}^n)^* \to \mathbb{R}^n$ such that $\langle \tau_x(\varphi), v\rangle = \varphi(v)$ where $v \in \mathbb{R}^n$ and $\varphi \in (\mathbb{R}^n)^*$.

Let $\{v_k^m : k = 1, \dots, \ell\}$ be a basis of $E_i(f^m(x))$ and $\{w_k^m : k = 1, \dots, \ell\}$ the dual basis of $E_i^*(f^m(x))$. We have $\tau_{f^m(x)}(w_k^m) = v_k^m$. Denote by $A_{m,j}^i$ and $B_{m,j}^i$ the matrices corresponding to the linear maps $\mathcal{A}_{if^j(x)}^m$ and $\mathcal{B}_{if^j(x)}^m$ with respect to the above bases. We have that

$$A_{j,0}^i (B_{j,0}^i)^* = \mathrm{Id}$$

where * stands for matrix transposition. Hence, for every $m > 0$ the matrix corresponding to the map $\mathcal{A}_{if^j(x)}^m$ is

$$A_{m,j}^i = A_{m+j,0}^i (A_{j,0}^i)^{-1} = A_{m+j,0}^i (B_{j,0}^i)^*.$$

Therefore, in view of (5.13), we obtain that if $m > 0$ then

$$\|\mathcal{A}_{if^j(x)}^m\| \le D(x,\varepsilon)^2 e^{(\chi_i+\varepsilon)(m+j)+(-\chi_i+\varepsilon)j} = D(x,\varepsilon)^2 e^{2\varepsilon j} e^{(\chi_i+\varepsilon)m},$$

$$\|\mathcal{A}_{if^j(x)}^m\| \ge D(x,\varepsilon)^{-2} e^{(\chi_i-\varepsilon)(m+j)+(-\chi_i-\varepsilon)j} = D(x,\varepsilon)^{-2} e^{-2\varepsilon j} e^{(\chi_i-\varepsilon)m},$$

if $m > 0$ and $j - m \ge 0$ then

$$\|\mathcal{A}_{if^j(x)}^{-m}\| \le D(x,\varepsilon)^2 e^{(\chi_i+\varepsilon)(j-m)+(-\chi_i+\varepsilon)j} = D(x,\varepsilon)^2 e^{2\varepsilon j} e^{(-\chi_i+\varepsilon)m},$$

$$\|\mathcal{A}_{if^j(x)}^{-m}\| \ge D(x,\varepsilon)^{-2} e^{(\chi_i-\varepsilon)(j-m)+(-\chi_i-\varepsilon)j} = D(x,\varepsilon)^{-2} e^{-2\varepsilon j} e^{(-\chi_i-\varepsilon)m},$$

and if $m > 0$ and $m - j \ge 0$ then

$$\|\mathcal{A}_{if^j(x)}^{-m}\| \le D(x,\varepsilon)^2 e^{(\chi_i+\varepsilon)(m-j)+(-\chi_i+\varepsilon)j} = D(x,\varepsilon)^2 e^{2\varepsilon j} e^{(-\chi_i+\varepsilon)m},$$

$$\|\mathcal{A}_{if^j(x)}^{-m}\| \ge D(x,\varepsilon)^{-2} e^{(\chi_i-\varepsilon)(m-j)+(-\chi_i-\varepsilon)j} = D(x,\varepsilon)^{-2} e^{-2\varepsilon j} e^{(-\chi_i-\varepsilon)m},$$

Similar inequalities hold for the maps $\mathcal{B}_{if^j(x)}^m$. Comparing this with the inequalities (5.13) applied to the point $f^j(x)$ and using (5.14) we conclude that if $j \ge 0$, then

$$D(f^j(x),\varepsilon) \le D(x,\varepsilon)^2 e^{2\varepsilon j}. \tag{5.15}$$

Similar arguments show that if $j \le 0$, then

$$D(f^{-j}(x),\varepsilon) \le D(x,\varepsilon)^2 e^{-2\varepsilon j}. \tag{5.16}$$

It follows from (5.15) and (5.16) that if $j \in \mathbb{Z}$, then

$$D(f^j(x),\varepsilon) \le D(x,\varepsilon)^2 e^{2\varepsilon |j|}.$$

thus establishing the subexponential behavior of the constant along the trajectory necessary for nonuniform hyperbolicity.

Another important manifestation of the Oseledets–Pesin Reduction Theorem is a crucial property of the Lyapunov inner norms. It states that the function

$x \mapsto \|v(x)\|'_x/\|v(x)\|_x$ is tempered on the set of regular points for every measurable vector field $X \ni x \mapsto v(x) \in \mathbb{R}^n \setminus \{0\}$. We recall that a positive function $K\colon X \to \mathbb{R}$ is called *tempered* on a set $Z \subset X$ if for any $x \in Z$,

$$\lim_{m\to\pm\infty} \frac{1}{m} \log K(f^m(x)) = 0. \tag{5.17}$$

Theorem 5.12 (see [135]). *For every measurable vector field $X \ni x \mapsto v(x) \in \mathbb{R}^n \setminus \{0\}$, the function $x \mapsto \|v(x)\|'_x/\|v(x)\|_x$ is tempered on the set of regular points.*

The proof uses a technical but crucial statement known as the Tempering-Kernel Lemma.

Lemma 5.13 ([135]). *Let $f\colon X \to X$ be a measurable transformation. If $K\colon X \to \mathbb{R}$ is a positive measurable function tempered on some subset $Z \subset X$, then for any $\varepsilon > 0$ there exists a positive measurable function $K_\varepsilon\colon Z \to \mathbb{R}$ such that $K(x) \le K_\varepsilon(x)$ and if $x \in Z$ then*

$$e^{-\varepsilon} \le \frac{K_\varepsilon(f(x))}{K_\varepsilon(x)} \le e^{\varepsilon}.$$

Note that if f preserves a Lebesgue measure ν on the space X, then any positive function $K\colon X \to \mathbb{R}$ with $\log K \in L^1(X,\nu)$ satisfies (5.17). The following is now an immediate consequence of Theorem 5.12.

Theorem 5.14. *Given $\varepsilon > 0$ there is a positive measurable function $K_\varepsilon\colon X \to \mathbb{R}$ such that if $x \in X$ is a regular point then:*

1. $K_\varepsilon(x)e^{-\varepsilon|m|} \le K_\varepsilon(f^m(x)) \le K_\varepsilon(x)e^{\varepsilon|m|}$ *for every $m \in \mathbb{Z}$;*
2. $n^{-1/2}\|v\|_x \le \|v\|'_x \le K_\varepsilon(x)\|v\|_x$ *for every $v \in \mathbb{R}^n$.*

5.6. **The case of flows.** We briefly discuss counterparts to the results in the above sections for flows. Let (X,ν) be a Lebesgue space.

The measurable map $\varphi\colon \mathbb{R} \times X \to X$ is called a *measurable flow* on X if

$$\varphi_0 = \mathrm{Id}, \text{ and } \varphi_t \circ \varphi_s = \varphi_{t+s} \text{ for every } t,\, s \in \mathbb{R}. \tag{5.18}$$

A measurable flow $\varphi\colon \mathbb{R} \times X \to X$ is called a *measure preserving preserving flow* if $\varphi_t \stackrel{\text{def}}{=} \varphi(t,\cdot)$ is ν-invariant for every $t \in \mathbb{R}$.

We note that given a family $\{\varphi_t : t \in \mathbb{R}\}$ of measurable maps $\varphi_t\colon X \to X$ satisfying (5.18) one can define a measurable flow $\varphi\colon \mathbb{R}\times X \to X$ by $\varphi(t,x) = \varphi_t(x)$.

A measurable function $\mathcal{A}\colon X \times \mathbb{R} \to GL(n,\mathbb{R})$ is called a *linear multiplicative cocycle over φ* or simply a *cocycle* if for every $x \in X$ the following properties hold:

1. $\mathcal{A}(x,0) = \mathrm{Id}$;
2. if $t,\, s \in \mathbb{R}$ then

$$\mathcal{A}(x,t+s) = \mathcal{A}(\varphi_t(x),s)\mathcal{A}(x,t).$$

The cocycle $\mathcal{A}$ induces *linear extensions* $F_t\colon X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ by the formula

$$F_t(x,v) = (\varphi_t(x), \mathcal{A}(x,t)v).$$

Given $(x,v) \in X \times \mathbb{R}^n$, the *forward Lyapunov exponent of (x,v) (with respect to the cocycle $\mathcal{A}$)* given by

$$\chi^+(x,v) = \chi^+(x,v,\mathcal{A}) = \varlimsup_{t\to+\infty} \frac{1}{t} \log\|\mathcal{A}(x,t)v\|.$$

For every $x \in X$, there exist a positive integer $p^+(x) \le n$, a collection of values

$$\chi_1^+(x) < \chi_2^+(x) < \cdots < \chi_{p^+(x)}^+(x),$$

and linear spaces

$$\{0\} = V_0^+(x) \subsetneqq V_1^+(x) \subsetneqq \cdots \subsetneqq V_{p^+(x)}^+(x) = \mathbb{R}^n,$$

such that:

1. $V_i^+(x) = \{v \in \mathbb{R}^n : \chi^+(x,v) \le \chi_i^+(x)\}$;
2. if $v \in V_i^+(x) \setminus V_{i-1}^+(x)$, then $\chi^+(x,v) = \chi_i^+(x)$.

The number

$$k_i^+(x) = \dim V_i^+(x) - \dim V_{i-1}^+(x)$$

is the *multiplicity* of the value $\chi_i^+(x)$. In a similar way the quantity

$$\chi^-(x,v) = \chi^-(x,v,\mathcal{A}) = \overline{\lim_{t\to-\infty}} \frac{1}{|t|} \log\|\mathcal{A}(x,t)v\|$$

is the *backward Lyapunov exponent of* (x,v) *(with respect to the cocycle* $\mathcal{A}$*)*. There exist a positive integer $p^-(x) \le n$, a collection of *values*

$$\chi_1^-(x) > \cdots > \chi_{p^-(x)}^-(x)$$

and the *filtration* $\mathcal{V}_x^-$ of $\mathbb{R}^n$ associated with χ^- at x,

$$\mathbb{R}^n = V_1^-(x) \supsetneqq \cdots \supsetneqq V_{p^-(x)}^-(x) \supsetneqq V_{p^-(x)+1}^-(x) = \{0\},$$

where $V_i^-(x) = \{v \in \mathbb{R}^n : \chi^-(x,v) \le \chi_i^-(x)\}$. The number

$$k_i^-(x) = \dim V_i^-(x) - \dim V_{i+1}^-(x)$$

is the *multiplicity* of the value $\chi_i^-(x)$.

Write $\mathcal{V}^+ = \{\mathcal{V}_x^+\}_{x\in X}$ and $\mathcal{V}_x^- = \{\mathcal{V}_x^-\}_{x\in X}$. The filtrations $\mathcal{V}^+$ and $\mathcal{V}^-$ *comply* at the point $x \in X$ if the following properties hold:

1. $p^+(x) = p^-(x) \overset{\text{def}}{=} p(x)$;
2. there exists a decomposition

   $$\mathbb{R}^n = \bigoplus_{i=1}^{p(x)} E_i(x)$$

   into subspaces $E_i(x)$ such that $\mathcal{A}(x,t)E_i(x) = E_i(\varphi_t x)$ for every $t \in \mathbb{R}$ and

   $$V_i(x) = \bigoplus_{j=1}^{i} E_j(x) \quad \text{and} \quad V_i^-(x) = \bigoplus_{j=i}^{p(x)} E_j(x);$$

3. $\chi_i^+(x) = -\chi_i^-(x) \overset{\text{def}}{=} \chi_i(x)$;
4. if $v \in E_i(x) \setminus \{0\}$ then

   $$\lim_{t\to\pm\infty} \frac{1}{t} \log\|\mathcal{A}(x,t)v\| = \chi_i(x),$$

   with uniform convergence on $\{v \in E_i(x) : \|v\| = 1\}$.

A point x is *forward regular* for $\mathcal{A}$ if the following limit exists

$$\lim_{t\to+\infty}\frac{1}{t}\log|\det\mathcal{A}(x,t)| = \sum_{i=1}^{p^+(x)}\chi_i^+(x)k_i^+(x).$$

and is *backward regular* for $\mathcal{A}$ if the following limit exists

$$\lim_{t\to-\infty}\frac{1}{|t|}\log|\det\mathcal{A}(x,t)| = \sum_{i=1}^{p^-(x)}\chi_i^-(x)k_i^-(x).$$

Finally, a point x is *Lyapunov regular* or simply *regular* for $\mathcal{A}$ if

1. x is simultaneously forward and backward regular for $\mathcal{A}$;
2. the filtrations $\mathcal{V}^+$ and $\mathcal{V}^-$ comply at x.

Theorem 5.15 (Multiplicative Ergodic Theorem for flows)**.** *Let φ be a measure preserving flow of a Lebesgue space (X,ν) such that φ_t is invertible for every $t\in\mathbb{R}$. Let also $\mathcal{A}$ be a measurable cocycle over φ such that*

$$\sup_{-1\le t\le 1}\log^+\|\mathcal{A}(\cdot,t)\|\in L^1(X,\nu). \tag{5.19}$$

Then the set of regular points for $\mathcal{A}$ has full ν-measure.

Given $\varepsilon>0$ and a regular point $x\in X$, we introduce a family of inner products $\langle\cdot,\cdot\rangle_x$ on $\mathbb{R}^n$ by setting

$$\langle u,v\rangle_x' = \int_{\mathbb{R}}\langle\mathcal{A}(x,t)u,\mathcal{A}(x,t)v\rangle e^{-2\chi_i(x)t-2\varepsilon|t|}\,dt$$

if $u,\ v\in E_i(x)$, and $\langle u,v\rangle_x'=0$ if $u\in E_i(x)$ and $v\in E_j(x)$ with $i\ne j$. We call $\langle\cdot,\cdot\rangle_x'$ a *Lyapunov inner product at x*, and the corresponding norm $\|\cdot\|_x'$ a *Lyapunov norm at x*. One can show that there exists a tempered function $K_\varepsilon\colon X\to\mathbb{R}$ such that if $x\in X$ is a regular point and $v\in\mathbb{R}^n$ then

$$n^{-1/2}\|v\|_x\le\|v\|_x'\le K_\varepsilon(x)\|v\|_x.$$

We recall that a positive function $K\colon X\to\mathbb{R}$ is called *tempered* on a set $Z\subset X$ if for any $x\in Z$,

$$\lim_{t\to\pm\infty}\frac{1}{m}\log K(\varphi_t x)=0.$$

Theorem 5.16 (Reduction Theorem for flows)**.** *Let φ be a measure preserving flow of a Lebesgue space (X,ν) such that φ_t is invertible for every $t\in\mathbb{R}$. Let also $\mathcal{A}$ be a measurable cocycle over φ. Given $\varepsilon>0$ and a regular point x, there exists a Lyapunov change of coordinates C_ε with the following properties:*

1. *the cocycle $\mathcal{A}_\varepsilon(x,t)=C_\varepsilon(\varphi_t x)^{-1}\mathcal{A}(x,t)C_\varepsilon(x)$ has the block form*
$$\mathcal{A}_\varepsilon(x,t)=\begin{pmatrix}\mathcal{A}_\varepsilon^1(x,t) & & \\ & \ddots & \\ & & \mathcal{A}_\varepsilon^{p(x)}(x,t)\end{pmatrix},$$
where each block $\mathcal{A}_\varepsilon^i(x,t)$ is a $k_i(x)\times k_i(x)$ matrix, and the entries are zero above and below the matrices $\mathcal{A}_\varepsilon^i(x,t)$;
2. *each block $\mathcal{A}_\varepsilon^i(x)$ satisfies*
$$e^{\chi_i(x)t-\varepsilon|t|}\le\|\mathcal{A}_\varepsilon^i(x,t)^{-1}\|^{-1}\le\|\mathcal{A}_\varepsilon^i(x,t)\|\le e^{\chi_i(x)t+\varepsilon|t|};$$

3. *if the integrability condition* (5.19) *holds then the map* C_ε *is tempered* ν*-almost everywhere, and the spectra of* $\mathcal{A}$ *and* $\mathcal{A}_\varepsilon$ *coincide* ν*-almost everywhere.*

5.7. **The case of noninvertible dynamical systems.** Consider a measure preserving transformation $f\colon X \to X$ of a Lebesgue space (X,ν) (the map f need not be invertible). We assume that ν is a probability measure. Given a measurable function $A\colon X \to GL(n,\mathbb{R})$ and $x \in X$, define the *one-sided cocycle* $\mathcal{A}\colon X \times \mathbb{N} \to GL(n,\mathbb{R})$ by

$$\mathcal{A}(x,m) = A(f^{m-1}(x)) \cdots A(f(x))A(x).$$

Note that the cocycle equation (4.1) holds for every m, $k \in \mathbb{N}$. Given $(x,v) \in X \times \mathbb{R}^n$, define the *forward Lyapunov exponent* of (x,v) (with respect to $\mathcal{A}$) by

$$\chi^+(x,v) = \chi^+(x,v,\mathcal{A}) = \varlimsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(x,m)v\|.$$

However, since the map f and the matrices $A(x)$ may not be invertible, one may not in general define a backward Lyapunov exponent. Therefore, we can only discuss the forward regularity for $\mathcal{A}$. One can establish a Multiplicative Ergodic Theorem in this case.

Theorem 5.17. *Let* f *be a measure preserving transformation of a Lebesgue space* (X,ν)*, and* $\mathcal{A}$ *a measurable cocycle over* f *such that* $\log^+\|A\| \in L^1(X,\nu)$*. Then the set of forward regular points for* $\mathcal{A}$ *has full* ν*-measure and for* ν*-almost every* $x \in X$ *and every subspace* $F \subset E_i^+(x)$ *such that* $F \cap E_{i-1}^+(x) = \{0\}$ *we have*

$$\lim_{m\to+\infty} \frac{1}{m} \log \inf_v \|\mathcal{A}(x,m)v\| = \lim_{m\to+\infty} \frac{1}{m} \log \sup_v \|\mathcal{A}(x,m)v\| = \chi_i^+(x),$$

with the infimum and supremum taken over $\{v \in F : \|v\| = 1\}$.

When the matrix $A(x)$ is invertible for every $x \in X$ and $\log^+\|A\|$, $\log^+\|A^{-1}\| \in L^1(X,\nu)$ for some f-invariant Lebesgue measure ν, one can show that for the cocycle induced by $\mathcal{A}$ on the inverse limit of f the set of *regular* points has full ν-measure.

5.8. **The case of nonpositively curved spaces.** Karlsson and Margulis [128] obtained an extension of the noninvertible case of the Multiplicative Ergodic Theorem 5.5 to some nonpositively curved spaces.

Let (Y,ρ) be a complete metric space. Y is called:

1. *convex* if any two points x, $y \in Y$ have a *midpoint*, i.e., a point z for which
$$\rho(z,x) = \rho(z,y) = \frac{1}{2}\rho(x,y);$$
2. *uniformly convex* if it is convex and there is a strictly decreasing continuous function g on $[0,1]$ such that $g(0) = 1$ and for any $x,y,w \in Y$ and midpoint m_{xy} of x and y,
$$\frac{\rho(m_{xy},w)}{R} \le g\Big(\frac{\rho(m_{xy},w)}{R}\Big),$$
where $R = \max\{\rho(x,w),\rho(y,w)\}$;
3. *nonpositively curved* (in the sense of Busemann) if it is convex and for any x, y, $z \in Y$ and any midpoints m_{xz} of x and z and m_{yz} of y and z,
$$\rho(m_{xz},m_{yz}) \le \frac{1}{2}\,\rho(x,y).$$

If Y is uniformly convex then midpoints are unique.

Examples of nonpositively curved spaces include uniformly convex Banach spaces (e.g., Hilbert spaces or L^p for $1 < p < \infty$), Cartan–Hadamard manifolds (e.g., Euclidean spaces, hyperbolic spaces or $GL(n, \mathbb{R})/O(n, \mathbb{R})$), and more generally CAT(0) spaces (e.g., Euclidean buildings or $\mathbb{R}$-trees).

A continuous map $\gamma\colon I \to Y$ (I is an interval) is called a (unit speed minimizing) *geodesic* if for any $s, t \in I$,

$$\rho(\gamma(s), \gamma(t)) = |s - t|.$$

If Y is convex then any two points can be joined by a geodesic and if Y is uniformly convex then this geodesic is unique.

A geodesic $\gamma\colon [0, \infty) \to Y$ is called a *ray* if the limit $\lim_{t\to\infty} \gamma(t)$ does not exist. The two rays γ_1 and γ_2 are called *asymptotic* if $\rho(\gamma_1(t), \gamma_2(t)) \leq \text{const}$ for $t \geq 0$. We denote by $[\gamma]$ the set of all rays asymptotic to γ and by $Y(\infty)$ the ideal boundary of Y , i.e., the set of all classes of asymptotic rays.

Let $D \subset Y$ be a nonempty subset. A map $\varphi\colon D \to D$ is called a *semicontraction (or nonexpanding)* if $\rho(\varphi(v), \varphi(z)) \leq d(y, z)$ for all $y, z \in D$. Isometries are semicontractions.

Let us fix a semigroup S of semicontractions and equip it with the Borel σ-algebra associated with the compact-open topology on S. Fix $y \in Y$. Consider a cocycle $\mathcal{A}$ with values in S over an ergodic transformation f of a measure space (X, μ). Let $A\colon X \to X$ be the generator.

Theorem 5.18 (Karlsson and Margulis [128]). *Assume that*

$$\int_X \rho(y, A(x)y)\, d\mu(x) < \infty.$$

Then for almost every $x \in X$ the following limit exists

$$\lim_{m\to\infty} \frac{1}{m} d(y, \mathcal{A}(x, m)y) = a \tag{5.20}$$

and if $a > 0$ then for almost every $x \in X$ there exists a unique geodesic ray $\gamma(\cdot, x)$ in Y starting at y such that

$$\lim_{m\to\infty} \frac{1}{m} d(\gamma(an, x), \mathcal{A}(x, m)y) = 0$$

and hence, $\mathcal{A}(\cdot, m)y$ converges to $[\gamma]$ in $Y \cup Y(\infty)$.

The existence of the limit in (5.20) is an easy corollary of Kingman's Sub-additive Ergodic Theorem.

Consider the symmetric space $Y = GL(n, \mathbb{R})/O(n, \mathbb{R})$ and a cocycle $\mathcal{A}$ with values in $O(n, \mathbb{R})$ over an ergodic transformation f of a measure space (X, μ). Let $A\colon X \to X$ be the generator. Fix a point $y \in O(n, \mathbb{R})$. For $g \in GL(n, \mathbb{R})$ let λ_i be the eigenvalues of $(gg^*)^{1/2}$ where g^* is the transpose of g. The distance in Y between y and gy is

$$\rho(y, gy) = \Big(\sum_{i=1}^{n} (\log \lambda_i)^2\Big)^{1/2}.$$

A geodesic starting at y is of the form $\gamma(t) = e^{tH}y$ where H is a symmetric matrix. Then $\Lambda = e^H$ is a positive definite symmetric matrix. We have that

$$\lim_{n\to\infty} \frac{1}{m} d(\Lambda^{-m}y, \mathcal{A}(m, x)^{-1}y) = 0$$

for almost every $x \in X$. In view of (3.7) this means that x is forward regular.

Theorem 5.18 has interesting applications to random walks and Hilbert–Schmidt operators (see [128]). It is shown in [128] with an explicit example that there is no invertible version of Theorem 5.18, i.e., there is in general no two-sided geodesic approximating both the forward and backward orbits $m \mapsto \mathcal{A}(x, \pm m)y$.

5.9. **Notes.** The term "Multiplicative Ergodic Theorem" was introduced by Oseledets in [191] where he presented the first proof of the theorem.

In [182], Millionshchikov announced a somewhat independent proof of the Multiplicative Ergodic Theorem which is based on some subtle properties of the action of the differential with respect to the Lyapunov exponents.[1] Mañé used similar properties in his proof of the entropy formula (see Section 12.2).

Other proofs of the Multiplicative Ergodic Theorem were obtained by Ruelle [215], by Mañé [174] (see also [172]),[2] and by Goldsheid and Margulis [98]. A simpler version of the Multiplicative Ergodic Theorem was considered by Johnson, Palmer and Sell [126],[3] and related topics were discussed by Sacker and Sell [221, 222, 226] and by Johnson [125].

In [141], Kifer established a "random" version of the Multiplicative Ergodic Theorem—for compositions of independent identically distributed transformations of a measurable vector bundle. His proof is built on the work of Furstenberg and Kifer [94] (see also Chapter III in [142]). Under more restrictive conditions a similar result was obtained by Carverhill [59]. See the book by Arnold [12] for a detailed description of various versions of the Multiplicative Ergodic Theorem and related questions in the random dynamical systems setup.

There are also infinite-dimensional versions of the Multiplicative Ergodic Theorem. Namely, it was extended by Ruelle [216] to Hilbert spaces (following closely his finite-dimensional approach in [215]), and by Mañé in [172] to compact transformations in Banach spaces (see also Thieullen [238] for the case of not necessarily compact transformations). The proof due to Goldsheid and Margulis also extends to the infinite-dimensional case (see [98]).

6. Cocycles over smooth dynamical systems

6.1. **The derivative cocycle.** Let $f\colon M \to M$ be a diffeomorphism of a smooth n-dimensional Riemannian manifold. Given $x \in M$, set $X = \{f^m(x)\}_{m\in\mathbb{Z}}$. Identifying the tangent spaces $T_{f^m(x)}M$ with $\mathbb{R}^n$ one can introduce the cocycle $\mathcal{A}_x = \{d_{f^m(x)}f\}_{m\in\mathbb{Z}}$ over the transformation $f\colon X \to X$. It is called the *derivative cocycle* associated with the diffeomorphism f and the point x. The Lyapunov exponent χ^+ of x specified by the derivative cocycle is the Lyapunov exponent specified by the diffeomorphism f at the point x.

The "individual" derivative cocycles $\mathcal{A}_x$ depend on the individual trajectories $\{f^m(x)\}_{m\in\mathbb{Z}}$. We now introduce the "global" cocycle associated with f. One can

---

[1] Millionshchikov's proof was never published as a solid piece; instead, it is scattered through a series of papers with cross-references and is difficult to comprehend.

[2] In both [215] and [174] a slightly weaker version of Lyapunov regularity, then the one we introduced in Section 5.1, is considered but the proofs contain arguments which are indeed, sufficient to establish a stronger version.

[3] They established some but not all properties of Lyapunov regularity referring the reader to the original work of Oseledets.

represent M as a finite union $\bigcup_i \Delta_i$ of differentiable copies Δ_i of the n-simplex such that:

1. in each Δ_i one can introduce local coordinates in such a way that $T\Delta_i$ can be identified with $\Delta_i \times \mathbb{R}^n$;
2. all the nonempty intersections $\Delta_i \cap \Delta_j$, for $i \neq j$, are $(n-1)$-dimensional manifolds.

In each Δ_i the derivative of f can be interpreted as a linear cocycle. This implies that $df\colon M \to \mathbb{R}^n$ can be interpreted as a measurable linear cocycle $\mathcal{A}$ with d_xf to be the matrix representation of d_xf in local coordinates. We call $\mathcal{A}$ the *derivative cocycle* specified by the diffeomorphism f. It does not depend on the choice of the decomposition $\{\Delta_i\}$. Indeed, if we choose another decomposition $\{\Delta_i'\}$, then the coordinate change in $\Delta_i \cap \Delta_j'$ sending one representation to the other one is effected by maps which are uniformly bounded together with their derivatives, their inverses, and the inverses of their derivatives. Hence, by Proposition 4.2, the coordinate change is tempered and the two cocycles corresponding to the two decompositions $\{\Delta_i\}$ and $\{\Delta_i'\}$ are equivalent.

We remark that if ν is an f-invariant Borel probability measure on M then the decomposition $\{\Delta_i\}$ can be chosen such that $\nu(\partial\Delta_i) = 0$ for every i.

6.2. **Nonuniformly hyperbolic diffeomorphisms.** We say that a diffeomorphism f is *nonuniformly partially hyperbolic in the broad sense* if so is the derivative cocycle generated by f. More precisely, this means[4] that f possesses an invariant Borel subset $\Lambda \subset M$ such that there exist: (a) numbers λ and μ, $0 < \lambda < \mu$, $\lambda < 1$; (b) a sufficiently small number $\varepsilon > 0$ and Borel functions $C, K\colon \Lambda \to (0, \infty)$; (c) subspaces $E_1(x)$ and $E_2(x)$, $x \in \Lambda$, which satisfy the following conditions:

1. the subspaces $E_1(x)$ and $E_2(x)$ depend measurably on x and form an invariant splitting of the tangent space, i.e.,
$$\begin{gathered} T_xM = E_1(x) \oplus E_2(x), \\ d_xfE_1(x) = E_1(f(x)), \quad d_xfE_2(x) = E_2(f(x)); \end{gathered} \tag{6.1}$$
2. for $v \in E_1(x)$ and $n > 0$,
$$\|d_xf^nv\| \leq C(x)\lambda^n e^{\varepsilon n}\|v\|; \tag{6.2}$$
3. for $v \in E_2(x)$ and $n < 0$,
$$\|d_xf^nv\| \leq C(x)\mu^n e^{\varepsilon|n|}\|v\|; \tag{6.3}$$
4. the angle
$$\angle(E_1(x), E_2(x)) \geq K(x); \tag{6.4}$$
5. for $n \in \mathbb{Z}$,
$$C(f^n(x)) \leq C(x)e^{\varepsilon|n|}, \quad K(f^n(x)) \geq K(x)e^{-\varepsilon|n|}. \tag{6.5}$$

Condition (6.5) means that estimates (6.2), (6.3) and (6.4) may deteriorate along the trajectory with subexponential rate. We stress that the rates of contraction along stable subspaces and expansion along unstable subspaces are exponential and hence, prevail.

Furthermore, f is *nonuniformly partially hyperbolic* on an f-invariant Borel subset $\Lambda \subset M$ if there exist: (a) numbers λ, λ', μ, and μ' such that $0 < \lambda < 1 < \mu$

[4]For simplicity, we consider here only one of the nested subsets in the definition of nonuniformly hyperbolic cocycles; see Section 4.4.

and $\lambda < \lambda' \le \mu' < \mu$; (b) a sufficiently small number $\varepsilon > 0$ and Borel functions C, $K\colon \Lambda \to (0, \infty)$; (c) subspaces $E^s(x)$, $E^c(x)$, and $E^u(x)$, $x \in \Lambda$, which satisfy the following conditions:

1'. the subspaces $E^s(x)$, $E^c(x)$, and $E^u(x)$ depend measurably on x and form an invariant splitting of the tangent space, i.e.,
$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x),$$
$$d_xfE^s(x) = E^s(f(x)), \quad d_xfE^c(x) = E^c(f(x)),$$
$$d_xfE^u(x) = E^u(f(x));$$
2'. the subspaces $E^s(x)$ and $E^u(x)$ satisfy (6.2) and (6.3); in addition, for $v \in E^c(x)$ and $n \in \mathbb{Z}$,
$$C(x)^{-1}(\lambda')^n e^{-\varepsilon n}\|v\| \le \|d_xf^nv\| \le C(x)(\mu')^n e^{\varepsilon n}\|v\|;$$
3'. the subspaces $E^s(x)$ and $E^u(x)$ satisfy (6.4); in addition, $\angle(E^s(x), E^c(x)) \ge K(x)$ and $\angle(E^u(x), E^c(x)) \ge K(x)$;
4'. the functions $C(x)$ and $K(x)$ satisfy (6.5).

In the case $E^c(x) = 0$ we say that f is *nonuniformly (completely) hyperbolic* on Λ.

Throughout this chapter we deal with three types of nonuniform hyperbolicity: the partial hyperbolicity in the broad sense, its stronger version of partial hyperbolicity (sometimes called partial hyperbolicity in the narrow sense), and yet the stronger complete hyperbolicity (sometimes simply called nonuniform hyperbolicity). We shall refer to subspaces $E_1(x)$ and respectively, $E^s(x)$ as *stable* subspaces, to $E^c(x)$ as *central* subspaces and to $E^u(x)$ as *unstable* subspaces. In the case of general nonuniform partial hyperbolicity in the broad sense the subspaces $E_2(x)$ may not be unstable as some vectors may contract under the action of df.

It should be stressed that principle results describing local behavior of the system (such as Stable Manifold theorem 8.8 and Absolute Continuity theorems 10.1 and 11.1) as well as some results of a global nature (such as construction of global invariant manifolds in Section 9 and of the pseudo-π-partition in Theorem 11.16 and the lower bound for the metric entropy in Theorem 12.11) need only nonuniform partial hyperbolicity in the broad sense. On the other hand, more advanced results describing ergodic and topological properties of the system require stronger nonuniform complete hyperbolicity, see Sections 11–16.

Consider a diffeomorphism f which is nonuniformly partially hyperbolic in the broad sense on an invariant set Λ. Given $\ell > 0$, we introduce the *regular set* (of level ℓ) by
$$\Lambda^\ell = \left\{ x \in \Lambda : C(x) \le \ell,\, K(x) \ge \frac{1}{\ell} \right\}.$$
Without loss of generality we may assume that the sets Λ^ℓ are closed (otherwise they can be replaced by their closures $\overline{\Lambda^\ell}$).

We describe a special inner product in the tangent bundle $T\Lambda$ which is known as the *Lyapunov inner product*. It provides a convenient technical tool in studying nonuniform hyperbolicity. Choose numbers $0 < \lambda' < \mu' < \infty$ such that
$$\lambda e^\varepsilon < \lambda', \quad \mu' < \mu e^{-\varepsilon}.$$

We define a new inner product $\langle\cdot,\cdot\rangle'_x$, as follows. Set

$$\langle v, w\rangle'_x = \sum_{k=0}^{\infty} \langle df^k v, df^k w\rangle_{f^k(x)} {\lambda'}^{-2k}$$

if v, $w \in E_1(x)$, and

$$\langle v, w\rangle'_x = \sum_{k=0}^{\infty} \langle df^{-k} v, df^{-k} w\rangle_{f^{-k}(x)} {\mu'}^{2k}$$

if v, $w \in E_2(x)$.

Using (6.2) and (6.3) one can verify that each series converges. We extend $\langle\cdot,\cdot\rangle'_x$ to all vectors in T_xM by declaring the subspaces $E_1(x)$ and $E_2(x)$ to be mutually orthogonal with respect to $\langle\cdot,\cdot\rangle'_x$, i.e., we set

$$\langle v, w\rangle'_x = \langle v_1, w_1\rangle'_x + \langle v_2, w_2\rangle'_x,$$

where $v = v_1 + v_2$ and $w = w_1 + w_2$ with v_1, $w_1 \in E_1(x)$ and v_2, $w_2 \in E_2(x)$.

The norm induced by the Lyapunov inner product is called the *Lyapunov norm* and is denoted by $\|\cdot\|'$. We emphasize that the Lyapunov inner product, and hence, the norm $\|\cdot\|'$ depend on the choice of numbers λ' and μ'.

The Lyapunov inner product has several important properties:

1. the angle between the subspaces $E_1(x)$ and $E_2(x)$ in the inner product $\langle\cdot,\cdot\rangle'_x$ is $\pi/2$ for each $x \in \Lambda$;
2. $\|A_x\|' \le \lambda'$ and $\|{B_x}^{-1}\|' \le (\mu')^{-1}$;
3. the relation between the Lyapunov inner product and the Riemannian inner product is given by

$$\frac{1}{\sqrt{2}}\|w\|_x \le \|w\|'_x \le D(x)\|w\|_x,$$

where $w \in T_xM$ and

$$D(x) = C(x)K(x)^{-1}[(1-\lambda e^{\varepsilon}/\lambda')^{-1} + (1-\mu'/(\mu e^{-\varepsilon}))^{-1}]^{1/2}$$

is a measurable function satisfying (in view of (6.5))

$$D(f^m(x)) \le D(x)e^{2\varepsilon|m|}, \quad m \in \mathbb{Z}. \tag{6.6}$$

Properties (1) and (2) show that the action of the differential df is *uniformly* partially hyperbolic in the broad sense with respect to the Lyapunov inner product.

For a partially hyperbolic in the broad sense $C^{1+\beta}$ diffeomorphism f the subspaces $E_1(x)$ and $E_2(x)$ depend continuously on the point x in a regular set. Indeed, one can prove a stronger statement.

Theorem 6.1. *The distribution $E_1(x)$ depends Hölder continuously on $x \in \Lambda^\ell$, i.e.,*

$$d(E_1(x), E_2(y)) \le C\rho(x,y)^\alpha,$$

where $C > 0$ and $\alpha \in (0,1]$ are constants, and d is the distance in the Grassmannian bundle of TM generated by the Riemannian metric.

This theorem is a particular case of a more general result which we now state.

A k-dimensional distribution E on a subset Λ of a differentiable manifold M is a family of k-dimensional subspaces $E(x) \subset T_xM$, $x \in \Lambda$. A Riemannian metric on M naturally induces distances in TM and in the space of k-dimensional subspaces in TM. The Hölder continuity of a distribution E can be defined using these distances. However, by the Whitney Embedding Theorem, every manifold M can be embedded in $\mathbb{R}^N$ with a sufficiently large N. If M is compact, the Riemannian metric on M is equivalent to the distance $\|x-y\|$ induced by the embedding. The Hölder exponent does not change if the Riemannian metric is changed for an equivalent smooth metric, while the Hölder constant may change. We assume in Theorem 6.2, without loss of generality, that the manifold is embedded in $\mathbb{R}^N$.

For a subspace $A \subset \mathbb{R}^N$ and a vector $v \in \mathbb{R}^N$, set

$$\operatorname{dist}(v, A) = \min_{w \in A} \|v - w\|.$$

i.e., $\operatorname{dist}(v, A)$ is the length of the difference between v and its orthogonal projection to A. For subspaces A, B in $\mathbb{R}^N$, define

$$\operatorname{dist}(A, B) = \max\left\{ \max_{v \in A, \|v\|=1} \operatorname{dist}(v, B), \max_{w \in B, \|w\|=1} \operatorname{dist}(w, A) \right\}.$$

A k-dimensional *distribution* E defined on a set $\Lambda \subset \mathbb{R}^N$ is called *Hölder continuous* with *Hölder exponent* $\alpha \in (0, 1]$ and *Hölder constant* $L > 0$ if there exists $\varepsilon_0 > 0$ such that

$$\operatorname{dist}(E(x), E(y)) \le L\|x - y\|^\alpha$$

for every x, $y \in \Lambda$ with $\|x - y\| \le \varepsilon_0$.

The subspaces E_1, $E_2 \subset \mathbb{R}^N$ are said to be *κ-transverse* if $\|v_1 - v_2\| \ge \kappa$ for all unit vectors $v_1 \in E_1$ and $v_2 \in E_2$.

Theorem 6.2 (Brin [48]). *Let M be a compact m-dimensional C^2 submanifold of $\mathbb{R}^N$ for some $m < N$, and $f\colon M \to M$ a $C^{1+\beta}$ map for some $\beta \in (0, 1)$. Assume that there exist a set $\Lambda \subset M$ and real numbers $0 < \lambda < \mu$, $c > 0$, and $\kappa > 0$ such that for each $x \in \Lambda$ there are κ-transverse subspaces $E_1(x)$, $E_2(x) \subset T_xM$ with the following properties:*

1. *$T_xM = E_1(x) \oplus E_2(x)$;*
2. *$\|d_xf^nv_1\| \le c\lambda^n\|v_1\|$ and $\|d_xf^nv_2\| \ge c^{-1}\mu^n\|v_2\|$ for every $v_1 \in E_1(x)$, $v_2 \in E_2(x)$, and every positive integer n.*

Then for every $a > \max_{z \in M} \|d_zf\|^{1+\beta}$, the distribution E_1 is Hölder continuous with exponent

$$\alpha = \frac{\log \mu - \log \lambda}{\log a - \log \lambda}\beta.$$

6.3. **Regularity of the derivative cocycle.** We say that a point $x \in M$ is *Lyapunov forward f-regular* (or simply *forward regular*), *Lyapunov backward f-regular* (or simply *backward f-regular*), or *Lyapunov f-regular* (or simply *regular*), respectively, if it is forward regular, backward regular, or regular with respect to the cocycle $\mathcal{A}_x$.

We recall that for any regular point $x \in M$ there exist an integer $s(x) \le n$, numbers $\chi_1(x) < \cdots < \chi_{s(x)}(x)$ and a decomposition

$$T_xM = \bigoplus_{i=1}^{s(x)} E_i(x) \tag{6.7}$$

into subspaces $E_i(x)$ such that for $v \in E_i(x) \setminus \{0\}$ and $i = 1, \ldots, s(x)$,

$$\lim_{m \to \pm\infty} \frac{1}{m} \log\|d_xf^mv\| = \chi_i(x)$$

with uniform convergence on $\{v \in E_i(x) : \|v\| = 1\}$. Write $k_i(x) = \dim H_i(x)$.

Assume that there exists $C > 0$ such that $\|d_x f\|, \|d_x f^{-1}\| \le C$ for every $x \in M$. Note that this property holds when M is compact. Then the derivative cocycle satisfies the condition (5.3), and by the Multiplicative Ergodic Theorem 5.5 the set of regular points (as well as the sets of forward and backward regular points) is nonempty. Moreover, the following statement is an immediate consequence of Theorem 5.5.

Theorem 6.3. *Let f be a diffeomorphism of a smooth Riemannian manifold. Then the set of regular points has full measure with respect to any f-invariant Borel probability measure with compact support.*

The set of points which are not regular is negligible from the measure-theoretical point of view, since it has zero measure with respect to any Borel invariant measure. However, this set may be large with respect to other characteristics. For example, it may have positive Lebesgue measure, positive Hausdorff dimension, or positive topological entropy.

Theorem 6.3 does not allow one to determine whether a given trajectory is regular (or forward regular or backward regular). We now present some criteria which guarantee forward and backward regularity of individual trajectories.

Let us first notice that if x is a fixed point or a periodic point for f then the cocycle $\mathcal{A}_x$ is rigid with generator $A = d_x f$ (if x is a fixed point) or $A = d_x f^p$ (if x is a periodic point of period p).

We now consider the case of an arbitrary point x.

Proposition 6.4. *Let f be a diffeomorphism of a smooth Riemannian manifold M.*

1. *If $x \in M$ is such that*
$$\chi^+(x, v_1, \dots, v_k) = \lim_{m\to+\infty} \frac{1}{m} \log V(d_x f^m v_1, \dots, d_x f^m v_k)$$
(that is, $\chi^+(x, v_1, \dots, v_k)$ is exact), for any choice of linearly independent vectors $v_1, \dots, v_k \in T_x M$ and $k = 1, \dots, n$, then x is forward regular.
2. *If $x \in M$ is such that*
$$\chi^-(x, v_1, \dots, v_k) = \lim_{m\to-\infty} \frac{1}{|m|} \log V(d_x f^m v_1, \dots, d_x f^m v_k)$$
(that is, $\chi^-(x, v_1, \dots, v_k)$ is exact), for any choice of linearly independent vectors $v_1, \dots, v_k \in T_x M$ and $k = 1, \dots, n$, then x is backward regular.

We also formulate a criterion for regularity.

Proposition 6.5. *Let f be a diffeomorphism of a smooth Riemannian manifold M and $x \in M$. Assume that:*

1. *$\chi^+(x, v_1, \dots, v_k)$ and $\chi^-(x, v_1, \dots, v_k)$ are exact for any choice of linearly independent vectors $v_1, \dots, v_k \in T_x M$ and $k = 1, \dots, n$;*
2. *$s^+(x) = s^-(x) \stackrel{\text{def}}{=} s(x)$ and $\chi_i^+(x) = -\chi_i^-(x)$ for $i = 1, \dots, s(x)$;*
3. *$\bigoplus_{i=1}^{s(x)} (V_i^+(x) \cap V_i^-(x)) = \mathbb{R}^n$ where $\{V_i^+\}$ and $\{V_i^-\}$ are filtrations associated with the Lyapunov exponents χ^+ and χ^-.*

Then x is regular.

The diffeomorphism f acts on the cotangent bundle T^*M by its codifferential
$$d_x^* f \colon T^*_{f(x)} M \to T^*_x M$$

defined by

$$d_x^* f\varphi(v) = \varphi(d_x f v), \quad v \in T_x M, \quad \varphi \in T_{f(x)}^* M.$$

We denote the inverse map by

$$d_x' f = (d_x^* f)^{-1} \colon T_x^* M \to T_{f(x)}^* M.$$

Let ν be an ergodic f-invariant Borel measure. There exist numbers $s = s^\nu$, $\chi_i = \chi_i^\nu$, and $k_i = k_i^\nu$ for $i = 1, \ldots, s$ such that

$$s(x) = s, \quad \chi_i(x) = \chi_i, \quad k_i(x) = k_i \tag{6.8}$$

for ν-almost every x. The collection of pairs

$$\operatorname{Sp} \chi(\nu) = \{(\chi_i, k_i) : 1 \le i \le s\}$$

is called the *Lyapunov spectrum* of the measure ν.

A diffeomorphism f is a *dynamical system with nonzero Lyapunov exponents* if there exists an ergodic f-invariant Borel probability measure ν on M – a *hyperbolic measure* – such that the set

$$\Lambda = \{x \in \mathcal{L} : \text{there exists } 1 \le k(x) < s(x) \\ \text{with } \chi_{k(x)}(x) < 0 \text{ and } \chi_{k(x)+1}(x) > 0\}$$

has full measure.

Consider the set $\widetilde{\Lambda} = \widetilde{\Lambda}^\nu$ of those points in Λ which are Lyapunov regular and satisfy (6.8). By the Multiplicative Ergodic Theorem 5.5, we have $\nu(\widetilde{\Lambda}) = 1$. For every $x \in \widetilde{\Lambda}$, set

$$E^s(x) = \bigoplus_{i=1}^{k} E_i(x) \quad \text{and} \quad E^u(x) = \bigoplus_{i=k+1}^{s} E_i(x).$$

Theorem 6.6. *The subspaces $E^s(x)$ and $E^u(x)$, $x \in \widetilde{\Lambda}$, have the following properties:*

1. *they depend Borel measurably on x;*
2. *they form a splitting of the tangent space, i.e., $T_x M = E^s(x) \oplus E^u(x)$;*
3. *they are invariant,*

$$d_x f E^s(x) = E^s(f(x)) \quad \textit{and} \quad d_x f E^u(x) = E^u(f(x));$$

Furthermore, there exist $\varepsilon_0 > 0$, Borel functions $C(x, \varepsilon) > 0$ and $K(x, \varepsilon) > 0$, $x \in \widetilde{\Lambda}$ and $0 < \varepsilon \le \varepsilon_0$ such that

4. *the subspace $E^s(x)$ is* stable*: if $v \in E^s(x)$ and $n > 0$, then*

$$\|d_x f^n v\| \le C(x, \varepsilon) e^{(\chi_k + \varepsilon)n} \|v\|;$$

5. *the subspace $E^u(x)$ is* unstable*: if $v \in E^u(x)$ and $n < 0$, then*

$$\|d_x f^n v\| \le C(x, \varepsilon) e^{(\chi_{k+1} - \varepsilon)n} \|v\|;$$

6.

$$\angle(E^s(x), E^u(x)) \ge K(x, \varepsilon);$$

7. *for every $m \in \mathbb{Z}$,*

$$C(f^m(x), \varepsilon) \le C(x, \varepsilon) e^{\varepsilon|m|} \quad \textit{and} \quad K(f^m(x), \varepsilon) \ge K(x, \varepsilon) e^{-\varepsilon|m|}.$$

We remark that Condition (7) is crucial and is a manifestation of the regularity property.

It follows from Theorem 6.6 that f is nonuniformly completely hyperbolic on $\widetilde{\Lambda}$.

6.4. **Cocycles over smooth flows.** Let φ_t be a smooth flow on a smooth n-dimensional Riemannian manifold M. It is generated by the vector field $\mathfrak{X}$ on M given by

$$\mathfrak{X}(x) = \frac{d\varphi_t(x)}{dt}|_{t=0}.$$

For every $x_0 \in M$ the trajectory $\{x(x_0,t) = \varphi_t(x_0) : t \in \mathbb{R}\}$ represents a solution of the nonlinear differential equation

$$\dot{v} = \mathfrak{X}(v)$$

on the manifold M. This solution is uniquely determined by the initial condition $x(x_0,0) = x_0$.

Given a point $x \in M$ and the trajectory $\{\varphi_t(x) : t \in \mathbb{R}\}$ passing through x we introduce the *variational differential equation*

$$\dot{w}(t) = A(x,t)w(t), \tag{6.9}$$

where

$$A(x,t) = d\mathfrak{X}(\varphi_t(x)).$$

This is a linear differential equation along the trajectory $\{\varphi_t(x) : t \in \mathbb{R}\}$ known also as the *linear variational equation*.

The Lyapunov exponent generated by the cocycle $\mathcal{A}$ is defined by

$$\chi^+(x,v) = \overline{\lim_{t\to+\infty}} \frac{1}{t} \log \|w(t)\|,$$

where $w(t)$ is the solution of (6.9) with initial condition $w(0) = v$, and is called the *Lyapunov exponent of the flow* φ_t. In particular, one can speak of trajectories which are *forward* or *backward regular*, and *(Lyapunov) regular*.

Note that every periodic trajectory is regular. However, this is not true in general for nonperiodic trajectories. For example, consider a flow on the unit sphere with the North and the South poles to be, respectively, attracting and repelling points, and without other fixed points. If the coefficients of contraction and expansion are different then every trajectory of the flow (except for the North and the South poles) is nonregular.

One can establish a criterion for regularity of individual trajectories (see [24]). However, it is not a simple task to apply this criterion and check whether a given trajectory is regular. On the other hand, let ν be a Borel measure which is invariant under the flow φ_t. It is easy to see that the derivative cocycle $\mathcal{A}(x,t)$ satisfies

$$\sup_{-1\le t\le 1} \log^+ \|\mathcal{A}(\cdot,t)\| \in L^1(M,\nu).$$

The Multiplicative Ergodic Theorem for flows (see Theorem 5.15) implies that almost every trajectory with respect to ν is Lyapunov regular.

We say that a smooth flow φ_t is *nonuniformly hyperbolic* if it possesses an invariant Borel subset $\Lambda \subset M$ such that there exist: (a) numbers $0 < \lambda < 1 < \mu$; (b) a sufficiently small number $\varepsilon > 0$ and Borel functions $C, K\colon \Lambda \to (0,\infty)$; (c) subspaces $E^s(x)$ and $E^u(x)$, $x \in \Lambda$, which satisfy Conditions (1')–(4') in the definition of nonuniform partial hyperbolicity with $E^c(x) = \mathfrak{X}(x)$. Note that for every

t the diffeomorphism φ_t is nonuniformly partially hyperbolic with one-dimensional central subspace.

Assume that a smooth flow φ_t possesses an invariant Borel subset Λ and an invariant Borel measure ν with $\nu(\Lambda) = 1$ such that $\chi(x, v) \neq 0$ for almost every $x \in \Lambda$ and every $v \in T_xM$ not colinear with $\mathcal{X}$. Assume also that for these x there are vectors $v, w \in T_xM$ such that $\chi(x, v) > 0$ and $\chi(x, w) < 0$. Then the flow φ_t is nonuniformly hyperbolic on Λ.

7. Methods for estimating exponents

The absence of zero Lyapunov exponents implies nonuniform hyperbolicity. In fact, this seems to be one of the most "practical" universal ways to establish weak hyperbolic behavior. We discuss a powerful method which allows one to verify that Lyapunov exponents do not vanish. It was suggested by Wojtkowski in [248] and is a significant generalization of the initial approach by Alexeyev (see [3, 4, 5]) to build an invariant family of unstable cones.

The *cone* of size $\gamma > 0$ centered around $\mathbb{R}^{n-k}$ in the product space $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ is

$$C_\gamma = \big\{(v, w) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : \|v\| < \gamma\|w\|\big\} \cup \{(0, 0)\}.$$

Note that $\{0\} \times \mathbb{R}^{n-k} \subset C_\gamma$ for every γ.

Consider a cocycle $\mathcal{A}$ over an invertible measurable transformation $f \colon X \to X$ preserving a Borel probability measure ν on X, and let $A \colon \mathbb{R}^n \to GL(n, \mathbb{R})$ be its generator. Assume that there exist $\gamma > 0$ and $a > 1$ such that for ν-almost every $x \in \mathbb{R}^n$:

1. $A(x)C_\gamma \subset C_\gamma$;
2. $\|A(x)v\| \geq a\|v\|$ for every $v \in C_\gamma$.

Then the largest Lyapunov exponent can be shown to be positive ν-almost everywhere. Indeed, $n - k$ values of the Lyapunov exponent, counted with their multiplicities, are positive.

Wojtkowski's great insight is that Condition 1 alone is in fact sufficient to establish positivity of the values of the Lyapunov exponent. The importance of this observation is that Condition 1 is of pure qualitative nature and thus, no estimates on the growth of vectors inside the cone are required.

It turns out that Wojtkowski's approach can be described in a more general and more convenient framework elaborated by Burns and Katok in [132]. This approach, in turn, is a further development of that by Lewowicz in [160, 161] and Markarian in [178] and is based on the notion of infinitesimal Lyapunov function (see Section 7.1 below; see also Section 7.2 for the version of this approach in the case of cocycles with values in the symplectic group).

In the later work Wojtkowski himself strengthened his original approach and, using results of Potapov on monotone operators of a linear space generated by a quadratic form, obtained estimates of Lyapunov exponents for cocycles with values in the semigroup of matrices preserving the form (see [249]). These results apply to estimate Lyapunov exponents for Hamiltonian dynamical systems as well as to the Boltzmann-Sinai gas of hard spheres and the system of falling balls in one dimension (see [249] for more details and references therein).

7.1. **Cone and Lyapunov function techniques.** Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function which is homogeneous of degree one (i.e., $Q(\alpha v) = \alpha Q(v)$ for any $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$) and takes on both positive and negative values. The subset

$$C^+(Q) \stackrel{\text{def}}{=} \{0\} \cup Q^{-1}(0, +\infty) \subset \mathbb{R}^n \tag{7.1}$$

is called the *positive (generalized) cone associated with* Q or simply the *positive cone of* Q. Similarly,

$$C^-(Q) \stackrel{\text{def}}{=} \{0\} \cup Q^{-1}(-\infty, 0) \subset \mathbb{R}^n \tag{7.2}$$

is the *negative (generalized) cone associated to* Q or the *negative cone of* Q. The maximal dimension of a linear subspace $L \subset \mathbb{R}^n$ such that $L \subset C^+(Q)$ (respectively, $L \subset C^-(Q)$) is called *positive* (respectively, *negative*) *rank* of Q and is denoted by $r^+(Q)$ (respectively, $r^-(Q)$). We clearly have $r^+(Q) + r^-(Q) \le n$, and since Q takes on both positive and negative values, we have $r^+(Q) \ge 1$ and $r^-(Q) \ge 1$. We call the function Q *complete* if

$$r^+(Q) + r^-(Q) = n. \tag{7.3}$$

For example, consider the function

$$Q(v) = \text{ sign } K(v, v) \cdot |K(v, v)|^{1/2}, \tag{7.4}$$

where K is a nondegenerate indefinite quadratic form. Q is complete and its positive and negative ranks are equal to the number of positive and negative eigenvalues of the quadratic form K, respectively.

More generally, let λ be a positive real number and K_λ a real function on $\mathbb{R}^n$ which is homogeneous of degree λ (i.e., $K_\lambda(\alpha v) = \alpha^\lambda K_\lambda(v)$ for any $v \in \mathbb{R}^n$ and $\alpha > 0$) and takes on both positive and negative values. Define a homogeneous function Q of degree one by

$$Q(v) = \operatorname{sign} K_\lambda(v) \cdot |K_\lambda(v)|^{1/\lambda}.$$

We say that K_λ is *complete* if Q is complete, and we define the positive and negative cones, and positive and negative ranks of K_λ as those of Q.

Let $\mathcal{A}\colon X \times \mathbb{Z} \to GL(n, \mathbb{R})$ be a cocycle, and $F\colon X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ its linear extension defined by

$$F(x, v) = (f(x), A(x)v)$$

where $A(x) = \mathcal{A}(x, 1)$ is the generator of $\mathcal{A}$.

A real-valued measurable function Q on $X \times \mathbb{R}^n$ is called a *Lyapunov function for the extension* F or *for the cocycle* $\mathcal{A}$ (with respect to a measure ν in X) if there exist positive integers r_Q^+ and r_Q^- such that for ν-almost every $x \in X$,

1. the function Q_x given by $Q_x(v) = Q(x, v)$ is continuous, homogeneous of degree one and takes on both positive and negative values;
2. Q_x is complete and $r^+(Q_x) = r_Q^+$ and $r^-(Q_x) = r_Q^-$;
3. $$Q_{f(x)}(A(x)v) \ge Q_x(v) \text{ for all } v \in \mathbb{R}^n. \tag{7.5}$$

The numbers r_Q^+ and r_Q^- are called the *positive* and *negative ranks* of Q.

When Q is a Lyapunov function, it follows from (7.5) that for ν-almost every $x \in X$,

$$A(x)C^+(Q_x) \subset C^+(Q_{f(x)}), \quad A(f^{-1}(x))^{-1}C^-(Q_x) \subset C^-(Q_{f^{-1}(x)}). \tag{7.6}$$

A Lyapunov function Q on $X \times \mathbb{R}^n$ is called *strict* if the inequality in (7.5) is strict for every $v \neq 0$ and *eventually strict* if for ν-almost every $x \in X$ there exists a positive integer $m = m(x)$ such that for every $v \in \mathbb{R}^n \setminus \{0\}$,

$$Q_{f^m(x)}(\mathcal{A}(x,m)v) > Q_x(v) \tag{7.7}$$

and

$$Q_{f^{-m}(x)}(\mathcal{A}(x,-m)v) < Q_x(v). \tag{7.8}$$

If a Lyapunov function Q is eventually strict then by (7.5), for ν-almost every $x \in X$ the inequalities (7.7) and (7.8) hold for all $m \geq m(x)$.

When Q is a strict Lyapunov function, it follows from (7.5) that

$$A(x)\overline{C^+(Q_x)} \subsetneqq C^+(Q_{f(x)}), \quad A(f^{-1}(x))^{-1}\overline{C^-(Q_x)} \subsetneqq C^-(Q_{f^{-1}(x)}) \tag{7.9}$$

for ν-almost every $x \in X$. Furthermore, if Q is an eventually strict Lyapunov function it follows from (7.7) and (7.8) that

$$\mathcal{A}(x,m)\overline{C^+(Q_x)} \subsetneqq C^+(Q_{f^m(x)}), \quad \mathcal{A}(x,-m)^{-1}\overline{C^-(Q_x)} \subsetneqq C^-(Q_{f^{-m}(x)}) \tag{7.10}$$

for ν-almost every $x \in X$ and every $m \geq m(x)$.

The following result establishes a criterion for nonvanishing Lyapunov exponents.

Theorem 7.1 (Burns and Katok [132]). *If $\mathcal{A}$ possesses an eventually strict Lyapunov function Q then*

1. *$\mathcal{A}$ has ν-almost everywhere r_Q^+ positive and r_Q^- negative values of the Lyapunov exponent counted with their multiplicities;*
2. *for ν-almost every $x \in X$ we have*

$$E^+(x) = \bigcap_{m=1}^{\infty} \mathcal{A}(f^{-m}(x),m)\overline{C^+(Q_{f^{-m}(x)})} \subset C^+(Q_x)$$

and

$$E^-(x) = \bigcap_{m=1}^{\infty} \mathcal{A}(f^{m}(x),-m)\overline{C^-(Q_{f^{m}(x)})} \subset C^-(Q_x).$$

Lyapunov functions are intimately related to the invariant families of cones. Here we give a detailed description of this relationship.

A *(generalized) cone* C in $\mathbb{R}^n$ is a homogeneous set (i.e., $\alpha v \in C$ whenever $v \in C$ and $\alpha \in \mathbb{R}$) such that $C \setminus \{0\}$ is open. In particular, C need not be convex and $\operatorname{int} C$ need not be connected. The *rank* of C is the maximal dimension of a linear subspace $L \subset \mathbb{R}^n$ which is contained in C. We denote it by $r(C)$. The *complimentary cone* $\widehat{C}$ to C is defined by

$$\widehat{C} = \left(\mathbb{R}^n \setminus \overline{C}\right) \cup \{0\}.$$

Obviously the complementary cone to $\widehat{C}$ is C. We have $r(C) + r(\widehat{C}) \leq n$ and this inequality may be strict (this is the case for example, when $C \neq \mathbb{R}^n$ but $\overline{C} = \mathbb{R}^n$). A pair of complementary cones C and $\widehat{C}$ is called *complete* if $r(C) + r(\widehat{C}) = n$.

Let $\mathcal{A}\colon X \times \mathbb{Z} \to GL(n,\mathbb{R})$ be a cocycle over X with generator $A\colon X \to GL(n,\mathbb{R})$. Consider a measurable family of cones $C = \{C_x : x \in X\}$ in $\mathbb{R}^n$. Given a measure ν in X, we say that

1. C is *complete* if the pair of complementary cones $(C_x, \widehat{C}_x)$ is complete for ν-almost every $x \in X$;

2. C is $\mathcal{A}$-*invariant* if for ν-almost every $x \in X$,

$$A(x)C_x \subset C_{f(x)}, \quad A(f^{-1}(x))^{-1}\widehat{C}_x \subset \widehat{C}_{f^{-1}(x)}.$$

Let C be an $\mathcal{A}$-invariant measurable family of cones. We say that

1. C is *strict* if for ν-almost every $x \in X$,

$$A(x)\overline{C_x} \subsetneqq C_{f(x)}, \quad A(f^{-1}(x))^{-1}\overline{\widehat{C}_x} \subsetneqq \widehat{C}_{f^{-1}(x)};$$

2. C is *eventually strict* if for ν-almost every $x \in X$ there exists $m = m(x) \in \mathbb{N}$ such that

$$\mathcal{A}(x, m)\overline{C_x} \subsetneqq C_{f^m(x)}, \quad \mathcal{A}(x, -m)^{-1}\overline{\widehat{C}_x} \subsetneqq \widehat{C}_{f^{-m}(x)}.$$

Let C be a complete $\mathcal{A}$-invariant measurable family of cones in $\mathbb{R}^n$. Any Lyapunov function $Q\colon X \times \mathbb{R}^n \to \mathbb{R}$ satisfying

$$C_x = C^+(Q_x), \quad \widehat{C}_x = C^-(Q_x) \text{for } \nu\text{-almost every } x \in X$$

is called a *Lyapunov function associated with* C. Any complete $\mathcal{A}$-invariant measurable family of cones has an associated Lyapunov function. It is given by

$$Q(x, v) = \begin{cases} d(v/\|v\|, \partial C_x)\|v\| & \text{if } v \in C_x \\ -d(v/\|v\|, \partial C_x)\|v\| & \text{if } v \in \widehat{C}_x \end{cases}.$$

Furthermore, if a complete invariant family of cones is strict (respectively eventually strict) then any of its associated Lyapunov functions is strict (respectively eventually strict).

The above discussion allows us to rephrase Theorem 7.1 in the following fashion.

Theorem 7.2. *If* (5.3) *holds for some f-invariant measure ν, and there exists a complete $\mathcal{A}$-invariant measurable family of cones $C = \{C_x : x \in X\}$, then*

1. *$\mathcal{A}$ has ν-almost everywhere r_Q^+ positive and r_Q^- negative values of the Lyapunov exponent counted with their multiplicities;*
2. *for ν-almost every $x \in X$ we have*

$$E^+(x) = \bigcap_{m=1}^{\infty} \mathcal{A}(f^{-m}(x), m)\overline{C_{f^{-m}(x)}} \subset C_x$$

and

$$E^-(x) = \bigcap_{m=1}^{\infty} \mathcal{A}(f^{m}(x), -m)\overline{\widehat{C}_{f^{m}(x)}} \subset \widehat{C}_x.$$

Let Q be a Lyapunov function on $X \times \mathbb{R}^n$ for a cocycle $\mathcal{A}$. We consider the family of cones $C = \{C_x : x \in X\}$ in $\mathbb{R}^n$ given by

$$C_x = C^+(Q_x).$$

Conditions 2 and 3 in the definition of Lyapunov function imply that C is complete and $\mathcal{A}$-invariant. Note that the complementary cone $\widehat{C}_x$ is not always equal to the cone $C^-(Q_x)$, and thus Q may not be a Lyapunov function associated with C. However, we have $\widehat{C}_x = C^-(Q_x)$ provided that for each v such that $Q_x(v) = 0$ one can find w arbitrarily close to v such that $Q_x(w) > 0$. Furthermore, if Q is strict (respectively eventually strict) then C is strict (respectively eventually strict).

7.2. **Cocycles with values in the symplectic group.** Let $\mathcal{A}\colon X\times\mathbb{Z}\to GL(n,\mathbb{R})$ be a cocycle and Q a homogeneous function of degree one on $X\times\mathbb{Z}$. Consider the corresponding families of cones $C^+(Q_x)$ and $C^-(Q_x)$ given by (7.1) and (7.2). If Q is complete (see (7.3)) and (7.6) holds, then Q is a Lyapunov function. Moreover, if (7.9) (respectively (7.10)) holds then Q is strict (respectively eventually strict). On the other hand, if Condition (7.6) is satisfied only with respect to the family of cones $C^+(Q_x)$ then Q may not be a Lyapunov function. However, this does occur for some interesting classes of cocycles and cones. The most important case for applications involves cocycles with values in the symplectic group $Sp(2m,\mathbb{R})$, $m\ge 1$ and the so-called symplectic cones which we define later.

We begin with the simple case of $SL(2,\mathbb{R})$ cocycles.

We call a cone in $\mathbb{R}^n$ *connected* if its projection to the projective space $\mathbb{R}P^{n-1}$ is a connected set. A connected cone in $\mathbb{R}^2$ is simply the union of two opposite sectors bounded by two different straight lines intersecting at the origin plus the origin itself. By a linear coordinate change such a cone can always be reduced to the following standard cone

$$S=\{(v,w)\in\mathbb{R}: vw>0\}\cup\{(0,0)\}.$$

Theorem 7.3. *If a cocycle with values in $SL(2,\mathbb{R})$ has an eventually strictly invariant family of connected cones $C=\{C_x : x\in X\}$, then it has an eventually strict Lyapunov function Q such that for ν-almost every $x\in X$ the function Q_x has the form* (7.4) *and its zero set coincides with the boundary of the cone C_x.*

Let us now proceed with the general symplectic case. We denote by ω the standard symplectic form in $\mathbb{R}^{2m}$,

$$\omega(v,w)=\sum_{i=1}^{m}(v_iw_{m+i}-w_iv_{m+i}),$$

and by K the following nondegenerate quadratic form of signature zero:

$$K(v)=\sum_{i=1}^{m}v_iv_{m+i}.$$

The cone

$$S=\{v\in\mathbb{R}^{2m}: K(v)>0\}\cup\{0\}$$

is called the *standard symplectic cone*. The image of this cone under an invertible linear symplectic map (i.e., a map with values in $Sp(2m,\mathbb{R})$) is called a *symplectic cone*.

Let L_1 and L_2 be two transverse Lagrangian subspaces in a $2m$-dimensional symplectic space (H,ω), i.e., complementary m-dimensional subspaces on which the symplectic form ω vanishes identically. Then for any $v\in H$ there is a unique decomposition

$$v=v_1+v_2 \text{ with } v_i\in L_i \text{ for } i=1,\,2.$$

Let

$$K_{L_1,L_2}(v)=\omega(v_1,v_2) \text{ and } C_{L_1,L_2}=K^{-1}_{L_1,L_2}((0,\infty))\cup\{0\}.$$

Then C_{L_1,L_2} is a symplectic cone and K_{L_1,L_2} is the corresponding quadratic form.

It is easy to see (for example, by a direct calculation for the case of standard cones), that for a given symplectic cone C in a symplectic space there are exactly two isolated Lagrangian subspaces L_1 and L_2 that belong to the boundary of C

and that $C = C_{L_1,L_2}$ or $C = C_{L_2,L_1}$. Thus, the cone C canonically determines the form K

$$K(C) = K_{L_1,L_2} \text{ or } K(C) = K_{L_2,L_1},$$

depending on whether the form K_{L_1,L_2} or the form K_{L_2,L_1} is positive on C.

For example, the standard cone S is C_{L_1,L_2}, where

$$L_1 = \{(x,0) : x \in \mathbb{R}^m\} \text{ and } L_2 = \{(0,x) : x \in \mathbb{R}^m\}.$$

Proposition 7.4. *Let H and H' be two $2m$-dimensional spaces, L_1, $L_2 \subset H$ and L'_1, $L'_2 \subset H'$ pairs of transverse Lagrangian subspaces and $T\colon H \to H'$ a symplectic linear transformation such that $T\overline{C_{L_1,L_2}} \subset C_{L_1,L_2}$. Then for all $v \in H \setminus \{0\}$ we have*

$$K_{L'_1,L'_2}(Tv) > K_{L_1,L_2}(v).$$

Proposition 7.4 immediately implies the following relation between invariant cone families and Lyapunov functions.

Theorem 7.5. *Let $A\colon X \to Sp(2m,\mathbb{R})$ be a cocycle over a measurable transformation $f\colon X \to X$ which preserves a measure ν. If A has an eventually strictly invariant family of symplectic cones $C = \{C_x : x \in X\}$, then it also has an eventually strict Lyapunov function Q such that for ν-almost every $x \in X$ the function Q_x has the form* (7.4) *with a quadratic form $K = K_x$ of signature zero. Furthermore, the zero set of the function Q_x coincides with the boundary of the cone C_x.*

Combining Theorem 7.5 with Theorem 7.1 we immediately obtain the following.

Corollary 7.6. *If a cocycle $A\colon X \to Sp(2m,\mathbb{R})$ satisfies* (5.3) *and has an eventually strictly invariant family of symplectic cones, then the linear extension F of f has ν-almost everywhere m positive and m negative values of the Lyapunov exponent.*

7.3. **Lyapunov exponents estimates for some particular cocycles.** The cone techniques provide some general methodology for establishing positivity of Lyapunov exponents for cocycles and in particular, for dynamical systems. However, in some particular cases one can use more effective tools and obtain sharper estimates of Lyapunov exponents.

7.3.1. *Herman's method.* We describe a method due to Herman [110] for obtaining a lower bound for the maximal Lyapunov exponent of a holomorphic cocycle with values in a Banach algebra, in particular, with values in $\mathbb{C}^p$. This method is based on some properties of pluri-subharmonic functions.

For $r > 0$, let

$$B^n(0,r) = \{(z_1,\dots,z_n) \in \mathbb{C}^n : |z_i| \le r,\ 1 \le i \le n\},$$

be the closed ball and

$$\mathbb{T}^n_r = \{(z_1,\dots,z_n) \in \mathbb{C}^n : |z_i| = r,\ 1 \le i \le n\}.$$

the torus in C^n. Let also $f\colon U \to \mathbb{C}^n$ be a holomorphic function in a neighborhood U of $B^n(0,r)$ satisfying $f(0) = 0$, $f(B^n(0,r)) \subset B^n(0,r)$, and $f(\mathbb{T}^n_r) \subset \mathbb{T}^n_r$. We also consider a Banach algebra $\mathcal{B}$ over $\mathbb{C}$ and a cocycle $\mathcal{A}\colon \mathbb{T}^n_r \times \mathbb{N} \to \mathcal{B}$ over f with values in $\mathcal{B}$. We have

$$\mathcal{A}(z,m) = A(f^{m-1}(z)) \cdots A(f(z))A(z),$$

where $A\colon X \to \mathcal{B}$ is the generator of the cocycle. Denote by

$$\rho(B) = \lim_{m\to\infty} \|B^m\|^{1/m} = \inf_{m\geq 1} \|B^m\|^{1/m}$$

the spectral radius of the element $B \in \mathcal{B}$ (where $\|\cdot\|$ is the norm in $\mathbb{C}$).

Theorem 7.7. *If f preserves the Lebesgue measure μ in $\mathbb{T}^n_r$, and A is a holomorphic map in a neighborhood of $B^n(0,r)$ with values in a Banach algebra $\mathcal{B}$, then the cocycle $\mathcal{A}$ over f with generator A satisfies*

$$\varliminf_{m\to\infty} \frac{1}{m} \int_{\mathbb{T}^n_r} \log\|\mathcal{A}(z,m)\|\, d\mu(z) \geq \log \rho(A(0)).$$

To see this set

$$a_m = \int_{\mathbb{T}^n_r} \log\|\mathcal{A}(z,m)\|\, d\mu(z).$$

Since the function $z \mapsto \log\|\mathcal{A}(z,m)\|$ is pluri-subharmonic for each m (see [117]),

$$a_m \geq \log\|\mathcal{A}(0,z)\| = \log\|A(0)^m\|.$$

Therefore,

$$\inf_{m\geq 1} \frac{a_m}{m} \geq \log \rho(A(0)).$$

Since f preserves μ, the sequence a_m is subadditive and thus,

$$\varliminf_{m\to\infty} \frac{a_m}{m} = \lim_{m\to\infty} \frac{a_m}{m} = \inf_{m\geq 1} \frac{a_m}{m}$$

and the desired result follows.

7.3.2. *Parameter-exclusion techniques.* In [258], Young considered a C^1 family of cocycles over irrational rotations $R_\alpha(x)$ by $2\pi\alpha$ with generators $A_t\colon S^1 \to SL(2,\mathbb{R})$ such that $|A_t(x)| \approx \chi$ (uniformly in t and x) where $\chi > 0$ is a number. The cocycles are not uniformly hyperbolic. The statement is that *for sufficiently large χ and for a generic family the set of parameters (α, t), for which the Lyapunov exponents of (R_α, A_t) are $\approx \pm\chi$, has nearly full measure.* The proof exploits a parameter-exclusion procedure which goes back to the work of Jacobson [123] and of Benedicks and Carleson [29]: inductively, one identifies certain regions of criticality, studies orbit segments that begin and end near those regions and tries to concatenate long blocks of matrices that have been shown to be hyperbolic; parameters are deleted to ensure the hyperbolicity of the concatenated blocks, and the induction moves forward.

The parameter-exclusion techniques is used to study hyperbolic and ergodic properties of Hénon-like attractors, see Section 14.4.

7.3.3. *Open set of nonuniformly hyperbolic cocycles with values in $SL(2,\mathbb{R})$.* In [257], Young constructed an open set, in the C^1 topology, of cocycles with values in $SL(2,\mathbb{R})$ over a hyperbolic automorphism T of the 2-torus $\mathbb{T}^2$ such that every cocycle in this set has positive Lyapunov exponent but is not uniformly hyperbolic.

Choose $\lambda > \sqrt{\mu} + 1$ where $\mu > 1$ is the eigenvalue of the matrix T (the other eigenvalue is μ^{-1}). Given $\varepsilon > 0$, we define a cocycle over T with the generator $A_\varepsilon\colon \mathbb{T}^2 \to SL(2,\mathbb{R})$ as follows. Let $0 < \beta < 2\pi$ be a number, $J_\varepsilon \subset S^1$ an interval, and $\varphi_\varepsilon\colon \mathbb{T}^2 \to \mathbb{R}/2\pi\mathbb{R}$ a C^1 function such that

1. $\varphi_\varepsilon \equiv 0$ outside of $J_\varepsilon \times S^1$;
2. on $J_\varepsilon \times S^1$, φ_ε increases monotonically from 0 to 2π along the leaves of W^u;

3. on $\varphi_\varepsilon^{-1}[\beta, 2\pi-\beta]$, the directional derivatives of φ_ε along the leaves of W^u are $\geq \frac{1}{\varepsilon}$.

The cocycle A_ε is defined to be

$$A_\varepsilon(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \circ R_{\varphi_\varepsilon(x)},$$

where R_θ is the rotation by the angle θ. The statement is that *one can choose β and, for all sufficiently small ε, the interval J_ε and a neighborhood $\mathfrak{U}_\varepsilon$ of A_ε in $C^1(\mathbb{T}^2, SL(2,\mathbb{R}))$ such that for any $B \in \mathfrak{U}_\varepsilon$ the cocycle over T with the generator B is not uniformly hyperbolic and has a positive Lyapunov exponent with respect to the Lebesgue measure.*

7.3.4. *Cocycles associated with the Jacobi–Perron (JP) algorithm.* This algorithm is a higher-dimensional generalization of the continued fraction algorithm and is used to construct simultaneous rational approximations of real numbers (see [225], [152]). The map f defining the JP algorithm acts on the d-dimensional cube I^d by the formula

$$f(x) = \Big(\frac{x_2}{x_1} \bmod 1, \dots, \frac{x_d}{x_1} \bmod 1, \frac{1}{x_1} \bmod 1\Big)$$

provided $x_1 \neq 0$. The map f preserves a probability measure ν which is absolutely continuous with respect to the Lebesgue measure in the cube and is ergodic with respect to ν.

The JP algorithm associates to almost every point $x \in I^d$ a matrix $A(x)$ such that x can be expressed as $x = a_1 \circ \cdots \circ a_n \circ f^n(x)$ where a_n are the projective maps defined by the matrices $A_n = A(f^{n-1}(x))$ in the space $\mathbb{R}^n \subset \mathbb{P}^n$. The $d+1$ points

$$J_n = a_1 \circ \cdots \circ a_n(0), \dots, J_{n+d} = a_1 \circ \cdots \circ a_{n+d}(0)$$

form a simplex $\sigma_n(xa)$ in $\mathbb{R}^d$ which contains x. Its asymptotic form turns out to be determined by the Lyapunov exponents of the measure ν. The latter are closely related to the Lyapunov exponents χ_i, $i = 1, \dots, d+1$, of the cocycle over f generated by the matrix function $A = A(x)$.

In [49], Broise-Alamichel and Guivarc'h showed that for the JP algorithm:

1. $\sum_{i=1}^{d+1} \chi_i = 0$ and $\chi_1 > \chi_2 > \cdots > \chi_{d+1}$;
2. $\chi_1 + \chi_{d+1} > 0$.

In the case $d = 2$ we have that $\chi_2 < 0$.

7.3.5. *Partially hyperbolic cocycles over locally maximal hyperbolic sets.* Let f be a diffeomorphism of a compact smooth manifold possessing a locally maximal hyperbolic set Λ. Assume that $f|\Lambda$ is topologically transitive. Let μ be an equilibrium measure on Λ corresponding to a Hölder continuous potential φ.

Consider a cocycle $\mathcal{A}$ over f with values in $SL(p,\mathbb{R})$ and let A be the generator of the cocycle. We assume that A depends smoothly on x and that it is dominated by the hyperbolicity of f, i.e., $A(x)$ expands vectors less than the minimum expansion induced by $d_x f$ on the unstable subbundle and $A(x)$ contracts vectors less than the minimum contraction induced by $d_x f$ on the stable subbundle. In other words, the cocycle is partially hyperbolic on $X \times \mathbb{R}^p$.

In [39], Bonatti, Gómez-Mont and Viana showed that the maximal Lyapunov exponent of μ, χ_μ, is zero only in the following very special situation: there exists a continuous family of probability measures m_x, $x \in \Lambda$, on the projective space

$\mathbb{C}P^{p-1}$ which is simultaneously invariant under f, and under the holonomies along the strongly stable and strongly unstable foliations. One can deduce from here that the set of cocycles with a nonzero upper Lyapunov exponent with respect to all the equilibrium measures is an open and dense set in the C^1 topology. It is also shown that for generic C^1 families of cocycles with finitely many parameters, the set of parameters for which the upper Lyapunov exponent is zero for some equilibrium measure is discrete.

8. Local manifold theory

We consider the problem of local stability of trajectories for nonuniformly partially and completely hyperbolic systems. This includes constructing local stable and unstable manifolds and studying their properties. Let us emphasize that the construction of stable (unstable) manifolds can be carried out if only one nonuniformly hyperbolic trajectory is present, i.e., the nonuniformly partially (or completely) hyperbolic set Λ consists of a single trajectory. In particular, the construction does not involve any invariant measure.

There are two well-known methods of building local stable manifolds originated in works of Hadamard [103] and Perron [193, 194]. Hadamard's approach is more geometrical and can be effected for Lipschitz (not necessarily differentiable) maps while Perron's approach allows more flexibility.

These methods work well in the case of uniform hyperbolicity and extending them to nonuniformly hyperbolic systems faces substantial problems. One of them is that the size of local stable manifolds may deteriorate along the trajectory and indeed, may become arbitrarily small. The crucial requirement (6.5) in the definition of nonuniform hyperbolicity provides a control of the deterioration: it can occur with at most subexponential rate.

Both Hadamard and Perron methods allow substantial generalizations to sequences of local diffeomorphisms (instead of iterations of a single diffeomorphism) or maps of Banach spaces (instead of Euclidean spaces), etc.

8.1. **Nonuniformly hyperbolic sequences of diffeomorphisms.** Let $f_m\colon U_m \to \mathbb{R}^n$, $m \in \mathbb{Z}$ ($U_m \subset \mathbb{R}^n$ is an open set) be a (two-sided) sequence of C^1 local diffeomorphisms, and $\{\langle\cdot,\cdot\rangle_m\}_{m\in\mathbb{Z}}$ a (two-sided) sequence of metrics. Write $\mathcal{F} = \{f_m\}_{m\in\mathbb{Z}}$. We assume that $f_m(0) = 0$,

We say that $\mathcal{F}$ is *nonuniformly hyperbolic* if so is the sequence of matrices $\{A_m\}_{m\in\mathbb{Z}} = \{d_0 f_m\}_{m\in\mathbb{Z}}$.

Let $\mathbb{R}^n = E^1_m \oplus E^2_m$ be the invariant splitting associated with nonuniform hyperbolic structure. For every $m \in \mathbb{Z}$ and $(x, y) \in U_m$ one can write f_m in the form

$$f_m(x, y) = (A_m x + g^1_m(x, y), B_m y + g^2_m(x, y)),$$

where $A_m = d_0 f_m|E^1_m$ and $B_m = d_0 f_m|E^2_m$ are linear invertible transformations and $g_m = (g^1_m, g^2_m)\colon U_m \to \mathbb{R}^n$ are C^1 maps satisfying $g_m(0) = 0$, $d_0 g_m(0) = 0$.

Set

$$\sigma_m = \sup\left\{\|d_{(x,y)} g_m\| : (x, y) \in U_m\right\}, \quad \sigma = \sup\{\sigma_m : m \in \mathbb{Z}\}.$$

Note that σ need not be finite in general.

Let $\mathcal{A} = \{A_m\}_{m\in\mathbb{Z}}$ and $\mathcal{B} = \{B_m\}_{m\in\mathbb{Z}}$. Define new sequences of matrices $\{\mathcal{A}_m\}_{m\in\mathbb{Z}}$ and $\{\mathcal{B}_m\}_{m\in\mathbb{Z}}$ using

$$\mathcal{A}_m = \begin{cases} A_{m-1}\cdots A_1 A_0 & \text{if } m>0 \\ \mathrm{Id} & \text{if } m=0 \\ (A_m)^{-1}\cdots (A_{-2})^{-1}(A_{-1})^{-1} & \text{if } m<0 \end{cases}.$$

We also set

$$\mathcal{F}_m = \begin{cases} f_{m-1}\circ\cdots\circ f_1\circ f_0 & \text{if } m>0 \\ \mathrm{Id} & \text{if } m=0 \\ (f_m)^{-1}\circ\cdots\circ (f_{-2})^{-1}\circ(f_{-1})^{-1} & \text{if } m<0 \end{cases},$$

whenever it is defined. The map $(\mathcal{A}_m, \mathcal{B}_m)$ is a linear approximation of $\mathcal{F}_m$ in a neighborhood of 0. We shall describe in the following sections how the stability of the linear approximation effects the stability of the sequence of C^1 local diffeomorphisms.

8.2. **Admissible manifolds and the graph transform.** Let $\gamma > 0$, k, $n \in \mathbb{N}$, $k < n$ be given. A map $\varphi\colon U \to \mathbb{R}^{n-k}$ with $U \subset \mathbb{R}^k$ is called *γ-Lipschitz* if for every x, $x' \in U$,

$$\|\varphi(x) - \varphi(x')\| \le \gamma\|x - x'\|.$$

A set $V \subset \mathbb{R}^n$ is said to be

1. an *admissible* (s,γ)*-set* if there exists a γ-Lipschitz map $\varphi\colon U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ such that
$$V = \mathrm{Graph}(\varphi) = \{(x, \varphi(x)) : x \in U\}.$$
If, in addition, φ is differentiable then V is called an *admissible* (s,γ)*-manifold*.
2. an *admissible* (u,γ)*-set* if there exists a γ-Lipschitz map $\varphi\colon U \subset \mathbb{R}^{n-k} \to \mathbb{R}^{k}$ such that
$$V = \mathrm{Graph}(\varphi) = \{(\varphi(x), x) : x \in U\}.$$
If, in addition, φ is differentiable then V is called an *admissible* (u,γ)*-manifold*.

Given $\gamma > 0$, let $\Gamma(u,\gamma)$ be the space of sequences $\{V_m\}_{m\in\mathbb{Z}}$ of admissible (u,γ)-sets such that $0 \in V_m$. We define a metric on $\Gamma(u,\gamma)$ by

$$d_{\Gamma(u,\gamma)}(\{V_{1m}\}_{m\in\mathbb{Z}}, \{V_{2m}\}_{m\in\mathbb{Z}}) = \sup\{d_m(\varphi_{1m}, \varphi_{2m}) : m \in \mathbb{Z}\},$$

where

$$d_m(\varphi_{1m}, \varphi_{2m}) = \sup\left\{\frac{\|\varphi_{1m}(x) - \varphi_{2m}(x)\|}{\|x\|} : x \in U \setminus \{0\}\right\}$$

and $V_{im} = \mathrm{Graph}(\varphi_{im})$ for each $m \in \mathbb{Z}$ and $i = 1, 2$. Since $\varphi_{1m}(0) = \varphi_{2m}(0) = 0$, and φ_{1m} and φ_{2m} are γ-Lipschitz we have $d_m(\varphi_{1m}, \varphi_{2m}) \le 2\gamma$ and the metric $d_{\Gamma(u,\gamma)}$ is well-defined. One can verify that $\Gamma(u,\gamma)$ is a complete metric space.

We define the *graph transform* $G\colon \Gamma(u,\gamma) \to \Gamma(u,\gamma)$ *induced by* $\mathcal{F}$ *on* $\Gamma(u,\gamma)$ by $G(\{V_m\}_{m\in\mathbb{Z}}) = \{f_m(V_m)\}_{m\in\mathbb{Z}}$.

Proposition 8.1. *If V_m is an admissible (u,γ)-set such that*

$$\sigma_m \le \frac{(\mu' - \lambda')\gamma}{(1+\gamma)^2}, \tag{8.1}$$

then $f_m V_m$ is an admissible (u,γ)-set.

It follow that under assumption (8.1) the map G is well-defined.

Proposition 8.2. *Assume that*

$$\sigma < \frac{\mu' - \lambda'}{2(1+\gamma)}.$$

Then the graph transform G is a contraction on $\Gamma(u, \gamma)$.

As an immediate corollary we obtain existence of (u, γ)-sets.

Theorem 8.3. *Assume that*

$$\sigma \le \frac{(\mu' - \lambda')\gamma}{(1+\gamma)^2} \quad \text{and} \quad \sigma < \frac{\mu' - \lambda'}{2(1+\gamma)}. \tag{8.2}$$

Then there exists a unique family $\{V^u_m\}_{m\in\mathbb{Z}}$ of admissible (u, γ)-sets such that $0 \in V^u_m$ and $f_m(V^u_m) = V^u_{m+1}$.

Note that for $\gamma < 1$ the second inequality in (8.2) follows from the first one.

We now briefly describe how to obtain similar results for (s, γ)-manifolds. For every $m \in \mathbb{Z}$ and $(x, y) \in f_m(U_m)$ one can write ${f_m}^{-1}$ in the form

$${f_m}^{-1}(x, y) = ({A_m}^{-1}x + h^1_m(x, y), {B_m}^{-1}y + h^2_m(x, y)),$$

where $h_m = (h^1_m, h^2_m)\colon f_m(U_m) \to \mathbb{R}^n$ is a C^1 map satisfying $h_m(0) = 0$ and $d_0h_m = 0$. Let

$$\tau_m = \sup\left\{\|d_{(x,y)}h_m\| : (x, y) \in f_mU_m\right\}, \quad \tau = \sup\{\tau_m : m \in \mathbb{Z}\}.$$

Theorem 8.4. *Assume that*

$$\tau \le \frac{(\mu' - \lambda')\gamma}{\lambda'\mu'(1+\gamma)^2} \quad \text{and} \quad \tau < \frac{\mu' - \lambda'}{2\lambda'\mu'(1+\gamma)}.$$

Then there exists a unique family $\{V^s_m\}_{m\in\mathbb{Z}}$ of admissible (s, γ)-sets such that $0 \in V^s_m$ and $f_m(V^s_m) = V^s_{m+1}$.

The following theorem substantially strengthen the above result by claiming that (s, γ) and (u, γ)-sets are indeed smooth manifolds.

Theorem 8.5 (see [135]). *Let $\{f_m\}_{m\in\mathbb{Z}}$ be a nonuniformly hyperbolic sequence of C^1 local diffeomorphisms defined on the whole $\mathbb{R}^n$. Given $\gamma > 0$ and a sufficiently small $\sigma > 0$, there exist a unique family $\{V^s_m\}_{m\in\mathbb{Z}}$ of C^1 admissible (s, γ)-manifolds and a unique family $\{V^u_m\}_{m\in\mathbb{Z}}$ of C^1 admissible (u, γ)-manifolds such that:*

1. *$0 \in V^s_m \cap V^u_m$;*
2. *$f_m(V^s_m) = V^s_{m+1}$ and $f_m(V^u_m) = V^u_{m+1}$;*
3. *$T_0V^s_m = E^s_m$ and $T_0V^u_m = E^u_m$;*
4. *if $(x, y) \in V^s_m$ then*

   $$\|f_m(x, y)\| \le (1+\gamma)(\lambda + \sigma_m)\|(x, y)\|$$

   and if $(x, y) \in V^u_m$ then

   $$\|f_m(x, y)\| \ge (\mu/(1+\gamma) - \sigma_m)\|(x, y)\|,$$

   where $0 < \lambda < 1 < \mu$;

5. *for every* $(1+\gamma)(\lambda+\sigma) < \nu < \mu/(1+\gamma) - \sigma$ *and* $(x, y) \in \mathbb{R}^n$*, if there exists* $C > 0$ *such that*
$$\|\mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y)\| \le C\nu^k \|(x, y)\|$$
for every $k \ge 0$ *then* $(x, y) \in V_m^s$*, and if there exists* $C > 0$ *such that*
$$\|\mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y)\| \le C\nu^k \|(x, y)\|$$
for every $k \le 0$ *then* $(x, y) \in V_m^u$*.*

Notice that an admissible (s, γ)-manifold (respectively, (u, γ)-manifold) is also an admissible (s, γ')-manifold (respectively, (u, γ')-manifold) for every $\gamma' > \gamma$. Therefore, the uniqueness property in Theorem 8.5 implies that both families $\{V_m^s\}_{m\in\mathbb{Z}}$ and $\{V_m^u\}_{m\in\mathbb{Z}}$ are independent of γ. These families are called, respectively, *family of invariant s-manifolds* and *family of invariant u-manifolds*. They can be characterized as follows.

Proposition 8.6. *For each* $\gamma \in (0, \sqrt{\mu'/\lambda'} - 1)$ *and each sufficiently small* $\sigma > 0$*:*

1. *if*
$$\nu \in ((1+\gamma)(\lambda+\sigma), \mu/(1+\gamma) - \sigma)$$
then
$$V_m^s = \left\{(x, y) \in \mathbb{R}^n : \overline{\lim_{k\to+\infty}} \frac{1}{k} \log\|\mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y)\| < \log \nu\right\}$$
and
$$V_m^u = \left\{(x, y) \in \mathbb{R}^n : \overline{\lim_{k\to-\infty}} \frac{1}{|k|} \log\|\mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y)\| < -\log \nu\right\};$$
2. *if* $(1+\gamma)(\lambda+\sigma) < 1 < \mu/(1+\gamma) - \sigma$ *then*
$$\begin{aligned} V_m^s &= \left\{(x, y) \in \mathbb{R}^n : \overline{\lim_{k\to+\infty}} \frac{1}{k} \log\|\mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y)\| < 0\right\} \\ &= \left\{(x, y) \in \mathbb{R}^n : \sup_{k\ge 0}\|\mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y)\| < \infty\right\} \\ &= \left\{(x, y) \in \mathbb{R}^n : \mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y) \to 0 \text{ as } k \to +\infty\right\} \end{aligned}$$
and
$$\begin{aligned} V_m^u &= \left\{(x, y) \in \mathbb{R}^n : \overline{\lim_{k\to-\infty}} \frac{1}{|k|} \log\|\mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y)\| < 0\right\} \\ &= \left\{(x, y) \in \mathbb{R}^n : \sup_{k\le 0}\|\mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y)\| < \infty\right\} \\ &= \left\{(x, y) \in \mathbb{R}^n : \mathcal{F}_{m+k} \circ \mathcal{F}_m{}^{-1}(x, y) \to 0 \text{ as } k \to -\infty\right\}. \end{aligned}$$

The following result provides some additional information on higher differentiability of (s, γ)- and (u, γ)-manifolds.

Theorem 8.7. *Let* $\mathcal{F}$ *be a sequence of* C^r *local diffeomorphisms, for some* $r > 0$*. Then the unique family* $\{V_m^u\}_{m\in\mathbb{Z}}$ *of admissible* (u, γ)*-sets given by Theorem 8.3 is composed of* C^r *manifolds.*

8.3. **Hadamard–Perron Theorem: Perron's method.** We describe a version of Perron's approach to the proof of the Stable Manifold Theorem which originated in [196] and allows one to construct stable (and unstable) invariant manifolds along a single nonuniformly partially hyperbolic trajectory in the broad sense.[5]

Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and $x \in M$. Assume that f is nonuniformly partially hyperbolic in the broad sense on the set $\Lambda = \{f^n(x)\}_{n\in\mathbb{Z}}$ (see Section 6.2). We obtain the local stable manifold in the form

$$V(x) = \exp_x\{(x, \psi(x)) : x \in B_1(r)\}, \tag{8.3}$$

where $\psi\colon B_1(r) \to E_2(x)$ is a smooth map, satisfying $\psi(0) = 0$ and $d\psi(0) = 0$, $E_1(x)$, $E_2(x)$ are invariant distributions in the tangent space (see (6.1)), and $B_1(r) \in E_1(x)$ is the ball of radius r centered at the origin. The number $r = r(x)$ is called the *size* of the local stable manifold.

We now describe how to construct the function ψ. Fix $x \in M$ and consider the map

$$\widetilde{f}_x = \exp^{-1}_{f(x)} \circ f \circ \exp_x\colon B_1(r) \times B_2(r) \to T_{f(x)}M,$$

which is well-defined if r is sufficiently small. Here $B_2(r)$ is the ball of radius r in $E_1(x)$ centered at the origin. The map $\widetilde{f}$ can be written in the following form:

$$\widetilde{f}_x(v_1, v_2) = (A_x v_1 + g_{1x}(v_1, v_2), B_x v_2 + g_{2x}(v_1, v_2)),$$

where $v_1 \in E_1(x)$ and $v_2 \in E_2(x)$. Furthermore,

$$A_x\colon E_1(x) \to E_2(f(x)) \quad \text{and} \quad B_x\colon E_2(x) \to E_2(f(x))$$

are linear maps. The map A_x is a contraction and the map B_x is an expansion. Since f is of class $C^{1+\alpha}$ we also have for $g = (g_1, g_2)$,

$$\|g_x(v)\| \le C_1\|v\|^{1+\alpha} \tag{8.4}$$

and

$$\|dg_x(v) - dg_x(w)\| \le C_1\|v - w\|^{\alpha}, \tag{8.5}$$

where $C_1 > 0$ is constant (which may depend on x).

In other words the map $\widetilde{f}_x$ can be viewed as a small perturbation of the linear map $(v_1, v_2) \mapsto (A_x v_1, B_x v_2)$ by the map $g_x(v_1, v_2)$ satisfying Conditions (8.4) and (8.5) in a small neighborhood U_x of the point x.

Note that size of U_x depends on x and may decay along the trajectory of x with subexponential rate (see (6.6)). This requires a substantial modification of the classical Perron's approach.

Proceeding further with Perron's approach we identify each of the tangent spaces $T_{f^m(x)}M$ with $\mathbb{R}^p = \mathbb{R}^k \times \mathbb{R}^{p-k}$ (recall that $p = \dim M$ and $1 \le k < p$) via an isomorphism τ_m such that $\tau_m(E_1(x)) = \mathbb{R}^k$ and $\tau_m(E_2(x)) = \mathbb{R}^{p-k}$. The map $\widetilde{F}_m = \tau_{m+1} \circ F_m \circ {\tau_m}^{-1}$ is of the form

$$\widetilde{F}_m(v_1, v_2) = (A_m v_1 + g_{1m}(v_1, v_2), B_m v_2 + g_{2m}(v_1, v_2)), \tag{8.6}$$

[5] In [196], the system is assumed to preserve a hyperbolic smooth measure. However, the proof does not use this assumption and readily extends to the case of a single nonuniformly partially hyperbolic trajectory in the broad sense. This was observed in [215].

where $A_m : \mathbb{R}^k \to \mathbb{R}^k$ and $B_m : \mathbb{R}^{p-k} \to \mathbb{R}^{p-k}$ are linear maps, and $g : \mathbb{R}^n \to \mathbb{R}^k$ is a nonlinear map defined for each $v_1 \in B_1(r_0) \subset \mathbb{R}^k$ and $v_2 \in B_2(r_0) \subset \mathbb{R}^{p-k}$. With respect to the Lyapunov inner product these maps satisfy:

$$\|A_m\|' \le \lambda', \quad (\|{B_m}^{-1}\|')^{-1} \ge \mu', \text{ where } 0 < \lambda' < \min\{1, \mu'\} \tag{8.7}$$

and

$$g_m(0) = 0, \quad dg_m(0) = 0,$$
$$\|dg_m(v) - dg_m(w)\|' \le C_2\gamma^{-m}\|v - w\|'^{\alpha},$$

where

$$\lambda'^{\alpha} < \gamma < 1, \quad 0 < \alpha \le 1, \quad C > 0$$

(see (8.5)). We now state a general version of the Stable Manifold Theorem.

Theorem 8.8 (Pesin [196])**.** *Let κ be any number satisfying*

$$\lambda' < \kappa < \min\{\mu', \gamma^{\frac{1}{\alpha}}\}. \tag{8.8}$$

There exist $D > 0$ and $r_0 > r > 0$, and a map $\psi : B_1(r) \to \mathbb{R}^{p-k}$ such that:

1. *ψ is of class $C^{1+\alpha}$ and $\psi(0) = 0$, $d\psi(0) = 0$;*
2. *$\|d\psi(v) - d\psi(w)\|' \le D\|v - w\|'^{\alpha}$ for any v, $w \in B_1(r)$;*
3. *if $m \ge 0$ and $v \in B_1(r)$ then*
$$\left(\prod_{i=0}^{m-1} \widetilde{F}_i\right)(v, \psi(v)) \in B_1(r) \times B_2(r),$$
$$\left\|\left(\prod_{i=0}^{m-1} \widetilde{F}_i\right)(v, \psi(v))\right\|' \le D\kappa^m \|(v, \psi(v))\|',$$
where $\prod_{i=0}^{m-1} \widetilde{F}_i$ denotes the composition $\widetilde{F}_{m-1} \circ \cdots \circ \widetilde{F}_0$ (with the convention that $\prod_{i=0}^{-1} \widetilde{F}_i = \mathrm{Id}$;
4. *given $v \in B_1(r)$ and $w \in B_2(r)$, if there is a number $K > 0$ such that*
$$\left(\prod_{i=0}^{m-1} \widetilde{F}_i\right)(v, w) \in B_1(r) \times B_2(r), \qquad \left\|\left(\prod_{i=0}^{m-1} \widetilde{F}_i\right)(v, w)\right\|' \le K\kappa^m$$
for every $m \ge 0$, then $w = \psi(v)$;
5. *the numbers D and r depend only on the numbers λ', μ', γ, α, κ, and C.*

We outline the proof of the theorem. Consider the linear space Γ_κ of sequences of vectors $z = \{z(m) \in \mathbb{R}^p\}_{m \in \mathbb{N}}$ satisfying

$$\|z\|_\kappa = \sup_{m \ge 0} (\kappa^{-m}\|z(m)\|') < \infty.$$

Γ_κ is a Banach space with the norm $\|z\|_\kappa$. Given $r > 0$, set

$$W = \{z \in \Gamma_\kappa : z(m) \in B_1(r) \times B_2(r) \text{ for every } m \in \mathbb{N}\}.$$

Since $0 < \kappa < 1$ the set W is open. Consider the map $\Phi_\kappa : B_1(r_0) \times W \to \Gamma_\kappa$ given by

$$\Phi_\kappa(y, z)(0) = \left(y, -\sum_{k=0}^{\infty} \left(\prod_{i=0}^{k} B_i\right)^{-1} g_{2k}(z(k))\right),$$

and for $m > 0$,

$$\Phi_\kappa(y,z)(m) = -z(m) + \left(\left(\prod_{i=0}^{m-1} A_i\right) y, 0\right)$$
$$+ \left(\sum_{n=0}^{m-1}\left(\prod_{i=n+1}^{m-1} A_i\right) g_{1n}(z(n)), -\sum_{n=0}^{\infty}\left(\prod_{i=0}^{n} B_{i+m}\right)^{-1} g_{2\,n+m}(z(n+m))\right).$$

Using Conditions (8.7)–(8.8) one can show that the map Φ_κ is well-defined, continuously differentiable over y and z and $\Phi_\kappa(0,0) = (0,0)$. Moreover, Φ_κ is of class C^1 with partial derivatives given by

$$\partial_y \Phi_\kappa(y,z)(m) = \mathcal{A}_\kappa(z) - \mathrm{Id},$$

where

$$(\mathcal{A}_\kappa(z))t(m) = \left(\sum_{n=0}^{m-1}\left(\prod_{i=n+1}^{m-1} A_i\right) dg_{1n}(z(n))t(n),\right.$$
$$\left. -\sum_{n=0}^{\infty}\left(\prod_{i=0}^{n} B_{i+m}\right)^{-1} dg_{2\,n+m}(z(n+m))t(m+n)\right).$$

Furthermore,

$$\|\mathcal{A}_\kappa(z_1) - \mathcal{A}_\kappa(z_2)\| \le C\|z_1 - z_2\|_\kappa{}^\alpha, \tag{8.9}$$

where $C > 0$ is a constant. We have, in particular, that $\partial_z\Phi_\kappa(y,0) = -\mathrm{Id}$ and the map $\partial_z\Phi_\kappa(y,z)$ is continuous. Therefore, the map Φ_κ satisfies the conditions of the Implicit Function Theorem, and hence, there exist a number $r \le r_0$ and a map $\varphi\colon B_1(r) \to W$ of class C^1 with

$$\varphi(0) = 0 \quad \text{and} \quad \Phi_\kappa(y,\varphi(y)) = 0. \tag{8.10}$$

Note that the derivatives $\partial_y\Phi_\kappa$ and $\partial_z\Phi_\kappa$ are Hölder continuous. It is clear for the former and follows for the latter in view of (8.9):

$$\begin{aligned}\|\partial_z\Phi_\kappa(y_1,z_1) - \partial_z\Phi_\kappa(y_2,z_2)\| &\le \|\partial_z\Phi_\kappa(y_1,z_1) - \partial_z\Phi_\kappa(y_1,z_2)\| \\ &\quad + \|\partial_z\Phi_\kappa(y_1,z_2) - \partial_z\Phi_\kappa(y_2,z_2)\| \\ &= 2\|\mathcal{A}_\kappa(z_1) - \mathcal{A}_\kappa(z_2)\| \le CM\|z_1 - z_2\|_\kappa{}^\alpha.\end{aligned}$$

There is a special version of the Implicit Function Theorem for maps with Hölder continuous derivatives (see [24]) which enables one to obtain an explicit estimate of the number r and to show that it depends only on λ', μ', γ, α, κ, and C.

We now describe some properties of the map φ. Differentiating the second equality in (8.10) with respect to y we obtain

$$d\varphi(y) = -[\partial_z\Phi_\kappa(y,\varphi(y))]^{-1}\partial_y\Phi_\kappa(y,\varphi(y)).$$

Setting $y = 0$ in this equality yields

$$d\varphi(0)(m) = \left(\prod_{i=0}^{m-1} A_i, 0\right).$$

One can write the vector $\varphi(y)(m)$ in the form

$$\varphi(y)(m) = (\varphi_1(y)(m), \varphi_2(y)(m)),$$

where $\varphi_1(y)(m) \in \mathbb{R}^k$ and $\varphi_2(y)(m) \in \mathbb{R}^{p-k}$. It follows from (8.10) that if $m \geq 0$ then

$$\varphi_1(y)(m) = \left(\prod_{i=0}^{m-1} A_i\right) y + \sum_{n=0}^{m-1} \left(\prod_{i=n+1}^{m-1} A_i\right) g_{1n}(\varphi(y)(n)), \tag{8.11}$$

and

$$\varphi_2(y)(m) = -\sum_{n=0}^{\infty} \left(\prod_{i=0}^{n} B_{i+m}\right)^{-1} g_{2\,n+m}(\varphi(y)(n+m)). \tag{8.12}$$

These equalities imply that

$$\varphi_1(y)(m+1) = A_m\varphi_1(y)(m) + g_{1m}(\varphi_1(y)(m), \varphi_2(y)(m)),$$
$$\varphi_2(y)(m+1) = B_m\varphi_2(y)(m) + g_{2m}(\varphi_1(y)(m), \varphi_2(y)(m)).$$

Indeed, iterating the first equality "forward" one easily obtains (8.11). Rewriting the second equality in the form

$$\varphi_2(y)(m) = B_m^{-1}\varphi_2(y)(m+1) - B_m^{-1}g_{2m}(\varphi_1(y)(m), \varphi_2(y)(m))$$

and iterating it "backward" yields (8.12).

Thus, we obtain that the function $\varphi(y)$ is invariant under the family of maps $\widetilde{F}_m$, i.e.,

$$\widetilde{F}_m(\varphi(y)(m)) = \varphi(y)(m+1).$$

The desired map ψ^s is now defined by $\psi(v) = \varphi_2(v)(0)$ for each $v \in B^s(r)$.

Applying the above result to a diffeomorphism f which is nonuniformly partially hyperbolic in the broad sense along the trajectory of a point $x \in M$ we obtain the following version of the Stable Manifold Theorem.

Theorem 8.9. *There exists a local stable manifold $V(x)$ such that $x \in V(x)$, $T_xV(x) = E_1(x)$, and for $y \in V(x)$ and $n \geq 0$,*

$$\rho(f^n(x), f^n(y)) \leq T(x)\lambda^n e^{\varepsilon n}\rho(x, y), \tag{8.13}$$

where $T\colon \Lambda \to (0, \infty)$ is a Borel function satisfying

$$T(f^m(x)) \leq T(x)e^{10\varepsilon|m|}, \quad m \in \mathbb{Z}. \tag{8.14}$$

In [207], Pugh constructed an explicit example of a nonuniformly completely hyperbolic diffeomorphism of a 4-dimensional manifold of class C^1 (and not of class $C^{1+\alpha}$ for any $\alpha > 0$) for which the statement of Theorem 8.9 fails. More precisely, there exists no manifold tangent to $E_1(x)$ such that (8.13) holds on some open neighborhood of x. This example illustrates that the assumption $\alpha > 0$ in Theorem 8.9 is crucial. The situation is different for systems with continuous time: Barreira and Valls [27] has shown that there is a class of C^1 vector fields that are not $C^{1+\alpha}$ for any $\alpha > 0$ whose nonuniformly hyperbolic trajectories possess stable manifolds.

One can obtain a more refined information about smoothness of local stable manifolds. More precisely, let f be a diffeomorphism of class $C^{p+\alpha}$, with $p \geq 1$ and $0 < \alpha \leq 1$. Assume that f is nonuniformly partially hyperbolic in the broad sense along a trajectory of a point $x \in M$. Then the local stable manifold $V(x)$ is of class C^p; in particular, if f is of class C^p for some $p \geq 2$, then $V(x)$ is of class C^{p-1} (and even of class $C^{p-1+\alpha}$ for any $0 < \alpha < 1$). These results are immediate consequences of the following version of Theorem 8.8.

Theorem 8.10. *Assume that the conditions of Theorem 8.8 hold. In addition, assume that:*

1. *g_m are of class C^p for some $p \geq 2$;*
2. *there exists $K > 0$ such that for $\ell = 1, \ldots, p$,*
$$\sup_{z \in B} \|d^\ell g_m(z)\|' \leq K\gamma^{-m}, \quad \sup_{z \in B} \|d^\ell h_m(z)\|' \leq K\gamma^{-m},$$
where $B = B_1(r_0) \times B_2(r_0)$ (see (8.6));
3. *for $z_1, z_2 \in B$ and some $\alpha \in (0,1)$,*
$$\|d^p g_m(z_1) - d^p g_m(z_2)\|' \leq K\gamma^{-m}(\|z_1 - z_2\|')^\alpha.$$

If $\psi(u)$ is the map constructed in Theorem 8.8, then there exists a number $N > 0$, which depends only on the numbers λ', μ', γ, α, κ, and K, such that:

1. *ψ is of class $C^{p+\alpha}$;*
2. *$\sup_{u \in B_1(r)} \|d^\ell \psi(u)\|' \leq N$ for $\ell = 1, \ldots, p$.*

In [208], Pugh and Shub strengthened the above result and showed that in fact, if f is of class C^p for some $p \geq 2$, then $V(x)$ is also of class C^p.

In the case of diffeomorphisms which are nonuniformly partially hyperbolic, in particular, nonuniformly completely hyperbolic, there is a symmetry between the objects marked by the index "s" and those marked by the index "u". Namely, when the time direction is reversed the statements concerning objects with index "s" become the statements about the corresponding objects with index "u". In these cases we shall denote the local stable manifold at x by $V^s(x)$. We can also construct the local unstable manifolds.

Theorem 8.11 (Unstable Manifold Theorem)**.** *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M which is nonuniformly partially hyperbolic along the trajectory of a point $x \in M$. Then there exists a local unstable manifold $V^u(x)$ such that $x \in V^u(x)$, $T_x V^u(x) = E^u(x)$, and if $y \in V^u(x)$ and $n \leq 0$ then*
$$\rho(f^n(x), f^n(y)) \leq T(x)\mu^n e^{\varepsilon|n|}\rho(x,y),$$
where $T \colon \Lambda \to (0, \infty)$ is a Borel function satisfying (8.14)*.*

Stable Manifold Theorem 8.9 was first established by Pesin in [196]. His proof is built upon classical work of Perron. Katok and Strelcyn [138] extended Stable Manifold Theorem to smooth maps with singularities (see Section 18). They essentially followed Pesin's approach. Ruelle [215] obtained another proof of Theorem 8.9, based on his study of perturbations of the matrix products in the Multiplicative Ergodic Theorem 5.5. Fathi, Herman, and Yoccoz [84] provided a detailed exposition of Theorem 8.9 which essentially follows the approaches of Pesin and Ruelle. Pugh and Shub [208] proved Stable Manifold Theorem for nonuniformly partially hyperbolic systems using graph transform techniques.

On another direction, Liu and Qian [164] established a version of Theorem 8.9 for random maps (see the article by Kifer and Liu [143] in this volume). One can extend the Stable Manifold Theorem 8.9 to infinite-dimensional spaces. Ruelle [216] proved this theorem for Hilbert spaces, closely following his approach in [215], and Mañé [172] considered Banach spaces (under certain compactness assumptions on the dynamics).

8.4. **Stable Manifold Theorem for flows.** Let φ_t be a smooth flow on a compact smooth Riemannian manifold M. The following is an analog of Theorem 8.9 for flows.

Theorem 8.12. *Assume that φ_t is nonuniformly hyperbolic along a trajectory $\varphi_t(x)$. Then there exists a local stable manifold $V^s(x)$ satisfying: (a) $x \in V^s(x)$, (b) $T_xV^s(x) = E^s(x)$, (c) if $y \in V^s(x)$ and $t > 0$ then*

$$\rho(\varphi_t(x), \varphi_t(y)) \le T(x)\lambda^t e^{\varepsilon t}\rho(x, y),$$

where $T\colon \Lambda \to (0, \infty)$ is a Borel function such that for $s \in \mathbb{R}$,

$$T(\varphi_s(x)) \le T(x)e^{10\varepsilon|s|}.$$

The proof of Theorem 8.12 can be obtained by applying Theorem 8.9 to the diffeomorphism $f = \varphi_1$ (that is nonuniformly partially hyperbolic). We call $V^s(x)$ a *local stable manifold* at x.

By reversing the time one can construct a *local unstable manifold* $V^u(x)$ at x. It has the properties similar to those of the stable manifold.

8.5. **Continuity and sizes of local manifolds.** Recall that the size of the local stable manifold $V(x)$ at a point $x \in \Lambda$ (with Λ as in Section 8.3) is the number $r = r(x)$ that is determined by Theorem 8.8 and such that (8.3) holds. It follows from Statement 5 of Theorem 8.8 that the sizes of the local stable manifold at a point x and any point $y = f^m(x)$ along the trajectory of x are related by

$$r(f^m(x)) \ge Ke^{-\varepsilon|m|}r(x), \tag{8.15}$$

where $K > 0$ is a constant.

Assume now that f is nonuniformly partially hyperbolic in the broad sense on an invariant set Λ, and let ν be an f-invariant ergodic Borel measure with $\nu(\Lambda) = 1$. For all sufficiently large ℓ the regular set Λ^ℓ has positive measure. Therefore, the trajectory of almost every point visits Λ^ℓ infinitely many times. It follows that for typical points x the function $r(f^m(x))$ is an oscillating function of m which is of the same order as $r(x)$ for many values of m. Nevertheless, for some integers m the value $r(f^m(x))$ may become as small as it is allowed by (8.15). Let us emphasize that the rate with which the sizes of the local stable manifolds $V(f^m(x))$ decreases as $m \to +\infty$ is smaller than the rate with which the trajectories $\{f^m(x)\}$ and $\{f^m(y)\}$, $y \in V(x)$ approach each other.

It follows from Statement 5 of Theorem 8.8 that the sizes of local manifolds are bounded from below on any regular set Λ^ℓ, i.e., there exists a number $r_\ell > 0$ that depends only on ℓ such that

$$r(x) \ge r_\ell \text{ for } x \in \Lambda^\ell. \tag{8.16}$$

Local stable manifolds depend uniformly continuously on $x \in \Lambda^\ell$ in the C^1 topology, i.e., if $x_n \in \Lambda^\ell$ is a sequence of points converging to x then $d_{C^1}(V(x_n), V(x)) \to 0$ as $n \to \infty$. Furthermore, by the Hölder continuity of stable distributions, local stable manifolds depend Hölder continuously on $x \in \Lambda^\ell$. More precisely, for every $\ell \ge 1$, $x \in \Lambda^\ell$, and points z_1, $z_2 \in V(x)$,

$$d(T_{z_1}V(x), T_{z_2}V(x)) \le C\rho(z_1, z_2)^\alpha,$$

where $C > 0$ is a constant depending only on ℓ.

In the case when f is nonuniformly partially hyperbolic on an invariant subset Λ we have, for almost every $x \in \Lambda$, the local stable and unstable manifolds. Their sizes vary along the trajectory according to (8.15) and are bounded below by (8.16) on any regular set Λ^ℓ.

Finally, if f is nonuniformly completely hyperbolic on an invariant subset Λ then continuity of local stable and unstable manifolds on a regular set Λ^ℓ implies that there exists a number $\delta_\ell > 0$ such that for every $x \in \Lambda^\ell$ and $y \in \Lambda^\ell \cap B(x, \delta_\ell)$ the intersection $V^s(x) \cap V^u(y)$ is nonempty and consists of a single point which depends continuously (and in fact, Hölder continuously) on x and y.

8.6. **Graph transform property.** There is a version of the Stable Manifold Theorem known as Graph Transform Property (usually referred to as Inclination Lemma or λ-Lemma).

Consider a $C^{1+\alpha}$ diffeomorphism f which is nonuniformly partially hyperbolic in the broad sense along the trajectory of a point $x \in M$. Choose numbers r_0, b_0, and c_0 and for every $m \geq 0$, set

$$r_m = r_0 e^{-\varepsilon m}, \quad b_m = b_0 \mu^{-m} e^{\varepsilon m}, \quad c_m = c_0 e^{-\varepsilon m}.$$

Consider the class Ψ of $C^{1+\alpha}$ functions on $\{(m, v) : m \geq 0, v \in B_1(r_m)\}$ with values $\psi(m, v) \in E_2(f^{-m}(x))$ (where $B_1(r_m)$ is the ball in $E_1(f^{-m}(x))$ centered at 0 of radius r_m) satisfying the following conditions:

$$\|\psi(m, 0)\| \leq b_m, \qquad \max_{v \in B_1(r_m)} \|d\psi(m, v)\| \leq c_m.$$

Theorem 8.13. *There are positive constants r_0, b_0, and c_0 such that for every $\psi \in \Psi$ one can find a function $\widetilde{\psi} \in \Psi$ for which*

$$F_m^{-1}(\{(v, \psi(m, v)) : v \in B_1(r_m)\}) \supset \{(v, \widetilde{\psi}(m+1, v)) : v \in B_1(r_{m+1})\}$$

for all $m \geq 0$.

8.7. **Regular neighborhoods.** Let $f\colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth n-dimensional Riemannian manifold M which is nonuniformly completely hyperbolic on an invariant set Λ. Viewing df as a linear cocycle over f we shall use the theory of linear extensions of cocycles (see Section 4) to construct a special coordinate system for every regular point $x \in \Lambda$. Applying the Reduction Theorem 5.10, given $\varepsilon > 0$ and a regular point $x \in M$, there exists a linear transformation $C_\varepsilon(x)\colon \mathbb{R}^n \to T_x M$ such that:

1. the matrix
$$A_\varepsilon(x) = C_\varepsilon(fx)^{-1} \circ d_x f \circ C_\varepsilon(x).$$
has the Lyapunov block form (5.12) (see Theorem 5.10);
2. $\{C_\varepsilon(f^m(x))\}_{m \in \mathbb{Z}}$ is a tempered sequence of linear transformations.

For every regular point $x \in M$ there is a neighborhood $N(x)$ of x such that f acts in $N(x)$ very much like the linear map $A_\varepsilon(x)$ in a neighborhood of the origin.

Denote by Λ the set of regular points for f and by $B(0, r)$ the standard Euclidean r-ball in $\mathbb{R}^n$ centered at the origin.

Theorem 8.14 (see [135]). *For every $\varepsilon > 0$ the following properties hold:*

1. *there exists a tempered function $q\colon \Lambda \to (0, 1]$ and a collection of embeddings $\Psi_x\colon B(0, q(x)) \to M$ for each $x \in \Lambda$ such that $\Psi_x(0) = x$ and $e^{-\varepsilon} < q(fx)/q(x) < e^\varepsilon$; these embeddings satisfy $\Psi_x = \exp_x \circ C_\varepsilon(x)$, where $C_\varepsilon(x)$ is the Lyapunov change of coordinates;*

2. *if* $f_x \stackrel{\text{def}}{=} \Psi_{fx}^{-1} \circ f \circ \Psi_x \colon B(0, q(x)) \to \mathbb{R}^n$, *then* $d_0 f_x$ *has the Lyapunov block form* (5.12)*;*
3. *the* C^1 *distance* $d_{C^1}(f_x, d_0 f_x) < \varepsilon$ *in* $B(0, q(x))$*;*
4. *there exist a constant* $K > 0$ *and a measurable function* $A \colon \Lambda \to \mathbb{R}$ *such that for every* $y, z \in B(0, q(x))$,
$$K^{-1} \rho(\Psi_x y, \Psi_x z) \le \|y - z\| \le A(x) \rho(\Psi_x y, \Psi_x z)$$
with $e^{-\varepsilon} < A(fx)/A(x) < e^{\varepsilon}$.

We note that for each $x \in \Lambda$ there exists a constant $B(x) \ge 1$ such that for every y, $z \in B(0, q(x))$,
$$B(x)^{-1} \rho(\Psi_x y, \Psi_x z) \le \rho_x'(\exp_x y, \exp_x z) \le B(x) \rho(\Psi_x y, \Psi_x z),$$
where $\rho_x'(\cdot, \cdot)$ is the distance on $\exp_x B(0, q(x))$ with respect to the Lyapunov metric $\|\cdot\|_x'$. By Lusin's Theorem, given $\delta > 0$ there exists a set of measure at least $1 - \delta$ where $x \mapsto B(x)$ as well as $x \mapsto A(x)$ in Theorem 8.14 are bounded.

For each regular point $x \in \Lambda$ the set
$$R(x) \stackrel{\text{def}}{=} \Psi_x(B(0, q(x)))$$
is called a *regular neighborhood* of x or a *Lyapunov chart* at x.

We stress that the existence of regular neighborhoods uses the fact that f is of class $C^{1+\alpha}$ in an essential way.

9. Global manifold theory

Let $f \colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold M which is nonuniformly partially hyperbolic in the broad sense on an invariant set $\Lambda \subset M$. Starting with local stable manifolds we will construct global stable manifolds for f.

In the case of uniformly partially hyperbolic systems (in the broad sense) global manifolds are integral manifolds of the stable distribution E_1. The latter is, in general, continuous but not smooth and hence, the classical Frobenius method fails. Instead, one can *glue* local manifolds to obtain leaves of the foliation.

In the case of nonuniformly hyperbolic systems (in the broad sense) the stable distribution E_1 may not even be continuous but measurable. The resulting "foliation" is measurable in a sense but has smooth leaves.

9.1. **Global stable and unstable manifolds.**

Given a point $x \in \Lambda$, the *global stable manifold* is given by
$$W(x) = \bigcup_{n=0}^{\infty} f^{-n}(V(f^n(x))). \tag{9.1}$$
This is a finite-dimensional immersed smooth submanifold of class $C^{r+\alpha}$ if f is of class $C^{r+\alpha}$. It has the following properties which are immediate consequences of the Stable Manifold Theorem 8.8.

Theorem 9.1. *If* x, $y \in \Lambda$, *then:*
1. $W(x) \cap W(y) = \varnothing$ *if* $y \notin W(x)$*;*
2. $W(x) = W(y)$ *if* $y \in W(x)$*;*
3. $f(W(x)) = W(f(x))$*;*

4. *$W(x)$ is characterized as follows*

$$W(x) = \{y \in M\colon \varlimsup_{n\to\infty} \frac{1}{n} \log \rho(f^n(x), f^n(y)) < \log \lambda\}$$

(see section 6.2 for the definition of λ).

Note that local stable manifolds are not uniquely defined. Indeed, one can choose a "smaller" submanifold containing x and lying inside $V(x)$, and view it as a "new" local manifold at x. However, such variations in the choice of local manifolds do not effect the global stable manifolds in the following sense. Fix $x \in \Lambda$. Consider its trajectory $f^n(x)$. For each $n \geq 0$, choose a ball $B_n \subset V(f^n(x))$ centered at $f^n(x)$ of radius $r_n > 0$.

Theorem 9.2. *Assume that $r_{n+1} > r_n e^{-\varepsilon n}$. Then*

$$W(x) = \bigcup_{n=0}^{\infty} f^{-n}(B_n).$$

We give another useful characterization of global stable manifolds in the case when the diffeomorphism f possesses an invariant measure μ. Given $\ell > 1$, consider the regular set Λ^ℓ. For $x \in \Lambda^\ell$, denote by $n_i(x) > 0$ the successive moments of time for which $f^{n_i(x)}(x) \in \Lambda^\ell$. For almost every $x \in \Lambda^\ell$ the sequence $\{n_i(x)\}$ is unbounded.

Theorem 9.3 (Pesin [197])**.** *For almost every $x \in \Lambda^\ell$,*

$$W(x) = \bigcup_{n=0}^{\infty} f^{-n_i(x)}(V(f^{n_i(x)}(x))).$$

We recall that a partition W of M is called a *foliation of M with smooth leaves* if there exist $\delta > 0$, $q > 0$, and $k \in \mathbb{N}$ such that for each $x \in M$,

1. the element $W(x)$ of the partition W containing x is a smooth k-dimensional immersed submanifold; it is called the *(global) leaf* of the foliation at x; the connected component of the intersection $W(x) \cap B(x, \delta)$ that contains x is called the *local leaf* at x and is denoted by $V(x)$;
2. there exists a continuous map $\varphi_x\colon B(x, q) \to C^1(D, M)$ (where $D \subset \mathbb{R}^k$ is the unit ball) such that for every $y \in B(x, q)$ the manifold $V(y)$ is the image of the map $\varphi_x(y)\colon D \to M$.

The function $\Phi_x(y, z) = \varphi_x(y)(z)$ is called the *foliation coordinate chart*. This function is continuous and has continuous derivative $\frac{\partial \Phi_x}{\partial z}$.

In this section we deal only with foliations with smooth leaves and simply call them foliations. One can extend the notion of foliation to compact subsets of M (see [113] for more details).

In view of Theorem 9.1 global stable manifolds form a partition of Λ. When f is *uniformly* (partially) hyperbolic on Λ (which is compact), this partition is a foliation. When f is *nonuniformly* (partially) hyperbolic this partition is a "measurable" foliation in a certain sense (note that the partition by global manifolds may **not** be a measurable partition). We shall not discuss measurable foliations in this section (see Section 11.3 where we consider a very special class of such partitions).

Assume now that f is nonuniformly hyperbolic in the narrow sense on a set Λ. For every $x \in \Lambda$ we define the *global stable manifold* $W^s(x)$ as well as *global unstable* *manifold* by

$$W^u(x) = \bigcup_{n=0}^{\infty} f^n(V^u(f^{-n}(x))).$$

This is a finite-dimensional immersed smooth submanifold (of class $C^{r+\alpha}$ if f is of class $C^{r+\alpha}$) invariant under f.

Theorem 9.4 (Pesin [197])**.** *If $x,\ y \in \Lambda$, then:*

1. $W^u(x) \cap W^u(y) = \varnothing$ *if* $y \notin W^u(x)$*;*
2. $W^u(x) = W^u(y)$ *if* $y \in W^u(x)$*;*
3. $W^u(x)$ *is characterized as follows*

$$W^u(x) = \{y \in M\colon \overline{\lim_{n\to\infty}} \frac{1}{n} \log \rho(f^{-n}(x), f^{-n}(y)) < -\log \mu\}$$

(see Section 6.2 for the definition of μ).

We describe global manifolds for nonuniformly hyperbolic flows. Let φ_t be a smooth flow on M which is nonuniformly partially hyperbolic on an invariant set Λ. For every $x \in \Lambda$ we define the *global stable manifold* at x by

$$W^s(x) = \bigcup_{t>0} \varphi_{-t}(V^s(\varphi_t(x))). \tag{9.2}$$

This is a finite-dimensional immersed smooth submanifold of class $C^{r+\alpha}$ if φ_t is of class $C^{r+\alpha}$. It satisfies Statements 1, 2, and 3 of Theorem 9.1. Furthermore, for every $y \in W^s(x)$ we have $\rho(\varphi_t(x), \varphi_t(y)) \to 0$ as $t \to +\infty$ with an exponential rate.

We also define the *global weakly stable manifold* at x by

$$W^{sc}(x) = \bigcup_{t\in\mathbb{R}} W^s(\varphi_t(x)).$$

It follows from (9.2) that

$$W^{sc}(x) = \bigcup_{t\in\mathbb{R}} \varphi_t(W^s(x)).$$

Furthermore, for every $x \in \Lambda$ define the *global unstable manifold* at x by

$$W^u(x) = \bigcup_{t>0} \varphi_t(V^u(\varphi_{-t}(x))).$$

These are finite-dimensional immersed smooth submanifolds of class $C^{r+\alpha}$ if φ_t is of class $C^{r+\alpha}$. They satisfy Statements 1, 2, and 3 of Theorem 9.1.

We also define the *global weakly unstable manifold* at x by

$$W^{uc}(x) = \bigcup_{t\in\mathbb{R}} W^u(\varphi_t(x)).$$

It follows from (9.2) that

$$W^{sc}(x) = \bigcup_{t\in\mathbb{R}} \varphi_t(W^s(x)), \quad W^{uc}(x) = \bigcup_{t\in\mathbb{R}} \varphi_t(W^u(x)).$$

Global (weakly) stable and unstable manifolds form partitions of the set Λ.

9.2. **Filtrations of stable manifolds.** Given a point $x \in \Lambda$, consider the Oseledets decomposition at x,

$$T_x M = \bigoplus_{j=1}^{p(x)} E_j(x).$$

Set $s(x) = \max\{j : \chi_j(x) < 0\}$ and for $i = 1, \dots, s(x)$,

$$F_i(x) = \bigoplus_{j=1}^{i} E_j(x).$$

The Stable Manifold Theorem 8.9 applies to the distribution $F_i(x)$ and provides a $C^{1+\alpha}$ local stable manifold $V_i(x)$. It is characterized as follows: there exists $r(x) > 0$ such that

$$V_i(x) = \left\{ y \in B(x, r(x)) : \varlimsup_{n\to+\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < \chi_i(x) \right\}.$$

Local stable manifolds form the *filtration of local stable manifolds* at x:

$$x \in V_1(x) \subset V_2(x) \subset \cdots \subset V_{s(x)}(x). \tag{9.3}$$

We define the *i-th global stable manifold* at x by

$$W_i(x) = \bigcup_{n=0}^{\infty} f^{-n}(V_i(f^n(x))).$$

It is a finite-dimensional immersed smooth submanifold of class $C^{r+\alpha}$ if f is of class $C^{r+\alpha}$. It does not depend on the particular choice of local stable manifolds in the sense of Theorem 9.2 and has the following properties which are immediate corollaries of the Stable Manifold Theorem 8.8.

Theorem 9.5. *If x, $y \in \Lambda$, then:*

1. $W_i(x) \cap W_i(y) = \varnothing$ *if* $y \notin W_i(x)$*;*
2. $W_i(x) = W_i(y)$ *if* $y \in W_i(x)$*;*
3. $f(W_i(x)) = W_i(f(x))$*;*
4. $W_i(x)$ *is characterized by*

$$W_i(x) = \left\{ y \in M : \varlimsup_{n\to+\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < \chi_i(x) \right\},$$

For each $x \in \Lambda$ we have the *filtration of global stable manifolds*

$$x \in W_1(x) \subset W_2(x) \subset \cdots \subset W_{s(x)}(x),$$

Consider the case when f is a nonuniformly partially hyperbolic diffeomorphism on an f-invariant set Λ. In a similar way, let $u(x) = \min\{j : \chi_j(x) > 0\}$ and for $i = u(x), \dots, p(x)$,

$$G_i(x) = \bigoplus_{j=i}^{p(x)} E_j(x).$$

The Unstable Manifold Theorem 8.11 applies to the distribution $G_i(x)$ and provides a $C^{1+\alpha}$ local manifold $V_i(x)$. It is characterized as follows: there exists $r(x) > 0$ such that

$$V_i(x) = \left\{ y \in B(x, r(x)) : \varlimsup_{n\to-\infty} \frac{1}{|n|} \log d(f^n(x), f^n(y)) < -\chi_i(x) \right\}.$$

We obtain the *filtration of local unstable manifolds* at x:

$$x \in V_{u(x)}(x) \subset V_{u(x)+1}(x) \subset \cdots \subset V_{p(x)}(x).$$

Finally, we have $V^s(x) = V_{s(x)}(x)$ and $V^u(x) = V_{u(x)}(x)$.[6]

We define the *i-th global unstable manifold* at x by

$$W_i(x) = \bigcup_{n=0}^{\infty} f^n(V_i(f^{-n}(x))).$$

It is a finite-dimensional immersed smooth submanifold of class $C^{r+\alpha}$ if f is of class $C^{r+\alpha}$. It does not depend on the particular choice of local unstable manifolds in the sense of Theorem 9.2 and is characterized as follows

$$W_i(x) = \left\{ y \in M : \overline{\lim_{n\to-\infty}} \frac{1}{|n|} \log d(f^n(x), f^n(y)) < -\chi_i(x) \right\}.$$

For each $x \in \Lambda$ we have the *filtration of global unstable manifolds*

$$x \in W_{u(x)}(x) \subset W_{u(x)+1}(x) \subset \cdots \subset W_{p(x)}(x).$$

Finally, consider a diffeomorphism f which is a nonuniformly completely hyperbolic on an f-invariant set Λ.

Given $r \in (0, r(x))$ we denote by $B_i(x, r) \subset V_i(x)$ the ball centered at x of radius r with respect to the induced metric on $V_i(x)$.

By Theorem 8.14 there exists a special Lyapunov chart at x associated with the Oseledets decomposition at x.

1. there exists a local diffeomorphism $\varphi_x \colon U_x \to \mathbb{R}^n$ with the property that the spaces $\mathbb{E}_i = \varphi_x(\exp_x E_i(x))$ form an orthogonal decomposition of $\mathbb{R}^n$;
2. the subspaces $\mathbb{F}_k = \varphi_x(\exp_x F_k(x))$ and $\mathbb{G}_k = \varphi_x(\exp_x G_k(x))$ are independent of x;
3. if $i = 1, \ldots, p(x)$ and $v \in E_i(x)$ then
$$\begin{aligned} e^{\lambda_i(x)-\tau}\|\varphi_x(\exp_x v)\| &\le \|\varphi_{f(x)}(\exp_{f(x)} d_x f v)\| \\ &\le e^{\lambda_i(x)+\tau}\|\varphi_x(\exp_x v)\|; \end{aligned}$$
4. there is a constant K and a tempered function $A \colon \Lambda \to \mathbb{R}$ such that if y, $z \in U_x$ then
$$K\|\varphi_x y - \varphi_x z\| \le d(y, z) \le A(x)\|\varphi_x y - \varphi_x z\|;$$
5. there exists $\widetilde{r}(x) \in (0, r(x))$ such that $B_i(x, \widetilde{r}(x)) \subset V_i(x) \cap U_x$ for every $x \in \Lambda$ and $i = 1, \ldots, k(x)$ with $\lambda_i(x) \neq 0$. Moreover, for $1 \le i \le s(x)$, the manifolds $\varphi_x(V_i(x))$ are graphs of smooth functions $\psi_i \colon \mathbb{F}_i \to \mathbb{F}_{i+1}$ and for $u(x) \le i \le p(x)$, of smooth functions $\psi_i \colon \mathbb{G}_i \to \mathbb{G}_{i-1}$; the first derivatives of ψ_i are bounded by $1/3$.

It follows that for $1 \le i \le s(x)$,

$$f(V_i(x) \cap U_x) \subset V_i(f(x)) \cap U_{f(x)}$$

[6] This notation is a bit awkward as the supscripts s and u stand for the words "stable" and "unstable", while $s(x)$ and $u(x)$ are numbers. It may get even more confusing since the functions $s(x)$ and $u(x)$, being measurable and invariant, are constant almost everywhere with respect to any invariant measure and the constant value is often denoted by s and u. We hope the reader will excuse us for such an abuse of notation.

and for $u(x) \le i \le p(x)$,

$$f^{-1}(V_i(x) \cap U_x) \subset V_i(f^{-1}(x)) \cap U_{f^{-1}(x)}.$$

9.3. **Lipschitz property of intermediate stable manifolds.** Local manifold $V_k(y)$ in (9.3) depends Lipschitz continuously on $y \in V_{k+1}(x) \cap \Lambda^\ell$ for every $k < s(x)$. In order to state this result explicitly we shall first introduce the holonomy maps associated with families of local stable manifolds. Fix $\ell \ge 1$ and $x \in \Lambda^\ell$. Given transversals T^1, $T^2 \subset V_{k+1}(x)$ to the family of local stable manifolds

$$\mathcal{L}_k(x) = \{V_k(w) : w \in \Lambda^\ell \cap B(x,r)\},$$

we define the *holonomy map*

$$\pi_k \colon Q^\ell(x) \cap T^1 \to Q^\ell(x) \cap T^2$$

using the relation

$$\pi_k(y) = T^2 \cap V_k(w), \text{ where } y = T^1 \cap V_k(w) \text{ and } w \in Q^\ell(x) \cap B(x,r).$$

Theorem 9.6 (Barreira, Pesin and Schmeling [25])**.** *Given $\ell \ge 1$, $x \in \Lambda^\ell$, and transversals T^1, $T^2 \subset V_{k+1}(x)$ to the family $\mathcal{L}_k(x)$, the holonomy map π_k is Lipschitz continuous with Lipschitz constant depending only on ℓ.*

The set Λ can be decomposed into sets Λ_β in which the numbers $k(x)$, $\dim E_i(x)$, and $\lambda_i(x)$ are constant for each i. For every ergodic measure μ invariant under f there exists a unique β for which the set Λ_β has full μ-measure. From now on we restrict our consideration to a subset $\Lambda_\beta \subset \Lambda$ and set $k(x) = k$, $s(x) = s$, $u(x) = u$, and $\lambda_i(x) = \lambda_i$ for each i and $x \in \Lambda_\beta$.

Given $\ell > 0$, consider the set $\Lambda'_{\beta\ell}$ defined by

$$\left\{ x \in \Lambda_\beta : \rho(x) > 1/\ell,\ A(x) < \ell,\ \angle(E_i(x), \bigoplus_{j\ne i} E_j(x)) > \frac{1}{\ell},\ i = 1\dots,k \right\}.$$

Let $\Lambda_{\beta\ell}$ be the closure of $\Lambda'_{\beta\ell}$. For each $x \in \Lambda'_{\beta\ell}$ there exists an invariant decomposition $T_xM = \bigoplus_{i=1}^{p(x)} E_i(x)$, filtration of local stable manifolds $V_i(x)$ and Lyapunov chart (U_x, φ_x) at x (see the previous section). In particular, the functions $\rho(x)$ and $A(x)$ can be extended to $\Lambda'_{\beta\ell}$ such that $\rho(x) > 1/\ell$, $A(x) < \ell$, and $\angle(E_i(x), \bigoplus_{j\ne i} E_j(x)) > 1/\ell$ for $i = 1\dots,k$. The set $\Lambda_{\beta\ell}$ is compact and $\Lambda_{\beta\ell} \subset \Lambda_{\beta(\ell+1)}$, $\Lambda_\beta = \bigcup_{\ell>0} \Lambda_{\beta\ell}$ (mod 0).

Let us fix $c > 0$, $\ell > 0$, $x \in \Lambda_{\beta\ell}$, and $y' \in \Lambda_{\beta\ell} \cap B_{i+1}(x, c/\ell)$. For each $i < s$, consider two local smooth manifolds T_x and $T_{y'}$ in $V_{i+1}(x)$, containing x and y', respectively and transverse to $V_i(z)$ for all $z \in \Lambda_{\beta\ell} \cap B_{i+1}(x, c/\ell)$. The holonomy map

$$\pi_i = \pi_i(T_x, T_{y'}) \colon T_x \cap \Lambda_{\beta\ell} \cap B_{i+1}(x, c/\ell) \to T_{y'}$$

is given by

$$\pi_i(x') = V_i(x') \cap T_{y'}$$

with $x' \in T_x$. This map is well-defined if c is sufficiently small (c may depend on ℓ but does not depend on x and y).

Theorem 9.7. *Let f be a $C^{1+\alpha}$ diffeomorphism. For each $\ell > 0$, $i < s$, $x \in \Lambda_{\beta\ell}$, and $y' \in \Lambda_{\beta\ell} \cap B^i(x, c/\ell)$ the holonomy map $\pi_i(T_x, T_{y'})$ is Lipschitz continuous with the Lipschitz constant depending only on β and ℓ.*

10. Absolute continuity

Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M. We describe one of the most crucial properties of local stable and unstable manifolds which is known as *absolute continuity*.

Consider a foliation with smooth leaves W of M (see Section 9.2). Fix $x \in M$ and let ξ be the partition of the ball $B(x,q)$ by local manifolds $V(y)$, $y \in B(x,q)$.

The absolute continuity property addresses the following question:

> *If $E \subset B(x,q)$ is a Borel set of positive volume, can the intersection $E \cap V(y)$ have zero Lebesgue measure (with respect to the Riemannian volume on $V(y)$) for almost every $y \in E$?*

If the foliation W is indeed, smooth then due to the Fubini theorem, the intersection $E \cap V(y)$ has positive measure for almost all $y \in B(x,q)$. If the foliation is only continuous the absolute continuity property may not hold. A simple example which illustrates this paradoxical phenomenon was constructed by Katok (see below). A continuous but not absolutely continuous foliation does not satisfy the conditions of the Fubini theorem—a set of full Lebesgue measure may meet almost every leaf of the foliation at a single point—the phenomenon known as "Fubini's nightmare". Such pathological foliations appears generically in the stable ergodicity theory (see Section 13.8).

A celebrated result by Anosov claims that the stable and unstable invariant foliations for Anosov diffeomorphisms are absolutely continuous. We stress that generically these foliations are not smooth and therefore, the absolute continuity property is not at all trivial and requires a deep study of the structure of these foliations.

In [10], Anosov and Sinai suggested an approach to absolute continuity which is based on the study of the holonomy maps associated with the foliation. To explain this, consider a foliation W. Given x, choose two transversals T^1 and T^2 to the family of local manifolds $V(y)$, $y \in B(x,q)$. The holonomy map associates to a point $z \in T^1$ the point $w = V(z) \cap T^2$. This map is a homeomorphism. If it is absolutely continuous (see the definition below) for all points x and transversals T^1 and T^2 then the absolute continuity property follows.

For nonuniformly hyperbolic diffeomorphisms the study of absolute continuity is technically much more complicated due to the fact that the global stable and unstable manifolds may not form foliations (they may not even exist for some points in M) and the sizes of local manifolds may vary wildly from point to point. In order to overcome this difficulty one should define and study the holonomy maps associated with local stable (or unstable) manifolds on regular sets.

10.1. **Absolute continuity of stable manifolds.** Let Λ be the set of nonuniformly partially hyperbolic points in the broad sense for f so that Conditions (6.2)–(6.5) hold. Let also $\{\Lambda^\ell : \ell \geq 1\}$ be the associated collection of regular sets. We assume that Λ is nonempty. Without loss of generality we may assume that each set Λ^ℓ is compact. We have $\Lambda^\ell \subset \Lambda^{\ell+1}$ for every ℓ. Furthermore, the stable subspaces $E_1(x)$ depend continuously on $x \in \Lambda^\ell$ and their sizes are bounded away from zero by a number r_ℓ (see (8.16)).

Fix $x \in \Lambda^\ell$, a number r, $0 < r \leq r_\ell$ and set

$$Q^\ell(x) = \bigcup_{w \in \Lambda^\ell \cap B(x,r)} V(w), \tag{10.1}$$

where $B(x,r)$ is the ball at x of radius r. Consider the family of local stable manifolds

$$\mathcal{L}(x) = \{V(w) : w \in \Lambda^\ell \cap B(x,r)\}$$

and a local open submanifold T which is *uniformly* transverse to it. For sufficiently small r we can chose T such that the set $\exp_x^{-1} T$ is the graph of a smooth map $\psi\colon B_2(q) \subset E_2(x) \to E_1(x)$ (for some $q > 0$) with sufficiently small C^1 norm. In this case T intersects each local stable manifold $V(w) \in \mathcal{L}(x)$ and this intersection is transverse. We will consider local open submanifolds constructed only in this way and call them *transversals to the family* $\mathcal{L}(x)$. We also say that the map ψ *represents* T.

Let T^1 and T^2 be two transversals to the family $\mathcal{L}(x)$. We define the *holonomy map*

$$\pi\colon Q^\ell(x) \cap T^1 \to Q^\ell(x) \cap T^2$$

by setting

$$\pi(y) = T^2 \cap V(w), \text{ if } y = T^1 \cap V(w) \text{ and } w \in Q^\ell(x) \cap B(x,r).$$

The holonomy map π is a homeomorphism onto its image. It depends on x, ℓ, T^1, and T^2. Set

$$\Delta(T^1, T^2) = \|\psi^1 - \psi^2\|_{C^1}, \tag{10.2}$$

where the maps ψ^1 and ψ^2 represent T^1 and T^2 respectively.

Given a smooth submanifold W in M, we denote by ν_W the Riemannian volume on W induced by the restriction of the Riemannian metric to W. We denote by $\mathrm{Jac}\,(\pi)(y)$ the Jacobian of the holonomy map π at the point $y \in Q^\ell(x) \cap T^1$ specified by the measures ν_{T^1} and ν_{T^2}.

Theorem 10.1 (Absolute Continuity)**.** *Given $\ell \geq 1$, $x \in \Lambda^\ell$, and transversals T^1 and T^2 to the family $\mathcal{L}(x)$, the holonomy map π is absolutely continuous (with respect to the measures ν_{T^1} and ν_{T^2}) and the Jacobian $Jac\,(\pi)$ is bounded from above and bounded away from zero.*

Remark 10.2. *(1) One can obtain an explicit formula for the Jacobian. Namely, for every $y \in Q^\ell(x) \cap T^1$,*

$$Jac\,(\pi)(y) = \prod_{k=0}^{\infty} \frac{Jac\,(d_{f^k(\pi(y))} f^{-1} | T_{f^k(\pi(y))} f^k(T^2))}{Jac\,(d_{f^k(y)} f^{-1} | T_{f^k(y)} f^k(T^1))}$$

(in particular, the infinite product on the right hand-side converges).

(2) In the case when f is nonuniformly hyperbolic on Λ, one can show that the Jacobian $Jac\,(\pi)$ satisfies

$$|Jac\,(\pi) - 1| \leq C \Delta(T^1, T^2), \tag{10.3}$$

where $C > 0$ is a constant and $\Delta(T^1, T^2)$ is given by (10.2)*.*

(3) If the holonomy map π is absolutely continuous then the foliation W has the absolute continuity property (see Theorem 11.1). However, the absolute continuity property of the foliation W does not necessarily imply that the the holonomy map π is absolutely continuous.

The first basic proof of the Absolute Continuity theorem for nonuniformly partially hyperbolic diffeomorphisms (in the broad sense) was obtained by Pesin in [196]. A more conceptual and lucid proof (but for a less general case of nonuniform complete hyperbolicity) can be found in [24]. A somewhat different approach to absolute continuity was suggested by Pugh and Shub (see [208]).

Let us outline the main idea of the proof following the line of Pesin's argument. To estimate the Jacobian Jac(π) choose a small open set $A \subset T^1$ and let $B = \pi(A) \subset T^2$. We need to compare the measures $\nu_{T^1}(A \cap \Lambda^\ell)$ and $\nu_{T^2}(B \cap \Lambda^\ell)$. Consider the images $f^m(A)$ and $f^m(B)$, $m > 0$ which are smooth submanifolds of M. When m increases the sets $A \cap \Lambda^\ell$ and $B \cap \Lambda^\ell$ may get stretched and/or shrunk in the "unstable" direction E_2. This may occur with at most an exponential uniform rate γ with some $\lambda < \gamma < \min\{1, \mu\}$. On the other hand, the distance between the sets $f^m(A \cap \Lambda^\ell)$ and $f^m(B \cap \Lambda^\ell)$ gets exponentially small with a uniform rate λ' where $\lambda < \lambda' < \gamma$.

We then cover the set $f^m(A \cap \Lambda^\ell)$ and $f^m(B \cap \Lambda^\ell)$ by specially chosen open sets whose sizes are of order γ^m such that the multiplicity of these covers is finite and depends only on the dimension of T^1. More precisely, given a point $w \in \Lambda^\ell \cap B(x, r)$, let $y_i = V(w) \cap T^i$, $i = 1, 2$. Fix a number $q > 0$. In view of Theorem 8.13 there exists an open neighborhood $T^i_m(w, q) \subset T^i_m$ of the point $f^m(y_i)$ such that

$$T^i_m(w, q) = \exp_{w_m}\{(\psi^i_m(v), v) \colon v \in B_2(q_m)\},$$

where the map $\psi^i_m \colon B_2(q_m) \to E_1(f^m(w))$ represents $T^i_m(w, q)$ and $B_2(q_m) \subset E_1(f^m(w))$ is the ball centered at zero of radius $q_m = q\gamma^m$. If $q = q(m)$ is sufficiently small then for any $w \in \Lambda^\ell \cap B(x, r)$ and $k = 0, \ldots, m$ we have that

$$f^{-1}(T^i_k(w, q)) \subset T^i_{k-1}(w, q),\ i = 1,\ 2.$$

We now compare the measures $\nu_{T^1_m}|T^1_m(w, q)$ and $\nu_{T^2_m}|T^2_m(w, q)$ for sufficiently large m.

Lemma 10.3. *There exists $C_1 > 0$ such that the following holds: for any $m > 0$ there exists $q_0 = q_0(m) > 0$ such that for any $0 < q \le q_0$ we have*

$$C_1^{-1} \le \frac{\nu_{T^1_m}(T^1_m(w, q))}{\nu_{T^2_m}(T^2_m(w, 2q))} \le C_1.$$

Lemma 10.4. *For any sufficiently large $m > 0$ there are points $w_j \in \Lambda^\ell \cap B(x, r)$, $j = 1, \ldots, p = p(m)$ and a number $q = q(m) > 0$ such that the sets $W^1_m(w_j, q)$ form an open cover of the set $f^m(Q^\ell(x) \cap T^1)$ (see (10.1)) of finite multiplicity which depends only on the dimension of T^1.*

For sufficiently large m the sets $T^2_m(w, 2q)$ cover the set $f^m(B \cap \Lambda^\ell)$. It follows from Lemmas 10.3 and 10.4 that the ratio of the measures of the sets $f^m(A \cap \Lambda^\ell)$ and $f^m(B \cap \Lambda^\ell)$ is bounded.

To return back to the measure $\nu_{T^1}(A \cap \Lambda^\ell)$ we use the well-known relation

$$\nu_{T^1}(A \cap \Lambda^\ell) = \int_{f^m(A\cap\Lambda^\ell)} \text{Jac}\,(df^{-m}|T_y f^m(T^1)) d\nu_{f^m(T^1)}(y).$$

Similar relation holds for the measures $\nu_{T^2}(B \cap \Lambda^\ell)$ and $\nu_{f^m(T^2)}(f^m(B \cap \Lambda^\ell))$. It remains to estimate the ratio of the Jacobians of the pullbacks $df^{-m}|T_y f^m(T^1)$ and $df^{-m}|T_{\pi(y)} f^m(T^2)$ for $y \in f^m(A \cap \Lambda^\ell)$. To do this choose a point $z \in f^{-m}(T^i_m(w, q))$ and set $z_m = f^m(z)$ and

$$D^i(z, m) = \text{Jac}\,(d_{z_m} f^{-m}|T_{z_m} T^i_m(w, q)).$$

Lemma 10.5. *There exist $C_2 > 0$ and $m_1(\ell) > 0$ such that for every $w \in \Lambda^\ell \cap B(x,r)$ and $m \geq m_1(\ell)$ one can find $q = q(m)$ such that*

$$C_2^{-1} \leq \left| \frac{D^2(y_m^2, m)}{D^1(y_m^1, m)} \right| \leq C_2,$$

and for $z \in f^{-m}(T_m^1(w,q))$,

$$C_2^{-1} \leq \left| \frac{D^1(z_m, m)}{D^1(y_m^1, m)} \right| \leq C_2.$$

This result allows one to compare the measures of the preimages under f^{-m} of $T_m^1(w,q)$ and $T_m^2(w,q)$. More precisely, the following statement holds.

Lemma 10.6. *There exist $C_3 > 0$ and $m_2(\ell) > 0$ such that if $w \in \Lambda^\ell \cap B(x,r)$ and $m \geq m_2(\ell)$, then one can find $q = q(m)$ such that*

$$C_3^{-1} \leq \frac{\nu_{T^1}(f^{-m}(T_m^1(w,q)))}{\nu_{T^2}(f^{-m}(T_m^2(w,q)))} \leq C_3.$$

10.2. **Non-absolutely continuous foliation.** We describe an example due to Katok of a nonabsolutely continuous foliation (another version of this example can be found in [183]; see also Section 6.2 of the Chapter "Partially hyperbolic dynamical systems" by B. Hasselblatt and Ya. Pesin in this volume [106]). Consider a hyperbolic automorphism A of the torus $\mathbb{T}^2$ and let $\{f_t : t \in S^1\}$ be a family of diffeomorphisms preserving the area m and satisfying the following conditions:

1. f_t is a small perturbation of A for every $t \in S^1$;
2. f_t depends smoothly on t;
3. the function $h(t) = h_m(f_t)$ is strictly monotone in a small neighborhood of $t = 0$ (here $h_m(f_t)$ is the metric entropy of the diffeomorphism f_t).

Note that for any family f_t the entropy is given by

$$h(t) = \int_{\mathbb{T}^2} \log \|d_x f_t | E_t^u(x)\| \, dm(x),$$

where $E_t^u(x)$ denotes the unstable subspace of f_t at the point x (see Section 14). Hence, one can modify A in a small neighborhood such that $h(t)$ is strictly monotone.

We introduce the diffeomorphism $F \colon \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$ by $F(x,t) = (f_t(x), t)$. Since f_t is sufficiently close to A, they are conjugate via a Hölder homeomorphism g_t, i.e., $f_t = g_t \circ A \circ g_t^{-1}$. Given $x \in \mathbb{T}^2$, consider the set

$$H(x) = \{(g_t(x), t) : t \in S^1\}.$$

It is diffeomorphic to the circle S^1 and the collection of these sets forms an F-invariant foliation H of $\mathbb{T}^2 \times S^1 = \mathbb{T}^3$ with $F(H(x)) = H(A(x))$. Note that $H(x)$ depends Hölder continuously on x. However, the holonomy maps associated with the foliation H are not absolutely continuous. To see this consider the holonomy map

$$\pi_{t_1,t_2} \colon \mathbb{T}^2 \times \{t_1\} \to \mathbb{T}^2 \times \{t_2\}.$$

We have that

$$\pi_{0,t}(x,0) = (g_t(x), t) \text{ and } F(\pi_{0,t}(x,0)) = \pi_{0,t}(A(x), 0).$$

If the map $\pi_{0,t}$ (with t being fixed) were absolutely continuous the measure $(\pi_{0,t})_* m$ would be absolutely continuous with respect to m. Note that each map f_t is ergodic (it is conjugate to the ergodic map A) and hence, m is the only absolutely continuous f_t-invariant probability measure. Thus, $(\pi_{0,t})_* m = m$. In particular, $h(t) = h(0)$. Since the entropy function $h(t)$ is strictly monotone in a small neighborhood of $t = 0$, the map g_t is not absolutely continuous for small t and so is the map $\pi_{0,t}$.

This example is a particular case of a more general situation of partially hyperbolic systems with nonintegrable central foliations, see [112].

11. Smooth invariant measures

In this section we deal with dynamical systems on compact manifolds which preserve smooth measures and are nonuniformly hyperbolic on some invariant subsets of positive measure (in particularly, on the whole manifold). We will present a sufficiently complete description of ergodic properties of the system. Note that most complete results (for example, on ergodicity, K-property and Bernoulli property) can be obtained when the system is completely hyperbolic. However, some results (for example, on Pinsker partition) hold true if only partial hyperbolicity in the broad sense is assumed. One of the main technical tools to in the study is the absolute continuity property of local stable and unstable invariant manifolds established in the previous section.

11.1. **Absolute continuity and smooth measures.** We begin with a more detailed description of absolute continuity of local stable and unstable manifolds for diffeomorphisms with respect to smooth measures.

Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold M without boundary and let ν be a *smooth measure*, i.e., a probability measure which is equivalent to the Riemannian volume m. Let also Λ be the set of nonuniformly partially hyperbolic points in the broad sense for f. We assume that $\nu(\Lambda) = 1$.

Consider a regular set Λ^ℓ of positive measure. For every $x \in \Lambda^\ell$ we have the filtration of stable subspaces at x:

$$0 \in F_1(x) \subset F_2(x) \subset \cdots \subset F_{s(x)}(x)$$

and the corresponding filtration of local stable manifolds at x:

$$x \in V_1(x) \subset V_2(x) \subset \cdots \subset V_{s(x)}(x)$$

(see Section 9.2). Since $V_k(x)$ depends continuously on $x \in \Lambda^\ell$, without loss of generality we may assume that $s(x) = s$, $\dim V_k(x) = d_k$ for every $x \in \Lambda^\ell$ and $1 \le s \le p$. Fix $x \in \Lambda^\ell$ and consider the family of local stable manifolds

$$\mathcal{L}_k^\ell(x) = \{V_k(y) : y \in B(x,r) \cap \Lambda^\ell\}.$$

For $y \in B(x,r) \cap \Lambda^\ell$, denote by $m^k(y)$ the Riemannian volume on $V_k(w)$ induced by the Riemannian metric on M. Consider the set

$$P_k^\ell(x,r) = \bigcup_{y \in B(x,r) \cap \Lambda^\ell} V_k(w)$$

and its partition ξ^k by local manifolds $V_k(y)$. Denote by $\nu^k(y)$ the conditional measure on $V_k(y)$ generated by the partition ξ^k and the measure ν. The factor space $P^\ell(x,r)/\xi^k$ can be identified with the subset

$$A_k(x) = \{w \in T : \text{ there is } y \in \Lambda^\ell \cap B(x,r) \text{ such that } w = T \cap V_k(y)\},$$

where T is a transverse to the family $\mathcal{L}_k^\ell$.

Theorem 11.1. *The following statements hold:*

1. *for ν-almost every $y \in \Lambda^\ell \cap B(x,r)$, the measures $\nu^k(y)$ and $m^k(y)$ are equivalent, i.e.,*
$$d\nu^k(y)(z) = \kappa_k(y,z)dm^k(y)(z),$$
where $\kappa_k(y,z)$, $z \in V_k(y)$ is the density function;
2. $$\kappa_k(y,z) = \prod_{i=0}^{\infty} \frac{Jac\,(df|F_k(f^i(z)))}{Jac\,(df|F_k(f^i(y)))};$$
3. *the function $\kappa_k(y,z)$ is Hölder continuous;*
4. *there is $C = C(\ell) > 0$ such that*
$$C^{-1}dm^k(y)(z) \le d\nu^k(y)(z) \le Cdm^k(y)(z);$$
5. *$m^k(x)(V^k(x) \setminus \Lambda) = 0$ for ν-almost every $x \in \Lambda$.*

We now consider the case when Λ is a nonuniformly completely hyperbolic set for f. The above results apply to the families of local stable and unstable manifolds. For $y \in B(x,r) \cap \Lambda^\ell$ let $m^s(y)$ and $m^u(y)$ be the Riemannian volumes on $V^s(y)$ and $V^u(y)$ respectively. Let also ξ^s and ξ^u be the partitions of $B(x,r)$ by local stable and unstable manifolds, and $\nu^s(y)$ (respectively $\nu^u(y)$) the conditional measures on $V^s(y)$ (respectively $V^u(y)$) generated by ν and the partitions ξ^s (respectively ξ^u). Finally, let $\hat\nu^s$ (respectively $\hat\nu^u$) be the factor measures.

Theorem 11.2. *The following statements hold:*

1. *for ν-almost every $y \in \Lambda^\ell \cap B(x,r)$ the measures $\nu^s(y)$ and $m^s(y)$ are equivalent; moreover, $d\nu^s(y)(z) = \kappa(y,z)dm^s(y)(z)$ where*
$$\kappa(y,z) = \prod_{i=0}^{\infty} \frac{Jac\,(df|E^s(f^i(z))}{Jac\,(df|E^s(f^i(y))};$$
2. *the factor measures $\hat\nu^s$ is equivalent to the measure $m^u(x)|A_k(x)$;*
3. *$m^s(x)(V^s(x) \setminus \Lambda) = 0$ for ν-almost every $x \in \Lambda$;*
4. *similar statements hold for the family of local unstable manifolds.*

11.2. **Ergodic components.** The following statement is one of the main results of smooth ergodic theory. It describes the decomposition of a hyperbolic smooth invariant measure into its ergodic components.

Theorem 11.3 (Pesin [197])**.** *Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold M and ν an f-invariant smooth (completely) hyperbolic measure on M. There exist invariant sets Λ_0, Λ_1, ... such that:*

1. *$\bigcup_{i\ge0} \Lambda_i = \Lambda$, and $\Lambda_i \cap \Lambda_j = \varnothing$ whenever $i \ne j$;*
2. *$\nu(\Lambda_0) = 0$, and $\nu(\Lambda_i) > 0$ for each $i \ge 1$;*
3. *$f|\Lambda_i$ is ergodic for each $i \ge 1$.*

The proof of this theorem exploits a simple yet deep argument due to Hopf [116]. Consider the regular sets Λ^ℓ of positive measure and let $x \in \Lambda^\ell$ be a Lebesgue point. For each $r > 0$ set
$$P^\ell(x,r) = \bigcup_{y\in\Lambda^\ell\cap B(x,r)} V^s(y).$$

Clearly, $P^\ell(x,r)$ has positive measure. It turns out that for a sufficiently small $r = r(\ell)$ the set

$$Q(x) = \bigcup_{n\in\mathbb{Z}} f^n(P^\ell(x,r))$$

is an ergodic component, i.e., the map $f|Q(x)$ is ergodic. Indeed, given an f-invariant continuous function φ, consider the functions

$$\overline{\varphi}(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \varphi(f^k(x)),$$

$$\varphi^+(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(f^k(x)), \quad \text{and} \quad \varphi^-(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(f^{-k}(x))$$

which are well-defined for ν-almost every point x. We also have that $\overline{\varphi}(x) = \varphi^+(x) = \varphi^-(x)$ outside a subset $N \subset M$ of zero measure.

Since $\rho(f^n(z), f^n(w)) \to 0$ as $n \to \infty$ and φ is continuous, we obtain

$$\overline{\varphi}(z) = \varphi^+(z) = \varphi^+(w) = \overline{\varphi}(w).$$

Notice that the continuous functions are dense in $L^1(M,\nu)$ and hence, the functions of the form $\overline{\varphi}$ are dense in the set of f-invariant Borel functions.

It remains to show that the function $\overline{\varphi}(z)$ is constant almost everywhere. By Theorem 11.2 there exists a point $y \in (\Lambda^\ell \cap B(x,r)) \setminus N$ such that $m^u(y)(V^u(y) \cap N) = 0$ (recall that $\nu^s(y)$ and $\nu^u(y)$ are, respectively, the measures induced on $V^s(y)$ and $V^u(y)$ by the Riemannian volume). Let

$$P^s = \bigcup V^s(w),$$

where the union is taken over all points $w \in \Lambda^\ell \cap B(x, r_\ell)$ for which, respectively, $V^s(w) \cap V^u(y) \in N$. By absolute continuity property, we have $\nu(P^s) = 0$.

Let z_1, $z_2 \in P^\ell(x,r) \setminus (P^s \cup N)$. There are points $w_i \in \Lambda^\ell \cap B(x,r)$ such that $z_i \in V^s(w_i)$ for $i = 1$, 2. Note that the intersection $V^s(w_i) \cap V^u(y)$ is nonempty and consists of a single point y_i, $i = 1$, 2. We have that

$$\overline{\varphi}(z)(z_1) = \overline{\varphi}(z)(y_1) = \overline{\varphi}(z)(y_2) = \overline{\varphi}(z)(z_2)$$

and the ergodicity of $f|Q(x)$ follows.

Since almost every point $x \in \Lambda$ is a Lebesgue point of Λ^ℓ for some ℓ, the invariant sets $Q(x)$ cover the set Λ (mod 0) and there is at most countable many such sets. We denote them by Q_1, Q_2, ... We have $\nu(Q_i) > 0$ for each $i \geq 1$, and the set $\Lambda_0 = \Lambda \setminus \bigcup_{i\geq 1} Q_i$ has zero measure. Since $f|Q_i$ is ergodic $Q_i \cap Q_j = \varnothing$ (mod 0) whenever $i \neq j$. If we set $\Lambda_n = Q_n \setminus \bigcup_{i=1}^{n-1} Q_i$ then $\Lambda_i \cap \Lambda_j = \varnothing$ and $\nu(Q_i) = \nu(\Lambda_i) > 0$.

We describe an example of a diffeomorphism with nonzero Lyapunov exponents that has more than one ergodic component. Consider the diffeomorphism $G_{\mathbb{T}^2}$ of the torus $\mathbb{T}^2$ constructed in Section 2.2. This map is ergodic. The punched torus $\mathbb{T}^2 \setminus \{0\}$ is C^∞-diffeomorphic to the manifold $\mathbb{T}^2 \setminus \overline{U}$, where U is a small open disk around 0 and $\overline{U}$ denotes its closure. Therefore, we obtain a C^∞ diffeomorphism $F_{\mathbb{T}^2}$ of the manifold $\mathbb{T}^2 \setminus U$ with $F_{\mathbb{T}^2}|\partial U = \mathrm{Id}$. We have that $F_{\mathbb{T}^2}$ preserves a smooth measure, has nonzero Lyapunov exponents, and is ergodic.

Let $(\widetilde{M}, \widetilde{F}_{\mathbb{T}^2})$ be a copy of $(M, F_{\mathbb{T}^2})$. By gluing the manifolds M and $\widetilde{M}$ along ∂U we obtain a smooth compact manifold $\mathcal{M}$ without boundary and a diffeomorphism

$\mathcal{F}$ of $\mathcal{M}$ which preserves a smooth measure and has nonzero Lyapunov exponents almost everywhere. However, the map $\mathcal{F}$ is not ergodic and has two ergodic components of positive measure (M and $\widetilde{M}$).

Similarly, one can obtain a diffeomorphism with nonzero Lyapunov exponents with n ergodic components of positive measure for an arbitrary n. However, it does not seem feasible to push this construction further and obtain a diffeomorphism with nonzero Lyapunov exponents with countably many ergodic components of positive measure. Such an example was constructed by Dolgopyat, Hu and Pesin in [77] using a different approach. It illustrates that Theorem 11.3 cannot be improved.

Example 11.4. *There exists a volume-preserving C^∞ diffeomorphism f of the three-dimensional torus $\mathbb{T}^3$ with nonzero Lyapunov exponents almost everywhere and countably many ergodic components which are open* (mod 0).

The construction starts with a linear hyperbolic automorphism $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ which has at least two fixed points p and p'. The desired map f is obtained as a perturbation of the map $F = A \times \mathrm{Id}$ of the three-dimensional torus $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$. More precisely, consider a countable collection of intervals $\{I_n\}_{n=1}^\infty$ on the circle S^1, where

$$I_{2n} = [(n+2)^{-1}, (n+1)^{-1}], \quad I_{2n-1} = [1-(n+1)^{-1}, 1-(n+2)^{-1}].$$

Clearly, $\bigcup_{n=1}^\infty I_n = (0,1)$ and $\operatorname{int} I_n$ are pairwise disjoint.

The main result in [77] states that for any $k \geq 2$ and $\delta > 0$, there exists a map g of the three-dimensional manifold $M = \mathbb{T}^2 \times I$ such that:

1. g is a C^∞ volume-preserving diffeomorphism of M;
2. $\|F - g\|_{C^k} \leq \delta$;
3. for all $0 \leq m < \infty$, $D^m g|\mathbb{T}^2 \times \{z\} = D^m F|\mathbb{T}^2 \times \{z\}$ for $z = 0$ and 1;
4. g is ergodic with respect to the Riemannian volume and has nonzero Lyapunov exponents almost everywhere.

Applying this result, for each n, one can construct a C^∞ volume-preserving ergodic diffeomorphism $f_n\colon \mathbb{T}^2 \times [0,1] \to \mathbb{T}^2 \times [0,1]$ satisfying

1. $\|F - f_n\|_{C^n} \leq e^{-n^2}$;
2. $D^m f_n|\mathbb{T}^2 \times \{z\} = D^m F|\mathbb{T}^2 \times \{z\}$ for $z = 0$ or 1 and all $0 \leq m < \infty$;
3. f_n has nonzero Lyapunov exponents μ-almost everywhere.

Let $L_n\colon I_n \to [0,1]$ be the affine map and $\pi_n = (\mathrm{Id}, L_n)\colon \mathbb{T}^2 \times I_n \to \mathbb{T}^2 \times [0,1]$. The desired map f is given by $f|\mathbb{T}^2 \times I_n = \pi_n^{-1} \circ f_n \circ \pi_n$ for all n and $f|\mathbb{T}^2 \times \{0\} = F|\mathbb{T}^2 \times \{0\}$. Note that for every $n > 0$ and $0 \leq m \leq n$,

$$\begin{aligned}\|D^m F|\mathbb{T}^2 \times I_n - \pi_n^{-1} \circ D^m f_n \circ \pi_n\|_{C^n} &\leq \|\pi_n^{-1} \circ (D^m F - D^m f_n) \circ \pi_n\|_{C^n} \\ &\leq e^{-n^2} \cdot (n+1)^n \to 0\end{aligned}$$

as $n \to \infty$. It follows that f is C^∞ on M and it has the required properties.

In the following section we describe a result (see Theorem 11.9) which provides some additional conditions guaranteeing that the number of ergodic component in Theorem 11.3 is finite. Roughly speaking one should require that: 1) the global stable (or unstable) foliation extends to a continuous foliation of the manifold and that 2) the Lyapunov exponents $\chi_i(x)$ are away from zero uniformly over x.

We now consider the case of a smooth flow φ_t on a compact manifold M preserving a smooth hyperbolic measure ν. We also assume that ν vanishes on the set of fixed points of φ_t.

Since the time-one map of the flow is nonuniformly partially hyperbolic we conclude that the families of local stable and unstable manifolds possess the absolute continuity property. This is a key fact which allows one to study the ergodic properties of nonuniformly hyperbolic flows.

Theorem 11.5 (Pesin [197]). *There exist invariant sets* $\Lambda_0, \Lambda_1, \dots$ *such that*

1. $\bigcup_{i\geq 0}\Lambda_i = \Lambda$, *and* $\Lambda_i \cap \Lambda_j = \varnothing$ *whenever* $i \neq j$*;*
2. $\nu(\Lambda_0) = 0$, *and* $\nu(\Lambda_i) > 0$ *for each* $i \geq 1$*;*
3. $\varphi_t|\Lambda_i$ *is ergodic for each* $i \geq 1$.

Using the flow described in Section 2.6 one can construct an example of a flow with nonzero Lyapunov exponents which has an arbitrary finite number of ergodic components.

11.3. **Local ergodicity.** Consider a $C^{1+\alpha}$ diffeomorphism of a compact manifold M preserving a smooth hyperbolic measure. In this section we discuss the *local ergodicity problem.* – under what conditions ergodic components are open (up to a set of measure zero).

In this connection the following two problems are of interest:

Problem 11.6. *Is there a volume-preserving diffeomorphism which has nonzero Lyapunov exponents almost everywhere such that some (or even all) of its ergodic components with positive measure are not open* (mod 0)*?*

Problem 11.7. *Is there a volume-preserving diffeomorphism which has nonzero Lyapunov exponents on an open* (mod 0) *and dense set* U *such that* U *has positive but not full measure? Is there a volume preserving diffeomorphism with the above property such that* $f|U$ *is ergodic?*

The main obstacles for local ergodicity are the following:

1. the stable and unstable distributions are measurable but not necessarily continuous;
2. the global stable (or unstable) leaves may not form a foliation;
3. the unstable leaves may not *expand* under the action of f^n (note that they are defined as being exponentially *contracting* under f^{-n}, so that they are determined by the *negative* semi-trajectory); the same is true for stable leaves with respect to the action of f^{-n}.

There are three different ways to obtain sufficient conditions for local ergodicity. Each of them is based on requirements which eliminate one or more of the above mentioned obstacles.

1. The first one is due to Pesin [197]. It requires a special structure of the global stable or unstable manifolds and is used to establish local ergodicity of geodesic flows (see Section 17).
2. The second one is due to Katok and Burns [132]. Its main advantage is that it relies on requirements on the local behavior of the system.
3. The third one is due to Liverani and Wojtkovski [167]. It deals with symplectic dynamical systems and is an adaptation of the Sinai method (that was developed for billiard dynamical systems; see [233]) to nonuniformly hyperbolic dynamical systems (both smooth and smooth with singularities; see Section 18).

1. We first describe the approach in [197]. Roughly speaking it requires that the stable (or unstable) leaves form a foliation of a measurable subset of full measure in M. First, we extend the notion of foliation of M with smooth leaves, introduced in Section 9.2, to foliation of a measurable subset.

Given a subset $X \subset M$, we call a partition ξ of X a *(δ, q)-foliation of X with smooth leaves* or simply a *(δ, q)-foliation of X* if there exist continuous functions $\delta\colon X \to (0, \infty)$ and $q\colon X \to (0, \infty)$ and an integer $k > 0$ such that for each $x \in X$:

1. there exists a smooth immersed k-dimensional submanifold $W(x)$ containing x for which $\xi(x) = W(x) \cap X$ where $\xi(x)$ is the element of the partition ξ containing x; the manifold $W(x)$ is called the *(global) leaf* of the (δ, q)-foliation at x; the connected component of the intersection $W(x) \cap B(x, \delta(x))$ that contains x is called the *local leaf* at x and is denoted by $V(x)$;
2. there exists a continuous map $\varphi_x\colon B(x, q(x)) \to C^1(D, M)$ ($D \subset \mathbb{R}^k$ is the open unit ball) such that for every $y \in X \cap B(x, q(x))$ the manifold $V(y)$ is the image of the map $\varphi_x(y)\colon D \to M$.

For every $x \in X$ and $y \in B(x, q(x))$ we set $U(y) = \varphi(y)(D)$ and we call it the *local leaf* of the (δ, q)-foliation at y. Note that $U(y) = V(y)$ for $y \in X$.

The following result establishes the local ergodicity property in the case when the stable (or unstable) foliation for f extends to a continuous foliation of M with smooth leaves.

Theorem 11.8 (Pesin [197])**.** *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure ν and nonuniformly hyperbolic on an invariant set Λ. Assume that $\nu(\Lambda) > 0$ and that there exists a (δ, q)-foliation W of Λ such that $W(x) = W^s(x)$ for every $x \in \Lambda$ (where $W^s(x)$ is the global stable manifold at x; see Section 8). Then every ergodic component of f of positive measure is open* (mod 0) *in Λ (with respect to the induced topology).*

This theorem provides a way to establish the ergodicity of the map $f|\Lambda$. Namely, under the conditions of Theorem 11.8 every ergodic component of f of positive measure that lies in Λ is open (mod 0), hence, the set Λ is open (mod 0) and, if $f|\Lambda$ is topologically transitive, then $f|\Lambda$ is ergodic.

In general, a diffeomorphism f preserving a smooth hyperbolic measure may have countably many ergodic components which are open (mod 0) (see Example 11.4). We describe a criterion which guarantees that the number of open (mod 0) ergodic components is finite.

Theorem 11.9. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure ν and nonuniformly hyperbolic on an invariant set Λ. Assume that $\nu(\Lambda) > 0$ and that there exists a continuous foliation W of M with smooth leaves such that $W(x) = W^s(x)$ for every $x \in \Lambda$. Assume, in addition, that there exists a number $a > 0$ such that for almost every $x \in M$,*

$$|\chi_i(x)| > a. \tag{11.1}$$

Then $f|\Lambda$ has at most finitely many ergodic components of positive measure.

To see this observe that Assumption (11.1) allows one to apply Proposition 13.16 and find a number $r > 0$ with the following property: for almost every $x \in \Lambda$ there is $n = n(x)$ such that the size of a local unstable manifold $V^u(f^n(x))$ is at least r.

Let x be a density point of Λ. Consider the set

$$P(x,r) = \bigcup_{y \in V^u(f^n(x))} B^s(y,r),$$

where $B^s(y,r)$ is the ball in $W^s(y)$ centered at y of radius r. This set is contained in an ergodic component. It is also open and contains a ball of radius $\varepsilon > 0$ which does not depend on x. Thus, every ergodic component contains a ball of radius ε.

For a general diffeomorphism preserving a smooth hyperbolic measure, one should not expect the unstable (and stable) leaves to form a (δ, q)-foliation for some functions $\delta(x)$ and $q(x)$. In order to explain why this can happen consider a local unstable manifold $V^u(x)$ passing through a point $x \in \Lambda$. For a typical x and sufficiently large ℓ, the set $V^u(x) \cap \Lambda^\ell$ has positive Riemannian volume (as a subset of the smooth manifold $V^u(x)$) but is, in general, a Cantor-like set. When the local manifold is moved forward a given time n one should expect a sufficiently small neighborhood of the set $V^u(x) \cap \Lambda^\ell$ to expand. Other pieces of the local manifold (corresponding to bigger values of ℓ) will also expand but with smaller rates. As a result the global leaf $W^u(x)$ (defined by (9.1)) may bend "uncontrollably"—the phenomenon that is yet to be observed but is thought to be "real" and even "typical" in some sense. As a result the map $x \mapsto \varphi_x$ in the definition of a (δ, q)-foliation may not be, indeed, continuous.

Furthermore, the global manifold $W^u(x)$ may be "bounded", i.e., it may not admit an embedding of an arbitrarily large ball in $\mathbb{R}^k$ (where $k = \dim W^u(x)$). This phenomenon is yet to be observed too.

The local continuity of the global unstable leaves often comes up in the following setting. Using some additional information on the system one can build an invariant foliation whose leaves contain local unstable leaves. This alone may not yet guarantee that global unstable leaves form a foliation. However, one often may find that the local unstable leaves expand in a "controllable" and somewhat uniform way when they are moved forward. We will see below that this guarantees the desired properties of unstable leaves. Such a situation occurs, for example, for geodesic flows on compact Riemannian manifolds of nonpositive curvature (see Section 17.1).

We now state a formal criterion for local ergodicity.

Theorem 11.10 (Pesin [197])**.** *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold, preserving a smooth hyperbolic measure ν, and nonuniformly hyperbolic on an invariant set Λ of full measure. Let also W be a (δ, q)-foliation of Λ with the following properties:*

1. *$W(x) \supset V^s(x)$ for every $x \in \Lambda$;*
2. *there exists a number $\delta_0 > 0$ and a measurable function $n(x)$ on Λ such that for almost every $x \in \Lambda$ and any $n \ge n(x)$,*

$$f^{-n}(V^s(x)) \supset B_W(f^{-n}(x), \delta_0).$$

Then every ergodic component of f of positive measure is open (mod 0).

In the case of one-dimensional (δ, q)-foliations the second condition of Theorem 11.10 holds automatically and hence, can be omitted.

Theorem 11.11 (Pesin [197])**.** *Let W be a one-dimensional (δ, q)-foliation of Λ, satisfying the following property: $W(x) \supset V^s(x)$ for every $x \in \Lambda$. Then every* *ergodic component of f of positive measure is open* (mod 0)*. Moreover, $W^s(x) = W(x)$ for almost every $x \in \Lambda$.*

One can readily extend Theorems 11.10 and 11.11 to the case when the set Λ is open (mod 0) and has positive (not necessarily full) measure as well as to dynamical systems with continuous time.

Theorem 11.12. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold preserving a smooth measure ν and nonuniformly hyperbolic on an invariant set Λ. Assume that Λ is open* (mod 0) *and has positive measure. Let also W be a (δ, q)-foliation of Λ which satisfies properties 1 and 2 in Theorem 11.10. Then every ergodic component of $f|\Lambda$ of positive measure is open* (mod 0)*.*

Theorem 11.13. *Let φ_t be a smooth flow of a compact smooth Riemannian manifold preserving a smooth measure ν and nonuniformly hyperbolic on an invariant set Λ. Assume that Λ is open* (mod 0) *and has positive measure. Let also W be a (δ, q)-foliation of Λ with the following properties:*

1. *$W(x) \supset V^s(x)$ for every $x \in \Lambda$;*
2. *there exists a number $\delta_0 > 0$ and a measurable function $t(x)$ on Λ such that for almost every $x \in \Lambda$ and any $t \geq t(x)$,*
$$\varphi_{-t}(V^s(x)) \supset B_W(\varphi_{-t}(x), \delta_0).$$

Then every ergodic component of the flow $\varphi_t|\Lambda$ of positive measure is open (mod 0)*.*

2. We now describe the approach in [132] to study the local ergodicity. A continuous function $Q: TM \to \mathbb{R}$ is called an *infinitesimal eventually strict Lyapunov function* for f over a set $U \subset M$ if:

1. for each $x \in U$ the function $Q_x = Q|T_xM$ is homogeneous of degree one, and takes on both positive and negative values;
2. there exist continuous distributions $D^s_x \subset C^s(x)$ and $D^u_x \subset C^u(x)$ such that $T_xM = D^s_x \oplus D^u_x$ for all $x \in U$, where
$$C^s(x) = Q^{-1}((-\infty, 0)) \cup \{0\} \quad \text{and} \quad C^u(x) = Q^{-1}((0, \infty)) \cup \{0\};$$
3. for every $x \in U$, $n \in \mathbb{N}$, $f^n(x) \in U$, and $v \in T_xM$,
$$Q_{f^n(x)}(d_xf^nv) \geq Q_x(v);$$
4. for ν-almost every $x \in U$ there exist $k = k(x)$, $\ell = \ell(x) \in \mathbb{N}$ such that $f^k(x) \in U$, $f^{-\ell}(x) \in U$, and for $v \in T_xM \setminus \{0\}$,
$$Q_{f^k(x)}(d_xf^kv) > Q_x(v) \quad \text{and} \quad Q_{f^{-\ell}(x)}(d_xf^{-\ell}v) < Q_x(v).$$

A function Q is called an *infinitesimal eventually uniform Lyapunov function* for f over a set $U \subset M$ if it satisfies Conditions 1–3 and the following condition: there exists $\varepsilon > 0$ such that for ν-almost every $x \in M$ one can find $k = k(x)$, $\ell = \ell(x) \in \mathbb{N}$ for which $f^k(x) \in U$, $f^{-\ell}(x) \in U$, and if $v \in T_xM \setminus \{0\}$ then
$$Q_{f^k(x)}(d_xf^kv) > Q_x(v) + \varepsilon\|v\|$$
and
$$Q_{f^{-\ell}(x)}(d_xf^{-\ell}v) < Q_x(v) - \varepsilon\|v\|.$$
The following result gives a criterion for local ergodicity in terms of infinitesimal Lyapunov functions.

Theorem 11.14 (Katok and Burns [132])**.** *The following properties hold:*

1. *If f possesses an infinitesimal eventually strict Lyapunov function Q over an open set $U \subset M$, then almost every ergodic component of f on the set $\bigcup_{n\in\mathbb{Z}} f^n(U)$ is open* (mod 0).
2. *If f possesses an infinitesimal eventually uniform Lyapunov function Q over an open set $U \subset M$, then every connected component of the set $\bigcup_{n\in\mathbb{Z}} f^n(U)$ belongs to one ergodic component of f. Moreover, if U is connected then $f|U$ is a Bernoulli transformation.*

This theorem was first proved by Burns and Gerber [54] for flows in dimension 3.

We sketch the proof of this theorem. When Q is an infinitesimal eventually strict Lyapunov function, given a compact set $K \subset U$, one can use the uniform continuity of $x \mapsto Q_x$ on the set K, and Requirement 3 in the definition of Lyapunov function to show that the size of the stable and unstable manifolds on K is uniformly bounded away from zero. Furthermore, using Requirement 4 one can show that for ν-almost every point $z \in M$ there exist $\theta = \theta(z) > 0$ and a neighborhood N of z such that for ν-almost every $x \in N$ and $y \in V^u(x) \cap N$ the tangent space $T_yV^u(x)$ is in the θ-interior of $C^u(y)$. A similar statement holds for stable manifolds.

Together with Requirement 2 this implies that the stable and unstable manifolds have almost everywhere a "uniform" product structure; namely, for almost every $x \in U$ there exist a neighborhood $N(x)$ of x and $\delta > 0$ such that:

1. $V^s(y)$ and $V^u(y)$ have size at least δ for almost every $y \in N(x)$;
2. $V^s(y) \cap V^u(z) \neq \varnothing$ for $\nu \times \nu$-almost every $(y, z) \in N(x) \times N(x)$.

The proof of Statement 1 follows now by applying the Hopf argument.

When Q is an infinitesimal eventually uniform Lyapunov function, the function $\theta(z)$ is uniformly bounded away from zero. This can be used to establish that for every x (and not only almost every x) there exists a neighborhood $N(x)$ of x and $\delta > 0$ with the above properties. A similar argument now yields the first claim in Statement 2. The last claim is an immediate consequence of Theorem 11.19.

3. Finally we outline the approach in [167] to study the local ergodicity in the symplectic case. This approach is built upon a method which was developed by Sinai [233] in his pioneering work on billiard systems. It has been later improved by Sinai and Chernov [234] and by Krámli, Simányi and Szász [150] who considered semidispersing billiards.

Let M be a smooth compact symplectic manifold of dimension $2d$ with the symplectic form ω. Let also $f\colon M \to M$ be a symplectomorphism (i.e., a diffeomorphism of M which preserves the symplectic structure).

Fix $x \in M$. A subspace $V \subset T_xM$ is called *Lagrangian* if V is a maximal subspace on which ω vanishes (it has dimension d). Given two transverse Lagrangian subspaces V_1 and V_2 define the *sector* between them by

$$C = C(V_1, V_2) = \{v \in T_xM : \omega(v_1, v_2) \geq 0 \text{ for } v = v_1 + v_2,\ v_i \in V_i,\ i = 1, 2\}.$$

Define the quadratic form associated with an ordered pair of transverse Lagrangian subspaces V_1 and V_2 by

$$Q(v) = Q(V_1, V_2, v) = \omega(v_1, v_2) \text{ for } v = v_1 + v_2,\ v_i \in V_i,\ i = 1, 2.$$

Using this quadratic form we can write the cone $C(V_1, V_2)$ in the form

$$C(V_1, V_2) = \{v \in T_xM : Q(v) \geq 0\}.$$

We define the interior of the cone by

$$\operatorname{int} C(V_1, V_2) = \{v \in T_x M : Q(v) > 0\}.$$

We assume that two continuous subbundles of transverse Lagrangian subspaces are chosen in an open (not necessarily dense) subset $U \subset M$. We denote them by $\{V_1(x)\}_{x\in U}$ and $\{V_2(x)\}_{x\in U}$ respectively. For $x \in U$ let $C(x) = C(V_1(x), V_2(x))$ and $C'(x) = C(V_2(x), V_1(x))$.

If $x \in U$ and $f^n(x) \in U$ let us define

$$\sigma(d_x f^n) = \inf_{v \in \operatorname{int} C(x)} \sqrt{\frac{Q(V_1(x), V_2(x), d_x f^n v)}{Q(V_1(x), V_2(x), v)}}.$$

Theorem 11.15. *Assume that the following conditions hold:*

1. *Monotonicity condition: if $x \in U$ and $f^k(x) \in U$ for $k \geq 0$ then*
$$d_x f^k C(x) \subset C(f^k(x));$$
2. *Strict monotonicity condition: for almost every point $x \in U$ there exist $n > 0$ and $m < 0$ such that $f^n(x)$, $f^m(x) \in U$ and*

$$d_x f^n C(x) \subset \operatorname{int} C(f^n(x)) \cup \{0\}, \quad d_x f^m C'(x) \subset \operatorname{int} C'(f^m(x)) \cup \{0\}. \tag{11.2}$$

Then for any $n \geq 1$ and any point $x \in U$ such that $f^n(x) \in U$ and $\sigma(d_x f^n) > 1$ there is a neighborhood of x which is contained in one ergodic component of f.

It follows from this theorem that if U is connected and every point in it is strictly monotone (i.e., (11.2) holds) then $\bigcup_{k\in\mathbb{Z}} f^k(U)$ belongs to one ergodic component of f. This is a symplectic version of Theorem 11.14. We observe that Theorem 11.15 is a particular case of a more general result by Liverani and Wojtkowski for smooth dynamical systems with singularities (see Section 18).

11.4. **Pinsker partition, K-property and Bernoulli property.** In the ergodic theory there is a hierarchy of ergodic properties of which ergodicity (or the description of ergodic components) is the first and weakest one. Among the stronger properties are (weak and strong) mixing, K-property (including the description of the Pinsker or π-partition) and the strongest among them – the Bernoulli property (or the description of Bernoulli components). The latter means essentially that the system is isomorphic in the measure-theoretical sense to the classical Bernoulli scheme.

We shall see that dynamical systems with nonzero Lyapunov exponents (nonuniformly hyperbolic systems) have all of these properties with respect to smooth invariant measures.

Let $f\colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold M preserving a smooth measure ν. Assume that f is nonuniformly partially hyperbolic in the broad sense on an invariant set Λ of positive measure. For every $x \in \Lambda$ we have that

$$\chi_1(x) < \cdots < \chi_{s(x)}(x) < 0 \leq \chi_{s(x)+1}(x) < \cdots < \chi_{p(x)}(x),$$

where $\chi_i(x)$, $i = 1, \ldots, p(x)$ are the distinct values of the Lyapunov exponent at x each with multiplicity $k_i(x)$. We also have the filtration of local (stable) manifolds at x

$$x \in V_1(x) \subset V_2(x) \subset \cdots \subset V_{s(x)}(x) \tag{11.3}$$

as well as the filtration of global (stable) manifolds at x

$$x \in W_1(x) \subset W_2(x) \subset \cdots \subset W_{s(x)}(x) \tag{11.4}$$

(see Section 9.2). Fix $j > 0$ and $m > 0$ and consider the sets

$$\Lambda_{j,m} = \{x \in \Lambda : \dim W_j(x) = m\}, \quad \hat{\Lambda}_{j,m} = \bigcup_{x \in \Lambda_{j,m}} W_j(x). \tag{11.5}$$

For some j and m we have that $\nu(\Lambda_{j,m}) > 0$. Note that $W_j(x) \subset \Lambda_{j,m} \pmod 0$ for almost every $x \in \Lambda_{j,m}$. Hence, $\hat{\Lambda}_{j,m} = \Lambda_{j,m} \pmod 0$.

Consider the partition W_j of $\hat{\Lambda}_{j,m}$ by global manifolds $W_j(x)$. In general, this partition is not measurable. However, one can construct a special sub-partition of W_j which we call *pseudo* π-partition for $f|\Lambda_{j,m}$—when f is nonuniformly *completely* hyperbolic on the set Λ this partition is the π-partition for $f|\Lambda_{j,m}$, i.e., the maximal partition with zero entropy.

We denote the measurable hull of a partition ξ by $\mathcal{H}(\xi)$ and we use the notation ε for the partition by points.

Theorem 11.16. *There exists a measurable partition $\eta = \eta_{j,m}$ of $\hat{\Lambda}_{j,m}$ with the following properties:*

1. *for almost every $x \in \Lambda_{j,m}$ the set $C_\eta(x)$ is an open* (mod 0) *subset of $W_j(x)$;*
2. *$f\eta \geq \eta$;*
3. *$\eta^+ = \bigvee_{i=0}^{\infty} f^i\eta = \varepsilon$;*
4. *$\bigwedge_{i>-\infty}^{0} f^i\eta = \mathcal{H}(W_j)$;*
5. *if f is nonuniformly completely hyperbolic on Λ then $\mathcal{H}(W_j) = \pi(f|\Lambda_{j,m})$.*

Sinai [232, Theorem 5.2] proved this theorem for a class of dynamical systems with *transverse foliation*. Pesin [197] adapted this approach for nonuniformly hyperbolic dynamical systems.

We stress that the measurable hull $\mathcal{H}(W_j)$ does not depend on j (see Statement 1 of Theorem 11.17); this is a manifestation of the Lipschitz property of intermediate stable manifolds (see Theorem 9.6). One can estimate the entropy of f with respect to η from below (see Theorem 12.11).

In order to construct the partition η, given $\ell > 1$, consider the regular set Λ^ℓ. For a sufficiently small $r = r(\ell) > 0$ and $x \in \Lambda^\ell$, set

$$P_j^\ell(x) = \bigcup_{y \in \Lambda^\ell \cap B(x,r)} V_j(y), \quad Q(x) = \bigcup_{n=-\infty}^{\infty} f^n(P_j^\ell(x)). \tag{11.6}$$

It suffices to construct the partition η on the set $Q(x)$. Consider the partition $\widetilde{\xi}$ of $P_j^\ell(x)$ by local manifolds $V_j(y)$, $y \in \Lambda^\ell \cap B(x,r)$. Adding the element $Q(x) \setminus P_j^\ell(x)$ we obtain a partition of $Q(x)$ which we denote by ξ. The partition

$$\eta = \xi^- = \bigvee_{i \leq 0} f^i\xi$$

has the desired properties.

In [158], Ledrappier and Young constructed a special countable partition of M of finite entropy which is a refinement of the partition η. We describe this partition in Section 16.3.

An important manifestation of Theorem 11.16 is the establishment of the K-property of a $C^{1+\alpha}$ diffeomorphism f which preserves a smooth measure ν and is nonuniformly completely hyperbolic on an invariant set Λ of positive measure. By Theorem 11.3 the set Λ can be decomposed into ergodic components Λ_i, $i = 1, 2, \dots$ of positive measure. Fix i and denote by η_j the measurable partition of Λ_i associated with the foliation W_j, see Theorem 11.16.

Theorem 11.17 (Pesin [197]). *The following properties hold:*

1. $\mathcal{H}(W_{j_1}|\hat{\Lambda}_i) = \mathcal{H}(W_{j_2}|\hat{\Lambda}_i) = \pi(f|\Lambda_i)$ *for any* $1 \le j_1 < j_2 \le s$ *or* $s+1 \le j_1 < j_2 \le p$*;*
2. *the* π*-partition of* $f|\Lambda_i$ *is finite and consists of* n_i *elements* Λ_i^k, $k = 1, \dots, n_i$ *such that* $f(\Lambda_i^k) = \Lambda_i^{k+1}$, $k = 1, \dots, n_i - 1$ *and* $f(\Lambda_i^n) = \Lambda_i^1$*;*
3. $f^{n_i}|\Lambda_i^k$ *is a* K*-automorphism.*

We now discuss the Bernoulli property. There are examples in general ergodic theory of systems which have K-property but fail to be Bernoulli. This cannot happen for smooth systems with nonzero exponents: Bernoulli property holds automatically as long as the system has the K-property (indeed, the mixing property is already sufficient).

Theorem 11.18. *Let* f *be a* $C^{1+\alpha}$ *diffeomorphism of a smooth compact Riemannian manifold* M *preserving a smooth hyperbolic measure* ν*. Assume that* f *is weakly mixing with respect to* ν*. Then* f *is a Bernoulli automorphism.*

Ornstein and Weiss [190] established the Bernoulli property for geodesic flows on compact manifolds of negative curvature. Pesin [197] used a substantially more general version of their approach to prove Theorem 11.18. The proof exploits the characterization of a Bernoulli map in terms of *very weakly Bernoulli partitions* (see [190]). More precisely, there is a finite measurable partition α of the manifold M whose elements have piecewise smooth boundaries and arbitrarily small diameter. Indeed, one can construct a sequence of such partitions $\alpha_1 \le \alpha_2 \le \cdots$ such that $\alpha_n \to \varepsilon$. The proof goes to show that each partition α_n is very weakly Bernoulli and the result follows. An important technical tool of the proof is the refined estimate (10.3) of the Jacobian of the holonomy map.

Combining Theorems 11.17 and 11.18 we obtain the following Spectral Decomposition Theorem for systems with nonzero Lyapunov exponents preserving smooth measures.

Theorem 11.19. *For each* $i \ge 1$ *the following properties hold:*

1. Λ_i *is a disjoint union of sets* Λ_i^j, *for* $j = 1, \dots, n_i$, *which are cyclically permuted by* f, *i.e.,* $f(\Lambda_i^j) = \Lambda_i^{j+1}$ *for* $j = 1, \dots, n_i - 1$, *and* $f(\Lambda_i^{n_i}) = \Lambda_i^1$*;*
2. $f^{n_i}|\Lambda_i^j$ *is a Bernoulli automorphism for each* j*.*

We consider the case of dynamical systems with continuous time. Let φ_t be a C^2 flow on a smooth compact Riemannian manifold M preserving a smooth measure ν and nonuniformly hyperbolic on M. By Theorem 11.5, M can be decomposed into ergodic components Λ_i, $i = 1, 2, \dots$ of positive measure. Applying Theorem 11.16 to the nonuniformly partially hyperbolic diffeomorphism $\varphi_1|\Lambda_i$ we obtain the following result.

Theorem 11.20 (Pesin [197]). *There exists a partition* $\eta = \eta_i$ *of* Λ_i *for which:*

1. *for almost every* $x \in \Lambda_i$ *the element* $C_\eta(x)$ *is an open* (mod 0) *subset of* $W^s(x)$*;*

2. $\varphi_1\eta \geq \eta$;
3. $\bigvee_{i=0}^{\infty} \varphi_i\eta = \varepsilon$;
4. $\bigwedge_{i>-\infty}^{0} \varphi_i\eta = \mathcal{H}(W^s) = \pi(\varphi_i|\Lambda_i)$.

The following result establishes the K-property of the flow φ_t on the set Λ_i. For simplicity we will drop the index i. We remind the reader that a flow φ_t is a *K-flow* if and only if the diffeomorphism φ_t is a K-automorphism for every $t \in \mathbb{R}$.

Theorem 11.21 (Pesin [197])**.** *Assume that the flow $\varphi_t|\Lambda$ has continuous spectrum. Then it is a Bernoulli flow and in particular, a K-flow.*

The following result is an immediate consequence of this theorem.

Corollary 11.22. *Let φ_t be a smooth flow on a compact smooth Riemannian manifold M preserving a smooth measure ν. Assume that ν is hyperbolic and that φ_t is mixing with respect to ν. Then φ_t is a Bernoulli flow.*

12. Metric entropy

A crucial idea in Smooth Ergodic Theory is that sufficient instability of trajectories yields rich ergodic properties of the system. The entropy formula is in a sense a "quantitative manifestation" of this idea and is yet another pearl of Smooth Ergodic Theory. It expresses the Kolmogorov–Sinai entropy $h_\nu(f)$ of a diffeomorphism, preserving a smooth hyperbolic measure, in terms of the values of the Lyapunov exponent.

12.1. **Margulis–Ruelle inequality.** Let f be a C^1 diffeomorphism of a compact smooth manifold M. The following very general result provides an upper bound for the entropy of f with respect to any Borel invariant probability measure ν.

Theorem 12.1 (Margulis–Ruelle Inequality)**.** *The following estimate holds*

$$h_\nu(f) \leq \int_M \Sigma_+ \, d\nu(x), \tag{12.1}$$

where

$$\Sigma_+ = \sum_{i:\chi_i(x)>0} k_i(x)\chi_i(x).$$

In the case of volume-preserving diffeomorphisms this estimate was obtained by Margulis (unpublished). The inequality in the general case was established by Ruelle in [212] (see also [24] and [174]).

We sketch the proof of the theorem. By decomposing ν into its ergodic components we may assume without loss of generality that ν is ergodic. Then $s(x) = s$ and $k_i(x) = k_i$, $\chi_i(x) = \chi_i$ are constant ν-almost everywhere for each $1 \leq i \leq s$. Fix $m > 0$. Since M is compact, there exists $t_m > 0$ such that for every $0 < t \leq t_m$, $y \in M$, and $x \in B(y,t)$ we have

$$\frac{1}{2} d_x f^m \left(\exp_x^{-1} B(y,t)\right) \subset \exp_{f^m x}^{-1} f^m (B(y,t) \subset 2 d_x f^m \left(\exp_x^{-1} B(y,t)\right),$$

where for a set $A \subset T_zM$ and $z \in M$, we write $\alpha A = \{\alpha v : v \in A\}$.

There is a special partition of the manifold M which is described in the following statement.

Lemma 12.2. *Given $\varepsilon > 0$, there is a partition ξ of M such that:*

1. $\operatorname{diam}\xi \le t_m/10$ *and* $h_\nu(f^m,\xi) \ge h_\nu(f^m) - \varepsilon$;
2. *for every element* $C \in \xi$ *there exist balls* $B(x,r)$ *and* $B(x,r')$, *such that* $r < 2r' \le t_m/20$ *and* $B(x,r') \subset C \subset B(x,r)$;
3. *there exists* $0 < r < t_m/20$ *such that if* $C \in \xi$ *then* $C \subset B(y,r)$ *for some* $y \in M$, *and if* $x \in C$ *then*

$$\frac{1}{2} d_x f^m \left(\exp_x^{-1} B(y,r)\right) \subset \exp_{f^m x}^{-1} f^m C \subset 2 d_x f^m \left(\exp_x^{-1} B(y,r)\right).$$

To construct such a partition, given $\alpha > 0$, consider a maximal α-separated set Γ, i.e., a finite set of points for which $d(x,y) > \alpha$ whenever $x, y \in \Gamma$. For $x \in \Gamma$ set

$$\mathcal{D}_\Gamma(x) = \{y \in M : d(y,x) \le d(y,z) \text{ for all } z \in \Gamma \setminus \{x\}\}.$$

Obviously, $B(x,\alpha/2) \subset \mathcal{D}_\Gamma(x) \subset B(x,\alpha)$. Note that the sets $\mathcal{D}_\Gamma(x)$ corresponding to different points $x \in \Gamma$ intersect only along their boundaries, i.e., at a finite number of submanifolds of codimension greater than zero. Since ν is a Borel measure, if necessary, we can move the boundaries slightly so that they have zero measure. Thus, we obtain a partition ξ with $\operatorname{diam}\xi \le \alpha$ which can be chosen to satisfy

$$h_\nu(f^m,\xi) > h_\nu(f^m) - \varepsilon \quad \text{and} \quad \operatorname{diam}\xi < t_m/10.$$

This guarantees the properties in the lemma.

Continuing with the proof of the theorem observe that

$$\begin{aligned} h_\nu(f^m,\xi) &= \lim_{k\to\infty} H_\nu(\xi | f^m\xi \vee \cdots \vee f^{km}\xi) \\ &\le H_\nu(\xi|f^m\xi) = \sum_{D\in f^m\xi} \nu(D) H(\xi|D) \\ &\le \sum_{D\in f^m\xi} \nu(D) \log \operatorname{card}\{C \in \xi : C \cap D \ne \varnothing\}, \end{aligned} \tag{12.2}$$

where $H(\xi|D)$ is the entropy of ξ with respect to the conditional measure on D induced by ν. The following is a uniform exponential estimate for the number of elements $C \in \xi$ which have nonempty intersection with a given element $D \in f^m\xi$.

Lemma 12.3. *There exists a constant* $K_1 > 0$ *such that for* $D \in f^m\xi$,

$$\operatorname{card}\{C \in \xi : D \cap C = \varnothing\} \le K_1 \sup\{\|d_x f\|^{mn} : x \in M\},$$

where $n = \dim M$.

This can be shown by estimating the volume of each element C and using Property 2 of the partition ξ.

We also have an exponential bound for the number of those sets $D \in f^m\xi$ which contain regular points. Namely, given $\varepsilon > 0$, let $R_m = R_m(\varepsilon)$ be the set of forward regular points $x \in M$ which satisfy the following condition: for $k > m$ and $v \in T_xM$,

$$e^{k(\chi(x,v)-\varepsilon)}\|v\| \le \|d_x f^k v\| \le e^{k(\chi(x,v)+\varepsilon)}\|v\|.$$

Lemma 12.4. *If* $D \in f^m\xi$ *has nonempty intersection with* R_m *then there exists a constant* $K_2 > 0$ *such that*

$$\operatorname{card}\{C \in \xi : D \cap C \ne \varnothing\} \le K_2 e^{\varepsilon m} \prod_{i:\chi_i>0} e^{m(\chi_i+\varepsilon)k_i}.$$

To establish the inequality note that

$$\operatorname{card}\{C \in \xi : D \cap C \neq \varnothing\} \leq \operatorname{vol}(B)(\operatorname{diam} \xi)^{-n},$$

where $\operatorname{vol}(B)$ denotes the volume of

$$B = \{y \in M : d(y, \exp_{f^m(x)}(d_x f^m(\exp_x^{-1} B'))) < \operatorname{diam} \xi\}$$

and $B' = B(x, 2 \operatorname{diam} C')$ for some $C' \in \xi$ such that $C' \cap R_m \neq \varnothing$ and $f^m(C') = D$, and some $x \in C' \cap f^{-m}(R_m)$. Up to a bounded factor, $\operatorname{vol}(B)$ is bounded by the product of the lengths of the axes of the ellipsoid $d_x f^m(\exp_x^{-1} B')$. Those of them that correspond to nonpositive exponents are at most subexponentially large. The remaining ones are of size at most $e^{m(\chi_i+\varepsilon)}$, up to a bounded factor, for all sufficiently large m. Thus,

$$\begin{aligned}\operatorname{vol}(B_1) &\leq K e^{m\varepsilon} (\operatorname{diam} B)^n \prod_{i:\chi_i>0} e^{m(\chi_i+\varepsilon)k_i} \\ &\leq K e^{m\varepsilon} (2 \operatorname{diam} \xi)^n \prod_{i:\chi_i>0} e^{m(\chi_i+\varepsilon)k_i},\end{aligned}$$

for some constant $K > 0$. The lemma follows.

By Lemmas 12.3 and 12.4 and (12.2), we obtain

$$\begin{aligned} m h_\nu(f) - \varepsilon = h_\nu(f^m) - \varepsilon &\leq h_\nu(f^m, \xi) \\ &\leq \sum_{D \cap R_m \neq \varnothing} \nu(D) \left(\log K_2 + \varepsilon m + m \sum_{i:\chi_i>0} (\chi_i + \varepsilon) k_i \right) \\ &\quad + \sum_{D \cap R_m = \varnothing} \nu(D) (\log K_1 + nm \log \sup\{\|d_x f\| : x \in M\}) \\ &\leq \log K_2 + \varepsilon m + m \sum_{i:\chi_i>0} (\chi_i + \varepsilon) k_i \\ &\quad + (\log K_1 + nm \log \sup\{\|d_x f\| : x \in M\}) \nu(M \setminus R_m). \end{aligned}$$

By the Multiplicative Ergodic Theorem 5.5, we have $\bigcup_{m \geq 0} R_m(\varepsilon) = M \pmod 0$ for every sufficiently small ε. It follows that

$$h_\nu(f) \leq \varepsilon + \sum_{i:\chi_i>0} (\chi_i + \varepsilon) k_i.$$

Letting $\varepsilon \to 0$ we obtain the desired upper bound.

As an immediate consequence of Theorem 12.1 we obtain an upper bound for the topological entropy $h(f)$ of a diffeomorphism f. Namely,

$$h(f) = \sup_\nu h_\nu(f) \leq \sup_\nu \int_M \Sigma_+ \, d\nu, \tag{12.3}$$

where the suprema are taken over all f-invariant Borel probability measures on M.

In general Inequalities (12.1) and (12.3) can be strict. In fact, as the following example shows, there are C^∞ diffeomorphisms for which $h(f) < \inf_\nu \int_M \Sigma_+ \, d\nu$, and hence, $h_\nu(f) < \int_M \Sigma_+ \, d\nu$ for *any* invariant measure ν.

Example 12.5 (Figure-Eight; Bowen and Katok (see [131])). *Let f be a diffeomorphism of the two-dimensional sphere S^2 with three repelling fixed points p_1, p_2, p_3 and one saddle fixed point q. Suppose that the stable and unstable manifolds of the point q form two loops γ_1, γ_2 that divide S^2 into three regions A_1, A_2, and A_3.*

For $i = 1, 2, 3$, we have $p_i \in A_i$ and any point in $A_i \setminus \{p_i\}$ tends, respectively, to γ_1, γ_2, and $\gamma_1 \cup \gamma_2$. Thus, any f-invariant finite *measure ν is supported on the finite set $\{p_1, p_2, p_3, q\}$. Therefore, $h_\nu(f) = 0$ while $\int_M \Sigma_+ \, d\nu > c > 0$ for some c independent of ν. In addition, we have*

$$h(f) = \sup_\nu h_\nu(f) < \inf_\nu \int_M \Sigma_+ \, d\nu$$

where the supremum and infimum are taken over all f-invariant Borel probability measures on S^2.

Example 12.6 (Two-dimensional Horseshoes)**.** *Let Λ be a basic set (i.e., a locally maximal hyperbolic set), of a topologically transitive Axiom A surface diffeomorphism of class C^1. McCluskey and Manning* [181] *showed that for every $x \in \Lambda$ the Hausdorff dimension of the set $W^u(x) \cap \Lambda$ is the unique root s of Bowen's equation*

$$P\big(-s \log\|df|E^u\|\big) = 0,$$

where P is the topological pressure on $f|\Lambda$. In particular, s is independent of x.

Assume that $s < 1$. Since $s \mapsto P\big(-s \log\|df|E^u\|\big)$ is decreasing, we obtain

$$P\big(-\log\|df|E^u\|\big) < 0.$$

By the Variational Principle for the topological pressure, for every f-invariant measure ν,

$$h_\nu(f) < \int_\Lambda \log\|d_x f|E^u(x)\| \, d\nu(x) = \int_\Lambda \Sigma_+ \, d\nu$$

(we use here Birkhoff's Ergodic Theorem and the fact that $\dim E^u = 1$*).*

Note that $h_\nu(f^{-1}) = h_\nu(f)$ and the Lyapunov exponents of f^{-1} are those of f taken with opposite sign. Therefore, it follows from Theorem 12.1 that

$$h_\nu(f) \le -\int_M \sum_{i:\chi_i(x)<0} \chi_i(x) k_i(x) \, d\nu(x).$$

Set

$$a = \int_M \sum_{i:\chi_i(x)>0} \chi_i(x) k_i(x) \, d\nu(x)$$

and

$$b = -\int_M \sum_{i:\chi_i(x)<0} \chi_i(x) k_i(x) \, d\nu(x).$$

In Example 12.5 one can choose the eigenvalues of df at the critical points, and the measure ν to guarantee any of the relations: $a < b$ or $a = b$ or $a > b$. One can also show that if ν is the Riemannian volume on M, then $a = b$.

An important manifestation of Margulis–Ruelle's inequality is that positivity of topological entropy implies the existence of at least one nonzero Lyapunov exponent.

Corollary 12.7. *If the topological entropy of a C^1 diffeomorphism f of a compact manifold is positive, then there exists an ergodic f-invariant measure with at least one positive and one negative Lyapunov exponent.*

For surface diffeomorphisms, Corollary 12.7 means that any diffeomorphism with positive topological entropy possesses an ergodic invariant measure whose Lyapunov exponents are all nonzero.

Let us point out that the positivity of topological entropy can sometimes be determined using pure topological information. For example, theorems of Manning [176], Misiurewicz and Przytycki [186, 185], and Yomdin [253, 254] relate the topological entropy to the action of the diffeomorphism on the homology groups (see also [129]); see Section 15.5.

Other immediate consequences of Theorem 12.1 are as follows.

Corollary 12.8. *Let ν be a measure which is invariant under a C^1 diffeomorphism f of a compact manifold. If $h_\nu(f) > 0$ then ν has at least one positive and one negative Lyapunov exponent.*

For surface diffeomorphisms, Corollary 12.8 implies that if $h_\nu(f) > 0$ then the Lyapunov exponents of ν are all nonzero, i.e., ν is hyperbolic (see Sections 10.1 and 15).

Corollary 12.9. *We have*

$$\begin{aligned} h(f) &\le \dim M \times \inf_{m\ge 1} \frac{1}{m} \log^+ \sup_{x\in M} \|d_x f^m\| \\ &= \dim M \times \lim_{m\to\infty} \frac{1}{m} \log^+ \sup_{x\in M} \|d_x f^m\|. \end{aligned}$$

12.2. **The entropy formula.** Let $f\colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism, $\alpha > 0$ and ν an f-invariant measure which is absolutely continuous with respect to the Riemannian volume. The main result of this section is the *Pesin entropy formula* which expresses the entropy of f with respect to ν via its Lyapunov exponents. It was first proved by Pesin in [197]. The proof relies on properties of the unstable foliation and in particular, absolute continuity. Another proof of the entropy formula was obtained by Mañé in [171] (see also [174]). It does not involve directly the existence of stable and unstable foliations but instead uses some subtle properties of the action of the differential df with respect to the Lyapunov exponents in the presence of a smooth invariant measure.

Theorem 12.10 (Pesin [197])**.** *The following formula holds true:*

$$h_\nu(f) = \int_M \Sigma_+ \, d\nu. \tag{12.4}$$

In view of the Margulis–Ruelle inequality we only need to establish the lower bound

$$h_\nu(f) \ge \int_M \sum_{i:\chi_i(x)>0} k_i(x)\chi_i(x)\, d\nu(x),$$

or equivalently (by replacing f by f^{-1} and using Theorem 5.5)

$$h_\nu(f) \ge -\int_M \sum_{i:\chi_i(x)<0} k_i(x)\chi_i(x)\, d\nu(x).$$

This inequality is a corollary of a more general result which we now state.

Let $f\colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a smooth compact Riemannian manifold M preserving a smooth measure ν and nonuniformly partially hyperbolic in the broad sense on an invariant set Λ of positive measure. For every $x \in \Lambda$ we have that

$$\chi_1(x) < \cdots < \chi_{s(x)}(x) < 0 \le \chi_{s(x)+1}(x) < \cdots < \chi_{p(x)}(x),$$

where $\chi_i(x)$, $i = 1, \dots, p(x)$ are the distinct values of the Lyapunov exponent at x each with multiplicity $k_i(x)$. We also have the filtration of local (stable) manifolds (11.3) as well as the filtration of global (stable) manifolds (11.4) at x. Given $j > 0$ and $m > 0$, consider the sets (11.5). Note that $\nu(\Lambda_{j,m}) > 0$ for some j and m and $W_j(x) \subset \Lambda_{j,m} \pmod 0$ for almost every $x \in \Lambda_{j,m}$. Hence, $\hat{\Lambda}_{j,m} = \Lambda_{j,m} \pmod 0$.

Consider the partition $\eta = \eta_{j,m}$ of $\hat{\Lambda}_{j,m}$ constructed in Theorem 11.16.

Theorem 12.11 (Pesin [198])**.** *The entropy of f with respect to η admits the following estimate from below*

$$h_\nu(f, \eta) \ge -\int_{\Lambda_{j,m}} \sum_{i=1}^{j} k_i(z)\chi_i(z)\, d\nu(z).$$

We shall sketch the proof of the theorem. Given $\ell > 0$ consider the regular set Λ^ℓ. For sufficiently small $r = r(\ell)$ and $x \in \Lambda^\ell$ consider also the sets $P^{\ell,j}(x)$ and $Q(x)$ defined by (11.6). Let $\widetilde{\nu}$ be the measure on $Q(x)$ given for any measurable subset $A \subset Q(x)$ by $\widetilde{\nu}(A) = \nu(A)(\nu(Q(x)))^{-1}$. It suffices to show that

$$h(f|Q(x), \eta) \ge -\int_{Q(x)} \sum_{i=1}^{j} k_i(z)\chi_i(z)\, d\widetilde{\nu}(z). \tag{12.5}$$

Consider the function

$$g(z) = \prod_{i=1}^{j} \exp(\chi_i(z))^{k_i(z)}.$$

Given $\varepsilon > 0$, let $Q_p = \{z \in Q(x) : p\varepsilon < g(z) \le (p+1)\varepsilon\}$. It suffices to show the inequality (12.5) for the restriction $\bar{f} = f|Q_p$ and the measure $\bar{\nu}$ defined by $\bar{\nu}(A) = \nu(A)(\nu(Q_p))^{-1}$ for any measurable subset $A \subset Q_p$.

Set $J_n(z) = \mathrm{Jac}\,(df^n|T_zW^j(z))$. It follows from the Multiplicative Ergodic Theorem 5.5 that there exists a positive Borel function $T(z, \varepsilon)$, $z \in Q_p$ and $\varepsilon > 0$ such that for $n > 0$,

$$J_n(z) \le T(z, \varepsilon)g(z)\exp(\varepsilon n).$$

Set for $t \ge 0$,

$$Q_p^t = \{z \in Q_p : \ T(z, \varepsilon) \le t\}.$$

We have that for any $\alpha > 0$ and all sufficiently large t,

$$\bar{\nu}(Q_p^t) \ge 1 - \alpha. \tag{12.6}$$

It follows from Theorem 10.1 that there exists $C_1 = C_1(t) > 0$ such that for any $z \in Q_p^t$ and $n > 0$,

$$\nu^j(z)(f^n(C_\eta(z))) \le C_1 J^n(z). \tag{12.7}$$

Denote by $B_\eta(z, r)$ the ball in $C_\eta(z)$ centered at z of radius r.

Lemma 12.12. *For any $\beta > 0$ there exists $q = q(t)$ and a subset $A^t \subset Q_p^t$ such that:*

1. $\bar{\nu}(Q_p^t \setminus A^t) \le \beta$;
2. *for any $z \in A^t$ the element $C_\eta(z)$ contains the ball $B_\eta(z, q)$.*

Denote by $\nu_\eta(z)$ the conditional measure on the element $C_\eta(z)$ of the partition η generated by the measure ν. For every $z \in A^t$,

$$C_4^{-1} \le \frac{d\nu_\eta(z)}{dm^j(z)} \le C_4, \tag{12.8}$$

where $C_4 = C_4(t) > 0$ is a constant independent of z. For any $n > 0$,

$$h_{\bar\nu}(f) = \frac{1}{n} h_{\bar\nu}(f^n) \ge \frac{1}{n} H(f^n\eta|\eta).$$

We use here the fact that

$$\eta = \eta^- = \bigvee_{i\le 0} f^i\eta$$

(see Theorem 11.16). It follows from (12.6), (12.7), and (12.8) that for every $x \in A^t$ and $n > 0$,

$$\begin{aligned} H(\bar f^n\eta|C_\eta(z)) &= -\int_{C_\eta(x)} \frac{\nu_\eta(C_\eta(x)\cap C_{f^n\eta}(z))}{\nu_\eta(C_\eta(y))}\, d\nu_\eta(y) \\ &\ge -\log[{C_4}^2 C_1 t((p+1)\varepsilon)^n e^{\varepsilon n} V(B_\eta(z,q(t)))^{-1}] = I_n, \end{aligned} \tag{12.9}$$

where $V(B_\eta(z,q(t)))$ is the Riemannian volume of the ball $B_\eta(z,q(t))$. We have that $V(B_\eta(z,q(t))) \ge C_5 q^m(t)$ where $C_5 > 0$ is a constant. It follows that

$$\begin{aligned} I_n &\ge -\log({C_4}^2C_1t)(C_5q^m(t))^{-1} - n\,(\log((p+1)\varepsilon) + \varepsilon) \\ &\ge C_6 - n\,(\log g(z) + \varepsilon). \end{aligned} \tag{12.10}$$

By (12.6) and Statement 1 of Lemma 12.12, we obtain that $\bar\nu(Q_p \setminus A^t) \le \alpha + \beta$. Therefore, integrating inequality (12.9) over the elements $C_\eta(x)$ and taking (12.10) into account we conclude that

$$\begin{aligned} \frac{1}{n} H(\bar f^n\eta|\eta) \ge \frac{1}{n} I_n \bar\nu(A^t) &\ge \frac{1}{n}\int_{Q_p} I_n\, d\bar\nu(1-\alpha\beta) \\ &\ge \int_{Q_p} \sum_{i=1}^{j} k_i(z)\chi_i(z)\, d\bar\nu(z) - \gamma, \end{aligned}$$

where γ can be made arbitrary small if ε, α, and β are chosen sufficiently small and n sufficiently large. The desired result follows.

In the two-dimensional case the assumption that $f \in C^{1+\alpha}$ can be relaxed for a residual set of diffeomorphisms.

Theorem 12.13 (Tahzibi [237])**.** *Let M be a compact smooth surface. There exists a residual subset $\mathcal{G}$ in the space* $\mathrm{Diff}^1(M,m)$ *of C^1 volume preserving diffeomorphisms of M such that every $f \in \mathcal{G}$ satisfies the entropy formula* (12.4)*. Moreover, $\mathcal{G}$ contains all volume-preserving diffeomorphisms of class $C^{1+\alpha}$.*

The main idea of the proof is the following. In the two dimensional case a volume-preserving diffeomorphism f has at most one positive Lyapunov exponents $\chi^+(x)$ almost everywhere. For $f \in \mathrm{Diff}^1(M,m)$ set $L(f) = \int_M \chi^+(x)\,d\mu$. One can show that the set of continuity points of the functions $L(f)$ and $h_m(f)$ is residual in the C^1 topology. Let f be a continuity point. One obtains the entropy formula for f by approximating f by a sequence f_n of $C^{1+\alpha}$ diffeomorphisms for which the entropy formula (12.4) holds.

Ledrappier and Strelcyn [138] extended the entropy formula to SRB-measures invariant under $C^{1+\alpha}$ diffeomorphisms (see Section 14) and Ledrappier and Young

[158] obtained a general version of the entropy formula for arbitrary C^2 diffeomorphisms (see Section 16.1).

13. Genericity of systems with nonzero exponents

13.1. **Existence of diffeomorphisms with nonzero exponents.** Presence of an Anosov diffeomorphism f on a compact Riemannian manifold M imposes strong conditions on the topology of the manifold. For example, M should admit two foliations with smooth leaves (invariant under f). Anosov diffeomorphisms are only known to exist on multi-dimensional tori or more generally on factors of nilpotent Lie groups. On the contrary nonuniform hyperbolicity imposes no restrictions on the topology of M.

Theorem 13.1 (Dolgopyat and Pesin [78])**.** *Given a compact smooth Riemannian manifold $M \neq S^1$ there exists a C^∞ volume-preserving Bernoulli diffeomorphism f of M with nonzero Lyapunov exponents almost everywhere.*

Let us comment on the proof of this theorem.

1. Katok [130] proved this theorem in the two dimensional case. His argument goes as follows. Consider the diffeomorphism G_{S^2} of the sphere, constructed in Section 2.3. It has four *singularity* points $p_i = \zeta(x_i)$. Let ξ be a C^∞ map which blows up the point p_4. Consider the map $G_{D^2} = \xi \circ G_{S^2} \circ \xi^{-1}$ of the closed unit disk D^2. It is a C^∞ diffeomorphism which preserves the area, has the Bernoulli property and nonzero Lyapunov exponents almost everywhere.

The disk D^2 can be embedded into any surface. This is a corollary of a more general statement (see [130]).

Proposition 13.2. *Given a p-dimensional compact C^∞ manifold M and a smooth measure μ on M, there exists a continuous map $h\colon D^p \to M$ (D^p is the unit ball in $\mathbb{R}^p$) such that*

1. *the restriction $h|int\ D^p$ is a diffeomorphic embedding;*
2. *$h(D^p) = M$;*
3. *$\mu(M \setminus h(D^p)) = 0$;*
4. *$h_* m = \mu$ where m is the volume in $\mathbb{R}^p$.*

Note that G_{D^2} is identity on the boundary ∂D^2. Moreover, one can choose the function ψ in the construction of maps $G_{\mathbb{T}^2}$ and G_{S^2} such that the map G_{D^2} is "sufficiently flat" near the boundary of the disk.

More precisely, let $\rho = \{\rho_n\}$ be a sequence of nonnegative real-valued continuous functions on D^p which are strictly positive inside the disc. Let $C^\infty_\rho(D^p)$ be the set of all C^∞ functions on D^p satisfying the following condition: for any $n \geq 0$ there exists a sequence of numbers $\varepsilon_n > 0$ such that for all $(x_1, \dots, x_p) \in D^p$ for which $x_1^2 + \dots + x_p^2 \geq (1 - \varepsilon_n)^2$ we have

$$\left| \frac{\partial^n h(x_1, \dots, x_p)}{\partial^{i_1} x_1 \dots \partial^{i_p} x_p} \right| < \rho_n(x_1, \dots, x_p),$$

where $i_1, \dots i_p$ are nonnegative integers and $i_1 + \dots i_p = n$.

Any diffeomorphism G of the disc D^p can be written in the form $G(x_1, \dots, x_p) = (G_1(x_1, \dots, x_p), \dots G_p(x_1, \dots, x_p))$. Set

$$\mathrm{Diff}^\infty_\rho(D^p) = \{g \in \mathrm{Diff}^\infty(D^p)\colon G_i(x_1, \dots, x_p) - x_i \in C^\infty_\rho(D^p), \quad i = 1, \dots, p\}.$$

Proposition 13.3 (Katok [130])**.** *Given a compact C^∞ Riemannian manifold M there exists a sequence of functions ρ such that for any $G \in \mathrm{Diff}^\infty_\rho(D^p)$ the map g defined as $g(x) = h(G(h^{-1}(x)))$ for $x \in h(\text{int } D^p)$ and $g(x) = x$ otherwise, is a C^∞ diffeomorphism of M (the map h is from Proposition 13.2).*

The function ψ can be chosen so that $G_{D^2} \in \mathrm{Diff}^\infty_\rho(D^2)$ and hence, the map f, defined as $f(x) = h(G_{D^2}(h^{-1}(x)))$ for $x \in h(\text{int } D^p)$ and $f(x) = x$ otherwise, has all the desired properties: it preserves area, has nonzero Lyapunov exponents and is a Bernoulli map.

2. For any smooth compact Riemannian manifold M of dimension $p = \dim M \geq 5$, Brin [47] constructed a C^∞ volume-preserving Bernoulli diffeomorphism which has all but one nonzero Lyapunov exponents. His construction goes as follows.

Let A be a volume-preserving hyperbolic automorphism of the torus $\mathbb{T}^{p-3}$ and φ_t the suspension flow over A with the roof function

$$H(x) = H_0 + \varepsilon \widetilde{H}(x),$$

where H_0 is a constant and the function $\widetilde{H}(x)$ is such that $|\widetilde{H}(x)| \leq 1$. The flow φ_t is an Anosov flow on the phase space Y^{p-2} which is diffeomorphic to the product $\mathbb{T}^{p-3} \times [0,1]$, where the tori $\mathbb{T}^{p-3} \times \{0\}$ and $\mathbb{T}^{p-3} \times \{1\}$ are identified by the action of A. Consider the skew product map R of the manifold $N = D^2 \times Y^{n-2}$ given by

$$R(z) = R(x,y) = (G_{D^2}(x), \varphi_{\alpha(x)}(y)), \quad z = (x,y), \tag{13.1}$$

where $\alpha \colon D^2 \to \mathbb{R}$ is a nonnegative C^∞ function which is equal to zero in a small neighborhood U of the *singularity set* $\{q_1, q_2, q_3\} \cap \partial D^2$ and is strictly positive otherwise. The map R is of class C^∞ and preserves volume. One can choose the function $\widetilde{H}(x)$ such that R is a Bernoulli diffeomorphism which has all but one nonzero Lyapunov exponents (the zero exponent corresponds to the direction of the flow φ_t).

Brin proved that there exists a smooth embedding of the manifold Y^{p-2} into $\mathbb{R}^p$. It follows that there is a smooth embedding $\chi_1 \colon D^2 \times Y^{p-2} \to D^p$ which is a diffeomorphism except for the boundary $\partial D^2 \times Y^{p-2}$. Using Proposition 13.2 one can find a smooth embedding $\chi \colon D^p \to M$ which is a diffeomorphism except for the boundary ∂D^p. Since the map R is identity on the boundary $\partial D^2 \times Y^{p-2}$ the map $h = (\chi_1 \circ \chi) \circ R \circ (\chi_1 \circ \chi)^{-1}$ has all the desired properties.

3. Dolgopyat and Pesin [78] constructed the required map P as a small perturbation of the map R (defined by (13.1)). The diffeomorphism P can be found in the form $P = \varphi \circ R$ where $\varphi(x,y) = (x, \varphi_x(y))$ and $\varphi_x \colon Y^{p-2} \to Y^{p-2}$, $x \in N$ is a family of volume preserving C^∞ diffeomorphisms satisfying $d_{C^1}(\varphi_x, \mathrm{Id}) \leq \varepsilon$. To construct such a family fix a sufficiently small number $\gamma > 0$, any point $y_0 \in Y^{p-2}$, and a point $x_0 \in D^2$ such that

$$G^j_{D^2}(B(x_0,\gamma)) \cap B(x_0,\gamma) = \varnothing, \quad -N < j < N, j \neq 0,$$
$$G^j_{D^2}(B(x_0,\gamma)) \cap \partial D^2 = \varnothing, \quad -N < j < N.$$

Set $\Delta = B(x_0,\gamma) \times B(q_0,\gamma)$ and choose a coordinate system $\{\xi_1, \xi_2, \eta_1, \dots, \eta_{p-2}\}$ in Δ such that $x = (\xi_1, \xi_2)$, $y = (\eta_1, \dots, \eta_{p-2})$, $dm = dxdy$ (recall that m is the volume) and

$$E^c_{\varphi_t}(y_0) = \frac{\partial}{\partial \eta_1}, \quad E^s_{\varphi_t}(y_0) = \left(\frac{\partial}{\partial \eta_2}, \dots, \frac{\partial}{\partial \eta_k}\right), \quad E^u_{\varphi_t}(y_0) = \left(\frac{\partial}{\partial \eta_{k+1}}, \dots, \frac{\partial}{\partial \eta_{p-2}}\right)$$

for some k, $2 \le k < p-2$. Let $\psi(t)$ be a C^∞ function with compact support. Set

$$\tau = \frac{1}{\gamma^2}(\|\xi_1\|^2 + \|\xi_2\|^2 + \|\eta_1\|^2 + \cdots + \|\eta_{p-2}\|^2)$$

and define

$$\begin{aligned}\varphi_x^{-1}(y) =& (\xi_1, \xi_2, \eta_1 \cos(\varepsilon\psi(\tau)) + \eta_2 \sin(\varepsilon\psi(\tau)),\\ & - \eta_1 \sin(\varepsilon\psi(\tau)) + \eta_2 \cos(\varepsilon\psi(\tau)), \eta_3, \dots, \eta_{p-2}).\end{aligned}$$

The family φ_x determines the map φ so that the map $P = \varphi \circ R$ is a volume-preserving Bernoulli diffeomorphism with nonzero Lyapunov exponents.

4. We discuss the case $\dim M = 3$. Consider the manifold $N = D^2 \times S^1$ and the skew product map R

$$R(z) = R(x, y) = (G_{D^2}(x), R_{\alpha(x)}(y)), \quad z = (x, y), \tag{13.2}$$

where $R_{\alpha(x)}$ is the rotation by the angle $\alpha(x)$ and $\alpha\colon D^2 \to \mathbb{R}$ is a nonnegative C^∞ function which is equal to zero in a small neighborhood of the singularity set $\{q_1, q_2, q_3\} \cap \partial D^2$ and is strictly positive otherwise.

We define a perturbation P of R in the form $P = \varphi \circ R$. Consider a coordinate system $\xi = \{\xi_1, \xi_2, \xi_3\}$ in a small neighborhood of a point $z_0 \in N$ such that $dm = d\xi$ and

$$E^c_R(z_0) = \frac{\partial}{\partial \xi_1}, \quad E^s_R(z_0) = \frac{\partial}{\partial \xi_2}, \quad E^u_R(z_0) = \frac{\partial}{\partial \xi_3}.$$

Let $\psi(t)$ be a C^∞ function with compact support. Set $\tau = \|\xi\|^2/\gamma^2$ and define

$$\varphi^{-1}(\xi) = (\xi_1 \cos(\varepsilon\psi(\tau)) + \xi_2 \sin(\varepsilon\psi(\tau)), -\xi_1 \sin(\varepsilon\psi(\tau)) + \xi_2 \cos(\varepsilon\psi(\tau)), \xi_3). \tag{13.3}$$

One can choose the function $\alpha(x)$ and the point z_0 such that the map P has all the desired properties.

5. We now proceed with the case $\dim M = 4$. Consider the manifold $N = D^2 \times \mathbb{T}^2$ and the skew product map R defined by (13.2) where $R_{\alpha(x)}$ is the translation by the vector $\alpha(x)$ and the function $\alpha(x)$ is chosen as above. Consider a perturbation P of R in the form $P = \varphi \circ R$ and choose the map φ as above to ensure that

$$\int_N [\chi^c_1(z, P) + \chi^c_2(z, P)]\, dz < 0,$$

where $\chi^c_1(z, P) \ge \chi^c_2(z, P)$ are the Lyapunov exponents of P along the central subspace $E^c_P(z)$. One can further perturb the map P in the C^1 topology to a map $\bar{P}$ to guarantee that

$$\int_N \left[\chi^c_1(z, \bar{P}) + \chi^c_2(z, \bar{P})\right] dz < 0, \quad \int_N \left[\chi^c_1(z, \bar{P}) - \chi^c_2(z, \bar{P})\right] dz \le \varepsilon,$$

where $\chi^c_1(z, \bar{P}) \ge \chi^c_2(z, \bar{P})$ are the Lyapunov exponents of $\bar{P}$ along the central subspace $E^c_{\bar{P}}(z)$ and $\varepsilon > 0$ is sufficiently small. This can be done using the approach described in the proof of Theorem 13.8 (this is one of the reasons why $\bar{P}$ is close to P in the C^1 topology only). The map $\bar{P}$ has all the desired properties.

13.2. **Existence of flows with nonzero exponents.** In [121], Hu, Pesin and Talitskaya established a continuous time version of Theorem 13.1.

Theorem 13.4. *Given a compact smooth Riemannian manifold M of* $\dim M \ge 3$, *there exists a C^∞ volume-preserving Bernoulli flow φ_t such that at m-almost every point $x \in M$ it has nonzero Lyapunov exponent except for the exponent in the direction of the flow.*

We sketch the proof of this theorem. Assume first that $\dim M \geq 5$. Consider the map

$$R = G_{D^2} \times A\colon D^2 \times \mathbb{T}^{p-3} \to D^2 \times \mathbb{T}^{p-3},$$

where $p = \dim M$, G_{D^2} is the above constructed diffeomorphism of the two-dimensional disk with nonzero Lyapunov exponents and A is a linear automorphism of the torus $\mathbb{T}^{p-3}$.

Consider further the suspension flow g_t over R with the roof function $H = 1$ and the suspension manifold $K = D^2 \times \mathbb{T}^{p-3} \times [0,1]/\sim$, where $\sim$ is the identification $(x, y, 1) = (G_{D^2}(x), A(y), 0)$. Denote by Z the vector field of the suspension flow.

Finally, consider the suspension flow h_t over A with the roof function $H = 1$ and the suspension manifold $L = \mathbb{T}^{p-3} \times [0,1]/\sim$, where $\sim$ is the identification $(y, 1) = (Ay, 0)$. Let $N = D^2 \times \mathbb{T}^{p-3} \times [0,1]/\sim$, where $\sim$ is the identification $(x, y, 1) = (x, Ay, 0)$ for any $x \in D^2$, $y \in \mathbb{T}^{p-3}$.

The proof goes by showing that there exists a volume-preserving C^∞ diffeomorphism $F\colon K \to N$ so that the vector field $Y = dFZ$ is divergence free and

$$Y(x, y, t) = (Y_1(x, y, t), 0, 1).$$

Choose a C^∞ function $a\colon D^2 \to [0,1]$ which vanishes on the boundary ∂D^2 with all its partial derivatives of any order, strictly positive otherwise and $a(x) = 1$ outside small neighborhood of the boundary. Define the vector field V on N by

$$V(x, y, t) = (Y_1(x, y, t), 0, a(x)).$$

The flow on K corresponding to the vector field $dF^{-1}VF$ is volume-preserving, has nonzero Lyapunov exponents (except for the exponent in the flow direction) and is Bernoulli. The manifold K can be embedded into M and this embedding carries over the flow into a flow on M with all the desired properties.

13.3. **Genericity conjecture.** Little is known about genericity of systems with nonzero Lyapunov exponents. On any manifold M of dimension $\dim M \geq 2$ and for sufficiently large r there are open sets of volume-preserving C^r diffeomorphisms of M which possess positive measure sets with all of the exponents to be zero: these sets consist of codimension one invariant tori on which the system is conjugate to a diophantine translation (see [60, 111, 251, 252]).

In this regard the following conjecture is of a great interest in the field.

Conjecture 13.5. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that f has nonzero Lyapunov exponents almost everywhere. Then there exists a neighborhood $\mathfrak{U}$ of f in* $\mathrm{Diff}^{1+\alpha}(M, \mu)$ *and a G_δ-set $\mathcal{A} \subset \mathfrak{U}$ such that any diffeomorphism $g \in \mathcal{A}$ has nonzero Lyapunov exponents on a set A_g of positive measure.*

13.4. **C^1-genericity for maps.** We stress that the assumption on the regularity of f (i.e., f is of class $C^{1+\alpha}$) is crucial: in the C^1 topology one should expect quite a different behavior. Let us describe some relevant results. We first consider the case of a compact surface M.

Theorem 13.6 (Bochi [35])**.** *There exists a residual subset $\mathcal{U}$ in the space of area preserving C^1 diffeomorphisms such that any $f \in \mathcal{U}$ is either Anosov or has zero Lyapunov exponents almost everywhere.*

This theorem was first announced by Mañé around 1983. Although the proof was never published a sketch of it appeared in [173] (see also [175] for a symplectic version of this result). A version of Theorem 13.6 for manifolds of higher dimension was obtained by Bochi and Viana in [37].

Let f be a volume-preserving ergodic C^1 diffeomorphism of a compact smooth Riemannian manifold M and x a Lyapunov regular point for f. Consider the Oseledets decomposition (6.7) along the orbit of x and two subspaces $E_i(x)$ and $E_j(x)$ corresponding to two distinct values of the Lyapunov exponent, $\chi_i > \chi_j$ (since f is ergodic these values do not depend on x). Given a point y in the orbit of x there is $m = m(y,i,j) \geq 1$ such that

$$\|df^m|E_i(y)\| \cdot \|df^m|E_j(y)\| \leq \frac{1}{2}.$$

Let $m(y) = \max_{i,j} m(y,i,j)$. We say that the Oseledets decomposition has the *dominated property* if $m(y)$ does not depend on y. In other words, the fact that df^n eventually expends $E_i(y)$ more than $E_j(y)$ can be observed in finite time *uniformly over the orbit of* f. The dominated property implies that the angles between the Oseledets subspaces are bounded away from zero along the orbit.

Theorem 13.7 ([37, 36, 38])**.** *Let M be a compact smooth Riemannian manifold. There exists a residual subset $\mathcal{U}$ in the space of volume-preserving C^1 diffeomorphisms such that for any $f \in \mathcal{U}$ and almost every $x \in M$ the Oseledets decomposition is either dominated along the orbit of x or is trivial, i.e., all Lyapunov exponents at x are zero.*

This theorem is a corollary of the following result that provides necessary conditions for continuity of Lyapunov exponents $\chi_i(f,x)$ over f. For $j = 1, \dots, p-1$ define

$$\mathrm{LE}_j(f) = \int_M [\chi_1(f,x) + \cdots + \chi_j(f,x)]\, dm(x).$$

It is well-known that the function

$$f \in \mathrm{Diff}^1(M,m) \to \mathrm{LE}_j(f)$$

is upper semi-continuous.

Theorem 13.8 (Bochi and Viana [37])**.** *Let $f_0 \in \mathrm{Diff}^1(M,m)$ be such that the map*

$$f \in \mathrm{Diff}^1(M,m) \to (LE_1(f), \dots, LE_{p-1}(f)) \in \mathbb{R}^{p-1}$$

is continuous at $f = f_0$. Then for almost every $x \in M$ the Oseledets decomposition is either dominated along the orbit of x or is trivial.

The main idea of the proof can be described as follows (we borrow this description from [37]). If the Oseledets decomposition is neither dominated nor trivial over a set of orbits of positive volume then for some i and arbitrary large m there exist infinitely many iterates $y_j = f^{n_j}(x)$ for which

$$\|df^m|E_i^-(y)\|\|(df^m|E_j^+(y))^{-1}\| > \frac{1}{2}, \tag{13.4}$$

where

$$E_i^+(y) = E_1(y) \oplus \cdots \oplus E_i(y)$$

and

$$E_i^-(y) = E_{i+1}(y) \oplus \cdots \oplus E_{p(y)}(y).$$

Applying a small perturbation one can move a vector originally in $E_i^+(y)$ to $E_i^-(y)$ thus "blending" different expansion rates.

More precisely, fix $\varepsilon > 0$, sufficiently large m and a point $x \in M$. For n much bigger than m choose an iterate $y = f^\ell(x)$ with $\ell \approx \frac{n}{2}$ as in (13.4). By composing df with small rotations near the first m iterates of y one can cause the orbit of some $df_x^\ell v \in E_i^+(y)$ to move to $E_i^-(z)$. This creates a perturbation $g = f \circ h$ which preserves the orbit segment $\{x, \dots, f^n(x)\}$ and is such that $dg_x^s v \in E_i^+$ during the first ℓ iterates and $dg_x^s v \in E_i^-$ during the last $n - \ell - m \approx \frac{n}{2}$ iterates. We wish to conclude that dg_x^n lost some expansion if compared to df_x^n. To this end we compare the kth exterior products of these linear maps with $k = \dim E_i^+$. We have

$$\| \wedge^p (dg_x^n)\| \le \exp(n(\chi_1 + \cdots + \chi_{k-1} + \frac{1}{2}(\chi_k + \chi_{k+1})),$$

where the Lyapunov exponents are computed at (f, x). Notice that $\chi_{k+1} = \hat{\lambda}_{i+1}$ is strictly smaller than $\chi_k = \hat{\lambda}_i$. This local procedure is then repeated for a positive volume set of points $x \in M$. Using the fact that

$$\mathrm{LE}_k(g) = \inf_n \frac{1}{n} \int_M \log \| \wedge^p (dg_x^n)\| \, dm$$

one can show that $\mathrm{LE}_k(g)$ drops under such arbitrary small perturbations contradicting continuity.

For the above construction to work one should arrange various intermediate perturbations around each $f^s(y)$ not to interfere with each other nor with other iterates of x in the time interval $\{0, \dots, n\}$. One can achieve this by rescaling the perturbation $g = f \circ h$ near each $f^s(y)$ if necessary to ensure that its support is contained in a sufficiently small neighborhood of the point. In a local coordinate w around $f^s(y)$ rescaling corresponds to replacing $h(w)$ by $rh(w/r)$ for some small $r > 0$. This does not affect the value of the derivative at $f^s(y)$ nor the C^1 norm of the perturbation and thus it can be made close to f in the C^1 topology. It is not clear whether the argument can be modified to work in C^q with $q > 1$.

One can establish a version of Theorem 13.7 in the symplectic case.

Theorem 13.9 (Bochi and Viana [37]). *Let M be a compact smooth Riemannian manifold. There exists a residual subset $\mathcal{U}$ in the space of C^1 symplectic diffeomorphisms such that every $f \in \mathcal{U}$ is either Anosov or has at least two zero Lyapunov exponents at almost every $x \in M$.*

13.5. **C^0-genericity for cocycles.** We now describe a version of Theorem 13.7 for linear cocycles.

Let $S \subset GL(n, \mathbb{R})$ be an embedded submanifold (with or without boundary). We say that S is *accessible* if it acts transitively on the projective space $\mathbb{R}P^{n-1}$. More precisely, for any $C > 0$, $\varepsilon > 0$ there are $m > 0$ and $\alpha > 0$ with the following property: given $\xi, \eta \in \mathbb{R}P^{n-1}$ with $\angle(\xi, \eta) \le \alpha$ and any $A_0, \dots, A_{m-1} \in S$ with $\|A_i^{\pm 1}\| \le C$ one can find $\widetilde{A}_0, \dots, \widetilde{A}_{m-1} \in S$ such that $\|A_i - \widetilde{A}_i\| \le \varepsilon$ and

$$\widetilde{A}_0, \dots, \widetilde{A}_{m-1}(\xi) = A_0, \dots, A_{m-1}(\eta).$$

Let X be a compact Hausdorff space and $f \colon X \to X$ a homeomorphism preserving a Borel probability measure μ. Let also $\mathcal{A} \colon X \times \mathbb{Z} \to GL(n, \mathbb{R})$ be the cocycle over f generated by a function $A \colon X \to GL(n, \mathbb{R})$.

Theorem 13.10 (Bochi and Viana [37]). *For any accessible set $S \subset GL(n,\mathbb{R})$ there exists a residual set $\mathcal{R} \subset C(X,S)$ such that for every $A \in \mathcal{R}$ and almost every $x \in X$ either all Lyapunov exponents of the cocycle $\mathcal{A}$, generated by A, are equal to each other or the Oseledets decomposition for $\mathcal{A}$ (see* (5.1)*) is dominated.*

This result applies to cocycles associated with Schrödinger operators. In this case $X = S^1$, $f\colon S^1 \to S^1$ is an irrational rotation, $f(x) = x + \alpha$, and the generator $A\colon S^1 \to SL(2,\mathbb{R})$ is given by

$$A(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix},$$

where $E \in \mathbb{R}$ is the total energy and $V\colon S^1 \to \mathbb{R}$ is the potential energy. The cocycle generated by A is a point of discontinuity for the Lyapunov exponents, as functions of $V \in C^0(S^1,\mathbb{R})$, if and only if the exponents are nonzero and E lies in the spectrum of the associated Schrödinger operator (E lies in the complement of the spectrum if and only if the cocycle is uniformly hyperbolic which for cocycles with values in $SL(2,\mathbb{R})$ is equivalent to domination; see also Ruelle [214], Bourgain [44] and Bourgain and Jitomirskaya [45]).

For $V \in C^r(S^1,\mathbb{R})$ with $r = \omega, \infty$, Avila and Krikorian [14] proved the following result on nonuniform hyperbolicity for Schrödinger cocycles: *if α satisfies the recurrent Diophantine condition (i.e, there are infinitely many $n > 0$ for which the nth image of α under the Gauss map satisfies the Diophantine condition with fixed constant and power) then for almost every E the Schrödinger cocycle either has nonzero Lyapunov exponents or is C^r-equivalent to a constant cocycle.*

For some C^1-genericity results on positivity of the maximal Lyapunov exponents see Sections 7.3.3 and 7.3.5.

13.6. **L^p-genericity for cocycles.** Let (X,μ) be a probability space and $f\colon X \to X$ a measure preserving automorphism. Consider the cocycle $\mathcal{A} : X \times \mathbb{Z} \to GL(n,\mathbb{R})$ over f generated by a measurable function $A\colon X \to GL(n,\mathbb{R})$. We endow the space $\mathcal{G}$ of these functions with a special L^p-like topology. Set for $1 \le p < \infty$,

$$\|A\|_p = \left(\int_X \|A(x)\|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

and

$$\|A\|_\infty = \operatorname{ess\,sup}_{x\in X} \|A\|.$$

We have $0 \le \|A\|_p \le \infty$. For $A, B \in \mathcal{G}$ let

$$\tau_p(A,B) = \|A - B\|_p + \|A^{-1} - B^{-1}\|_p$$

and

$$\rho_p(A,B) = \frac{\tau_p(A,B)}{1 + \tau_p(A,B)}.$$

Here we agree that $\|A - B\|_p = \infty$ or $\|A^{-1} - B^{-1}\|_p = \infty$ if and only if $\rho_p(A,B) = 1$. One can check that ρ_p is a metric on $\mathcal{G}$ and that the space $(\mathcal{G}, \rho_p)$ is complete.

Assume that f is ergodic. Following Arbieto and Bochi [11] we denote by $\mathcal{G}_{IC} \subset \mathcal{G}$ the subset of all maps A satisfying the integrability condition (5.3) and by $\mathcal{G}_{OPS}$ the subset of all those $A \in \mathcal{G}_{IC}$ which have one-point spectrum, i.e., for which the Lyapunov spectrum of the cocycle $\mathcal{A}$ consists of a single point. It turns out that the "one-point spectrum property" is typical in the following sense (see Arbieto and Bochi [11]; an earlier but weaker result is obtained by Arnold and Cong in [13]).

Theorem 13.11. *Assume that f is ergodic. Then $\mathcal{G}_{OPS}$ is a residual subset of $\mathcal{G}_{IC}$ in the L^p topology for any $1 \leq p \leq \infty$.*

The proof of this result is based upon the study of the functions $\Lambda_k \colon \mathcal{G}_{IC} \to \mathbb{R}$, $k = 1, \dots, n$ given by

$$\Lambda_k(A) = \int_X (\chi_1(A,x) + \cdots + \chi_k(A,x))\, d\mu(x)$$

Theorem 13.12. *The following statements hold:*

1. *the function Λ_k is upper semi-continuous (i.e., for any $A \in \mathcal{G}_{IC}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\Lambda_k(B) < \Lambda_k(A) + \varepsilon$ for any $B \in \mathcal{G}_{IC}$ with $\rho_p(A,B) < \delta$);*
2. *the function Λ_n is continuous;*
3. *if f is ergodic then Λ_k is continuous at $A \in \mathcal{G}_{IC}$ if and only if $A \in \mathcal{G}_{OPS}$.*

For some other results on genericity of cocycles with low differentiability see [42].

13.7. **Mixed hyperbolicity.** We consider the situation of *mixed* hyperbolicity, i.e., hyperbolicity is uniform throughout the manifold in some but not all directions. More precisely, we assume that f is partially hyperbolic, i.e., the tangent bundle TM is split into three df-invariant continuous subbundles

$$TM = E^s \oplus E^c \oplus E^u. \tag{13.5}$$

The differential df contracts uniformly over $x \in M$ along the *strongly stable* subspace $E^s(x)$, it expands uniformly along the *strongly unstable* subspace $E^u(x)$, and it can act either as nonuniform contraction or expansion with weaker rates along the *central* direction $E^c(x)$. More precisely, there exist numbers

$$0 < \lambda_s < \lambda_c' \leq 1 \leq \lambda_c'' < \lambda_u$$

such that for every $x \in M$,

$$\begin{aligned} \|d_x f(v)\| &\leq \lambda_s \|v\|, & v &\in E^s(x) \\ \lambda_c'\|v\| \leq \|d_x f(v)\| &\leq \lambda_c''\|v\|, & v &\in E^c(x) \\ \lambda_u\|v\| &\leq \|d_x f(v)\|, & v &\in E^u(x). \end{aligned}$$

We say that a partially hyperbolic diffeomorphism preserving a smooth measure μ has *negative central exponents* on a set A of positive measure if $\chi(x,v) < 0$ for every $x \in A$ and every nonzero $v \in E^c(x)$. The definition of *positive central exponents* is analogous. Partially hyperbolic systems with negative (positive) central exponents as explained above were introduced by Burns, Dolgopyat and Pesin [53] in connection to stable ergodicity of partially hyperbolic systems (see below). Their work is based upon earlier results of Alves, Bonatti and Viana [8] who studied SRB-measures for partially hyperbolic systems for which the tangent bundle is split into two invariant subbundles, one uniformly contracting and the other nonuniformly expanding (see Section 14.3).

For $x \in M$ one can construct local stable manifolds $V^s(x)$ and local unstable manifolds $V^u(x)$ and their sizes are bounded away from zero uniformly over $x \in M$. In addition, for $x \in A$ one can construct *local weakly stable manifolds* $V^{sc}(x)$ whose size varies with x and may be arbitrary close to zero.

Theorem 13.13 (Burns, Dolgopyat and Pesin [53])**.** *Let f be a C^2 diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that there exists an invariant subset $A \subset M$ with $\mu(A) > 0$ on which f has negative central exponents. Then every ergodic component of $f|A$ is open* (mod 0) *and so is the set A.*

To see this let us take a density point $x \in A$ and consider the sets

$$P(x) = \bigcup_{y \in V^{sc}} V^u(y), \quad Q(x) = \bigcup_{n \in \mathbb{Z}} f^n(P(x)). \tag{13.6}$$

$P(x)$ is open and so is $Q(x)$. Using absolute continuity of local unstable manifolds and repeating argument in the proof of Theorem 11.3 we obtain that $f|Q(x)$ is ergodic.

In general, one should not expect the set A to be of full measure nor the map $f|A$ to be ergodic. We introduce a sufficiently strong condition which guarantees this.

We call two points $p, q \in M$ *accessible* if there are points $p = z_0, z_1, \dots, z_{\ell-1}, z_\ell = q$, $z_i \in M$ such that $z_i \in V^u(z_{i-1})$ or $z_i \in V^s(z_{i-1})$ for $i = 1, \dots, \ell$. The collection of points $z_0, z_1, \dots, z_\ell$ is called a *us-path* connecting p and q. Accessibility is an equivalence relation. The diffeomorphism f is said to have the *accessibility property* if any two points p, $q \in M$ are accessible and to have the *essential accessibility property* if the partition into accessibility classes is trivial (i.e., a measurable union of equivalence classes must have zero or full measure).

A crucial manifestation of the essential accessibility property is that the orbit of almost every point $x \in M$ is dense in M. This implies the following result.

Theorem 13.14. *Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that f has negative central exponents on an invariant set A of positive measure and is essentially accessible. Then f has negative central exponents on the whole of M, the set A has full measure, f has nonzero Lyapunov exponents almost everywhere, and f is ergodic.*

Accessibility plays a crucial role in stable ergodicity theory. A C^2 diffeomorphism f preserving a Borel measure μ is called *stably ergodic* if any C^2 diffeomorphism g which is sufficiently close to f in the C^1 topology, which preserves μ, is ergodic. Volume-preserving Anosov diffeomorphisms are stably ergodic.

Theorem 13.15. *Under the assumption of Theorem 13.14, f is stably ergodic.*

One can show that indeed, f is stably Bernoulli, i.e., any C^2 diffeomorphism g which is sufficiently close to f in the C^1 topology, which preserves μ, is Bernoulli.

The proof of Theorem 13.15 is based upon some delicate properties of Lyapunov exponents for systems with mixed hyperbolicity which are of interest by themselves.

1. Since the map f is ergodic the values of the Lyapunov exponents are constant almost everywhere. Therefore,

$$\chi(x, v) \le a < 0 \tag{13.7}$$

uniformly over x and $v \in E^c(x)$. It follows that

$$\int_M \log \|df|E^c(x)\| \, d\mu \le a < 0.$$

Since the splitting (13.5) depends continually on the perturbation g of f we obtain that

$$\int_M \log \|df|E_g^c(x)\| \, d\mu \leq \frac{a}{2} < 0$$

(we assume that g is sufficiently close to f and preserves the measure μ). This, in turn, implies that g has negative central exponents on a set A_g of positive μ-measure.

2. Condition (13.7) allows one to estimate the sizes of global weakly stable manifolds along a typical trajectory of g.

Proposition 13.16. *Under the assumption* (13.7) *there is a number* $r > 0$ *such that for any* C^2 *diffeomorphism* g *which is sufficiently close to* f *in the* C^1 *topology and for any* $x \in A_g$ *one can find* $n \geq 0$ *such that the size of the global manifold* $W^{sc}(g^{-n}(x))$ *is at least* r.

The proof of this statement uses the notion of σ-hyperbolic times which is of interest by itself and provides a convenient technical tool in studying the behavior of local manifolds along trajectories. It was introduced by Alves in [6] (see also [8]) but some basic ideas behind this notion go back to the work of Pliss [204] and Mañé [174]. Given a partially hyperbolic diffeomorphism f and a number $0 < \sigma < 1$, we call the number n a *σ-hyperbolic time for f at x* if for every $0 \leq j \leq n$,

$$\prod_{k=1}^{j} \|df|E_f^c(f^{k-n}(x))\| \leq \sigma^j. \tag{13.8}$$

It is shown by Alves, Bonatti and Viana in [8] that if f satisfies (13.7) then any point $x \in A_f$ has infinitely many hyperbolic times. The proof of this statement is based on a remarkable result known as Pliss lemma. Although technical this lemma provides an important observation related to nonuniform hyperbolicity.

Lemma 13.17 (Pliss [204]; see also [174, Chapter IV.11])**.** *Let* $H \geq c_2 > c_1 > 0$ *and* $\zeta = (c_2 - c_1)/(H - c_1)$*. Given real numbers* $a_1, \dots, a_N$ *satisfying*

$$\sum_{j=1}^{N} a_j \geq c_2 N \text{ and } a_j \leq H \text{ for all } 1 \leq j \leq N,$$

there are $\ell > \zeta N$ *and* $1 < n_1 < \cdots < n_\ell \leq N$ *such that*

$$\sum_{j=n+1}^{n_j} a_j \geq c_1(N_i - n) \text{ for each } 0 \leq n < n_i,\ i = 1, \dots, \ell.$$

Alves and Araújo [7] estimated the frequency of σ-hyperbolic times. More precisely, given $\theta > 0$ and $x \in M$ we say that the frequency of σ-hyperbolic times $n_1 < n_2 < \cdots < n_\ell$ at x exceeds θ if for large n we have $n_\ell \leq n$ and $\ell \geq \theta n$. We also introduce the function h on M which is defined almost everywhere and assigns to $x \in M$ its first σ-hyperbolic time.

Theorem 13.18. *If for some* $\sigma \in (0, 1)$ *the function* h *is Lebesgue integrable then there are* $\hat{\sigma} > 0$ *and* $\theta > 0$ *such that almost every* $x \in M$ *has frequency of hyperbolic times bigger than* θ.

We return to the proof of the proposition. As we saw the map g also satisfies (13.7). Applying (13.8) to g we obtain that there is a number $r > 0$ such that for any σ-hyperbolic time n and $0 \le j \le n$,

$$\operatorname{diam}(g^j(B^{sc}(g^{-n}(x), r))) \le \sigma^j,$$

where $B^{sc}(y, r)$ is the ball in the global manifold $W^{sc}(y)$ centered at y of radius r. Since $\sigma < 1$ and the hyperbolic time n can be arbitrary large this ensures that $g^n(B^{sc}(g^{-n}(x), r))$ lies in the local manifold $V^{sc}(x)$ and hence, $B^{sc}(g^{-n}(x), r)$ is contained in the global manifold $W^{sc}(g^{-n}(x))$.

3. The perturbation g possesses the ε-*accessibility property* where $\varepsilon = d_{C^1}(f, g)$. This means that for any two points $p, q \in M$ there exists a us-path connecting p with the ball centered at q of radius ε. Although ε-accessibility is weaker then accessibility it still allows one to establish ergodicity of the perturbation g. Indeed, choosing a density point $x \in A_g$ and a number n such that the size of $W^{sc}(g^{-n}(x))$ is at least r we obtain that the set $P(x)$ (see (13.6)) contains a ball of radius $r \ge 2\varepsilon$.

13.8. **Open sets of diffeomorphisms with nonzero Lyapunov exponents.** It is shown in [40] that any partially hyperbolic diffeomorphism f_0 with one dimensional central direction preserving a smooth measure μ can be slightly perturbed such that the new map f is partially hyperbolic, preserves μ and has negative central exponents (hence, the results of the previous section apply to f). This result was first obtained by Shub and Wilkinson [229] in the particular case when f_0 is the direct product of a hyperbolic automorphism of two-torus and the identity map of the circle. The perturbation that remove zero exponents can be arranged in the form (13.3). The proof in the general case is a modification of the argument in [229] (see also [23] and [74]).

One can use this observation to obtain an open set of non-Anosov diffeomorphisms with nonzero Lyapunov exponents on multi-dimensional tori. Consider a diffeomorphism $f_0 = A \times Id$ of the torus $\mathbb{T}^p = \mathbb{T}^{p-1} \times S^1$, $p \ge 3$ where A is a linear hyperbolic automorphism of $\mathbb{T}^{p-1}$. It is partially hyperbolic and preserves volume. Let f be a small C^2 perturbation of f_0 preserving volume and having negative central exponents. One can arrange the perturbation f to have the accessibility property. Then any volume-preserving C^2 diffeomorphism g which is sufficiently close to f is ergodic and has nonzero Lyapunov exponents almost everywhere.

Note that g is partially hyperbolic and the central distribution E^c is one dimensional. By a result in [113] this distribution is integrable and the leaves W^c of the corresponding foliation are smooth closed curves which are diffeomorphic to circles. The foliation W^c is continuous (indeed, it is Hölder continuous) but is not absolutely continuous (see [229]). Moreover, there exists a set E of full measure and an integer $k > 1$ such that E intersects almost every leaf $W^c(x)$ at exactly k points (see [219]; the example in Section 10.2 is of this type).

14. SRB-measures

We shall consider hyperbolic invariant measures which are not smooth. This includes, in particular, dissipative systems for which the support of such measures is attracting invariant sets. A general hyperbolic measure may not have "nice" ergodic properties: its ergodic components may be of zero measure and it may have zero metric entropy. There is, however, an important class of hyperbolic measures known as SRB-measures (after Sinai, Ruelle, and Bowen). They appear naturally in applications due to the following observation.

Let f be a diffeomorphism of a smooth Riemannian p-dimensional manifold M. An open set $U \subset M$ is called a *trapping region* if $\overline{f(U)} \subset U$ (where $\bar{A}$ denotes the closure of the set A). The closed f-invariant set

$$\Lambda = \bigcap_{n \geq 0} f^n(U)$$

is an *attractor* for f so that f is dissipative in U.

Consider the evolution of the Riemannian volume m under f, i.e., the sequence of measures

$$\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k m, \tag{14.1}$$

where the measure $f_*^k m$ is defined by $f_*^k m(A) = m(f^{-1}(A))$ for any Borel set $A \subset F^k(U)$. Any limit measure ν of this sequence is supported on Λ. If indeed, the sequence (14.1) *converges* the limit measure ν is the *physical* (or *natural*) measure on Λ. The latter plays an important role in applications and is defined by the following property: for any continuous function φ on M, called *observable*, and m-almost every point $x \in U$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int_M \varphi \, d\nu. \tag{14.2}$$

We call ν an *SRB-measure* if there is a set $B = B(\nu) \subset U$ of positive Lebesgue measure such that for any continuous observable φ the identity (14.2) holds for $x \in B$ (in this case Λ is a *Milnor attractor*, see [184]). The set $B(\nu)$ is the *basin of attraction* of ν.

Assume that for m-almost every point $x \in U$ the Lyapunov exponents $\chi_i(x)$, $i = 1, \dots, p$ are not equal to zero. More precisely, there is a number $1 \leq k(x) < p$ such that $\chi_i(x) < 0$ for $i = 1, \dots, k(x)$ and $\chi_i(x) > 0$ for $i = k(x)+1, \dots, p$. It is not known whether *under this assumption the measure ν is hyperbolic*.

We stress that a physical measure need not be an SRB-measure as Example 12.5 of the figure-eight attractor shows. In the following sections we give another (equivalent) definition of SRB-measures in the case when these measures are hyperbolic. We also discuss their ergodic properties, and present some examples of systems with SRB-measures (for a somewhat less elaborated account of SRB-measures see [65]). In uniformly hyperbolic dynamics SRB-measures are examples of more general Gibbs measures (see the recent excellent survey on this topic by Ruelle [218]). It is an open problem to extend the theory of Gibbs measures to nonuniformly hyperbolic dynamical systems.

14.1. **Definition and ergodic properties of SRB-measures.** Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and ν a hyperbolic invariant measure for f. Denote by $\Lambda = \Lambda_\nu$ the set of points with nonzero Lyapunov exponents. We have that $\nu(\Lambda) = 1$. Fix a regular set Λ^ℓ of positive measure, a point $x \in \Lambda^\ell$, and a number $0 < r < r_\ell$ (see (8.16)). Set

$$R^\ell(x, r) = \bigcup_{y \in \Lambda^\ell \cap B(x,r)} V^u(y)$$

and denote by $\xi^\ell(x)$ the partition of $R^\ell(x,r)$ by local unstable manifolds $V^u(y)$, $y \in \Lambda^\ell \cap B(x,r)$.

A hyperbolic measure ν is called an *SRB-measure* if for every $\ell > 0$ and almost every $x \in \Lambda^\ell$, $y \in \Lambda^\ell \cap B(x,r)$, the conditional measure $\nu^u(y)$, generated by ν and partition $\xi^\ell(x)$ on $V^u(y)$, is absolutely continuous with respect to the Riemannian volume $m^u(y)$ on $V^u(y)$.

There is a measurable density function $\kappa(y,z)$, $z \in V^u(y)$ such that $d\nu^u(y)(z) = \kappa(y,z)\,dm^u(y)(z)$. The following result gives a description of the density function $\kappa(y,z)$.

Theorem 14.1. *For any $y \in \Lambda^\ell \cap B(x,r)$ and $z \in V^u(y)$,*

$$\kappa(y,z) = \prod_{i=1}^{\infty} \frac{J(df^{-1}|E^u(f^{-i}(z)))}{J(df^{-1}|E^u(f^{-i}(y)))}.$$

The density function $\kappa(y,z)$ is Hölder continuous and strictly positive.

SRB-measures have ergodic properties similar to those of smooth measures. The proofs of the corresponding results use Theorem 14.1 and are modifications of arguments in the case of smooth measures (those proofs in the latter case use only absolute continuity of local unstable manifolds), see [154].

Theorem 14.2. *There exist invariant sets Λ_0, Λ_1, ... such that:*

1. *$\bigcup_{i\ge 0} \Lambda_i = \Lambda$, and $\Lambda_i \cap \Lambda_j = \varnothing$ whenever $i \neq j$;*
2. *$\nu(\Lambda_0) = 0$, and $\nu(\Lambda_i) > 0$ for each $i \ge 1$;*
3. *$f|\Lambda_i$ is ergodic for each $i \ge 1$.*

Theorem 14.3. *There exists a measurable partition η of Λ with the following properties:*

1. *for almost every $x \in \Lambda$ the element $C_\eta(x)$ is open* (mod 0) *subset of $W^u(x)$;*
2. *$f\eta \ge \eta$;*
3. *$\eta^+ = \bigvee_{i=0}^{\infty} f^i\eta = \varepsilon$;*
4. *$\bigwedge_{i>-\infty}^{0} f^i\eta = \mathcal{H}(W^u) = \pi(f|\Lambda) = \nu$ (the trivial partition of Λ);*
5. *for each $i = 1,2,\dots$ the π-partition of $f|\Lambda_i$ is finite and consists of n_i elements Λ_i^k, $k = 1,\dots,n_i$ such that $f(\Lambda_i^k) = \Lambda_i^{k+1}$, $k = 1,\dots,n_i - 1$ and $f(\Lambda_i^n) = \Lambda_i^1$.*

Theorem 14.4.

1. *$f^{n_i}|\Lambda_i^k$ is a Bernoulli automorphism.*
2. *If the map $f|\Lambda$ is mixing then it is a Bernoulli automorphism.*
3. *(Ledrappier and Strelcyn* [156]*) The entropy of f is*

$$h_\nu(f) = h_\nu(f,\eta) = \int_M \sum_{i:\chi_i(x)>0} k_i(x)\chi_i(x)\,d\nu(x).$$

Let ν be an SRB-measure for a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and Λ the set of points with nonzero Lyapunov exponents. We have that $\nu(\Lambda) = 1$ and $V^u(x) \subset \Lambda \pmod 0$ for almost every $x \in \Lambda$. In view of the absolute continuity of local stable manifolds we obtain that the set $\bigcup_{x\in\Lambda} V^s(x)$ has positive volume. As an immediate corollary of this observation we have the following result.

Theorem 14.5. *A $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold possesses at most countably many ergodic SRB-measures. The basin of attraction of every SRB-measure has positive volume.*

14.2. **Characterization of SRB-measures.** It turns out that the entropy formula (see Statement 3 of Theorem 14.4) completely characterizes SRB-measures.

Theorem 14.6. *For a Borel measure ν invariant under a C^2 diffeomorphism, the entropy formula holds if and only if ν is an SRB-measure.*

This characterization was first established by Ledrappier [154] for systems with nonzero Lyapunov exponents and in the general case by Ledrappier and Young (see [157]; see also Section 16.1 for a discussion of this result). It is also shown in [157] that the Radon–Nikodym derivatives $d\nu^u(x)/dm^u(x)$ are strictly positive functions which are C^1 along unstable manifolds.

Qian and Zhu [210] extended the notion of SRB-measures to C^2 endomorphism via their inverse limits. They also established the entropy formula and the same characterization of SRB-measures as in the above theorem.

14.3. **Existence of SRB-measures I: some general results.** We describe here results on existence of SRB-measures in some general situations.

1. A topologically transitive Anosov diffeomorphism f possesses an ergodic SRB-measure: it is the limit of the sequence of measures (14.1). This result extends to uniformly hyperbolic attractors, i.e., attractors which are hyperbolic sets. For "almost Anosov" diffeomorphisms Hu [118] found conditions which guarantee existence of SRB-measures, while Hu and Young [120] described examples of such maps with no finite SRB-measures (see the articles [104, Section 3.6] and [65, Section 3] for relevant definitions and details).

2. More generally, consider a partially hyperbolic attractor Λ, i.e., an attractor such that $f|\Lambda$ is partially hyperbolic (see Section 9 in the Chapter "Partially hyperbolic dynamical systems" by B. Hasselblatt and Ya. Pesin in this volume [106]). Observe that $W^u(x) \subset \Lambda$ for every $x \in \Lambda$.

Let ν be an invariant Borel probability measure supported on Λ. Given a point $x \in \Lambda$, and a small number $r > 0$, set

$$R(x,r) = \bigcup_{y \in \Lambda \cap B(x,r)} V^u(y).$$

Denote by $\xi(x)$ the partition of $R(x,r)$ by $V^u(y)$, $y \in \Lambda \cap B(x,r)$. Following [203] we call ν a *u-measure* if for almost every $x \in \Lambda$ and $y \in \Lambda \cap B(x,r)$, the conditional measure $\nu^u(y)$, generated by ν and partition $\xi(x)$ on $V^u(y)$, is absolutely continuous with respect to $m^u(y)$.

Theorem 14.7 (Pesin and Sinai [203])**.** *Any limit measure of the sequence of measures* (14.1) *is a u-measure on* Λ.

Since partially hyperbolic attractors may not admit Markov partitions, the proof of this theorem exploits quite a different approach than the one used to establish existence of SRB-measures for classical hyperbolic attractors (see Section 19 where this approach is outlined).

In general, the sequence of measures (14.1) may not converge and some strong conditions are required to guarantee convergence.

Theorem 14.8 (Bonatti and Viana [41])**.** *Assume that:*

1. *every leaf of the foliation W^u is everywhere dense in Λ;*
2. *there exists a limit measure ν for the sequence of measures* (14.1) *with respect to which f has negative central exponents.*

Then the sequence of measures (14.1) *converges and the limit measure is the unique u-measure on Λ. It is an SRB-measure.*

Every SRB-measure on Λ is a u-measure. The converse statement is not true in general but it is true in the following two cases:

a. Λ is a (completely) hyperbolic attractor;
b. f has negative central exponents.

Here are two results in this direction.

Theorem 14.9 (Alves, Bonatti and Viana [8]). *Assume that f is nonuniformly expanding along the center-unstable direction, i.e.,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=1}^{n} \log \|df^{-1}|E^{cu}_{f^j(x)}\| < 0 \tag{14.3}$$

for all x in a positive Lebesgue measure set $A \subset M$. Then f has an ergodic SRB-measure supported in $\bigcap_{j=0}^{\infty} f^j(M)$. Moreover, if the limit in (14.3) *is bounded away from zero then A is contained* (mod 0) *in the union of the basins of finitely many SRB-measures.*

Theorem 14.10 (Burns, Dolgopyat and Pesin [53]). *Let ν be a u-measure on Λ. Assume that there exists an invariant subset $A \subset \Lambda$ with $\mu(A) > 0$ on which f has negative central exponents. Assume also that for every $x \in \Lambda$ the global unstable manifold $W^u(x)$ is dense in Λ. Then ν is the only u-measure for f and f has negative central exponents at ν-almost every $x \in \Lambda$; hence, (f, ν) is ergodic, ν is an SRB-measure and its basin contains the topological basin of Λ (mod 0).*

3. The following general statement links convergence of the sequence of measures (14.1) to the existence of SRB-measures.

Theorem 14.11 (Tsujii [241]). *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M and $A \subset M$ a set of positive volume such that for every $x \in A$ the sequence of measures*

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^k(x)}$$

converges weakly to an ergodic hyperbolic measure ν_x. If the Lyapunov exponents at x coincide with those of ν_x then ν_x is an SRB-measure for Lebesgue almost every $x \in A$.

4. In [259], Young suggested an axiomatic approach for constructing SRB-measures. It is built upon her work on tower constructions for nonuniformly hyperbolic systems and presents the system as a Markov extension (see Appendix). This approach is a basis to establish existence of SRB-measures for Hénon-type attractors as well as existence of absolutely continuous invariant measures for some piecewise hyperbolic maps and logistic maps.

Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M.

An embedded disk $\gamma \subset M$ is called an *unstable disk* if for any $x, y \in \gamma$ the distance $\rho(f^{-n}(x), f^{-n}(y)) \to 0$ exponentially fast as $n \to \infty$; it is called a *stable disk* if for any $x, y \in \gamma$ the distance $\rho(f^n(x), f^n(y)) \to 0$ exponentially fast as $n \to \infty$.

We say that a set Λ has a *hyperbolic product structure* if there exist a continuous family of unstable disks $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable disks $\Gamma^s = \{\gamma^s\}$ such that

1. $\dim \gamma^u + \dim \gamma^s = \dim M$;
2. the γ^u-disks are transversal to the γ^s-disks with the angles between them bounded away from zero;
3. each γ^u-disk meets each γ^s-disk at exactly one point;
4. $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$.

We impose some conditions on the map f (see Conditions (P1)–(P5) below) which guarantee the existence of an SRB-measure for f. Roughly speaking they mean that there exists a set Λ with a hyperbolic product structure and a return map f^R from Λ to itself such that f is a Markov extension over f^R (see Appendix). More precisely, we assume the following.

(P1) There exists $\Lambda \subset M$ with a hyperbolic product structure and such that $\mu_{\gamma^u}\{\gamma^u \cap \Lambda\} > 0$ for every $\gamma^u \in \Gamma^u$.

(P2) There are pairwise disjoint subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ with the properties that

1. each Λ_i has a hyperbolic product structure and its defining families can be chosen to be Γ^u and $\Gamma^s_i \subset \Gamma^s$; we call Λ_i an s-subset; similarly, one defines u-subsets;
2. on each γ^u-disk, $\mu_{\gamma^u}\{(\Lambda \setminus \cup\Lambda_i) \cap \gamma^u\} = 0$;
3. there exists $R_i \geq 0$ such that $f^{R_i}(\Lambda_i)$ is a u-subset of Λ; moreover, for all $x \in \Lambda_i$ we require that $f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x))$ and $f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x))$;
4. for each n, there are at most finitely many i's with $R_i = n$;
5. $\min R_i \geq R_0$ for some $R_0 > 0$ depending only on f.

Condition $(P2)$ means that the set Λ has the structure of a "horseshoe", however, infinitely many branches returning at variable times.

In order to state remaining Conditions (P3)–(P5) we assume that there is a function $s_0(x, y)$ – a *separation time* of the points x and y – which satisfy:

(i) $s_0(x, y) \geq 0$ and depends only on the γ^s-disks containing the two points;
(ii) the maximum number of orbits starting from Λ that are pairwise separated before time n is finite for each n;
(iii) for $x, y \in \Lambda$, $s_0(x, y) \geq R_i + s_0(f^{R_i}(x), f^{R_i}(y))$; in particular, $s_0(x, y) \geq R_i$;
(iv) for $x \in \Lambda_i$, $y \in \Lambda_j$, $i \neq j$ but $R_i = R_j$, we have $s_0(x, y) < R_i - 1$.

Conditions (iii) and (iv) describe the relations between $s_0(x, y)$ and returns to Λ, namely, that points in the same Λ_i must not separate before they return, while points in distinct Λ_i's must first separate if they are to return simultaneously.

We assume that there exist $C > 0$ and $\alpha < 1$ such that for all $x, y \in \Lambda$ the following conditions hold:

(P3) *contraction along γ^s-disks*: $\rho(f^n(x), f^n(y)) \leq C\alpha^n$ for all $n \geq 0$ and $y \in \gamma^s(x)$;

(P4) *backward contraction and distortion along* γ^u: for $y \in \gamma^u(x)$ and $0 \leq k \leq n < s_0(x, y)$, we have

(a) $\rho(f^n(x), f^n(y)) \leq C\alpha^{s_0(x,y)-n}$;
(b)
$$\log \prod_{i=k}^{n} \frac{\det df^u(f^i(x))}{\det df^u(f^i(y))} \leq C\alpha^{s_0(x,y)-n};$$

(P5) *convergence of $d(f^i|\gamma^u)$ and absolute continuity of Γ^s:*

(a) for $y \in \Gamma^s(x)$ and $n \geq 0$,

$$\log \prod_{i=k}^{\infty} \frac{\det df^u(f^i(x))}{\det df^u(f^i(y))} \leq C\alpha^n;$$

(b) for $\gamma, \gamma' \in \Gamma^u$ define $\Theta\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ by $\Theta(x) = \gamma^s(x) \cap \gamma'$. Then Θ is absolutely continuous and

$$\frac{d(\Theta_*^{-1}\mu_{\gamma'})}{d\mu_\gamma}(x) = \prod_{i=0}^{\infty} \frac{\det df^u(f^i(x))}{\det df^u(f^i(\Theta(x)))}.$$

In [259], Young showed that a map f satisfying Conditions (P1)–(P5) admits a Markov extension (see Appendix). As an important corollary one has the following result.

Theorem 14.12 (Young [259])**.** *Assume that for some $\gamma \in \Gamma^u$,*

$$\int_{\gamma\cap\Lambda} R\, d\mu_\gamma < \infty.$$

Then f admits an SRB-measure.

14.4. **Existence of SRB-measures II: Hénon attractors.** Constructing SRB-measures for nonuniformly hyperbolic dissipative systems is a challenging problem and few examples have been successfully studied.

In Section 19 we will discuss existence of SRB-measures for uniformly hyperbolic dissipative maps with singularities possessing generalized hyperbolic attractors. The behavior of trajectories in these systems is essentially nonuniformly hyperbolic.

An example of nonuniformly hyperbolic dissipative systems possessing SRB-measures is the Hénon map. It was introduced by Hénon in 1977 (see [108]) as a simplified model for the Poincaré first return time map of the Lorenz system of ordinary differential equations. The Hénon family is given by

$$H_{a,b}(x, y) = (1 - ax^2 + by, x).$$

Hénon carried out numerical studies of this family and suggested the presence of a "chaotic" attractor for parameter values near $a = 1.4$ and $b = 0.3$. Observe that for $b = 0$ the family $H_{a,b}$ reduces to the logistic family Q_a. By continuity, given $a \in (0, 2)$, there is a rectangle in the plane which is mapped by $H_{a,b}$ into itself. It follows that $H_{a,b}$ has an attractor provided b is sufficiently small. This attractor is called the *Hénon attractor*.

In the seminal paper [29], Benedicks and Carleson, treating $H_{a,b}$ as small perturbations of Q_a, developed highly sophisticated techniques to describe the dynamics near the attractor. Building on this analysis, Benedicks and Young [30] established existence of SRB-measures for the Hénon attractors and described their ergodic properties.

Theorem 14.13. *There exist $\varepsilon > 0$ and $b_0 > 0$ such that for every $0 < b \leq b_0$ one can find a set $\Delta_b \in (2 - \varepsilon, 2)$ of positive Lebesgue measure with the property that for each $a \in \Delta_b$ the map $H_{a,b}$ admits a unique SRB-measure $\nu_{a,b}$.*

In [31], Benedicks and Young studied ergodic properties of the measure $\nu_{a,b}$ showing that besides being Bernoulli this measure has exponential decay of correlations and satisfies a central limit theorem. More precisely, they proved the following result.

Let f be a transformation of a Lebesgue space X preserving a probability measure ν and $\mathfrak{L}$ be a class of functions on X. We say that f has *exponential decay of correlations* for functions in $\mathfrak{L}$ if there is a number $\tau < 1$ such that for every pair of functions $\varphi, \psi \in \mathfrak{L}$ there is a constant $C = C(\varphi, \psi) > 0$ such that for all $n \geq 0$,

$$\Big|\int \varphi(\psi \circ f^n)\, d\nu - \int \varphi\, d\nu \int \psi\, d\nu\Big| \leq C\tau^n$$

(see Appendix for more information on decay of correlations). Further, we say that f satisfies a *central limit theorem* for φ with $\int \varphi\, d\nu = 0$ if for some $\sigma > 0$ and all $t \in \mathbb{R}$,

$$\nu\Big\{\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ f^i < t\Big\} \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t} e^{-u^2/2\sigma^2}\, du$$

as $n \to \infty$.

Theorem 14.14. [31] *With respect to $\nu_{a,b}$ the map $H_{a,b}$*

1. *has exponential decay of correlations for Hölder continuous functions (the rate of decay may depend on the Hölder exponent);*
2. *satisfies the central limit theorem for Hölder continuous functions with zero mean; the standard deviation σ is strictly positive if and only if $\varphi \neq \psi \circ f - \psi$ for some $\psi \in L^2(\nu)$.*

In [246], Wang and Young introduced a 2-parameter family of maps of the plane to which the above results extend. This family is defined as follows.

Let $A = S^1 \times [-1, 1]$ and a 2-parameter family $T_{a,b} \colon A \to A$, $a \in [a_0, a_1]$, $b \in [0, b_1]$, be constructed via the following four steps.

Step I. Let $f \colon S^1 \to S^1$ satisfies the *Misiurewicz conditions*: if $C = \{x \colon f'(x) = 0\}$ then

1. $f'' \neq 0$ for all $x \in C$;
2. f has negative Schwarzian derivative on $S^1 \setminus C$;
3. $f^n(x) \neq x$ and $|(f^n)'(x)| \leq 1$ for any $x \in S^1$ and $n \in \mathbb{Z}$;
4. $\inf_{n \geq 0} d(f^n(x), C) > 0$ for all $x \in C$.

Observe that for $p \in S^1$ with $\inf_{n\geq 0} d(f^n(p), C) > 0$, and any g sufficiently close to f in the C^2 topology there is a unique point $p(g)$ having the same symbolic dynamics with respect to g as p does with respect to f. If f_a is a 1-parameter family through f with f_a sufficiently close to f in the C^2 topology for all a we let $p(a) = p(f_a)$. For $x \in C$ we denote by $x(a)$ the corresponding critical point of f_a.

Step II. Let $f_a \colon S^1 \to S^1$ be a 1-parameter family for which $f = f_{a^*}$ for some $a^* \in [a_0, a_1]$ with f as in Step I. We assume that the following *transversality condition* holds: for every $x \in C$ and $p = f(x)$,

$$\frac{d}{dx} f_a(x(a)) \neq \frac{d}{da} p(a) \text{ at } a = a^*.$$

Step III. Let $f_{a,b} \colon S^1 \times \{0\} \to A$ be a 2-parameter family which is an extension of the 1-parameter family in Step II, i.e., $f_{a,0} = f_a$ and $f_{a,b}$ is an embedding for $b > 0$.

Step IV. Let $T_{a,b}\colon A \to A$ be an extension of $f_{a,b}$ in such a way that $T_{a,0} \subset S^1 \to A$ and $T_{a,b}$ maps A diffeomorphically onto its image for $b > 0$. Assume also that the following *nondegeneracy condition* holds:

$$\partial_y T_{a^*,0}(x,0) \neq 0 \text{ whenever } f'_{a^*}(x) = 0.$$

For a version of this construction in higher dimensions see [247].

On another direction, Mora and Viana [187] modified Benedicks and Carleson's approach in a way which allowed them to treat Hénon-like maps using some techniques from the general bifurcation theory such as homoclinic tangencies. Later Viana [244] extended results from [187] to higher dimensions; see [169] for a more detailed account of these results and further references.

15. Hyperbolic measures I: topological properties

One can extend some techniques widely used in the theory of locally maximal hyperbolic sets to measures with nonzero exponents. These tools are not only important for applications but they provide a crucial nontrivial geometric structure to measures with nonzero exponents. In particular, one can close recurrent orbits, shadow pseudo-orbits, construct almost Markov covers, and determine the cohomology class of Hölder cocycles by periodic data.

Let f be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold M, for some $\alpha > 0$, and ν an f-invariant hyperbolic probability measure.

15.1. **Closing and shadowing lemmas.**

We address the following two fundamental problems:

1. Given a recurrent point x is it possible to find a nearby periodic point y which follows the orbit of x during the period of time that the points in the orbit of x return very close to x?
2. Given a sequence of points $\{x_n\}$ with the property that the image of x_n is very close to x_{n+1} for every n (such a sequence $\{x_n\}$ is called a *pseudo-orbit*), is it possible to find a point x such that $f^n(x)$ is close to x_n for every i? In other words, if a sequence of points $\{x_n\}$ resembles an orbit can one find a real orbit that shadows (or closely follows) the pseudo-orbit?

This sort of problems are known respectively as *closing problem* and *shadowing problem*, while the corresponding properties are called the *closing property* and the *shadowing property.*

The following result by Katok [131] establishes the closing property for nonuniformly hyperbolic diffeomorphisms.

Theorem 15.1. *For every $\ell > 0$ and $\eta > 0$ there exists $\delta = \delta(\ell,\eta) > 0$ with the following property: if $x \in \Lambda^\ell$ and $f^m(x) \in \Lambda^\ell$ with $d(f^m(x),x) < \delta$, then there exists $z = z(x)$ such that*

1. *z is a hyperbolic periodic point for f with $f^m(z) = z$;*
2. *for $i = 0, \dots, m$,*

$$d(f^i(z), f^i(x)) \leq \eta A_\ell \max\{e^{\varepsilon i}, e^{\varepsilon(m-i)}\},$$

where A_ℓ is a constant depending only on ℓ.

An immediate corollary of this result is the existence of periodic orbits in a regular set Λ^ℓ of a nonuniformly hyperbolic diffeomorphism. In fact, a stronger result holds. Denote by $\mathrm{Per}_h(f)$ the set of hyperbolic periodic points for f.

Theorem 15.2 (Katok [131])**.** *We have* $\operatorname{supp}\nu \subset \overline{\operatorname{Per}_h(f)}$.

The proof of Theorem 15.2 is an application of Theorem 15.1. Fix $x_0 \in \operatorname{supp}\mu$, $\alpha > 0$ and $\ell \geq 1$ such that $\mu(B(x_0, \alpha/2) \cap \Lambda^\ell) > 0$. Choose $\delta > 0$ according to Theorem 15.1 and such that $\eta A_\ell < \alpha/2$ and a set $B \subset B(x_0, \alpha/2) \cap \Lambda^\ell$ of positive measure and diameter at most δ. By the Poincaré Recurrence Theorem, for μ-almost every $x \in B$ there exists a positive integer $n(x)$ such that $f^{n(x)}(x) \in B$ and hence, $d(f^{n(x)}(x), x) < \delta$. By Theorem 15.1, there exists a hyperbolic periodic point z of period $n(x)$ such that $d(x, z) < \alpha/2$, and thus $d(x_0, z) < d(x_0, x) + d(x, z) < \alpha$.

A further application of Theorem 15.1 is the following statement.

Corollary 15.3. *For an ergodic measure ν, if all the Lyapunov exponents of f are negative (respectively, all are positive) on a set of full ν-measure, then* $\operatorname{supp}\nu$ *is an attracting (respectively, repelling) periodic orbit.*

See also Corollaries 15.7, 15.15, and 15.16 below for related results.

We now present an analog of the shadowing lemma for nonuniformly hyperbolic diffeomorphisms. Given $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z} \cup \{\infty\}$, a sequence $\{x_n\}_{a<n<b}$ is called an *ε-orbit* or *ε-pseudo-orbit* for f if $d(x_{n+1}, x_n) < \varepsilon$ for all $a < n < b$. It is *δ-shadowed* by the orbit of x if $d(x_n, f^n(x)) < \delta$ for all $a < n < b$.

Given $\ell > 0$, denote by

$$\widetilde{\Lambda}^\ell = \bigcup_{x \in \Lambda^\ell} R(x),$$

where $R(x)$ is a regular neighborhood of x (see Section 8.7).

Theorem 15.4 (Katok and Mendoza [135])**.** *For every sufficiently small $\alpha > 0$ there exists $\beta = \beta(\alpha, \ell)$ such that given a β-pseudo-orbit $\{x_m\} \subset \widetilde{\mathfrak{R}}^\ell$, there exists $y \in M$ such that its orbit α-shadows $\{x_m\}$.*

The following result is a nonuniformly hyperbolic version of the famous Livshitz theorem that determines the cohomology class of Hölder cocycles by periodic data.

Theorem 15.5 (Katok and Mendoza [135])**.** *Let $\varphi\colon M \to \mathbb{R}$ be a Hölder continuous function such that for each periodic point p with $f^m(p) = p$ we have $\sum_{i=0}^{m-1} \varphi(f^i(p)) = 0$. Then there exists a Borel measurable function h such that for ν-almost every x,*

$$\varphi(x) = h(f(x)) - h(x).$$

15.2. **Continuous measures and transverse homoclinic points.** In the neighborhood of any transverse homoclinic point there exists a hyperbolic horseshoe, that is, a (uniformly) hyperbolic invariant set obtained by a horseshoe-like construction (see, for example, [133, Theorem 6.5.5]). This phenomenon persists under small perturbations. It turns out that transverse homoclinic points are present whenever the diffeomorphism possesses hyperbolic continuous measures.

Theorem 15.6 ([131])**.** *Let ν be a continuous and nonatomic Borel invariant measure. Then*

1. $\operatorname{supp}\nu$ *is contained in the closure of the set of hyperbolic periodic points that have transverse homoclinic points;*
2. *if ν is ergodic, then* $\operatorname{supp}\nu$ *is contained in the closure of the set of transverse homoclinic points of exactly one hyperbolic periodic point.*

Let $P_m(f)$ be the number of periodic points of f of period m.

Corollary 15.7. *Let ν be a continuous and nonatomic Borel invariant measure. Then f has a compact f-invariant set $\Lambda \subset M$ such that*

1. $$\overline{\lim_{m\to\infty}} \frac{1}{m} \log P_m(f) \ge h(f|\Lambda) > 0; \tag{15.1}$$
2. *Λ is a horseshoe for f, i.e., Λ is a (uniformly) hyperbolic set for f and $f|\Lambda$ is topologically conjugate to a topological Markov chain.*

In particular, $h(f) > 0$ whenever there exists a continuous nonatomic hyperbolic invariant measure. One can strengthen Theorem 15.6 and obtain a Spectral Decomposition Theorem for hyperbolic measures.

Theorem 15.8 ([135])**.** *For each $\ell > 0$, the Pesin set Λ^ℓ can be decomposed into finitely many closed f-invariant sets Λ_i such that for each i there exists $x_i \in M$ with $\Lambda_i \subset \overline{\{f^n(x_i) : n \in \mathbb{Z}\}}$.*

Set now

$$\chi(x) = \min\{|\chi_i(x)| : 1 \le i \le s\},$$

where $\chi_i(x)$ are the values of the Lyapunov exponent at x. If ν is an ergodic hyperbolic measure, then $\chi(x) = \chi_\nu$, where χ_ν is a nonzero constant.

Theorem 15.9 ([135])**.** *Let ν be ergodic. If $x \in \operatorname{supp} \nu$, then for any $\rho > 0$, any neighborhoods V of x and W of $\operatorname{supp} \nu$, and any continuous functions $\varphi_1, \ldots, \varphi_k$, there exists a hyperbolic periodic point $z \in V$ such that:*

1. *the orbit of z is contained in W;*
2. *$\chi(z) \ge \chi_\nu - \rho$;*
3. *if $m(z)$ is the period of z, then for $i = 1, \ldots, k$,*

$$\left| \frac{1}{m(z)} \sum_{k=0}^{m(z)-1} \varphi_i(f^k(x)) - \int_M \varphi_i \, d\nu \right| < \rho.$$

Theorem 15.9 has the following consequence.

Corollary 15.10. *If $\{f_n\}_{n\ge1}$ is a sequence of $C^{1+\alpha}$ diffeomorphisms converging to f in the C^1 topology, then for each $n \ge 1$, f_n has a hyperbolic invariant probability measure ν_n such that $\{\nu_n\}_{n\ge1}$ converges weakly to ν. Furthermore, ν_n may be chosen such that $\operatorname{supp} \nu_n \subset \operatorname{Per}_h(f_n)$ for each $n \ge 1$.*

An application of Corollary 15.10 to a constant sequence of diffeomorphisms yields the following result.

Corollary 15.11. *For a $C^{1+\alpha}$ diffeomorphism $f \colon M \to M$ of a compact smooth manifold one of the following mutually exclusive alternatives holds:*

1. *the measures supported on hyperbolic periodic points are weakly dense in the set of hyperbolic measures;*
2. *there are no hyperbolic measures (and hence, there are no hyperbolic periodic points).*

Corollaries 15.10 and 15.11 suggest a "weak stability" of hyperbolic measures.

The estimate (15.1) can be strengthened in the following way to become somewhat a multiplicative estimate.

Theorem 15.12 (Ugarcovici [242]). *Assume that $h_\nu(f) > 0$. If ν is not a locally maximal ergodic measure in the class of f-invariant ergodic measures then there exist multiplicatively enough periodic orbits which are equidistributed with respect to ν. In other words, for any $r > 0$ and any collection of continuous functions $\varphi_1, \dots, \varphi_k$ there exist a sequence $n_m \to \infty$ and sets $P_{n_m} = P_{n_m}(r, \varphi_1, \dots, \varphi_k)$ of periodic orbits of period n_m such that for any $z \in P_{n_m}$,*

$$\left| \frac{1}{n_m} \sum_{i=1}^{n_m} \varphi_i(f^l(z)) - \int \varphi_i \, d\nu \right| < r$$

and

$$\overline{\lim_{m\to\infty}} \frac{\operatorname{card} P_{n_m}}{e^{n_m h_\nu(f)}} \geq 1.$$

15.3. **Entropy, horseshoes, and periodic points.** Recall that a set Λ is a horseshoe for a diffeomorphism f if there exist s, k and sets $\Lambda_0, \dots, \Lambda_{k-1}$ such that $\Lambda = \Lambda_0 \cup \dots \cup \Lambda_{k-1}$, $f^k(\Lambda_i) = \Lambda_i$, $f(\Lambda_i) = \Lambda_{i+1} \mod k$, and $f^k|\Lambda_0$ is conjugate to a full shift in s symbols. For a horseshoe Λ we set

$$\chi(\Lambda) = \inf\{\chi_\nu : \operatorname{supp} \nu \text{ is a periodic orbit on } \Lambda\}.$$

Theorem 15.13 (Katok and Mendoza [135]). *Assume that ν is ergodic and $h_\nu(f) > 0$. Then for any $\varepsilon > 0$ and any continuous functions $\varphi_1, \dots, \varphi_k$ on M, there exists a hyperbolic horseshoe Λ such that:*

1. $h(f|\Lambda) > h_\nu(f) - \varepsilon$;
2. *Λ is contained in an ε-neighborhood of* $\operatorname{supp} \nu$;
3. $\chi(\Lambda) > \chi_\nu - \varepsilon$;
4. *there exists a measure ν_0 supported on Λ such that for $i = 1, \dots, k$,*

$$\left| \int_M \varphi_i \, d\nu_0 - \int_M \varphi_i \, d\nu \right| < \delta.$$

We outline the proof of this result. Given $\ell \geq 1$, let ζ be a finite measurable partition of M refining the partition $\{\Lambda^\ell, M \setminus \Lambda^\ell\}$. Fix $r > 0$. For each $m \geq 1$, let Λ^ℓ_m be the set of points $x \in \Lambda^\ell$ such that $f^q(x) \in \zeta(x)$ for some $q \in [m, (1+r)m]$, and

$$\left| \frac{1}{s} \sum_{j=0}^{s-1} \varphi_i(f^j(x)) - \int_M \varphi \, d\nu \right| < \frac{r}{2}$$

for $s \geq m$ and $i = 1, \dots, k$. Using Birkhoff's Ergodic Theorem, one can show that $\nu(\Lambda^\ell_m) \to \nu(\Lambda^\ell)$ as $m \to \infty$. From now on we choose m such that $\nu(\Lambda^\ell_m) > \nu(\Lambda^\ell) - r$. Given $\delta > 0$, there exists a cover $\{R(x_1), \dots, R(x_t)\}$ of Λ^ℓ by closed rectangles (with $x_i \in \Lambda^\ell_m$) and numbers $\lambda \in (0, 1)$, satisfying $e^{-\chi_\nu - \delta} < \lambda < e^{-\chi_\nu + \delta}$, and $\gamma > 0$ such that

1. $\Lambda^\ell \subset \bigcup_{i=1}^t B(x_i, \delta)$, with $B(x_i, \delta) \subset \operatorname{int} R(x_i)$ for each i;
2. $\operatorname{diam} R(x_i) < r$ for each i;
3. if $x \in \Lambda^\ell \cap B(x_i, \delta)$ and $f^m(x) \in \Lambda^\ell \cap B(x_j, \delta)$ for some $m > 0$, then the connected component $\mathcal{C}(R(x_i) \cap f^{-m}(R(x_j)), x)$ of $R(x_i) \cap f^{-m}(R(x_j))$ containing x is an admissible (s, γ)-rectangle in $R(x_i)$ and $f^m(\mathcal{C}(R(x_i) \cap f^{-m}(R(x_j)), x))$ is an admissible (u, γ)-rectangle in $R(x_j)$;
4. for $k = 0, \dots, m$,

$$\operatorname{diam} f^k(\mathcal{C}(R(x_i) \cap f^{-m}(R(x_j)), x)) \leq 3 \operatorname{diam} R(x_i) \max\{\lambda^k, \lambda^{m-k}\}$$

Here an *admissible* (s,γ)*-rectangle* is the set of points

$$\{(v,u)\in[-h,h]^2 : u=\theta\psi_1(v)+(1-\theta)\psi_2(v), 0\le\theta\le 1\}$$

where ψ_1 and ψ_2 are two (s,γ)-curves (for some $h\le 1$ and some appropriate parametrization in each Lyapunov chart; see Section 8.2). The definition of (u,γ)-rectangles is analogous. The cover can be easily obtained from the behavior of (s,γ)- and (u,γ)-curves under iteration by f, and by using Theorem 15.1 to establish the last property.

Let $E_m\subset\Lambda^\ell_m$ be an (m,ε)-separated set of maximal cardinality. By the Brin–Katok formula for the metric entropy, there exist infinitely many m such that $\operatorname{card}E_m\ge e^{m(h_\nu(f)-r)}$. For each $q\in[m,(1+r)m]$, let $V_q=\{x\in E_m : f^q(x)\in\zeta(x)\}$ and let n be the value of q that maximizes $\operatorname{card}V_q$. Since $e^{mr}>mr$ we have $\operatorname{card}V_n\ge e^{m(h_\nu(f)-3r)}$. Consider now the value j for which $\operatorname{card}(V_n\cap R(x_j))$ is maximal. Then

$$\operatorname{card}(V_n\cap R(x_j))\ge\frac{1}{t}\operatorname{card}V_n\ge\frac{1}{t}e^{m(h_\nu(f)-3r)}. \tag{15.2}$$

Each point $x\in V_n\cap R(x_j)$ returns to the rectangle $R(x_j)$ in n iterations, and thus $\mathcal{C}(R(x_j)\cap f^n(R(x_j)),f^n(x))$ is an admissible (u,γ)-rectangle in $R(x_j)$ and $f^{-n}(\mathcal{C}(R(x_j)\cap f^n(R(x_j)),f^n(x))$ is an admissible (s,γ)-rectangle in $R(x_j)$. This follows from the fact that $d(x_j,x)<\delta$ and $d(f^n(x),x_j)<\delta$, and from Property 2 of the cover. If $y\in\mathcal{C}(R(x_j)\cap f^{-n}(R(x_j)),x)$ then by the last property of the cover, $d(f^i(x),f^i(y))\le 3r$ for $i=0,\ldots,n$. This implies that given a point $y\in\mathcal{C}(R(x_j)\cap f^{-n}(R(x_j)),x)\setminus\{x\}$, we must have $y\notin V_n$; otherwise it would contradict the separability of V_n. Hence, there exist $\operatorname{card}V_n$ disjoint admissible (s,γ)-rectangles mapped by f^n onto $\operatorname{card}V_n$ admissible (u,γ)-rectangles.

Let

$$\Lambda(m)=\bigcup_{l\in\mathbb{Z}}f^{nl}\left(\bigcup_{x\in V_n\cap R(x_j)}\mathcal{C}(R(x_j)\cap f^{-n}(R(x_j)),x)\right).$$

The map $f^n|\Lambda(m)$ is conjugate to the full shift on $\operatorname{card}(V_n\cap R(x_j))$ symbols. Now observe that for each $y\in\Lambda(m)$ its orbit remains in the union of the regular neighborhoods $R(x_j),\ldots,R(f^n(x_j))$, and thus $f^n|\Lambda(m)$ is a hyperbolic horseshoe.

The entropy of $f^n|\Lambda(m)$ equals $\log\operatorname{card}(V_n\cap R(x_j))$. By (15.2),

$$h(f|\Lambda(m))=\frac{1}{n}\log\operatorname{card}(V_n\cap R(x_j))\ge\frac{1}{n}\log\frac{1}{t}e^{m(h_\nu(f)-3r)}.$$

Since $m/n>1/(1+r)$, we obtain the desired properties.

The following are immediate consequences of Theorem 15.13.

Corollary 15.14. *Assume that ν is ergodic and $h_\nu(f)>0$. There exists a sequence of f-invariant measures ν_n supported on hyperbolic horseshoes Λ_n such that:*

1. *$\nu_n\to\nu$ in the weak* topology;*
2. *if $h_\nu(f)>0$ then $h_{\nu_n}(f)\to h_\nu(f)$.*

Corollary 15.15. *Assume that ν is ergodic and $h_\nu(f)>0$. Given $\varepsilon>0$,*

$$h_\nu(f)\le\varlimsup_{m\to\infty}\frac{1}{m}\log^+\operatorname{card}\{x\in M : f^m(x)=x \text{ and } \chi(x)\ge\chi(\nu)-\varepsilon\}.$$

In particular,

$$h(f)\le\varlimsup_{m\to\infty}\frac{1}{m}\log^+P_m(f).$$

In the two-dimensional case, any measure with positive entropy is hyperbolic (see Corollary 12.8). Therefore, Corollary 15.15 implies the following relation between periodic points and topological entropy.

Corollary 15.16. *For any $C^{1+\alpha}$ diffeomorphism f of a two-dimensional manifold,*

$$h(f) \le \overline{\lim_{m\to\infty}} \frac{1}{m} \log^+ P_m(f). \tag{15.3}$$

In the multi-dimensional case, the inequality (15.3) does not hold for arbitrary diffeomorphisms.

The following result shows that hyperbolic measures persist under C^1 perturbations. This is a consequence of the structural stability of hyperbolic measures.

Corollary 15.17. *Assume that ν is ergodic and $h_\nu(f) > 0$. Given $C^{1+\alpha}$ diffeomorphisms f_n for each $n \ge 1$ such that f_n converges to f in the C^1 topology, there exist f_n-invariant ergodic measures ν_n satisfying the following properties:*

1. *$\nu_n \to \nu$ in the weak topology;*
2. *$h_{\nu_n}(f_n) \to h_\nu(f)$;*
3. *$\chi_{\nu_n} \to \chi_\nu$.*

15.4. **Continuity properties of entropy.** It follows from Theorem 15.13 that

$$h(f) = \sup\{h(f|\Lambda) : \Lambda \text{ is a hyperbolic horseshoe}\}.$$

The following two results describe continuity-like properties of topological and metric entropies on the space of diffeomorphisms. The first result deals with diffeomorphisms of class $C^{1+\alpha}$ and follows from Corollary 15.17 and the structural stability of horseshoes.

Theorem 15.18. *The topological entropy on the space of $C^{1+\alpha}$ diffeomorphisms of a given surface is lower-semicontinuous.*

The second result deals with C^∞ diffeomorphisms.

Theorem 15.19. *For a C^∞ map $f\colon M \to M$ of a compact manifold*

1. *the map $\mu \mapsto h_\mu(f)$ is upper-semicontinuous on the space of f-invariant probability measures on M;*
2. *the map $f \mapsto h(f)$ is upper-semicontinuous.*

The first statement is due to Newhouse [189] and the second one was established independently by Newhouse [189] and Yomdin [253]. We refer to [189] for references in the case of interval maps. It follows from Theorems 15.18 and 15.19 that the topological entropy is continuous for C^∞ diffeomorphisms of a given compact surface.

15.5. **Yomdin-type estimates and the entropy conjecture.** In [227, §V], Shub conjectured that for any C^1 map $f\colon M \to M$ of a compact manifold,

$$h(f) \ge \log \sigma(f_*), \tag{15.4}$$

where $f_*\colon H_*(M,\mathbb{R}) \to H_*(M,\mathbb{R})$ is the linear map induced by f on the total homology of M,

$$H_*(M,\mathbb{R}) = \bigoplus_{i=0}^{\dim M} H_i(M,\mathbb{R})$$

and

$$\sigma(f_*) = \lim_{n\to\infty} \|f_*^n\|^{1/n} = \max\{\sigma(f_{*i}) : i = 0, \dots, \dim M\}$$

is the spectral radius of f_*. This is referred to as the *entropy conjecture*. For a $C^{1+\alpha}$ diffeomorphism f one could use (15.4) if available to establish positivity of the topological entropy and hence, existence of a measure with some positive Lyapunov exponents and the associated nontrivial stochastic behavior (see Section 15.2). We give here an account of the results in the direction of the conjecture (see also the survey by Katok [129] for the status of the conjecture prior to 1986).

In the case of the first homology $f_{*1}\colon H_1(M,\mathbb{R}) \to H_1(M,\mathbb{R})$ we have the following result for arbitrary continuous maps.

Theorem 15.20 (Manning [176]). *If f is a continuous map of a smooth compact manifold then $h(f) \geq \log\sigma(f_{*1})$.*

There exists a stronger version of Theorem 15.20 due to Katok [129] with the number $\log\sigma(f_{*1})$ replaced by the so-called algebraic entropy of the action induced by f on the (not necessarily commutative) fundamental group $\pi_1(M)$. It follows from Theorem 15.20 and Poincaré duality that the entropy conjecture holds for any homeomorphism of a manifold M with $\dim M \leq 3$ (see [176]).

In the case of the top homology group the following result holds (recall that $f_{*\dim M}$ is the same as multiplication by the degree $\deg f$).

Theorem 15.21 (Misiurewicz and Przytycki [185]). *If f is a C^1 map of a compact smooth manifold, then $h(f) \geq \log|\deg f|$.*

In particular, this implies that the entropy conjecture holds for any smooth map of a sphere (in any dimension) and any smooth map of a compact manifold with dimension at most 2.

On some manifolds the entropy conjecture turns out to hold for arbitrary continuous maps.

Theorem 15.22 (Misiurewicz and Przytycki [186]). *The entropy conjecture holds for any continuous map of a torus (in any dimension).*

Since any Anosov automorphism of the torus is topologically conjugate to an algebraic automorphism (see [133, Theorem 18.6.1]), we conclude that if f is an Anosov diffeomorphism of a torus, then $h(f) = \log\sigma(f_*)$.

Shub formulated the entropy conjecture in connection with the problem of defining the simplest diffeomorphisms in each isotopy class of diffeomorphisms. From this point of view, it is important to discuss the entropy conjecture for example for structurally stable diffeomorphisms. In [228], Shub and Sullivan described an open and dense subset (in the C^0-topology) of the set of structurally stable diffeomorphisms for which the entropy conjecture holds. Later Shub and Williams obtained a more general result which does no require the nonwandering set to have zero dimension.

Theorem 15.23 (Shub and Williams [230]). *The entropy conjecture holds for any axiom A no-cycles diffeomorphism.*

More recently Yomdin established the C^∞ version of the entropy conjecture with an approach using semi-algebraic geometry.

Theorem 15.24 (Yomdin [253, 254]). *The entropy conjecture holds for any C^∞ map of a compact manifold.*

Yomdin also proved more generally that for a C^k map $f\colon M \to M$ of a compact manifold, with $1 \le k \le \infty$, and $j = 0, \dots, \dim M$,

$$h(f) + \frac{j}{k} \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in M} \|d_x f^n\| \ge v_j(f) \ge \log \sigma(f_{*j}).$$

Here $v_j(f)$ is the exponential growth rate of j-volumes,

$$v_j(f) = \sup \overline{\lim_{n\to\infty}} \frac{1}{n} \log \operatorname{vol}(f^n(A)),$$

where the supremum is taken over all submanifolds $A \subset M$ of dimension j, and the volume is counted with multiplicities. Newhouse proved earlier in [188] that $h(f) \le \max_j v_j(f)$ for a $C^{1+\alpha}$ map of a compact manifold. In particular, we have $h(f) = \max_j v_j(f) = \log \sigma(f_*)$ for a C^∞ map.

16. Hyperbolic measures II: entropy and dimension

16.1. **Entropy formula.** We describe results of Ledrappier and Young [157, 158] including the general formula for the entropy of a diffeomorphism. Let f be a C^2 diffeomorphism of a compact smooth Riemannian manifold M preserving a Borel measure on M. For a regular point $x \in M$ and $i = 1, \dots, u(x) = \max\{i : \lambda_i(x) > 0\}$, consider the ith-unstable global manifold $W_i(x)$ of f at x (see Section 9.2). We introduce the notion of the entropy "along" the W_i-foliation.

For $n > 0$, and $\varepsilon > 0$ set

$$V_i(x, n, \varepsilon) = \{y \in W_i(x) : \rho_{W_i}(f^k(x), f^k(y)) < \varepsilon \text{ for } 0 \le k < n\}.$$

Consider a measurable partition ξ of M. We say that ξ is *subordinate* to the W_i-foliation if for ν-almost every $x \in M$ we have $\xi(x) \subset W_i(x)$ and $\xi(x)$ contains an open neighborhood of x in the topology of $W_i(x)$. Let $\{\nu_i(x)\}$ be the system of conditional measures associated with ξ. Define

$$\underline{h}_i(x, \xi) = \lim_{\varepsilon\to 0} \underline{\lim_{n\to\infty}} -\frac{1}{n} \log \nu_i(x)(V_i(x, n, \varepsilon)),$$

$$\overline{h}_i(x, \xi) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} -\frac{1}{n} \log \nu_i(x)(V_i(x, n, \varepsilon)).$$

Theorem 16.1 ([158])**.** *The following properties hold:*

1. $h_i(x) := \underline{h}_i(x, \xi) = \overline{h}_i(x, \xi)$ *for ν-almost every $x \in M$, independently of the choice of the partition ξ;*
2. $\int_M h_{u(x)}(x)\, d\nu(x) = h_\nu(f)$.

The number $h_i(x)$ is called the *local entropy* of f at x *along the W_i-foliation.*

We also consider the pointwise dimension of conditional measures "along" the W^i-foliation. Let $B_i(x, r)$ be the ball in $W_i(x)$ centered at x of radius ε and ξ a measurable partition subordinate to W_i. For a regular point $x \in M$ define

$$\underline{d}^i_\nu(x, \xi) = \underline{\lim_{r\to 0}} \frac{\log \nu_i(x)(B_i(x, r))}{\log r}, \quad \overline{d}^i_\nu(x, \xi) = \overline{\lim_{r\to 0}} \frac{\log \nu_i(x)(B_i(x, r))}{\log r}.$$

Theorem 16.2 (Ledrappier and Young [158])**.** *The following properties hold:*

1. $d^i_\nu(x) := \underline{d}^i_\nu(x, \xi) = \overline{d}^i_\nu(x, \xi)$ *for ν-almost every $x \in M$, independently of the choice of the partition ξ;*
2. $0 \le d^i_\nu(x) - d^{i-1}_\nu(x) \le \dim E_i(x)$ *for* $2 \le i \le u(x)$.

The number $d^i_\nu(x)$ is called the *pointwise dimension* of ν *along the* W_i*-foliation*. The number $d^i_\nu(x) - d^{i-1}_\nu(x)$ can be interpreted as a "transverse dimension" of ν on the quotient W_i/W_{i-1} (recall that each leaf of W_i is foliated by leaves of W_{i-1}).

In the particular cases $i = s$ and $i = u$ the quantities

$$d^s_\nu(x) \stackrel{\text{def}}{=} \lim_{r\to 0} \frac{\log \nu^s_x(B^s(x,r))}{\log r}, \quad d^u_\nu(x) \stackrel{\text{def}}{=} \lim_{r\to 0} \frac{\log \nu^u_x(B^u(x,r))}{\log r}$$

are called *stable* and *unstable local (pointwise) dimensions* of ν. They are well-defined for almost every $x \in M$ and are constant almost everywhere; we denote these constants by d^s_ν and d^u_ν. Set also $d^0_\nu(x) = 0$.

Theorem 16.3 (Ledrappier and Young [158]). *The metric entropy of a C^2 diffeomorphism f is expressed by the following formula*

$$h_\nu(f) = \int_M \sum_{i=1}^{u(x)} \lambda_i(x)(d^i_\nu(x) - d^{i-1}_\nu(x))\, d\nu(x).$$

The proof goes by showing that for ν-almost every $x \in M$ and $i = 2, \ldots, u(x)$,

$$h_1(x) = \lambda_1(x) d^1_\nu(x), \quad h_i(x) - h_{i-1}(x) = \lambda_i(x)(d^i_\nu(x) - d^{i-1}_\nu(x)).$$

To prove this Ledrappier and Young constructed a special countable partition $\mathcal{P}$ of M of finite entropy related to the Pinsker partition (see Theorem 11.16). Given integers $k, \ell \in \mathbb{N}$ we also consider the partition $\mathcal{P}^l_k = \bigvee_{n=-k}^{l} f^{-n}\mathcal{P}$.

Theorem 16.4 (Ledrappier and Young [157, 158]). *Let ν be ergodic. Given $0 < \varepsilon < 1$, there exists a set $\Gamma \subset M$ of measure $\nu(\Gamma) > 1 - \varepsilon/2$, an integer $n_0 \geq 1$, and a number $C > 1$ such that for every $x \in \Gamma$ and any integer $n \geq n_0$, the following statements hold:*

1. *for all integers k, $l \geq 1$,*

$$C^{-1}e^{-(l+k)h-(l+k)\varepsilon} \leq \nu(\mathcal{P}^l_k(x)) \leq Ce^{-(l+k)h+(l+k)\varepsilon}, \tag{16.1}$$

$$C^{-1}e^{-kh-k\varepsilon} \leq \nu^s_x(\mathcal{P}^0_k(x)) \leq Ce^{-kh+k\varepsilon}, \tag{16.2}$$

$$C^{-1}e^{-lh-l\varepsilon} \leq \nu^u_x(\mathcal{P}^l_0(x)) \leq Ce^{-lh+l\varepsilon}, \tag{16.3}$$

   where $h = h_\nu(f)$;

2.

$$\xi^s(x) \cap \bigcap_{n\geq 0} \mathcal{P}^n_0(x) \supset B^s(x, e^{-n_0}), \quad \xi^u(x) \cap \bigcap_{n\geq 0} \mathcal{P}^0_n(x) \supset B^u(x, e^{-n_0}); \tag{16.4}$$

3.

$$e^{-d^s n - n\varepsilon} \leq \nu^s_x(B^s(x, e^{-n})) \leq e^{-d^s n + n\varepsilon}, \tag{16.5}$$

$$e^{-d^u n - n\varepsilon} \leq \nu^u_x(B^u(x, e^{-n})) \leq e^{-d^u n + n\varepsilon}; \tag{16.6}$$

4.

$$\mathcal{P}^{an}_{an}(x) \subset B(x, e^{-n}) \subset \mathcal{P}(x), \tag{16.7}$$

$$\mathcal{P}^0_{an}(x) \cap \xi^s(x) \subset B^s(x, e^{-n}) \subset \mathcal{P}(x) \cap \xi^s(x), \tag{16.8}$$

$$\mathcal{P}^{an}_0(x) \cap \xi^u(x) \subset B^u(x, e^{-n}) \subset \mathcal{P}(x) \cap \xi^u(x), \tag{16.9}$$

   where a is the integer part of $2(1+\varepsilon)\max\{\lambda_1, -\lambda_p, 1\}$;

5. *if* $Q_n(x)$ *is defined by*

$$Q_n(x) = \bigcup \mathcal{P}_{an}^{an}(y)$$

where the union is taken over $y \in \Gamma$ *for which*

$$\mathcal{P}_0^{an}(y) \cap B^u(x, 2e^{-n}) \neq \varnothing \quad \textit{and} \quad \mathcal{P}_{an}^0(y) \cap B^s(x, 2e^{-n}) \neq \varnothing;$$

then

$$B(x, e^{-n}) \cap \Gamma \subset Q_n(x) \subset B(x, 4e^{-n}), \tag{16.10}$$

$$B^s(x, e^{-n}) \cap \Gamma \subset Q_n(x) \cap \xi^s(x) \subset B^s(x, 4e^{-n}), \tag{16.11}$$

$$B^u(x, e^{-n}) \cap \Gamma \subset Q_n(x) \cap \xi^u(x) \subset B^u(x, 4e^{-n}). \tag{16.12}$$

We outline here the construction of the partition $\mathcal{P}$, and its relation to the Pinsker partition (compare with Theorem 11.16). We proceed in a manner similar to that in Section 11.4. Consider a regular set Λ^ℓ with $\nu(\Lambda^\ell) > 0$. For a sufficiently small $r = r(\ell) > 0$ and $x \in \Lambda^\ell$, set

$$P^\ell(x) = \bigcup_{y \in \Lambda^\ell \cap B(x,r)} V^u(y), \quad Q(x) = \bigcup_{n=-\infty}^{\infty} f^n(P^\ell(x)).$$

Since ν is ergodic the set $Q(x)$ has full ν-measure. Let ξ be the partition of $Q(x)$ by local unstable manifolds $V^u(y)$, $y \in \Lambda^\ell \cap B(x,r)$, and the element $Q(x) \setminus P^\ell(x)$. Then $\xi^+ = \bigvee_{i \geq 0} f^i \xi$ is the Pinsker partition subordinate to the partition into global unstable manifolds.

Let now $\Lambda \subset M$ be the set of regular points, and $\Psi_x \colon B(0, q(x)) \to M$ a family of Lyapunov charts for $x \in \Lambda$ (see Theorem 8.14). Fix $\delta > 0$ and consider a partition $\mathcal{P}$ of finite entropy satisfying:

1. $\mathcal{P}$ is "adapted" to the Lyapunov charts in the sense that the elements of the partition $\mathcal{P}^+ = \bigvee_{n=0}^{\infty} f^n \mathcal{P}$ satisfy for each $x \in \Lambda$,

$$\mathcal{P}^+(x) \subset \Psi_x(\{y \in B(0, q(x)) : \|(\Psi_{f^{-n}(x)}^{-1} \circ f^{-n} \circ \Psi_x)(y)\| \leq \delta q(f^{-n}(x))\});$$

2. $h_\nu(f, \mathcal{P}) \geq h_\nu(f) - \varepsilon$;
3. the partition $\mathcal{P}$ refines $\{E, M \setminus E\}$ for some measurable set E of positive measure such that there exists a transversal T to W^u with the following property: if an element $C \in \xi^+$ intersects E, then T intersects C in exactly one point.

It is shown by Ledrappier and Young in [157] that a certain partition constructed by Mañé in [171] possesses Property 1. Property 3 is related to the construction of "transverse metrics" to ξ^+. Namely, consider the partitions $\eta_1 = \xi^+ \vee \mathcal{P}^+$ and $\eta_2 = \mathcal{P}^+$. Under the above properties one can define a metric on $\eta_2(x)/\eta_1$ for every $x \in \bigcup_{n \geq 0} f^n(E)$.

The inclusions (16.4) can be obtained from the fact that the partition $\mathcal{P}$ is adapted to the Lyapunov charts. Since the Lyapunov exponents at almost every point are constant, (16.7), (16.8), and (16.9) follow from (16.4) and an appropriate choice of a. The inequalities (16.5) and (16.6) are easy consequences of existence of the stable and unstable pointwise dimensions d_ν^s and d_ν^u (see Theorem 16.7). The inclusions (16.10) are based upon the continuous dependence of stable and unstable manifolds in the $C^{1+\alpha}$ topology on the base point in each regular set. The inclusions in (16.11) and (16.12) follow readily from (16.10).

Property (16.1) is an immediate corollary of Shannon–McMillan–Breiman's Theorem applied to the partition $\mathcal{P}$. Properties (16.2) and (16.3) follow from "leaf-wise" versions of this theorem. More precisely, Ledrappier and Young have shown (see [158, Lemma 9.3.1] and [157, Proposition 5.1]) that for ν-almost every x,

$$\lim_{n\to\infty} -\frac{1}{n}\log \nu_x^u(\mathcal{Q}_0^n(x)) = h_\nu(f),$$

where $\mathcal{Q}$ is any partition of finite entropy. Since $\mathcal{P}_0^n(x) \supset (\xi^+ \cap \mathcal{P})_0^n(x)$, we conclude that

$$\overline{\lim_{n\to\infty}} -\frac{1}{n}\log \nu_x^u(\mathcal{P}_0^n(x)) \le h_\nu(f) \tag{16.13}$$

for ν-almost every x. Moreover, using the fact that $\mathcal{P}$ is adapted to the Lyapunov charts one can show that the partition $\mathcal{P}$ additionally possesses the property that given $\delta > 0$ there exists $n_0 \ge 0$ such that $\mathcal{P}_0^n(x) \cap \xi^u(x) \subset V_{u(x)}(x, n, \delta)$ for ν-almost every x and every $n \ge n_0$ (see [158, Lemma 9.3.3]). It follows from Theorem 16.1 that for ν-almost every x,

$$\underline{\lim_{n\to\infty}} -\frac{1}{n}\log \nu_x^u(\mathcal{P}_0^n(x)) \ge \lim_{\delta\to 0}\underline{\lim_{n\to\infty}} -\frac{1}{n}\log \nu_x^u(V_{u(x)}(x,n,\delta)) = h_\nu(f). \tag{16.14}$$

Putting together (16.13) and (16.14) we obtain (16.3). A similar argument can be used to obtain (16.2).

Note that the Margulis–Ruelle inequality is an immediate corollary of Theorem 16.3 and so is the fact that any measure ν with absolutely continuous conditional measures on unstable manifolds satisfies Pesin's entropy formula.

16.2. **Dimension of measures. Local dimension.** For a Borel measure ν on a complete metric space X define the *Hausdorff dimension* $\dim_H\nu$, and *lower* and *upper box dimensions*, $\underline{\dim}_B\nu$, and $\overline{\dim}_B\nu$ by

$$\begin{aligned}\dim_H\nu &= \inf\{\dim_H Z : \nu(Z) = 1\},\\ \underline{\dim}_B\nu &= \lim_{\delta\to 0}\inf\{\underline{\dim}_B Z : \nu(Z) \ge 1-\delta\},\\ \overline{\dim}_B\nu &= \lim_{\delta\to 0}\inf\{\overline{\dim}_B Z : \nu(Z) \ge 1-\delta\},\end{aligned}$$

where $\dim_H Z$, $\underline{\dim}_B Z$ and $\underline{\dim}_B Z$ are respectively the Hausdorff dimension, lower and upper box dimensions of the set Z. It follows from the definition that

$$\dim_H\nu \le \underline{\dim}_B\nu \le \overline{\dim}_B\nu.$$

Another important characteristic of dimension type of ν is its *information dimension.* Given a partition ξ of X, define the *entropy of ξ with respect to ν* by

$$H_\nu(\xi) = -\sum_{C_\xi}\nu(C_\xi)\log\nu(C_\xi)$$

where C_ξ is an element of the partition ξ. Given a number $\varepsilon > 0$, set

$$H_\nu(\varepsilon) = \inf\{H_\nu(\xi)\colon \operatorname{diam}\xi \le \varepsilon\}$$

where $\operatorname{diam}\xi = \max\operatorname{diam}C_\xi$. We define the *lower* and *upper information dimensions of ν* by

$$\underline{I}(\nu) = \underline{\lim_{\varepsilon\to 0}}\frac{H_\nu(\varepsilon)}{\log(1/\varepsilon)}, \quad \overline{I}(\nu) = \overline{\lim_{\varepsilon\to 0}}\frac{H_\nu(\varepsilon)}{\log(1/\varepsilon)}.$$

Young established a powerful criterion that guarantees the coincidence of the Hausdorff dimension and lower and upper box dimensions of measures as well as their lower and upper information dimensions. Define the *local (pointwise) dimension* of ν by

$$d_\nu(x) = \lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r}, \tag{16.15}$$

where $B(x,r)$ is the ball centered at x of radius r (provided the limit exists). It was introduced by Young in [255] and characterizes the local geometrical structure of ν with respect to the metric in X. If the limit in (16.15) does not exist we consider the lower and upper limits and introduce respectively *the lower* and *upper local (pointwise) dimensions* of ν at x and we denote them by $\underline{d}_\nu(x)$ and $\overline{d}_\nu(x)$.

Theorem 16.5 (Young [255])**.** *Let X be a compact metric space of finite topological dimension and ν a Borel probability measure on X. Assume that*

$$\underline{d}_\nu(x) = \overline{d}_\nu(x) = d_\nu \tag{16.16}$$

for ν-almost every $x \in X$. Then

$$\dim_H \nu = \underline{\dim}_B \nu = \overline{\dim}_B \nu = \underline{I}(\nu) = \overline{I}(\nu) = d_\nu.$$

A measure ν satisfying (16.16) is called *exact dimensional*.

We will discuss the problem of existence of the limit in (16.15) for hyperbolic invariant measures. This problem is often referred to as the Eckmann–Ruelle conjecture. Its affirmative solution was obtained by Barreira, Pesin and Schmeling in [25].

Theorem 16.6. *Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth Riemannian manifold M without boundary, and ν an f-invariant compactly supported hyperbolic ergodic Borel probability measure. Then ν is exact dimensional and*

$$d_\nu = d^s_\nu + d^u_\nu.$$

In general, when the measure ν is not ergodic the stable and unstable local dimensions as well as the local dimension itself depend on the point x. In this case one can prove that for ν-almost every $x \in M$,

$$d_\nu(x) = d^s_\nu(x) + d^u_\nu(x).$$

Let us comment on the proof of Theorem 16.6. The upper bound for the pointwise dimension of *any* Borel f-invariant measure ν was obtained by Ledrappier and Young in [158].

Theorem 16.7. *Let f be a C^2 diffeomorphism of M. For ν-almost every $x \in M$,*

$$\overline{d}_\nu \le d^s_\nu + d^u_\nu + \dim E^c(x).$$

In the case when the measure ν is hyperbolic (i.e., $\dim E^c(x) = 0$ for ν-almost every $x \in M$) this result can be extended to $C^{1+\alpha}$ diffeomorphisms (not necessarily C^2). Namely it is shown in [25] that

$$\overline{d}_\nu \le d^s_\nu + d^u_\nu.$$

The lower bound for the pointwise dimension, $\underline{d}_\nu \ge d^s_\nu + d^u_\nu$, is an immediate corollary of Theorem 16.9.

Young proved Theorem 16.6 for surface diffeomorphisms.

Theorem 16.8 (Young [255]). *Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth compact surface M and ν a hyperbolic ergodic measure with Lyapunov exponents $\lambda_\nu^1 > 0 > \lambda_\nu^2$. Then*

$$\underline{d}_\nu = \overline{d}_\nu = h_\nu(f)\left(\frac{1}{\lambda_\nu^1} - \frac{1}{\lambda_\nu^2}\right).$$

Let us point out that neither of the assumptions of Theorem 16.6 can be omitted. Pesin and Weiss presented an example of a Hölder homeomorphism with Hölder constant arbitrarily close to 1 whose ergodic measure of maximal entropy is not exact dimensional (see [202]). Ledrappier and Misiurewicz [155] constructed an example of a smooth map of a circle preserving an ergodic measure with zero Lyapunov exponent which is not exact dimensional. Kalinin and Sadovskaya [127] strengthened this result by showing that for a residual set of circle diffeomorphisms with irrational rotation number the unique invariant measure has lower pointwise dimension 0 and upper pointwise dimension 1 for almost every point in S^1.

16.3. **Local product structure of hyperbolic measures.** The following principle result establishes a crucial property of hyperbolic measures: these measures have asymptotically "almost" local product structure.

Theorem 16.9 (Barreira, Pesin and Schmeling [25]). *Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth Riemannian manifold M without boundary, and ν an f-invariant compactly supported hyperbolic ergodic Borel probability measure. Then for every $\delta > 0$ there exist a set $\Lambda \subset M$ with $\nu(\Lambda) > 1-\delta$ such that for every $x \in \Lambda$ and every sufficiently small r (depending on x), we have*

$$r^\delta \nu_x^s(B^s(x,r))\nu_x^u(B^u(x,r)) \le \nu(B(x,r)) \le r^{-\delta}\nu_x^s(B^s(x,r))\nu_x^u(B^u(x,r)).$$

The proof of Theorem 16.9 uses the crucial Markov property of the special countable partition $\mathcal{P}$ of M constructed in Theorem 16.4.

Theorem 16.10 (Barreira, Pesin and Schmeling [25]). *For every $x \in \Gamma$ and $n \ge n_0$,*

$$\mathcal{P}_{an}^{an}(x) \cap \xi^s(x) = \mathcal{P}_{an}^{0}(x) \cap \xi^s(x);$$

$$\mathcal{P}_{an}^{an}(x) \cap \xi^u(x) = \mathcal{P}_{0}^{an}(x) \cap \xi^u(x).$$

Note that any SRB-measure possesses a stronger property of local product structure and so does any Gibbs measure on a locally maximal hyperbolic set.

We emphasize that Theorem 16.9 is not trivial even for measures supported on locally maximal uniformly hyperbolic sets. In this situation the stable and unstable foliations need not be Lipschitz (in fact, they are "generically" not Lipschitz), and in general, the measure need not have a local product structure despite the fact that the set itself does.

Let us illustrate Theorems 16.6 and 16.9 by considering the full shift σ on the space Σ_p of two-sided infinite sequences of numbers in $\{1,\dots,p\}$. This space is endowed with the usual "symbolic" metric d_β, for each fixed number $\beta > 1$, defined as follows:

$$d_\beta(\omega^1,\omega^2) = \sum_{i\in\mathbb{Z}} \beta^{-|i|}|\omega_i^1 - \omega_i^2|,$$

where $\omega^1 = (\omega_i^1)$ and $\omega^2 = (\omega_i^2)$.

Let ν be a σ-invariant ergodic measure on Σ_p. By Shannon–McMillan–Breiman's Theorem, for ν-almost every $\omega \in \Sigma_p$,

$$\lim_{n\to\infty} -\frac{1}{2n+1} \log \nu(C_n(\omega)) = h_\nu(\sigma), \tag{16.17}$$

were $C_n(\omega)$ is the cylinder at ω of "size" n. Since $C_n(\omega)$ is the ball in the symbolic metric centered at ω of radius β^n, the quantity in the right-hand side in (16.17) is the local dimension of ν at ω. Thus, Shannon–McMillan–Breiman's Theorem claims that the local dimension of ν is almost everywhere constant and that the common value is the measure-theoretical entropy of ν.

Further, fix $\omega = (\omega_i) \in \Sigma_p$. The cylinder $C_n(\omega)$ can be identified with the direct product $C_n^+(\omega) \times C_n^-(\omega)$ where

$$C_n^+(\omega) = \{\bar\omega = (\bar\omega_i) : \bar\omega_i = \omega_i \text{ for } i = 0, \dots, n\}$$

and

$$C_n^-(\omega) = \{\bar\omega = (\bar\omega_i) : \bar\omega_i = \omega_i \text{ for } i = -n, \dots, 0\}$$

are the "positive" and "negative" cylinders at ω of "size" n. Define measures

$$\nu_n^+(\omega) = \nu | C_n^+(\omega) \quad \text{and} \quad \nu_n^-(\omega) = \nu | C_n^-(\omega).$$

It follows from Theorem 16.9 that for every $\delta > 0$ there exist a set $\Lambda \subset \Sigma_p$ with $\nu(\Lambda) > 1 - \delta$ and an integer $m \geq 1$ such that for every $\omega \in \Lambda$ and every sufficiently large n (depending on ω), we have

$$\beta^{-\delta|n|} \nu_{n+m}^+(\omega) \times \nu_{n+m}^-(\omega) \leq \nu | C_n(\omega) \leq \beta^{\delta|n|} \nu_{n-m}^+(\omega) \times \nu_{n-m}^-(\omega).$$

17. Geodesic flows on manifolds without conjugate points

For a long time geodesic flows have played an important stimulating role in developing the hyperbolic theory. Already in the beginning of the 20th century Hadamard and Morse, while studying the statistics of geodesics on surfaces of negative curvature, observed that the local instability of trajectories is the prime reason for the geodesic flow to be ergodic and topologically transitive. The further study of geodesic flows has led researchers to introduce different classes of hyperbolic dynamical systems (Anosov systems, uniformly partially hyperbolic systems, and nonuniformly hyperbolic systems). On the other hand, geodesic flows always were one of the main areas for applying new advanced methods of the hyperbolic theory of dynamical systems. This in particular, has led to some new interesting results in differential and Riemannian geometry.

17.1. **Ergodic properties of geodesic flows.** Consider the geodesic flow g_t on a compact smooth Riemannian p-dimensional manifold M without conjugate points. The flow preserves the Liouville measure μ on the tangent bundle. Let the set Δ be given by (2.10). We assume that Δ is of positive Liouville measure. By Theorem 2.3 the geodesic flow is nonuniformly hyperbolic on Δ and hence, the results of Section 11.2 apply and show that ergodic components of $g_t|\Delta$ are of positive Liouville measure (see Theorem 11.3). Indeed, under some mild geometric assumptions the geodesic flow on Δ is ergodic. To see this we will first observe that every ergodic component of positive measure is open (mod 0) and then will use a remarkable result by Eberlein on topological transitivity of geodesic flows.

To establish local ergodicity of $g_t|\Delta$ we shall describe two invariant foliations (known as the stable and unstable horospherical foliations) of SM, W^- and W^+, such that $W^s(x) = W^-(x)$ and $W^u(x) = W^+(x)$ for almost every $x \in \Delta$.

We denote by H the universal Riemannian cover of M, i.e., a simply connected p-dimensional complete Riemannian manifold for which $M = H/\Gamma$ where Γ is a discrete subgroup of the group of isometries of H, isomorphic to $\pi_1(M)$. According to the Hadamard–Cartan theorem, any two points x, $y \in H$ are joined by a single geodesic which we denote by γ_{xy}. For any $x \in H$, the exponential map $\exp_x \colon \mathbb{R}^p \to H$ is a diffeomorphism. Hence, the map

$$\varphi_x(y) = \exp_x \left(\frac{y}{1 - \|y\|} \right) \tag{17.1}$$

is a homeomorphism of the open p-dimensional unit disk D onto H.

Two geodesics $\gamma_1(t)$ and $\gamma_2(t)$ in H are said to be *asymptotic* if

$$\sup_{t>0} \rho(\gamma_1(t), \gamma_2(t)) < \infty.$$

The asymptoticity is an equivalence relation, and the equivalence class $\gamma(\infty)$ corresponding to a geodesic γ is called a *point at infinity*. The set of these classes is denoted by $H(\infty)$ and is called the *ideal boundary* of H. Using (17.1) one can extend the topology of the space H to $\overline{H} = H \cup H(\infty)$ so that $\overline{H}$ becomes a compact space.

The map φ_x can be extended to a homeomorphism (still denoted by φ_x) of the closed p-dimensional disk $\overline{D} = D \cup S^{p-1}$ onto $\overline{H}$ by the equality

$$\varphi_x(y) = \gamma_y(+\infty), \qquad y \in S^{p-1}.$$

In particular, φ_x maps S^{p-1} homeomorphically onto $H(\infty)$.

For any two distinct points x and y on the ideal boundary there is a geodesic which joins them. This geodesic is uniquely defined if the Riemannian metric is of strictly negative curvature (i.e., if inequality (2.7) is strict). Otherwise, there may exist a pair of distinct points x, $y \in H(\infty)$ which can be joined by more than one geodesic. If the manifold has no focal points there exists a geodesic isometric embedding into H of an infinite strip of zero curvature which consists of geodesics joining x and y. This statement is know as the *flat strip theorem*.

The fundamental group $\pi_1(M)$ of the manifold M acts on the universal cover H by isometries. This action can be extended to the ideal boundary $H(\infty)$. Namely, if $p = \gamma_v(+\infty) \in H(\infty)$ and $\zeta \in \pi_1(M)$, then $\zeta(p)$ is the equivalent class of geodesics which are asymptotic to the geodesic $\zeta(\gamma_v(t))$.

We now describe the invariant foliations for the geodesic flow.

Fix a point $x \in H$ and a vector $v \in SH$. Consider a sequence of vectors $v_n \in SH$, $v_n \to v$, a sequence of points $x_n \in H$, $x_n \to x$ and a sequence of numbers $t_n \to \infty$. Denote by γ_n the geodesic joining the points x_n and $\gamma_{v_n}(t_n)$. Since the sequence o vectors $\dot\gamma_n(0)$ is compact the sequence of geodesics has a limit geodesic. Following [199] we say that the manifold M satisfies the *asymptoticity axiom* if for any choice of $x_n, x \in H$, $v_n, v \in SH$, $x_n \to x$, $v_n \to v$ and $t_n \to \infty$ any limit geodesic of the sequence of geodesic γ_n is asymptotic to the geodesic γ.

If the manifold M satisfies the asymptoticity axiom then the sequence γ_n, indeed, converges to γ. Moreover, given a geodesic γ and a point $x \in H$, there exists a unique geodesic γ' passing through x and asymptotic to γ.

Proposition 17.1 (Pesin [199]). *If the manifold M has no focal points then it satisfies the asymptoticity axiom.*

We consider the distributions E^- and E^+ introduced by (2.8) and (2.9).

Proposition 17.2 (Pesin [199]). *Assume that the manifold M satisfies the asymptoticity axiom. Then the distributions E^- and E^+ are integrable. Their integral manifolds form continuous foliations of SM with smooth leaves. These foliations are invariant under the geodesic flow.*

Denote by W^- and W^+ the foliations of SM corresponding to the invariant distributions E^- and E^+. These foliations can be lifted from SM to SH and we denote these lifts by $\widetilde{W}^-$ and $\widetilde{W}^+$ respectively.

Given $x \in H$ and $p \in H(\infty)$, set

$$L(x,p) = \pi(\widetilde{W}^-(v))$$

where $x = \pi(v)$ and $p = \gamma_v(\infty)$. The set $L(x,p)$ is called the *horosphere* through x centered at p.

We summarize the properties of the foliations and horospheres in the following statement.

Proposition 17.3. *The following statements hold:*

1. *for any $x \in H$ and $p \in H(\infty)$ there exists a unique horosphere $L(x,p)$ centered at p which passes through x; it is a limit in the C^1 topology of spheres $S^p(\gamma(t),t)$ as $t \to +\infty$ where γ is the unique geodesic joining x and p;*
2. *the leaf $W^-(v)$ is the framing of the horosphere $L(x,p)$ ($x = \pi(v)$ and $p = \gamma_v(+\infty)$) by orthonormal vectors which have the same direction as the vector v (i.e., they are "inside" the horosphere). The leaf $W^+(v)$ is the framing of the horosphere $L(x,p)$ ($x = \pi(v)$ and $p = \gamma_v(-\infty) = \gamma_{-v}(+\infty)$) by orthonormal vectors which have the same direction as the vector v (i.e., they are "outside" the horosphere);*
3. *for every $\zeta \in \pi_1(M)$,*
$$\zeta(L(x,p)) = L(\zeta(x),\zeta(p)),$$
$$d_v\zeta\widetilde{W}^-(v) = \widetilde{W}^-(d_v\zeta v), \quad d_v\zeta\widetilde{W}^+(v) = \widetilde{W}^+(d_v\zeta v);$$
4. *for every v, $w \in SH$, for which $\gamma_v(+\infty) = \gamma_w(+\infty) = p$, the geodesic $\gamma_w(t)$ intersects the horosphere $L(\pi(v),p)$ at some point.*

Theorem 17.4 (Pesin [199]). *Assume that the manifold M satisfies the asymptoticity axiom. Assume also that the set Δ has positive Liouville measure. Then for almost every $v \in SM$ we have that $W^-(v) = W^s(v)$ and $W^+(v) = W^u(v)$.*

By Theorem 11.8 we conclude that ergodic components of $g_t|\Delta$ of positive measure are open (mod 0). In particular, the set Δ is open (mod 0). See Theorem 2.5 that gives sufficient conditions for the set Δ to be of positive Liouville measure.

We describe the topological transitivity of geodesic flows. Following Eberlein [79] we say that the manifold M satisfies the *uniform visibility axiom* if for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that from each point $x \in H$ any geodesic segment γ with $d(x,\gamma) \geq R$ is visible at an angle less than ε.

Proposition 17.5 (Eberlein [79]). *The following statements hold:*

1. *if a manifold satisfies the uniform visibility axiom with respect to a Riemannian metric then it satisfies this axiom with respect to any other Riemannian metric with no conjugate points;*
2. *a manifold of nonpositive curvature satisfies the uniform visibility axiom if its universal cover does not admit an isometric geodesic embedding of* $\mathbb{R}^2$*;*
3. *a compact two-dimensional manifold* M *of genus* ≥ 1 *satisfies the uniform visibility axiom;*
4. *if* M *satisfies the uniform visibility axiom the it also satisfies the asymptoticity axiom.*

Theorem 17.6 (Eberlein [79])**.** *Assume that the compact manifold* M *satisfies the uniform visibility axiom. Then the geodesic flow* g_t *is topologically transitive.*

Theorem 17.7. *Let* M *be a compact smooth Riemannian manifold without focal points satisfying the uniform visibility axiom. If the set* Δ *has positive Liouville measure then it is open* (mod 0) *and is everywhere dense. The geodesic flow* $g_t|\Delta$ *is nonuniformly hyperbolic and ergodic. Indeed,* $g_t|\Delta$ *is Bernoulli.*

The Bernoulli property follows from Theorem 11.21 and from a result by Arnold (see [9, §23]) implying that the geodesic flow has continuous spectrum.

It is an open problem whether the set Δ has full Liouville measure. Brin and Burago have proved this under the additional assumption that the set of negative curvature in M has finitely many connected components. The same result was obtained by Hertz who used different methods. None of these results is published.

Further results on ergodic and topological properties of geodesic flows on manifolds of nonpositive curvature were obtained by Knieper [144, 145, 146]. His celebrated result establishes existence and uniqueness of the measure of maximal entropy thus extending the classical result by Margulis to nonpositively curved manifolds. Knieper also obtained multiplicative asymptotic bounds for the growth of volume of spheres (and hence, also that of balls) and the number of periodic orbits. For a detailed account of this work see the chapter [147].

In [101, 102], Gunesch strengthened Knieper's results and obtained precise asymptotic formulae for the growth of volume of spheres and the number of homotopy classes of periodic orbits for the geodesic flow on rank 1 manifolds of nonpositive curvature. This extends results by Margulis to the nonuniformly hyperbolic case.

Let M be a compact Riemannian manifold of nonpositive curvature. Given a tangent vector $v \in SM$, $\operatorname{rank}(v)$ is the dimension of the space of parallel Jacobi fields along the geodesic γ_v. The minimum of $\operatorname{rank}(v)$ over all $v \in SM$ is called the *rank* of M, $\operatorname{rank}(M)$. If $\operatorname{rank}(v) = \operatorname{rank}(M)$ the geodesic γ_v and the corresponding vector v are called *regular*. It is easy to see that $1 \leq \operatorname{rank}(M) \leq \dim M$.

The following result describes the fundamental rank rigidity for nonpositively curved manifolds. It was obtained independently by Ballmann [21] (see also [22]) and by Burns and Spatzier [55] (see also [81]).

Theorem 17.8. *Let* M *be a compact smooth Riemannian manifold of nonpositive curvature with irreducible universal cover* H*. Then the manifold has either rank* 1 *or* H *is a symmetric space of higher rank.*

In other words, the universal cover of a nonpositively curved manifold can be represented as a product of Euclidean, symmetric, and rank 1 spaces.

Theorem 17.9 (see [101, 102])**.**

1. *Given $x \in H$, let $b_r(x) = \mathrm{vol}(B(x,r))$ be the Riemannian volume of the ball centered at x of radius r in the universal cover H of M. Then*
$$b_r(x) \sim c(x)e^{hr},$$
where $c(x)$ is a continuous function on M and $h = h(g^t)$ is the topological entropy of the geodesic flow.
2. *Let $P(t)$ be the number of homotopy classes of periodic orbits of length at most t. Then*
$$P(t) \sim \frac{1}{ht}e^{ht}.$$

Observe that unlike in the case of negatively curved manifolds, for nonpositively curved manifolds there may be uncountably many periodic geodesics homotopic to a given one but they all have the same length.

17.2. **Entropy of geodesic flows.** For $v \in SM$ let $v^\perp$ be the set of vectors $w \in SM$ which are orthogonal to v. Consider the linear map $S_v \colon v^\perp \to v^\perp$ defined by the equality: $S_v w = K_\xi(w)$, where $\xi(w)$ is the vector in $E^-(v)$ such that $d\pi\xi(w) = w$.

Theorem 17.10. *For a Riemannian metric of class C^4 of nonpositive curvature S_v is a linear self-adjoint operator of the second quadratic form for the horosphere $L(\pi(v), \gamma_v(+\infty))$ at the point $\pi(v)$ (which is a submanifold in H of class C^2).*

Denote by $\{e_i(v)\}$, $i = 1, \dots, p-1$, the orthonormal basis in $v^\perp$ consisting of eigenvectors of S_v. Let $K_i(v)$ be the corresponding eigenvalues. The numbers $K_i(v)$ are called the *principal curvatures* and the directions determined by the vectors $e_i(v)$ the *directions of principal curvatures* for the limit sphere at $\pi(v)$.

Theorem 17.11 (Pesin [201], Freire and Mañé [91])**.**

1. *The entropy of the geodesic flow is*
$$h_\mu(g^1) = -\int_M \sum_{i=1}^{p-1} K_i(v)\, d\mu(v) = -\int_M \mathrm{tr}\, S_v\, d\mu(v),$$
where μ is the Liouville measure and $\mathrm{tr}\, S_v$ denotes the trace of S_v.
2. *Let ν be a g^t-invariant probability measure. Then*
$$h_\mu(g^1) \le -\int_M \mathrm{tr}\, S_v\, d\nu(v).$$

Statement 1 of Theorem 17.11 is analogous to a result of [234] for dispersing billiards.

For the topological entropy $h(g^t)$ of the geodesic flow on manifolds without conjugate points Freire and Mañé [91] established the following formula.

Theorem 17.12.
$$h(g^t) = \lim_{r\to\infty} \frac{\log \mathrm{vol}(B(x,r))}{r},$$
where $x \in H$ is a point in the universal cover of M (the limit exists and does not depend on x).

18. Dynamical systems with singularities: the conservative case

In this and the following sections we shall discuss how the core results of smooth ergodic theory can be extended to dynamical systems with singularities (where the map or its differential are discontinuous). We consider two cases: the conservative one when the system preserves volume and the dissipative one when the system possesses an attractor. The main example in the first case is billiards while the main example in the second case is the Lorenz attractor. In both cases the system is uniformly hyperbolic either on the whole phase space or in an neighborhood of the attractor. However, the presence of singularities may effect the behavior of trajectories in a crucial way so that along some trajectories Lyapunov exponents are zero. We shall describe some general conditions on the singularity set which guarantee that Lyapunov exponents along the "majority" of trajectories are nonzero and methods of nonuniform hyperbolicity theory apply.

Let M be a compact smooth Riemannian manifold and $S \subset M$ a closed set. Following Katok and Strelcyn [138] we call a map $f\colon M\setminus S \to f(M\setminus S)$ a *map with singularities* and the set S the *singularity set* for f if the following conditions hold:

(A1) f is a C^2 diffeomorphism;

(A2) there exist constants $C_1 > 0$ and $a \ge 0$ such that

$$\|d_x^2 f\| \le C_1 \rho(x,S)^{-a} \quad x \in M\setminus S,$$

$$\|d_x^2 f^{-1}\| \le C_1 \rho(x,S^-)^{-a}, \quad x \in f(M\setminus S),$$

where $S^- = \{y \in M :$ there are $z \in S$ and $z_n \in M\setminus S$ such that $z_n \to z, f(z_n) \to y\}$ is the *singularity set* for f^{-1}.

Let μ be a probability measure on M invariant under f. We assume that

(A3)

$$\int_M \log^+\|df\|\,d\mu < \infty \quad \text{and} \quad \int_M \log^+\|df^{-1}\|\,d\mu < \infty;$$

(A4) for every $\varepsilon > 0$ there exist constants $C_2 > 0$ and $b \in (0,1]$ such that

$$\mu(\{x \in M : \rho(x,S) < \varepsilon\}) \le C_2 \varepsilon^b.$$

Condition (A2) means that the derivative of f may grow with a "moderate" polynomial rate near the singularity set and Condition (A4) implies that $\mu(S) = 0$, i.e., the singularity set is "small".

Conditions (A1)–(A4) constitute the basis of the Katok-Strelcyn theory and allow one to extend results of smooth ergodic theory to smooth systems with singularities. In particular, at every Lyapunov regular point with nonzero Lyapunov exponents one can construct local stable and unstable manifolds, establish the crucial absolute continuity property, describe ergodic properties of the map with respect to a smooth hyperbolic invariant measure and obtain the entropy formula.

We shall now proceed with a formal description. Set $N^+ = \{x \in M\colon f^n(x) \notin S$ for all $n \ge 0\}$ and let $N = \bigcap_{n\ge 0} f^n(N^+)$. For each $\alpha \in (0,1)$ and $\gamma > 0$ set

$$\Omega^{\alpha,\gamma} = \{x \in N : \rho(f^n(x), A) \ge \gamma\alpha^{|n|} \text{ for every } n \in \mathbb{Z}\}.$$

Proposition 18.1. *We have that $\mu(N) = 0$ and $\mu(\Omega^{\alpha,\gamma}) \to 1$ as $\gamma \to 0$.*

To see this note that

$$N \setminus \Omega^{\alpha,\gamma} = \{x \in N : \rho(f^n(x), A) < \gamma\alpha^{|n|} \text{ for some } n \in \mathbb{Z}\}$$

and hence, by (A4),

$$\begin{aligned}\mu(N\setminus\Omega^{\alpha,\gamma}) &\le \sum_{n\in\mathbb{Z}}\mu(\{x\in N:\rho(f^n(x),A)<\gamma\alpha^{|n|}\})\\ &\le \sum_{n\in\mathbb{Z}}\mu(\{x\in X:\rho(x,A)<\gamma\alpha^{|n|}\})\\ &\le \sum_{n\in\mathbb{Z}}C_2\gamma^a\alpha^{a|n|}\le\frac{2C_2\gamma^a}{1-\alpha^a}.\end{aligned}$$

Let $\Lambda\subset M\setminus N$ be the set of points with nonzero Lyapunov exponents. We assume that $\mu(\Lambda)>0$ and we consider the collection of regular sets Λ^ℓ, $\ell\ge 1$ for f (see Section 4.5). Denote by

$$\Lambda^{\ell,\alpha,\gamma}=\Lambda^\ell\cap\Omega^{\alpha,\gamma}.$$

From now on we fix a sufficiently large $\ell>0$, $\alpha=\alpha(\ell)$ and $\gamma=\gamma(\ell)$ such that the set $A^\ell=\Lambda^{\ell,\alpha(\ell),\gamma(\ell)}$ has positive measure.

As an immediate corollary of Proposition 18.1 we obtain the following statement.

Theorem 18.2 (Stable manifold theorem)**.** *Let f be a diffeomorphism with singularities satisfying conditions (A1)–(A4). Then for every $x\in A^\ell$ there exists a local stable manifold $V^s(x)$ such that $x\in V^s(x)$, $T_xV^s(x)=E^s(x)$, and for $y\in V^s(x)$ and $n\ge 0$,*

$$\rho(f^n(x),f^n(y))\le T(x)\lambda^n e^{\varepsilon n}\rho(x,y),$$

where $T\colon X\to(0,\infty)$ is a Borel function satisfying

$$T(f^m(x))\le T(x)e^{10\varepsilon|m|},\quad m\in\mathbb{Z}.$$

Furthermore, for every $x\in A^\ell$ there exists a local unstable manifold $V^u(x)$ which have similar properties.

When a point moves under f its stable (and/or unstable) manifold may "meet" the singularity set and be cut by it into several pieces whose sizes, in general, may be *uncontrollably* small. It is Condition (A4) that allows one to control this process for almost every trajectory, see Proposition 18.1. In particular, the size of the local manifolds $V^s(x)$ and $V^u(x)$ may depend on the point x and may deteriorate along the trajectory but only with subexponential rate. Moreover, local manifolds satisfy the absolute continuity property, see Section 10.1. This provides the basis for obtaining a complete description of ergodic and topological properties of the system including: 1) descriptions of ergodic and K-components (see Theorems 11.3 and 11.17), 2) the entropy formula[7] (see Theorems 12.1 and 12.10), and 3) density of periodic points (see Theorem 15.2).

Liverani and Wojtkowski [167] designed a method which allows one to study local ergodicity of smooth systems with singularities. The systems to which this method applies are defined axiomatically by a number of conditions. They include some assumptions on the singularity set, existence of invariant cone families which are monotone and strictly monotone (see Section 11.3), and an adaptation of the Sinai–Chernov Ansatz for billiards (see [66]).

[7]To establish the upper bound for the entropy, one needs to assume, in addition to (A2), that $\|d_xf\|\le C_1\rho(x,S)^{-a}$ and $\|d_xf^{-1}\|\le C_1\rho(x,S^-)^{-a}$; see [138] for details.

Other results on local ergodicity of smooth systems with singularities were obtained by Chernov [61], Markarian [177], and Vaienti [243] (for some particular map).

In [250], Wojtkowski and Liverani introduced a special class of dynamical systems with singularities—conformally Hamiltonian systems with collisions. They are determined by a nondegenerate 2-form Θ and a function H (called Hamiltonian). The form does not have to be closed but $d\Theta = \gamma \wedge \Theta$ for some closed 1-form γ. This condition guarantees that, at least locally, the form Θ can be multiplied by a nonzero function to give a bona fide symplectic structure (such a structure is called *conformally symplectic*). Examples of systems with conformally symplectic structure include the Gaussian isokinetic dynamics and the Nosé–Hoover dynamics. The main result in [250] claims that the Lyapunov spectrum of the corresponding conformally Hamiltonian flow is symmetric. This recovers and generalizes results by Benettin, Galgani, Giorgilli and Strelcyn [32], Dettmann and Morriss [69, 70], Garrido and Galavotti [95], Dellago, Posch and Hoover [68], and Bonetto, Galavotti and Garrido [43].

18.1. **Billiards.** We consider billiards in the plane which form a special class of maps with singularities. Let Q be a compact connected subset of $\mathbb{R}^2$ such that ∂Q consists of a finite number of curves of class C^3. The *billiard flow* in Q is generated by the free motion of a particle in the interior of Q, with unit speed and elastic reflections at the boundary (reflection is elastic if the angle of reflection equals the angle of incidence). The flow acts on the unit tangent bundle SQ but is not well-defined in the corners of ∂Q. It can be shown that the billiard flow preserves the Liouville measure on SQ. We refer to [67] for more details.

Consider the set $X \subset SQ$ consisting of the unit vectors in SQ at the boundary ∂Q and pointing inside Q. The *billiard map* on Q is defined as the first return map $f\colon X \to X$ induced by the billiard flow. Given $(q, v) \in X$, its image $f(q, v)$ is the point $(q', v') \in X$, where q' and v' are the position and velocity of the particle with initial condition (q, v) immediately after the next reflection at ∂Q. We introduce the coordinates (s, θ) for a point $(q, v) \in X$ where s is the length of ∂Q up to q measured with respect to a given point in ∂Q and $\theta \in [-\pi/2, \pi/2]$ is the angle that the vector v makes with the inward normal of ∂Q at q. We endow X with the Riemannian metric $ds^2 + d\theta^2$. The billiard map preserves the measure $d\nu = (2c)^{-1} \cos\theta ds d\theta$ where c is the length of ∂Q.

The billiard map, in general, is not well-defined everywhere in X. Let Z be the set of corners of ∂Q, i.e., the points where ∂Q is not of class C^1, and let $q' \neq q$ be the first point of ∂Q where the particle with initial condition $(q, v) \in X$ hits ∂Q. Then f is not defined on the set

$$S^+ = \{(q, v) \in X : q' \in Z \text{ or the segment } \overline{qq'} \text{ is tangent to } \partial Q \text{ at } q'\}.$$

Thus, f is not defined at (q, v) if the particle with initial condition (q, v) either hits a corner of ∂Q or reflects at ∂Q with a null angle. Define $R\colon X \to X$ by $R(s, \theta) = (s, \pi - \theta)$ for $(s, \theta) \in X$. The map f^{-1} is not defined on the set $S^- = RS^+$. One can show that the sets S^+ and S^- consist of a finite number of curves of class C^2 that intersect only at their endpoints (see [138]). They are called respectively the *singularity sets* for f and f^{-1}. For each $n > 0$, the sets where f^n and f^{-n} are not defined are respectively

$$S_n^+ = S^+ \cup f^{-1}S^+ \cup \cdots \cup f^{-n+1}S^+,$$
$$S_n^- = S^- \cup fS^- \cup \cdots \cup f^{n-1}S^-.$$

Let

$$S_\infty^+ = \bigcup_{n>0} S_n^+, \quad S_\infty^- = \bigcup_{n>0} S_n^- \text{ and } S_\infty = S_\infty^- \cap S_\infty^+.$$

The points in S_∞^+ (respectively S_∞^-) hit a corner of ∂Q after a finite number of iterations of f (respectively f^{-1}). The points in S_∞ hit a corner of ∂Q after a finite number of iterations of f and after a finite number of iterations of f^{-1}, and thus they have orbits of finite length. Clearly, $\nu(S_n^+) = \nu(S_n^-) = 0$ for every $n > 0$.

According to the former observations, there exists an integer $m > 0$ such that the sets $X \setminus S^+$ and $X \setminus S^-$ consist both of a finite number of open connected sets, say $X_1^+, \dots, X_m^+$ and $X_1^-, \dots, X_m^-$, such that $f\colon X_i^+ \to X_i^-$ is a C^2 diffeomorphism for $i = 1, \dots, m$. Therefore, the billiard map f is a map with singularities on X (in this case $S = S^+$). Following Katok and Strelcyn [138] we will describe a sufficiently large class of billiards which satisfy Conditions (A1)–(A7).

Theorem 18.3. *Let f be a billiard map on Q. If ∂Q is piecewise C^2, has finite length, and has a uniformly bounded curvature, then Condition (A3) holds with respect to the measure ν.*

For example, any billiard whose boundary is a union of a finite number of closed arcs and closed curves of class C^2 satisfies the hypotheses of Theorem 18.3.

To establish Condition (A4) we consider the class P_k, $k \geq 2$ of billiards whose boundary is a union of a finite number of intervals and strictly convex or strictly concave C^k curves.

Theorem 18.4. *Let f be a billiard map of class P_k, $k \geq 2$. Then the singularity set for f is a union of a finite number of closed curves of class C^{k-1} of finite length and of a finite number of isolated points. In particular, f satisfies Condition (A4).*

We now discuss Condition (A2). Let γ be a strictly convex smooth curve in the plane. For each $p \in \gamma$, let ℓ_1 be the oriented tangent line to γ at p (one-sided tangent line if p is an endpoint of γ), and let ℓ_2 be the line through p orthogonal to ℓ_1. Orienting the line ℓ_2 in a suitable way, one can assume that in a neighborhood of p, with respect to the orthogonal coordinate system given by ℓ_1 and ℓ_2, the curve γ is the graph of a smooth strictly convex function (that we also denote by γ). We consider the class Π_k, $k \geq 2$ of billiards in P_k for which there exists $C > 0$ such that all strictly convex pieces of ∂Q satisfy

$$\frac{(s-t)\gamma'(s) - \gamma(s) + \gamma(t)}{\gamma(s) - \gamma(t) - (s-t)\gamma'(t)} \geq C$$

for every $s \neq t$ in a neighborhood of zero (at the endpoints we consider appropriate one-sided neighborhoods of zero). It is shown in [138] that the class Π_k includes the billiards in P_k for which the following holds: for each γ of class C^{m+2} as above, $\gamma^{(i)}(0) = 0$ for $2 \leq i \leq m-1$, and $\gamma^{(m)}(0) \neq 0$.

Theorem 18.5 ([138])**.** *Any billiard map $f \in \Pi_k$, $k \geq 3$ satisfies Condition (A2).*

It follows from Theorems 18.3–18.5 that billiard maps of class Π_k, $k \geq 3$, satisfy Conditions (A1)–(A4). Thus, the results of the previous section apply. Particular cases include dispersing (Sinai's) billiards and some semi-dispersing billiards. Recall that a curve Γ in the boundary ∂Q of a billiard is *dispersing*, *focusing*, or *flat* if Γ is respectively strictly concave outward (with respect to Q), strictly convex outward, or Γ is a straight segment. We denote by Γ_-, Γ_+, and Γ_0 the unions of the curves forming ∂Q that are respectively dispersing, focusing, and flat. A billiard is called *dispersing* if $\Gamma_+ = \Gamma_0 = \varnothing$, and *semi-dispersing* if $\Gamma_+ = \varnothing$, $\Gamma_- \neq \varnothing$.

We refer to the survey [52] for more details. See also the collection of surveys in [236].

19. Hyperbolic attractors with singularities

In this section we consider dissipative hyperbolic dynamical systems with singularities. They possess attractors and act uniformly hyperbolic in their vicinity. However, due to singularities the behavior of trajectories is effectively nonuniformly hyperbolic. We call these attractors generalized hyperbolic attractors. They were introduced by Pesin in [200]. Examples include Lorenz attractor, Lozi attractor and Belykh attractor. We describe a construction SRB-measures for these systems.

19.1. **Definitions and local properties.** Let M be a smooth Riemannian manifold, $K \subset M$ an open bounded connected set and $N \subset K$ a closed set. Let also $f : K \setminus N \to K$ be a map satisfying the following conditions:

(H1) f is a C^2 diffeomorphism from the open set $K \setminus N$ onto its image;

(H2) there exist constants $C > 0$ and $\alpha \geq 0$ such that

$$\|d_x f\| \leq C\rho(x, N^+)^{-\alpha}, \quad \|d_x^2 f\| \leq C\rho(x, N^+)^{-\alpha} \quad x \in K \setminus N,$$

$$\|d_x f^{-1}\| \leq C\rho(x, N^-)^{-\alpha}, \quad \|d_x^2 f^{-1}\| \leq C\rho(x, N^-)^{-\alpha}, \quad x \in f(K \setminus N),$$

where $N^+ = N \cup \partial K$ is the *singularity set* for f and $N^- = \{y \in K :$ there are $z \in N^+$ and $z_n \in K \setminus N^+$ such that $z_n \to z, f(z_n) \to y\}$ is the *singularity set* for f^{-1}.

Set

$$K^+ = \{x \in K : f^n(x) \notin N^+ \text{ for all } n \in \mathbb{N}\}$$

and

$$D = \bigcap_{n\in\mathbb{N}} f^n(K^+)$$

The set $\Lambda = \overline{D}$ is called the *attractor*. We have that

$$D = \Lambda \setminus \bigcup_{n\in\mathbb{Z}} f^n(N^+)$$

and that the maps f and f^{-1} are defined on D, with $f(D) = D$.

Let us fix $\varepsilon > 0$ and set for $\ell \geq 1$,

$$D^+_{\varepsilon,\ell} = \{z \in \Lambda : \rho(f^n(z), N^+) \geq \ell^{-1}e^{-\varepsilon n} \text{ for } n \geq 0\},$$

$$D^-_{\varepsilon,\ell} = \{z \in \Lambda : \rho(f^{-n}(z), N^-) \geq \ell^{-1}e^{-\varepsilon n} \text{ for } n \geq 0\},$$

$$D^\pm_\varepsilon = \bigcup_{\ell\geq1} D^\pm_{\varepsilon,\ell}, \quad D^0_\varepsilon = D^+ \cap D^-_\varepsilon.$$

The set D^0_ε is f- and f^{-1}-invariant and $D^0_\varepsilon \subset D$ for every ε. This set is an "essential part" of the attractor and in general, may be empty. Even if it is not it may not support any f-invariant Borel finite measure. We say that Λ is *observable* if

(H3) for all sufficiently small ε the set D_ε^0 supports an f-invariant Borel finite measure.

We shall provide some conditions that ensure that Λ is observable. Given $A \subset \Lambda$, write $f^{-1}(A) = \{z \in \Lambda \setminus N^+ : f(z) \in A\}$. We denote by $U(\varepsilon, N^+)$ the open ε-neighborhood (in K) of N^+, by $\mathcal{M}_f$ the family of f-invariant Borel probability measures on Λ and by $\varphi(z) = \rho(z, N^+)$.

Proposition 19.1. *The set Λ is observable if one of the following conditions holds:*

1. *there exists $\mu \in \mathcal{M}_f$ such that $\mu(D) > 0$ and $\int_\Lambda |\log \varphi| \, d\mu < \infty$;*
2. *there exist $C > 0$, $q > 0$ such that for any $\varepsilon > 0$ and $n \in \mathbb{N}$,*

$$\nu(f^{-n}(U(\varepsilon, N^+) \cap f^n(K^+))) \le C\varepsilon^q \tag{19.1}$$

(here ν is the Riemannian volume in K).

Let us stress that Condition (19.1) is similar to Condition (A4) for conservative systems with singularities.

Denote by $C(x, \alpha, P)$ the cone at $x \in M$ ($\alpha > 0$ is a real number and P is a linear subspace of T_xM), composed of all vectors $v \in T_xM$ for which

$$\angle(v, P) = \min_{w \in P} \angle(v, w) \le \alpha.$$

We say that Λ is a *generalized hyperbolic attractor* if there exist $C > 0$, $\lambda \in (0, 1)$, a function $\alpha(z)$, and two fields of subspaces $P^s(z)$, $P^u(z) \subset T_zM$, $\dim P^s(z) = q$, $\dim P^u(z) = p - q$ ($p = \dim M$) for $z \in K \setminus N^+$ such that the cones $C^s(z) = C^s(z, \alpha(z), P^s(z))$ and $C^u(z) = C(z, \alpha(z), P^u(z))$ satisfy the following conditions:

(H4) the angle between $C^s(x)$ and $C^u(x)$ is uniformly bounded away from zero over $x \in M \setminus S$; in particular, $C^s(x) \cap C^u(x) = \varnothing$;

(H5)

$$df(C^u(x)) \subset C^u(f(x)) \text{ for any } x \in M \setminus S;$$
$$df^{-1}(C^s(x)) \subset C^s(f^{-1}(x)) \text{ for any } x \in f(M \setminus S);$$

(H6) for any $n > 0$,

$$\|df^n v\| \ge C\lambda^{-n}\|v\| \text{ for any } x \in N^+, v \in C^u(x);$$
$$\|df^{-n} v\| \ge C\lambda^{-n}\|v\| \text{ for any } x \in f^n(N^+), v \in C^s(x).$$

Given $z \in D$, the subspaces

$$E^s(z) = \bigcap_{n \ge 0} df^{-n} C^s(f^n(z)), \quad E^u(z) = \bigcap_{n \ge 0} df^n C^u(f^{-n}(z))$$

satisfy

(E1) $T_zM = E^s(z) \oplus E^u(z)$, $E^s(z) \cap E^u(z) = \{0\}$;
(E2) the angle between $E^s(z)$ and $E^u(z)$ is uniformly bounded away from zero;
(E3) for each $n \ge 0$,

$$\|df^n v\| \le C\lambda^n \|v\|, \quad v \in E^s(z),$$
$$\|df^{-n} v\| \ge C^{-1}\lambda^{-n}\|v\|, \quad v \in E^u(z).$$

The subspaces $E^s(z)$ and $E^u(z)$ determine a uniform hyperbolic structure for f on the set D. One can construct local stable and unstable manifolds $V^s(z)$, $V^u(z)$, at every point $z \in D_\varepsilon^0$; in fact, local stable (respectively, unstable) manifolds can be constructed for every $z \in D_\varepsilon^+$ (respectively, for every $z \in D_\varepsilon^-$). Since f has singularities the "size" of local manifolds is a measurable (not continuous) function on D^0_ε, despite the fact that the hyperbolic structure on D is uniform. The size can deteriorate along trajectories but with subexponential rate; it is uniform over the points in $D^0_{\varepsilon,\ell}$.

To simplify our notations we drop the subscript ε from in $D^\pm_\varepsilon$, $D^\pm_{\varepsilon,\ell}$, $D^0_{\varepsilon,\ell}$, etc.

Proposition 19.2. *$V^u(z) \subset D^-$ for any $z \in D^-$.*

Let $A \subset \Lambda$. Define

$$\hat{f}(A) = f(A \setminus N^+), \quad \hat{f}^{-1}(A) = \hat{f}^{-1}(A \setminus N^-).$$

The sets $\hat{f}^n(A)$ and $\hat{f}^{-n}(A)$ for $n > 1$ are defined in the same way. Given $z \in D^0$ we set

$$W^s(z) = \bigcup_{n\ge 0} \hat{f}^{-n}(V^s(f^n(z))), \quad W^u(z) = \bigcup_{n\ge 0} \hat{f}^{n}(V^u(f^{-n}(z))).$$

The set $W^s(z)$ is a smooth embedded, but possibly not connected, submanifold in K. It is called the *global stable manifold* at z. If $y \in W^s(z)$ then all images $f^n(y)$, $n \ge 0$ are well-defined. Similar statements hold for $W^u(z)$, the *global unstable manifold* at z.

For $y \in W^s(z)$ denote by $B^s(y,r)$ the ball in $W^s(z)$ of radius r centered at y (we restrict ourselves to a connected component of $W^s(z)$). Fix $r > 0$ and take $y \in W^s(z)$, $w \in B^s(y,r)$, $n \ge 0$ (respectively $y \in W^u(z)$, $w \in B^u(y,r)$, $n \le 0$). We have

$$\rho^s(f^n(y), f^n(w)) \le C\mu^n \rho^s(y,w)$$

and respectively,

$$\rho^u(f^{-n}(y), f^{-n}(w)) \le C\mu^n \rho^u(y,w),$$

where $C = C(r) > 0$ is a constant, ρ^s and ρ^u are respectively the distances induced by ρ on $W^s(z)$ and $W^u(z)$.

19.2. **SRB-measures: existence and ergodic properties.** We outline a construction of SRB-measures for diffeomorphisms with generalized hyperbolic attractors. Denote by $J^u(z)$ the Jacobian of the map $df|E^u(z)$ at a point $z \in D^0$. Fix $\ell > 0$, $z \in D^0_\ell$, $y \in W^u(z)$, and $n > 0$, and set

$$\kappa_n(z,y) = \prod_{j=0}^{n-1} \frac{J^u(f^{-j}(z))}{J^u(f^{-j}(y))}.$$

Proposition 19.3. *The following properties hold:*

1. *For any $\ell \ge 1$ and $z \in D^0_\ell$, $y \in W^u(z)$ there exists the limit*

$$\kappa(z,y) = \lim_{n\to\infty} \kappa_n(z,y) > 0.$$

   Moreover, there is $r^1_\ell > 0$ such that for any $\varepsilon > 0$, $r \in (0, r^1_\ell)$ one can find $N = N(\varepsilon, r)$ such that for any $n \ge N$,

$$\max_{z\in D^0_\ell} \max_{y\in \overline{B}^u(z,r)} |\kappa_n(z,y) - \kappa(z,y)| \le \varepsilon.$$

2. *The function $\kappa(z,y)$ is continuous on D^0_ℓ.*
3. *For any $z \in D^0_\ell$ and y_1, $y_2 \in W^u(z)$,*

$$\kappa(z,y_1)\kappa(y_1,y_2) = \kappa(z,y_2).$$

Fix $\ell \geq 1$, $z \in D^0_\ell$ and let $B(z,r)$ be a ball in K centered at z of radius r. Define a *rectangle* at z by

$$\Pi = \Pi(z,r) = \bigcup_{y \in B(z,r) \cap D^0_\ell} B^u([y,z],r),$$

where $[y,z] = V^u(y) \cap V^s(z)$. Consider the partition $\xi = \xi(\Pi)$ of $\Pi(z,r)$ by the sets $C_\xi(y) = B^u([y,z],r)$, $y \in B(z,r) \cap D^0_\ell$. This partition is continuous and measurable with respect to any Borel measure μ on Λ.

Fix $z \in D^0_\ell$ and a rectangle $\Pi = \Pi(z,r)$ at z. Assume that $\mu(\Pi) > 0$ and denote by $\mu_\xi(y)$, $y \in B(z,r) \cap D^0_\ell$ the family of conditional measures on the sets $C_\xi(y)$. We say that μ is an *SRB-measure* if for any $\ell \geq 0$, $z \in D^0_\ell$, and $\Pi = \Pi(z,r)$ with $\mu(\Pi) > 0$,

$$d\mu_\xi(y') = r(y)\kappa([z,y],y')d\nu^u(y').$$

Here ν^u is the Riemannian volume on $W^u(z)$ induced by the Riemannian metric, $y \in B(z,r) \cap D^0_\ell$, $y' \in B^u([z,y],r)$ and $r(y)$ is the "normalizing factor",

$$r(y) = \left(\int_{B^u([y,z],r^3_\ell)} \kappa([z,y],y')\, d\nu^u(y') \right)^{-1}.$$

Denote by $\mathcal{M}'_f$ the family of measures $\mu \in \mathcal{M}_f$ for which $\mu(D^0) = 1$ and by $\mathcal{M}^u_f$ the family of SRB-measures in $\mathcal{M}'_f$. Any $\mu \in \mathcal{M}'_f$ is a measure with nonzero Lyapunov exponents $\chi^1(x)$, …, $\chi^p(x)$ and if μ is ergodic the function $\chi^i(x)$ are constant μ-almost everywhere. We denote the corresponding values by χ^i_μ and assume that

$$\chi^1_\mu \geq \cdots \geq \chi^q_\mu > 0 > \chi^{q+1}_\mu \geq \cdots \geq \chi^p_\mu.$$

Fix $z \in D^0_\ell$, $r > 0$ and set

$$U_0 = B^u(z,r), \quad \widetilde{U}_0 = U_0, \quad \widetilde{U}_n = f(U_{n-1}), \quad U_n = \widetilde{U}_n \setminus N^+,$$

and

$$c_0 = 1, \quad c_n = \left(\prod_{k=0}^{n-1} J^u(f^k(z)) \right)^{-1}.$$

We define measures $\widetilde{\nu}_n$ on U_n by

$$d\widetilde{\nu}_n(y) = c_n \kappa(f^n(z),y)d\nu^u(y), n \geq 0,$$

and measures ν_n on Λ by

$$\nu_n(A) = \widetilde{\nu}_n(A \cap U_n), n \geq 0 \tag{19.2}$$

for each Borel set $A \subset \Lambda$.

We say that the attractor Λ satisfies Condition (H7) if there exist a point $z \in D^0$ and constants $C > 0$, $t > 0$, $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ and $n \geq 0$,

$$\nu^u(V^u(z) \cap f^{-n}(U(\varepsilon, N^+))) \leq C\varepsilon^t.$$

If Λ satisfies Condition (H7) then $\nu_n(A) = \nu_0(f^{-n}A))$ for any $n > 0$ and any Borel set $A \subset \Lambda$.

Theorem 19.4 (Pesin [200]). *Assume that Λ is a generalized hyperbolic attractor satisfying Condition (H7). Then there exists a measure $\mu \in \mathcal{M}^u_f$ supported on D^0 which satisfies Conditions 1 and 2 of Proposition 19.1.*

We outline the proof of the theorem. Let $z \in D^0$ be the point mentioned in Condition (H7) and ν_k the sequence of measures on Λ as in (19.2). Consider the sequence of measures on Λ defined by

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \nu_k. \tag{19.3}$$

First, using Condition (H7), one can show that for any $\gamma > 0$ there exists $\ell_0 > 0$ such that $\mu_n(D^0_\ell) \geq 1 - \gamma$ for any $n > 0$ and $\ell \geq \ell_0$. It follows that some limit measure μ for the sequence of measures μ_n is supported on D^0. Next, one can prove that μ is f-invariant and an SRB-measure on Λ.

From now on we assume that Λ is a generalized hyperbolic attractor satisfying Condition (H7) and that $\mu \in \mathcal{M}^u_f$, $\mu(D^0) = 1$. We will describe the ergodic properties of μ.

Proposition 19.5. *For μ-almost every $z \in D^0$,*

$$\mu^u(D^0 \cap V^u(z)) = 1. \tag{19.4}$$

Fix $z \in D^0$ for which (19.4) holds and choose ℓ such that $\nu^u(D^0_\ell \cap V^u(z)) > 0$. Let W be a smooth submanifold in a small neighborhood of $V^u(z)$ of the form

$$W = \{\exp_z(w, \varphi(w)) : w \in I \subset E^u(z)\},$$

where I is an open subset and $\varphi\colon I \to E^s(z)$ is a diffeomorphism. W has the same dimension as $V^u(z)$ and is transverse to $V^s(y)$ for all $y \in D^0_\ell \cap V^u(z)$. Consider the map $p\colon D^0_\ell \cap V^u(z) \to W$ where $p(y)$ is the point of intersection of $V^s(y)$ and W. We denote by ν_W the measure on W induced by the Riemannian metric on W (considered as a submanifold of M). One can prove the following result using arguments in the proof of Theorem 10.1.

Proposition 19.6. *The measure $p_*\nu^u$ is absolutely continuous with respect to ν_W.*

Fix $z \in D^0$ and for each $\ell > 0$ set

$$Q(\ell, z) = \bigcup_{y \in D^0_\ell \cap V^u(z)} V^s(y) \cap \Lambda.$$

One can show that for μ-almost every $z \in \Lambda$ and any sufficiently large $\ell > 0$ we have $\mu(Q(\ell, z)) > 0$ and the set $Q = \bigcup_{n\in\mathbb{Z}} f^n(Q(\ell, z))$ is an ergodic component of positive measure for the map $f|\Lambda$. This implies the following description of the ergodic properties of the map $f|\Lambda$ with respect to the SRB-measure μ.

Theorem 19.7 (Pesin [200])**.** *Let $\mu \in \mathcal{M}^u_f$. Then there exist sets $\Lambda_i \subset \Lambda$, $i = 0, 1, 2, \ldots$ such that:*

1. $\Lambda = \bigcup_{i\geq 0} \Lambda_i$, $\Lambda_i \cap \Lambda_j = \varnothing$ *for* $i \neq j$, $i, j = 0, 1, 2, \ldots$;
2. $\mu(\Lambda_0) = 0$, $\mu(\Lambda_i) > 0$ *for* $i > 0$;
3. *for* $i > 0$, $\Lambda_i \subset D$, $f(\Lambda_i) = \Lambda_i$, $f|\Lambda_i$ *is ergodic;*
4. *for* $i > 0$, *there exists a decomposition* $\Lambda_i = \bigcup_{j=1}^{n_i} \Lambda_i^j$, $n_i \in \mathbb{N}$, *where*
   (a) $\Lambda_i^{j_1} \cap \Lambda_i^{j_2} = \varnothing$ *for* $j_1 \neq j_2$;
   (b) $f(\Lambda_i^j) = \Lambda_i^{j+1}$ *for* $j = 1, 2, \ldots, n_i - 1$, *and* $f(\Lambda_i^{n_i}) = \Lambda_i^1$;
   (c) $f^{n_i}|\Lambda_i^1$ *is isomorphic to a Bernoulli automorphism;*

5. *the metric entropy* $h_\mu(f|\Lambda)$ *satisfies*

$$h_\mu(f|\Lambda) = \int_\Lambda \sum_{i=1}^{u(x)} \chi_i(x)\, d\mu(x),$$

where $\chi_1(x),\ \chi_2(x),\ \dots,\ \chi_{u(x)}(x)$ *is the collection of positive values of the Lyapunov exponent, counted with multiplicities;*

6. *there exists a partition* η *of* Λ *with the following properties:*
   (a) *for* μ*-almost every* $x \in \Lambda$ *the element* $C_\eta(x)$ *of the partition* η *is an open subset in* $W^u(x)$*;*
   (b) $f\eta \geq \eta$, $\bigvee_{k\geq 0} f^k\eta = \varepsilon$, $\bigwedge_{k\geq 0} f^k\eta = \nu(W^u)$, *where* $\nu(W^u)$ *is the measurable hull of the partition of* Λ *consisting of single leaves* $W^u(x)$ *if* $x \in D^0$ *and single points* $\{x\}$ *if* $x \in \Lambda \setminus D^0$*;*
   (c) $h(f|\Lambda, \eta) = h_\mu(f|\Lambda)$.

Set

$$W^s(\Lambda) = \bigcup_{z\in D^0} W^s(z).$$

The following is a direct consequence of Proposition 19.1 and Theorem 19.7.

Theorem 19.8 (Pesin [200])**.** *Let* $\mu \in \mathcal{M}_f^u$*. Then for any set* Λ_i *with* $i > 0$ *as in Theorem 19.7 we have:*

1. *the Riemannian volume of* $W^s(\Lambda_i)$ *is positive;*
2. *there exists* $A_i \subset \Lambda$ *such that* $\mu(A_i) = \mu(\Lambda_i)$ *and for any* $z \in W^s(A_i)$ *and any continuous function* φ *on* M *there exists the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(z)) = \frac{1}{\mu(\Lambda_i)} \int_{\Lambda_i} \varphi\, d\mu.$$

Using the above results one can now describe the class of all SRB-measures on Λ.

Theorem 19.9 (Pesin [200])**.** *There exists sets* Λ_n, $n = 0, 1, 2, \dots$ *and measures* $\mu_n \in \mathcal{M}_f^u$, $n = 1, 2, \dots$ *such that:*

1. $\Lambda = \bigcup_{n\geq 0} \Lambda_n$, $\Lambda_n \cap \Lambda_m = \varnothing$ *for* $n \neq m$*;*
2. *the Riemannian volume of* $W^s(\Lambda_n) \cap W^s(\Lambda_m)$ *is zero for* $n \neq m$, n, $m > 0$*;*
3. *for* $n > 0$, $\Lambda_n \subset D$, $f(\Lambda_n) = \Lambda_n$, $\mu_n(\Lambda_n) = 1$, *and* $f|\Lambda_n$ *is ergodic with respect to* μ_n*;*
4. *for* $n > 0$*, there exist* $k_n > 0$ *and a subset* $A_n \subset \Lambda_n$ *such that*
   (a) *the sets* $A_{n,i} = f^i(A_n)$ *are pairwise disjoint for* $i = 1, \dots, k_n - 1$ *and* $A_{n,k_n} = A_{n,1}$, $\Lambda = \bigcup_{i=1}^{k_n-1} A_{n,i}$*;*
   (b) $f^{k_n}|A_{n,1}$ *is a Bernoulli automorphism with respect to* μ_n*;*
5. *if* $\mu \in \mathcal{M}_f^u$*, then* $\mu = \sum_{n>0} \alpha_n\mu_n$ *with* $\alpha_n \geq 0$ *and* $\sum_{n>0} \alpha_n = 1$*;*
6. *if* ν *is a measure on* K *absolutely continuous with respect to the Riemannian volume and* $\nu_n = \nu|W^s(\Lambda_n)$, $n > 0$*, then*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k-1} f_*^i \nu_n = \mu_n.$$

To conclude let us mention a connection between SRB-measures and Condition (H7). Notice that any accumulation point of the sequence of measures in (19.3) is an SRB-measure (this essentially follows from Theorem 19.4). We describe a special property of such measures.

Proposition 19.10. *If μ is the SRB-measure constructed in Theorem 19.4 then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $n > 0$,*

$$\mu(U(\varepsilon, N^+)) \leq C\varepsilon^t, \tag{19.5}$$

where $C > 0$, $t > 0$ are constants independent of ε and n.

We have seen that Condition (H7) is sufficient to prove the existence of an SRB-measure on a generalized hyperbolic attractor. We will now show that it is "almost" necessary.

Proposition 19.11. *Let $\mu \in \mathcal{M}_f^u$ (μ is an SRB-measure on Λ and $\mu(D^0) = 1$) satisfy* (19.5) *for some constants C, t, ε. Then for μ-almost every point $z \in D^0$ there exists $\varepsilon(z) > 0$ such that Condition (H7) holds with respect to z and any $\varepsilon \in (0, \varepsilon(z))$.*

19.3. **Examples.** We now consider a number of examples of maps with generalized hyperbolic attractor when M is a two-dimensional manifold. First we formulate some general assumptions which guarantee the validity of Properties (H3) and (H7). Let f be a map satisfying Condition (H1) and assume that:

(G1) $K = \bigcup_{i=1}^m K^i$, with K^i to be a closed sets, $\operatorname{int} K^i \cap \operatorname{int} K^j = \varnothing$ whenever $i \neq j$,
$$\partial K^i = \bigcup_{j=1}^{r_i} N_{ij} \cup \bigcup_{j=1}^{q_i} M_{ij},$$
where N_{ij} and M_{ij} are smooth curves, and
$$N = \bigcup_{i=1}^{m} \bigcup_{j=1}^{r_i} N_{ij}, \quad \partial K = \bigcup_{i=1}^{m} \bigcup_{j=1}^{q_i} M_{ij};$$
(G2) f is continuous, and differentiable on each K^i, $i = 1, \ldots, m$;
(G3) f possesses two families of stable and unstable cones $C^s(z)$ and $C^u(z)$, $z \in K \setminus \bigcup_{i=1}^m \partial K^i$;
(G4) the unstable cone $C^u(z)$ at z depends continuously on $z \in K^i$ and there exists $\alpha > 0$ such that for any $z \in N_{ij} \setminus \partial N_{ij}$, $v \in C^u(z)$, and any vector w tangent to N_{ij} we have $\angle(v, w) \geq \alpha$;
(G5) there exists $\tau > 0$ such that $f^k(N) \cap N = \varnothing$, $k = 0, \ldots, \tau$ and $a^\tau > 2$ where
$$a = \inf_{z \in K \setminus N} \inf_{v \in C^u(z)} \|d_z f v\| > 1.$$

Theorem 19.12 (Pesin [200])**.** *If f satisfies Conditions (H1) and (G1)–(G5), then it also satisfies Condition (H7) for any $z \in D^0$ and* (19.1) *(in particular, f satisfies Condition (H3)).*

Assume now that f satisfies Conditions (H1)–(H2), (G1)–(G2), (G4), and (instead of (G3) and (G5)) the following holds:

(G3') $\rho(f^k(N), n) \geq A \exp(-\gamma k)$, $k = 1, 2, \ldots$,

where $A > 0$ is a constant and $\gamma > 0$ is sufficiently small (when compared with λ; in particular, $f^k(N) \cap N = \varnothing$, $k = 1, 2, \ldots$). Then f satisfies Condition (H7) for any $z \in D^0$ and Condition (H3).

We now describe some particular two-dimensional maps with generalized hyperbolic attractors.

Lorenz type attractors. Let $I = (-1, 1)$ and $K = I \times I$. Let also $-1 = a_0 < a_1 < \cdots < a_q < a_{q+1} = 1$. Set

$$P_i = I \times (a_i, a_{i+1}), i = 0, \dots, q, \quad \ell = I \times \{a_0, a_1, \dots, a_q, a_{q+1}\}.$$

Let $T\colon K \setminus \ell \to K$ be an injective map,

$$T(x, y) = (f(x, y), g(x, y)), \quad x, y \in I, \tag{19.6}$$

where the functions f and g satisfy the following conditions:

(L1) f and g are continuous on $\overline{P}_i$ and

$$\lim_{y \nearrow a_i} f(x, y) = f_i^-, \lim_{y \nearrow a_i} g(x, y) = g_i^-,$$

$$\lim_{y \searrow a_i} f(x, y) = f_i^+, \lim_{y \searrow a_i} g(x, y) = g_i^+,$$

where $f_i^\pm$ and $g_i^\pm$ do not depend on x, $i = 1, 2, \dots, q$;

(L2) f and g have continuous second derivatives on P_i and if $(x, y) \in P_i$, $i = 1, \dots, q$, then

$$df(x, y) = B_i^1 (y - a_i)^{-\nu_i^1} (1 + A_i^1(x, y))$$
$$dg(x, y) = C_i^1 (y - a_i)^{-\nu_i^2} (1 + D_i^1(x, y))$$

whenever $y - a_i \le \gamma$, and

$$df(x, y) = B_i^2 (a_{i+1} - y)^{-\nu_i^3} (1 + A_i^2(x, y))$$
$$dg(x, y) = C_i^2 (a_{i+1} - y)^{-\nu_i^4} (1 + D_i^2(x, y))$$

whenever $a_{i+1} - y \le \gamma$, where $\gamma > 0$ is sufficiently small, B_i^1, B_i^2, C_i^1, C_i^2 are some positive constants, $0 \le \nu_i^1$, ν_i^2, ν_i^3, $\nu_i^4 < 1$, and $A_i^1(x, y)$, $A_i^2(x, y)$, $D_i^1(x, y)$, $D_i^2(x, y)$ are continuous functions, which tend to zero when $y \to a_i$ or $y \to a_{i+1}$ uniformly over x; furthermore, the norms of the second derivatives $\|f_{xx}\|$, $\|f_{xy}\|$, $\|g_{xy}\|$, and $\|g_{xx}\|$ are bounded;

(L3) we have the inequalities

$$\|f_x\| < 1, \quad \|g_y^{-1}\| < 1,$$

$$1 - \|g_y^{-1}\| \cdot \|f_x\| > 2\sqrt{\|g_y^{-1}\| \cdot \|g_x\| \cdot \|g_y^{-1} f_y\|},$$

$$\|g_y^{-1}\| \cdot \|g_x\| < (1 - \|f_x\|)(1 - \|g_y^{-1}\|),$$

where $\|\cdot\| = \max_{i=0,\dots,q} \sup_{(x,y) \in P_i} |\cdot|$.

The class of maps satisfying (L1)–(L3) was introduced in [2]. It includes the famous geometric model of the Lorenz attractor. The latter can be described as follows.

Theorem 19.13. *Assume that $\ell = I \times \{0\}$, $K = I \times I$, and that $T\colon \setminus \ell \to K$ is a map of the form* (19.6) *where the functions f and g are given by*

$$f(x, y) = (-B|y|^{\nu_0} + Bx \operatorname{sgn} y |y|^\nu + 1) \operatorname{sgn} y,$$

$$g(x, y) = ((1 + A)|y|^{\nu_0} - A) \operatorname{sgn} y.$$

If $0 < A < 1$, $0 < B < 1/2$, $\nu > 1$, and $1/(1 + A) < \nu_0 < 1$, then T satisfies Conditions (L1)–(L3).

The class of maps introduced above is somewhat representative.

Theorem 19.14. *On an arbitrary smooth compact Riemannian manifold of dimension at least* 3 *there exists a vector field X having the following property: there is a smooth submanifold S such that the first-return map T induced on S by the flow of X satisfies Conditions (L1)–(L3).*

We now describe the ergodic and topological properties of the maps with Lorenz type attractors.

Theorem 19.15 (Pesin [200])**.** *The following properties hold:*

1. *A map T satisfying (L1)–(L3) also satisfies Conditions (H1), (H2) and the attractor Λ for T is an observable generalized hyperbolic attractor; the stable (unstable) cone at each point $z \in K$ is the set of vectors having angle at most $\pi/6$ with the horizontal (vertical) line.*
2. *The stable lamination W^s can be extended to a continuous C^1-foliation in K.*
3. *Assume that one of the following condition holds:*
   (a) $\nu_i^j = 0$, $i = 1, \dots, q$, $j = 1, 2, 3, 4$;
   (b) $\rho(T^n(f_i^\pm, g_i^\pm), \ell) \ge C_i \exp(-\gamma n)$ *for any* $n \ge 0$, $i = 1, \dots, q$ ($C_i > 0$ *are constants independent of* n*;* γ *is sufficiently small).*

   Then T satisfies Conditions (G1)–(G5) (as well as (G1), (G2), (G3'), and (G4)). In particular, it satisfies Condition (H7) for any $z \in D^0$ and (19.1).

The existence of an SRB-measure for the classical geometric model of Lorenz attractor (when K is a square, and ℓ consists of a single interval) was shown in [51]. The proof uses Markov partitions. If the stable foliation W^s is smooth (it takes place, for example, when g does not depend on x) the existence of an SRB-measure follows from a well-known result in the theory of one-dimensional maps (one can show that Λ is isomorphic to the inverse limit of a one-dimensional piecewise expanding map for which (a_i, a_{i+1}), $i = 0, \dots, q$ are intervals of monotonicity; see [2] for details and references).

We now give an example of Lorenz type attractor for which the discontinuity set consists of countable number of intervals and the corresponding map has countable number of components of topological transitivity. Consider a one-dimensional map $g(y)$, $y \in [0, 1]$ given by

$$g(y) = \begin{cases} \frac{1}{n+2} + \frac{2}{2n+1} y & \text{if } \frac{1}{n+1} \le y < \frac{2n+1}{2(n+1)} \\ \frac{2n+1}{2(n+1)} + \frac{1}{2(n+1)} y & \text{if } \frac{2n+1}{2(n+1)} \le y < \frac{1}{n} \end{cases}$$

for $n = 1, 2, 3, \dots$. One can show that there exists a function $f(x, y)$ such that the map $T(x, y) = (f(x, y), g(y))$ satisfies Conditions (L1)–(L3). However, each set $\Lambda \cap I \times [1/(n+1), 1/n]$ is a component of topological transitivity for T.

Lozi type attractors. Let $c > 0$, $I = (0, c)$, $K = I \times I$, and $0 = a_0 < a_1 < \dots < a_q < a_{q+1} = c$. Set $\ell = \{a_0, a_1, \dots, a_q, a_{q+1}\} \times I$ and let $T\colon K \to K$ be an injective continuous map

$$T(x, y) = (f(x, y), g(x, y)), \quad x, y \in I$$

satisfying the following conditions:

Loz1. $T|(K \setminus \ell)$ is a C^2-diffeomorphism and the second derivatives of the maps T and T^{-1} are bounded from above;

Loz2. $\mathrm{Jac}(T) < 1$;

Loz3. $\inf\{(|\frac{\partial f}{\partial x}| - |\frac{\partial f}{\partial y}|) - (|\frac{\partial g}{\partial x}| + |\frac{\partial g}{\partial y}|)\} \ge 0$;

Loz4. $\inf\{|\frac{\partial f}{\partial x}| - |\frac{\partial f}{\partial y}|\} \overset{\text{def}}{=} u > 1$;

Loz5. $\sup\{(|\frac{\partial f}{\partial x}| + |\frac{\partial g}{\partial y}|)/(|\frac{\partial f}{\partial x}| - |\frac{\partial f}{\partial y}|)^2\} < 1$;
Loz6. there exists $N > 0$ such that $T^k(\ell) \cap \ell = \varnothing$ for $1 \le k \le N$ and $u^N > 2$.

This class of maps was introduced by Young in [256]. It includes the map

$$T(x, y) = (1 + by - a|x|, x) \tag{19.7}$$

which is obtained from the well-known Lozi map by a change of coordinates (see [159]). It is easy to verify that there exist open intervals of a and b such that (19.7) takes some square $[0, c] \times [0, c]$ into itself and satisfies Loz1–Loz6.

Theorem 19.16 (Pesin [200]). *The following properties hold:*

1. *A map T satisfying Loz1–Loz6 also satisfies Conditions (H1), (G1)–(G5), and the attractor Λ for T is an observable generalized hyperbolic attractor; the stable (respectively unstable) cone at each point $z \in K$ has a vertical (respectively horizontal) line as the center line. This map also satisfies Condition (H7) for any $z \in D^0$ and* (19.1)*.*
2. *The stable lamination W^s can be extended to a continuous C^0-foliation in K.*

Belykh type attractors. Let $I = [-1, 1]$, $K = I \times I$, and $\ell = \{(x, y) : y = kx\}$. Consider the map

$$T(x, y) = \begin{cases} (\lambda_1(x-1)+1, \lambda_2(y-1)+1) & \text{for } y > kx \\ (\mu_1(x+1)-1, \mu_2(y+1)-1) & \text{for } y < kx \end{cases}.$$

In the case $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$ this map was introduced by Belykh in [28] and was the simplest model in the so-called phase synchronization theory.

Theorem 19.17. *The following properties hold:*

1. *Assume that*

   $$0 < \lambda_1 < \frac{1}{2}, 0 < \mu_1 < \frac{1}{2}, 1 < \lambda_2 < \frac{2}{1-|k|}, 1 < \mu_2 < \frac{2}{1-|k|}, |k| < 1.$$

   Then T is a map from $K \setminus \ell$ into K satisfying Conditions (H1), (G1)–(G4), and the attractor Λ for T is a generalized hyperbolic attractor (the stable and unstable one-dimensional subspaces at each point $z \in D^0$ are respectively horizontal and vertical lines; the stable and unstable cones at each point $z \in K$ are the set of vectors having angle at most $\pi/4$ with the horizontal or vertical lines).
2. *If $\lambda_2 > 2$ and $\lambda_2 > 2$, then T satisfies Condition (G5), and hence, Condition (H7) for any $z \in D^0$ and* (19.1)*.*

Appendix A. Decay of correlations, by Omri Sarig

A.1. **Introduction.** One way of saying that a probability preserving transformation $(X, \mathcal{B}, m, T)$ has unpredictable dynamics is to claim that the results of a 'measurement at time zero' $f(x)$ and a 'measurement at time n' $g(T^n x)$ are correlated very weakly for large n. The correlation coefficient of two random variables f_1, f_2 is defined to be $\frac{Cov(f_1,f_2)}{\|f_1\|_2\|f_2\|_2}$, where $Cov(f_1, f_2) := \int f_1 f_2 - \int f_1 \int f_2$. This suggests the following definition:

Definition 1. *A probability preserving transformation $(X, \mathcal{B}, m, T)$ is called* strongly mixing *if for every $f, g \in L^2$, $Cov(f, g \circ T^n) := \int fg \circ T^n - \int f \int g \xrightarrow[n\to\infty]{} 0$.*

It is natural to ask for the speed of convergence (the faster it is the less predictable the system seems to be). Unfortunately, without extra assumptions, the convergence can be arbitrarily slow: For all sequences $\varepsilon_n \downarrow 0$ and all $0 \neq g \in L^2$ s.t. $\int g = 0$, $\exists f \in L^2$ with $Cov(f, g \circ T^n) \neq O(\varepsilon_n)$.[8]

We will therefore refine the question stated above and ask: *How fast does* $Cov(f, g \circ T^n) \to 0$ *for* f, g *in a given collection of functions* $\mathcal{L} \subsetneq L^2$*?* The collection $\mathcal{L}$ varies from problem to problem. In practice, the challenge often reduces to the problem of identifying a class of functions $\mathcal{L}$ which is large enough to generate $\mathcal{B}$, but small enough to admit analysis.

We discuss this problem below. The literature on this subject is vast, and cannot be covered in an appendix of this size. We will therefore focus on the *methods* used to attack the problem, rather than their actual application (which is almost always highly non-trivial, but also frequently very technical). The reader is referred to Baladi's book [17] for a more detailed account and a more complete bibliography.

In what follows, $(X, \mathcal{B}, m, T)$ is a probability preserving transformation, and $\mathcal{L}$ is a collection of square integrable functions. We assume for simplicity that T is *non*–invertible (the methods we describe below can be applied in invertible situations, but are easier to understand in the non-invertible setting). A key concept is:

Definition 2. *The* transfer operator *(or* dual operator*, or* Frobenius–Perron operator*) of* T *is* $\widehat{T} : L^1 \to L^1$ *where* $\widehat{T}f$ *is the unique* L^1*–function s.t.:*

$$\forall g \in L^\infty, \int g \cdot \widehat{T}f = \int g \circ T \cdot f.$$

The definition of $\widehat{T}$ is tailored to make the following statement correct: If $d\mu = f dm$, then $d\mu \circ T^{-1} = \widehat{T} f dm$. Thus, $\widehat{T}$ is the action of T on density functions.

It is easy to check that $\widehat{T}$ is a positive operator, a contraction (i.e. $\|\widehat{T}f\|_1 \leq \|f\|_1$) and that $\|\widehat{T}f\|_1 = \|f\|_1$ for all $f \geq 0$. The T–invariance of m implies that $\widehat{T}1 = 1$. The relation between $\widehat{T}$ and $Cov(f, g \circ T^n)$ is the following identity:

$$Cov(f, g \circ T^n) = \int \big[\widehat{T}^n f - \int f\big] g. \tag{A.1}$$

We see that the asymptotic behavior of $Cov(f, g \circ T^n)$ can be studied by analyzing the asymptotic behavior of $\widehat{T}^n$ as $n \to \infty$. This is the viewpoint we adopt here.

A.2. **Spectral gap and exponential decay of correlations.** Suppose $\mathcal{L}$ is a Banach space of square integrable functions s.t. $1 \in \mathcal{L}$, $\widehat{T}(\mathcal{L}) \subseteq \mathcal{L}$, and $\|\cdot\|_{\mathcal{L}} \geq \|\cdot\|_1$. We already mentioned that 1 is an eigenvalue of $\widehat{T}$. The operator $Pf := \int f$ is a projection onto its eigenspace.

We say that $\widehat{T}$ has a *spectral gap* in $\mathcal{L}$, if the spectrum of $\widehat{T} - P \in Hom(\mathcal{L}, \mathcal{L})$ is a proper subset of the open unit disc, or equivalently, if the $\mathcal{L}$–spectral radius of $\widehat{T} - P$, which we denote by $\rho_{\mathcal{L}}$, is strictly less than one.

To see the connection with decay of correlations, note that $\widehat{T}^n - P = (\widehat{T} - P)^n$, because $\widehat{T}P = P\widehat{T}$ and $P^2 = P$. Therefore, if $\rho_{\mathcal{L}} < \lambda < 1$, $f \in \mathcal{L}$ and $g \in L^\infty$, then $|Cov(f, g \circ T^n)| \leq \int \left| g(\widehat{T} - P)^n f \right| \leq \|g\|_\infty \|(\widehat{T} - P)^n f\|_{\mathcal{L}} = O(\lambda^n) \|f\|_{\mathcal{L}} \|g\|_\infty$. Thus, a spectral gap in $\mathcal{L}$ implies exponential decay of correlations in $\mathcal{L}$.

[8]Otherwise, the functionals $\varphi_n(f) := \frac{1}{\varepsilon_n} \int f g \circ T^n$ are pointwise bounded on L^2, whence by the Banach–Steinhaus theorem uniformly bounded. But $\|\varphi_n\| = \frac{1}{\varepsilon_n} \|g\|_2 \to \infty$. (Y. Shalom)

The question arises how to find a space $\mathcal{L}$ such that $\widehat{T}:\mathcal{L}\to\mathcal{L}$ has a spectral gap. We discuss two general methods. The first establishes a spectral gap directly, and the second indirectly.

A.2.1. *Double norm inequalities.* Consider the action of T on mass distributions fdm. If T is very chaotic, then its action will tend to flatten the mountains of fdm and to fill-up its crevices. After many iterations, the irregularity of the original mass distribution disappears, and the shape of $\widehat{T}^n fdm \equiv (fdm)\circ T^{-n}$ depends only on the size (total mass) of fdm, and not on its shape.

It is a deep insight of Doeblin & Fortet [71] that this phenomena is captured by the following double norm inequality, and that this inequality can be used to establish a spectral gap:

$$\|\widehat{T}^n f\|_{\mathcal{L}} \le \theta^n \|f\|_{\mathcal{L}} + M\|f\|_{\mathcal{C}} \quad (n\in\mathbb{N}).$$

Here $\|\cdot\|_{\mathcal{L}}$ measures regularity (Lipschitz, BV, etc.), $\|\cdot\|_{\mathcal{C}}$ measures size (L^∞, L^1, etc.), and $0<\theta<1$, $M>0$ are independent of n. We present the functional-analytic machinery in the form given by Ionescu-Tulcea & Marinescu [122] (see Hennion [107] for refinements):

Theorem A.1 (Doeblin & Fortet, Ionescu-Tulcea & Marinescu)**.** *Let $\mathcal{C}\supseteq\mathcal{L}$ be two Banach spaces such that $\mathcal{L}$–bounded sets are precompact in $\mathcal{C}$, and such that*

$$x_n\in\mathcal{L}, \sup\|x_n\|_{\mathcal{L}}<\infty,\ \|x_n-x\|_{\mathcal{C}}\to 0 \Rightarrow x\in\mathcal{L},\ \text{and } \|x\|_{\mathcal{L}}\le \sup\|x_n\|_{\mathcal{L}}.$$

Let S be a bounded linear operator on $\mathcal{L}$. If $\exists M,H>0$, $0<\theta<1$ s.t. for all $x\in\mathcal{L}$

$$\sup_{n\ge1}\|S^n x\|_{\mathcal{C}}\le H\|x\|_{\mathcal{L}} \text{ and } \|Sx\|_{\mathcal{L}}\le\theta\|x\|_{\mathcal{L}}+M\|x\|_{\mathcal{C}},$$

then $S=\sum_{i=1}^p \lambda_i P_i + N$ where $p<\infty$, $P_i^2=P_i$, $P_iP_j=0$ $(i\ne j)$, $P_iN=NP_i=0$, $\dim\operatorname{Im}(P_i)<\infty$, and $\|N^n\|=O(\kappa^n)$ for some $0<\kappa<1$.

In other words, the theorem gives sufficient conditions for the $\mathcal{L}$–spectrum of S to consist of a compact subset of the open unit disc, and a finite number of eigenvalues λ_i with finite multiplicity. The assumptions of the theorem clearly also imply that $|\lambda_i|\le 1$ for all i.

It follows that if S has no eigenvalues of modulus one other than a simple eigenvalue $\lambda=1$, then S has a spectral gap. This is always the case for the transfer operator as soon as $\mathcal{L}\subset L^1$ and $(X,\mathcal{B},m,T)$ is exact (i.e. $\bigcap_{n=1}^\infty T^{-n}\mathcal{B}=\{\varnothing,X\}\bmod m$). Indeed, a theorem of M. Lin [163] says that for exact systems $\|\widehat{T}^n f\|_1 \xrightarrow[n\to\infty]{} 0$ for all $f\in L^1$ with integral zero, so there can be no non-constant L^1–eigenfunctions with eigenvalue λ such that $|\lambda|=1$.

The key step in applying the double–norm method is the choice of Banach spaces $\mathcal{L}$ and $\mathcal{C}$: It is here that the specifics of the dynamics enter the picture. We indicate some typical choices (our list is by no means complete).

Maps with Markov partitions can be studied in terms of their symbolic dynamics using the sup norm for 'size' and the (symbolic) Hölder norm for 'regularity' (see Ruelle [213], Bowen [46] for finite partitions, and Aaronson & Denker [1] for infinite partitions). The resulting spaces depend on the Markov partition, and they therefore change when the map is perturbed. This makes the study of some stability questions difficult. In the case of Anosov diffeomorphisms, there is an alternative choice of Banach spaces due to Gouëzel & Liverani [100] and Blank, Keller & Liverani [34] which avoids symbolic dynamics, and is thus better suited to the study of such issues.

Without Markov partitions, it is not reasonable to expect the transfer operator to preserve Hölder continuity, and a different choice for $\mathcal{L}$ is needed. In one dimensional systems, one can sometime use the choice $\mathcal{L} = BV$, $\mathcal{C} = L^1$, see Lasota & Yorke [153], Rychlik [220], Hofbauer & Keller [114], Baladi & Keller [18], Keller [139], and Baladi [17] and references therein.

The multi-dimensional non-Markovian expanding case is more intricate, because of the absence of a canonical BV norm, and because of the difficulty in controlling the propagation of singularities in high iterates. Various generalizations of the BV norm have been suggested in this context, see Saussol [224] and Buzzi & Keller [56], and references therein.

Skew–products, i.e. maps of the form $(x, \xi) \mapsto (Tx, f_x(\xi))$, can also be treated using double norm inequalities, at least when the transfer operator of T is well–behaved. Additional conditions are required, however, to guarantee mixing: it is possible for the transfer operator of the skew product to have non-trivial eigenvalues of modulus one, even when T is mixing. We refer the reader to the works by Kowalski [149], Dolgopyat [76], Parry and Pollicott [192], Field, Melbourne & Török [88], and references therein.

A.2.2. *Cones.* This method is to find a cone of functions C such that $\widehat{T}(C) \subseteq C$. If $\widehat{T}(C)$ is sufficiently 'small' in C, then $\operatorname{span}\{\widehat{T}^n f\}$ converges exponentially fast to $\operatorname{span}\{1\}$ (the precise statements follow shortly). This convergence can then be used to derive a spectral gap on a suitable space, or to prove exponential decay of correlations in C directly.

We present the necessary machinery due to G. Birkhoff [33], and introduced to the study of decay of correlations by Liverani [165] (see also Ferrero & Schmitt [87] and Bakhtin [15, 16]).

A subset C of a normed vector space V is a *cone*, if $f \in C, \lambda > 0 \Rightarrow \lambda f \in C$. A cone is called *proper* if $C \cap -C = \varnothing$, *convex* if $f, g \in C \Rightarrow f + g \in C$, and *closed* if $C \cup \{0\}$ is closed. *Hilbert's projective metric* is the following pseudo-metric on C:

$$\Theta(f, g) := \log\left(\frac{\inf\{\mu > 0 : g \preceq \mu f\} \cup \{0\}}{\sup\{\lambda > 0 : \lambda f \preceq g\} \cup \{\infty\}}\right), \text{ where } f \preceq g \Leftrightarrow g - f \in C.$$

Alternatively, $\Theta(f, g) = \log \frac{\beta^*}{\alpha^*}$ where α^*, β^* are the best constants in the inequality $\alpha^* f \preceq g \preceq \beta^* f$. Observe that $\Theta(\alpha f, \beta g) = \Theta(f, g)$ for all $\alpha, \beta > 0$: Θ measures the distance between the directions generated by f, g, not between f, g themselves.

Theorem A.2 (G. Birkhoff)**.** *Let C be a closed convex proper cone inside a normed vector space $(V, \|\cdot\|)$, and let $S : V \to V$ be a linear operator such that $S(C) \subseteq C$. If $\Delta := \sup\{\Theta(Sf, Sg) : f, g \in C\} < \infty$, then S contracts Θ uniformly:*

$$\Theta(Sf, Sg) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta(f, g) \qquad (f, g \in C). \tag{A.2}$$

In particular, if we can find a closed convex proper cone $C \subset L^1$ which contains the constants and for which $\widehat{T}(C) \subset C$ and $\Delta < \infty$, then the iteration of $(A.2)$ gives for every $f \in L^1$, $\Theta(\widehat{T}^n f, Pf) = \Theta(\widehat{T}^n f, \widehat{T}^n Pf) \leq \tanh^{n-1}(\frac{\Delta}{4})\Delta$, and this tends to zero exponentially. (Recall that $Pf = \int f$.) We see that the Θ–distance between the rays determined by $\widehat{T}^n f$ and Pf tends to zero geometrically.

The next step is to estimate the L^1–distance between $\widehat{T}^n f$ and Pf. In general, this step depends on the cone in a non-canonical way, and cannot be described in a general terms. If we add the assumption that all functions in C are non-negative and that $f, g \in C, f \pm g \in C \Rightarrow \|f\|_1 \geq \|g\|_1$, then the situation simplifies considerably, because in this case (see e.g. [165]),

$$\left\| \frac{f}{\|f\|_1} - \frac{g}{\|g\|_1} \right\|_1 \leq e^{\Theta(f,g)} - 1 \qquad (f, g \in C).$$

Since $\|\widehat{T}^n f\|_1 = \|f\|_1 = \|Pf\|_1$ whenever $f \geq 0$, we see that for all $f \in C$, $\|\widehat{T}^n f - Pf\|_1 = \|f\|_1 \left\| \frac{\widehat{T}^n f}{\|\widehat{T}^n f\|_1} - \frac{Pf}{\|Pf\|_1} \right\|_1 \leq (e^{\Theta(\widehat{T}^n f, \widehat{T}^n Pf)} - 1)\|f\|_1 = O(\rho^n)\|f\|_1$ with $\rho = \tanh \frac{\Delta}{4}$. It now follows from (A.1) that $|Cov(f, g \circ T^n)| = O(\rho^n)\|f\|_1 \|g\|_\infty$ uniformly for $f \in C, g \in L^\infty$ and we proved exponential decay of correlations.

The assumption $f, g \in C, f \pm g \in C \Rightarrow \|f\|_1 \geq \|g\|_1$ is not satisfied in many dynamical situations of interest. In these cases other relations between the Banach distance and Hilbert distance occur, depending on the type of the cone that is used. We refer the reader to [165] for methods which handle this difficulty.

Finally, we mention that Birkhoff's inequality can be generalized for operators mapping one cone to another cone (see Liverani [165], Theorem 1.1). This is important in non-uniformly expanding situations, where one is forced to consider a chain of cones $\widehat{T}(C_i) \subsetneq C_{i+1}$, see Maume-Deschampes [180] for examples.

A.2.3. *Decay of correlations for flows.* We now turn from discrete time to continuous time.

Let $\sigma_t : X \to X$ be a strongly mixing probability preserving semi-flow on $(X, \mathcal{B}, m, T)$. The decay of correlations of σ_t is captured by the asymptotic behavior as $t \to \infty$ of the *correlation function*:[9]

$$\rho(t) := \int f \cdot g \circ \sigma_t \, d\mu - \int f d\mu \int g d\mu \qquad (t > 0).$$

In order to keep the exposition as simple as possible, we assume that the semi-flow is given as a *suspension* over a map $T : \Sigma \to \Sigma$ with *roof function* $r : \Sigma \to \mathbb{R}^+$:

$$\begin{aligned} &X = \{(x, \xi) \in \Sigma \times \mathbb{R} : 0 \leq \xi < r(x)\}, \\ &\sigma_t(x, \xi) = (x, \xi + t) \text{ with the identifications } (x, \xi) \sim \big(Tx, \xi - r(x)\big), \\ &dm(x, \xi) = \frac{1}{\int r d\mu} (\mu \times d\xi)|_X. \end{aligned}$$

The reader may want to think of $\Sigma \simeq \Sigma \times \{0\}$ as of a Poincaré section for the (semi)flow with section map $T : \Sigma \to \Sigma$, and first return time function $r : \Sigma \to \mathbb{R}$. This is the standard way to obtain such a representation.

The main difficulty in continuous time is that the decay of correlations of σ_t depends in a subtle way on the properties of $r : \Sigma \to \mathbb{R}_+$ and $T : \Sigma \to \Sigma$ as a *pair*. There are examples of Ruelle [217] and Pollicott [205] which show that σ_t may not have exponential decay of correlations, even when $T : \Sigma \to \Sigma$ does. In fact, they exhibit (strongly mixing) suspensions over the same section map which have exponential decay of correlations with one roof function, but not with another. In the other direction, there are examples by Kocergin [148] and Khanin & Sinai [140]

[9]This is a standard abuse of terminology: $\rho(t)$ is the covariance, not the correlation.

of mixing suspension flows built over *non*-mixing base transformations (see Fayad [86] for the decay of correlations for examples of this type).

It is only recently that Chernov [62] has identified the properties of T and r which are responsible for super-polynomial mixing for Anosov flows, and that Dolgopyat [72] has shown how to use these properties to show that the rate of mixing is in fact exponential for smooth observables, thus settling a problem that has remained open since the early days of hyperbolic dynamics.

Ruelle [217] and Pollicott [205, 206] suggested to study $\rho(t)$ as $t \to \infty$ by considering the analytic properties of its Fourier transform

$$\widehat{\rho}(s) := \int_{-\infty}^{\infty} e^{-ist}\rho(t)1_{[0,\infty)}(t)dt = \int_0^{\infty} e^{-ist}\rho(t)dt,$$

and then appealing to a suitable Tauberian theorem, for example ([231], IX.14):

Proposition A.3. *If $\widehat{\rho}(s)$ extends analytically to a strip $\{s = x+iy : |y| < \varepsilon\}$ and the functions $\mathbb{R} \ni x \mapsto \widehat{\rho}(x+iy)$ ($|y| < \varepsilon$) are absolutely integrable, with uniformly bounded L^1–norm, then $|\rho(t)| = O(e^{-\varepsilon_0 t})$ for every $0 < \varepsilon_0 < \varepsilon$.*

To apply this method, we must first find an analytic extension of $\widehat{\rho}$ to some horizontal strip, and then control the growth of this extension.

The starting point is a formula for $\widehat{\rho}(s)$ in terms of the transfer operator $\widehat{T}$ of T. To obtain such a formula we break $\int_0^{\infty} dt$ into $\int_0^{r(x)-\xi} + \sum_{n\geq 1}\int_{r_n(x)-\xi}^{r_{n+1}(x)-\xi}$ in accordance to the times t when the flow 'hits the roof' (here and throughout $r_n = \sum_{k=0}^{n-1} r\circ T^k$). Setting $E(s) := \int_X \int_0^{r(x)-\xi} e^{-ist} fg\circ\sigma_t dtdm$, and

$$\widehat{f}_s(x) := \int_0^{r(x)} e^{-is\xi} f(x,\xi)d\xi,\ \widehat{g}_s(x) := \int_0^{r(x)} e^{is\xi} g(x,\xi)d\xi,$$

and assuming f, g both have integral zero and $\int r d\mu = 1$, we obtain

$$\begin{aligned} \widehat{\rho}(s) &= E(s) + \sum_{n=1}^{\infty}\int_{\Sigma} \widehat{T}^n\left(e^{isr_n}\widehat{f}_s\right)\widehat{g}_s d\mu \\ &\equiv E(s) + \sum_{n=1}^{\infty}\int_{\Sigma} \widehat{T}_s^n(\widehat{f}_s)\widehat{g}_s d\mu,\ \text{where } \widehat{T}_s \text{ is defined by } \widehat{T}_s : F \mapsto \widehat{T}(e^{isr}F). \end{aligned}$$

The point of this representation is that, as long as r is bounded, $s \mapsto \widehat{T}_s$ has an obvious extension to $s \in \mathbb{C}$. When $\widehat{T}$ has a spectral gap, one can study the analyticity of this extension using the analytic perturbation theory of bounded linear operators (Pollicott [206]). The term $E(s)$ is of no importance, because it is an entire function of s.

The integrability conditions of proposition A.3 turn out to be more delicate. The problem is to control the infinite sum; the term $E(s)$ can be handled in a standard way under some reasonable assumptions on g. This sum is majorized by $\|\widehat{f}_s\|_\infty\|\widehat{g}_s\|_{\mathcal{L}}\sum_{n\geq 1}\|\widehat{T}_s^n\|$, so is natural to try to bound $\sum_{n\geq 1}\|\widehat{T}_s^n\|$ in some strip $S = \{s = x+iy : |y| < \varepsilon\}$, at least for $|x|$ large. This amounts to considering expressions of the form

$$\widehat{T}_s^n F = \widehat{T}^n(e^{ixr_n}e^{-yr_n}F) \qquad (s = x+iy \in S)$$

and showing that the cancellation effect of e^{ixr_n} is powerful enough to make $\widehat{T}_s^n$ small. It is at this point that the counterexamples of Ruelle and Pollicott we mentioned before behave badly, and where additional structure is required.

In the case of Anosov flows, Dolgopyat was able to carry out the estimate using Chernov's 'axiom of uniform non-integrability'[10]. We present his result in a special case, where this axiom is satisfied, and in a weaker form than that used in his paper. The reader is referred to [72] for more general statements.

Theorem A.4 (D. Dolgopyat)**.** *Let g^t be a geodesic flow on the unit tangent bundle of a smooth, compact, negatively curved surface M. There exist a Poincaré section Σ, a Banach space $\mathcal{L}$, and an $\varepsilon > 0$ s.t. $\sum_{n\geq 1} \|\widehat{T}_s^n\| = O(|\mathrm{Re}(s)|^\alpha)$ for some $0 < \alpha < 1$ and all $s \in \{s = x + iy : |y| < \varepsilon\}$ with $|\mathrm{Re}(s)|$ large.*

We get $|\widehat{\rho}(s)| \leq |E(s)| + \|\widehat{f}_s\|_\infty \|\widehat{g}_s\|_{\mathcal{L}} \sum_{n=1}^\infty \|\widehat{T}_s^n\| = |E(s)| + \|\widehat{f}_s\|_\infty \|\widehat{g}_s\|_{\mathcal{L}} O(|\mathrm{Re}(s)|^\alpha)$.

Under certain smoothness assumptions on f, g, $\|\widehat{f}_s\|_\infty, \|\widehat{g}_s\|_{\mathcal{L}}$, $|E(s)|$ can be shown to decay fast enough so that the integrability conditions of proposition A.3 hold. Exponential decay of correlations follows.

We end this section by mentioning the works of Pollicott [206] and Baladi & Vallée [19] for versions of Dolgopyat's estimate for semi-flows over piecewise expanding maps of the interval, Dolgopyat's study of exponential and rapid mixing for generic hyperbolic flows [73, 75], the paper by Stoyanov [235] for the case of open billiard flows, and the recent paper by Liverani [166] for an extension of Dolgopyat's work to contact Anosov flows.

A.3. **No spectral gap and sub-exponential decay of correlations.** There are examples (typically non-uniformly hyperbolic systems) where the decay of correlations is slower than exponential. Obviously, the transfer operator for these examples cannot have a spectral gap. We discuss two methods which can be used in this case.

Both methods rely on Kakutani's *induced transformation* construction, which we now review. Let $(X, \mathcal{B}, m, T)$ be a probability preserving transformation and fix $A \in \mathcal{B}$ with $m(A) \neq 0$. By Poincaré's Recurrence Theorem,

$$\varphi_A(x) := 1_A(x) \inf\{n \geq 1 : T^n x \in A\}$$

is finite a.e., so $T_A : A \to A$ given by $T_A(x) = T^{\varphi_A(x)}(x)$ is well-defined almost everywhere. The map T_A is called the *induced transformation* on A. It is known that if T preserves m, then T_A preserves the measure $m_A(E) := m(E|A)$.

Observe that one iteration of T_A corresponds to several iterations of T, so T_A is more 'chaotic' than T. As a result, $\widehat{T}_A$ averages densities much faster than $\widehat{T}$, and it is natural to expect it to behave better as an operator. The first method we describe applies when $\widehat{T}_A$ has better spectral properties than $\widehat{T}$. The second applies when it has better distortion properties.

A.3.1. *Renewal Theory.* This is a method for determining the asymptotic behavior of $\widehat{T}^n$ when $\widehat{T}$ has no spectral gap but $\widehat{T}_A$ does. Define the following operators on $L^1(A) = \{f \in L^1 : f \text{ is zero outside } A\}$:

$$T_n f := 1_A \widehat{T}^n (f 1_A) \text{ and } R_n f := 1_A \widehat{T}^n (f 1_{[\varphi_A = n]}).$$

[10]'Non-integrability' here refers to foliations, not functions.

Now form the generating functions $T(z) := I + \sum_{n\geq 1} z^n T_n$, $R(z) := \sum_{n\geq 1} z^n R_n$. Note that $R(1) = \widehat{T}_A$. The *renewal equation* is the following identity [223]:

$$T(z) = \big(I - R(z)\big)^{-1} \quad (|z| < 1).$$

The left hand side contains information on T_n which are almost the same as $\widehat{T}^n$ ($T_n f = \widehat{T}^n f$ on A whenever $f \in L^1(A)$), whereas the right hand side involves $R(z)$ which is a (singular) perturbation of $R(1) = \widehat{T}_A$.

The spectral gap of $R(1)$, if it exists, allows us to analyze $R(z)$ using perturbation theory. The analytic problem we are facing is how to translate information on $R(z)$ to information on $T(z)$. If $R(z)$ were an ordinary power series with non-negative coefficients, this problem would be covered by classical renewal theory. The following result [223] is an operator theoretic version of parts of this theory. In what follows, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$:

Theorem A.5 (O. Sarig)**.** *Let T_n be bounded linear operators on a Banach space $\mathcal{L}$ such that $T(z) = I + \sum_{n\geq 1} z^n T_n$ converges in* $\mathrm{Hom}(\mathcal{L},\mathcal{L})$ *for every $z \in \mathbb{D}$. Assume that:*

1. **Renewal Equation:** *for every $z \in \mathbb{D}$, $T(z) = (I - R(z))^{-1}$ where $R(z) = \sum_{n\geq 1} z^n R_n$, $R_n \in \mathrm{Hom}(\mathcal{L},\mathcal{L})$ and $\sum \|R_n\| < \infty$.*
2. **Spectral Gap:** *the spectrum of $R(1)$ consists of an isolated simple eigenvalue at 1 and a compact subset of $\mathbb{D}$.*
3. **Aperiodicity:** *the spectral radius of $R(z)$ is strictly less than one for all $z \in \overline{\mathbb{D}} \setminus \{1\}$.*

Let P be the eigenprojection of $R(1)$ at 1. If $\sum_{k>n} \|R_k\| = O(1/n^\beta)$ for some $\beta > 2$ and $PR'(1)P \neq 0$, then for all n

$$T_n = \frac{1}{\mu} P + \frac{1}{\mu^2} \sum_{k=n+1}^{\infty} P_n + E_n,$$

where μ is given by $PR'(1)P = \mu P$, $P_n = \sum_{\ell>n} PR_\ell P$, and $E_n \in \mathrm{Hom}(\mathcal{L},\mathcal{L})$ satisfy $\|E_n\| = O(1/n^{\lfloor\beta\rfloor})$.

Gouëzel has relaxed some of the conditions of this theorem, and has shown how to get higher order terms in this asymptotic expansion [99].

In the special case $T_n f = 1_A \widehat{T}^n(f 1_A)$, $R_n f = 1_A \widehat{T}^n(f 1_{[\varphi_A = n]})$, one checks that $\mu = \frac{1}{m(A)}$, $Pf = 1_A \frac{1}{m(A)} \int_A f\,dm$, $P_n f = 1_A \frac{1}{m(A)^2} \sum_{\ell>n} m[\varphi_A > \ell] \int_A f\,dm$. The theorem then implies that if f, g are supported inside A, $g \in L^\infty$, $f \in \mathcal{L}$, then

$$g\widehat{T}^n f = g \int f + g \sum_{k=n+1}^{\infty} m[\varphi_A > k] \int f + gE_n f.$$

It follows from (A.1) that if $\|\cdot\|_1 \leq \|\cdot\|_{\mathcal{L}}$, then

$$Cov(f, g\circ T^n) = \left(\sum_{k=n+1}^{\infty} m[\varphi_A > k]\right) \int f \int g + O(n^{-\lfloor\beta\rfloor}).$$

This is often enough to determine $Cov(f, g\circ T^n)$ up to asymptotic equivalence (see [223],[99] for examples). In particular, unlike the other methods we discuss here, the renewal method – when applicable – yields lower bounds, not just upper bounds for the decay of correlations.

A.3.2. *Coupling.* Fix a set A, and consider two positive functions f, g such that $\|f\|_1 = \|g\|_1$. The coupling method for estimating $\|\widehat{T}^n f - \widehat{T}^n g\|_1$ is based on the following heuristic: Suppose $\exists \varepsilon_1 > 0$ such that $\widehat{T} f = \varepsilon_1 1_A + f_1$, $\widehat{T} g = \varepsilon_1 1_A + g_1$ with f_1, g_1 positive. If $\delta_1 := 1 - \frac{\|f_1\|_1}{\|f\|_1}$ and $n > 1$, then

$$\widehat{T}^n f - \widehat{T}^n g \equiv \widehat{T}^{n-1} f_1 - \widehat{T}^{n-1} g_1, \text{ and } \|f_1\|_1 = \|g_1\|_1 = (1-\delta_1)\|f\|_1.$$

A fraction δ_1 of the total mass was 'coupled' and cancelled out. We now iterate this procedure. If this is possible, then $\exists f_k > 0$ and ε_k, δ_k such that $\widehat{T} f_k = f_{k+1} + \varepsilon_k 1_A$ and $\|f_k\|_1 = \|g_k\|_1 = \prod_{i=1}^k (1-\delta_i)\|f\|_1$, where $\delta_i = 1 - \frac{\|f_i\|_1}{\|f_{i-1}\|_1}$. For all $n > N$,

$$\|\widehat{T}^n f - \widehat{T}^n g\|_1 = \|\widehat{T}^{n-N} f_N - \widehat{T}^{n-N} g_N\|_1 \le$$
$$\le \|\widehat{T}^{n-N} f_N\|_1 + \|\widehat{T}^{n-N} g_N\|_1 = \|f_N\|_1 + \|g_N\|_1 = 2\prod_{i=1}^N (1-\delta_i)\|f\|_1.$$

If we start with $g = Pf$, we get an upper bound for $\|\widehat{T}^n f - Pf\|_1$ which we can then translate using (A.1) to an upper bound for $Cov(f, h \circ T^n)$ for all $h \in L^\infty$.

The upper bound that we get depends on how much we were able to 'couple' away at every stage. It is a deep insight of L.-S. Young [259, 260] that this can be done very efficiently in many important non-uniformly hyperbolic systems, if the set A is such that the induced transformation T_A is a piecewise onto map with uniform bounded distortion.

We describe the class of examples which can be treated this way abstractly. The reader interested in applications to 'real' systems is referred to Bálint & Tóth [20], Markarian [179], Chernov [63], Chernov & Young [64], Young [259] for a treatment of Billiard systems; Young [260], Bruin, van Strien & Luzzatto [50], and Holland [115] for interval maps; and Benedicks & Young [31] and Buzzi & Maume-Deschampes [57] for some higher dimensional examples.

A *L.-S. Young tower* is a non-singular conservative transformation $(\Delta, \mathcal{B}, m, F)$ equipped with a generating measurable partition $\{\Delta_{\ell,i} : i \in \mathbb{N}, \ell = 0, \dots, R_i - 1\}$ with the following properties:

- (T1) The measure of $\Delta_{\ell,i}$ is positive and finite for every i and ℓ, and $m(\Delta_0) < \infty$ where $\Delta_0 = \biguplus_{i \ge 1} \Delta_{0,i}$.
- (T2) g.c.d.$\{R_i : i = 1, 2, 3, \dots\} = 1$.
- (T3) If $\ell+1 < R_i$, then $F : \Delta_{\ell,i} \to \Delta_{\ell+1,i}$ is a measurable bijection, and $m|_{\Delta_{\ell+1,i}} \circ F|_{\Delta_{\ell,i}} = m|_{\Delta_{\ell,i}}$.
- (T4) If $\ell + 1 = R_i$, then $F : \Delta_{\ell,i} \to \Delta_0$ is a measurable bijection.
- (T5) Let $R : \Delta_0 \to \mathbb{N}$ be the function $R|_{\Delta_{0,i}} \equiv R_i$ and set $\varphi := \log \frac{dm|_{\Delta_0}}{dm|_{\Delta_0} \circ F^R}$. φ has an a.e. version for which $\exists C > 0, \theta \in (0,1)$ s.t. $\forall i$ and $\forall x, y \in \Delta_{0,i}$,

$$\left| \sum_{k=0}^{R(x)-1} \varphi(F^k x) - \sum_{k=0}^{R(y)-1} \varphi(F^k y) \right| < C\theta^{s(F^R x, F^R y)}$$

  where $s(x,y) = \min\{n \ge 0 : (F^R)^n x, (F^R)^n y \text{ lie in distinct } \Delta_{0,i}\}$.

Theorem A.6 (L.-S. Young)**.** *Suppose $(\Delta, \mathcal{B}, m, F)$ is a probability preserving L.-S. Young tower with θ as above. Set $\mathcal{L} := \{f : \Delta \to \mathbb{R} : \sup |f(x) - f(y)|/\theta^{s(x,y)} < \infty\}$, and define $\widehat{R}(x) := \inf\{n \ge 0 : F^n(x) \in \Delta_0\}$. For every $f \in \mathcal{L}$ and $g \in L^\infty$,*

1. *if* $m[\widehat{R} > n] = O(n^{-\alpha})$ *for some* $\alpha > 0$*, then* $|Cov(f, g \circ T^n)| = O(n^{-\alpha})$.
2. *if* $m[\widehat{R} > n] = O(\rho_0^n)$ *with* $0 < \rho_0 < 1$*, then* $|Cov(f, g \circ T^n) = O(\rho^n)$ *for some* $0 < \rho < 1$*;*
3. *if* $m[\widehat{R} > n] = O(\rho_0^{n_0^\gamma})$ *with* $0 < \rho_0 < 1$*,* $0 < \gamma_0 \leq 1$*, then* $|Cov(f, g \circ T^n)| = O(\rho^{n^\gamma})$ *for some* $0 < \rho < 1$*,* $0 < \gamma < \gamma_0$.

We remark that if $m[\widehat{R} > n] \asymp n^{-\alpha}$, then the bound in (1) was shown to be optimal in a particular example by Hu [119] and in the general case using the methods of the previous subsection by Sarig [223] and Gouëzel [99].

Acknowledgment. O. Sarig would like to thank D. Dolgopyat, M. Pollicott, and Ya. Pesin for helpful comments, and to acknowledge support by NSF grant 0400687.

References

[1] J. Aaronson and M. Denker, *Local limit theorems for partial sums of stationary sequences generated by Gibbs–Markov maps*, Stoch. Dyn. **1** (2001), 193–237.
[2] V. Afraimovich and Ya. Pesin, *Dimension of Lorenz type attractors*, Sov. Sci. Rev., Sect. C, Math. Phys. Rev. **6** (1987), 169–241.
[3] V. Alekseev, *Quasirandom dynamical systems. I. Quasirandom diffeomorphisms*, Math. USSR-Sb. **5** (1968), no. 1, 73–128.
[4] V. Alekseev, *Quasirandom dynamical systems. II. One-dimensional nonlinear vibrations in a periodically perturbed field*, Math. USSR-Sb. **6** (1968), no. 4, 505–560.
[5] V. Alekseev, *Quasirandom dynamical systems. III. Quasirandom vibrations of one-dimensional oscillators*, Math. USSR-Sb. **7** (1969), no. 1, 1–43.
[6] J. Alves, *SRB measures for nonhyperbolic systems with multidimensional expansion*, Ann. Sci. École Norm. Sup. (4) **33** (2000), 1–32.
[7] J. Alves and V. Araújo, *Hyperbolic times: frequency versus hyperbolicity*, Ergodic Theory Dynam. Systems **24** (2004), 329–346.
[8] J. Alves, C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding*, Invent. Math. **140** (2000), 351–298.
[9] D. Anosov, *Geodesic flows on closed Riemann manifolds with negative curvature*, Proc. Steklov Inst. Math. **90** (1969), 1–235.
[10] D. Anosov and Ya. Sinai, *Certain smooth ergodic systems*, Russian Math. Surveys **22** (1967), no. 5, 103–167.
[11] A. Arbieto and J. Bochi, *L^p-generic cocycles have one-point Lyapunov spectrum*, Stoch. Dyn. **3** (2003), 73–81.
[12] L. Arnold, *Random Dynamical Systems*, Monographs in Mathematics, Springer, 1998.
[13] L. Arnold and N. Cong, *Linear cocycles with simple Lyapunov spectrum are dense in L^∞*, Ergodic Theory Dynam. Systems **19** (1999), 1389–1404.
[14] A. Avila and R. Krikorian, *Reducibility or non-nuniform hyperbolicity for quasiperiodic Schrödinger cocycles*, Preprint, 2004.
[15] V. Bakhtin,*Random processes generated by a hyperbolic sequence of mappings I*, Russian Acad. Sci. Izv. Math. **44** (1995), 247–279.
[16] V. Bakhtin, *Random processes generated by a hyperbolic sequence of mappings II*, Russian Acad. Sci. Izv. Math. **44** (1995), 617–627.
[17] V. Baladi, *Positive transfer operators and decay of correlations*, Adv. Ser. in Nonlinear Dynam. 16, World Scientific, 2000.
[18] V. Baladi and G. Keller, *Zeta functions and transfer operators for piecewise monotonic maps of the interval*, Comm. Math. Phys. **127** (1990), 459–479.
[19] V. Baladi and B. Vallée, *Exponential decay of correlations for surface semi-flows without finite Markov partitions*, Proc. Amer. Math. Soc. **133** (2005), 865–874.
[20] P. Bálint, I. Tóth, *Correlation decay in certain soft billiards*, Comm. Math. Phys. **243** (2003), 55–91.
[21] W. Ballmann, *Nonpositively curved manifolds of higher rank*, Ann. of Math. (2) **122** (1985), 597–609.

[22] W. Ballmann, M. Brin and P. Eberlein, *Structure of manifolds of nonpositive curvature. I*, Ann. of Math. (2) **122** (1985), 171–203.

[23] A. Baraviera and C. Bonatti, *Removing zero Lyapunov exponents*, Ergodic Theory Dynam. Systems **23** (2003), 1655–1670.

[24] L. Barreira and Ya. Pesin, *Lyapunov Exponents and Smooth Ergodic Theory*, Univ. Lect. Series 23, Amer. Math. Soc., 2002.

[25] L. Barreira, Ya. Pesin and J. Schmeling, *Dimension and product structure of hyperbolic measures*, Ann. of Math. (2) **149** (1999), 755–783.

[26] L. Barreira and J. Schmeling, *Sets of "non-typical" points have full topological entropy and full Hausdorff dimension*, Israel J. Math. **116** (2000), 29–70.

[27] L. Barreira and C. Valls, *Smoothness of invariant manifolds for nonautonomous equations*, Comm. Math. Phys., to appear.

[28] V. Belykh, *Qualitative Methods of the Theory of Nonlinear Oscilaltions in Point Systems*, Gorki University Press, 1980.

[29] M. Benedicks and L. Carleson, *The dynamics of the Hénon map*, Ann. of Math. (2) **133** (1991), 73–169.

[30] M. Benedicks and L.-S. Young, *Sinai–Bowen–Ruelle measure for certain Hénon maps*, Invent. Math. **112** (1993), 541–576.

[31] M. Benedicks and L.-S. Young, *Markov extensions and decay of correlations for certain Hénon maps*, Astérisque, 2000.

[32] G. Benettin, L. Galgani, A. Giorgilli and J.-M. Strelcyn, *Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them*, Meccanica **15** (1980), 9–20.

[33] G. Birkhoff, *Extensions of Jentzsch's theorem*, Trans. Amer. Math. Soc. **85** (1957), 219–227.

[34] M. Blank, G. Keller and C. Liverani, *Ruelle–Perron–Frobenius spectrum for Anosov maps*, Nonlinearity **15** (2001), 1905–1973.

[35] J. Bochi, *Genericity of zero Lyapunov exponents*, Ergodic Theory Dynam. Systems **22** (2002), 1667–1696.

[36] J. Bochi and M. Viana, *Uniform (projective) hyperbolicity or no hyperbolicity: a dichotomy for generic concervative maps*, Ann. Inst. Henri Poincaré Anal. Non Linéaire **19** (2002), 113–123.

[37] J. Bochi and M. Viana, *The Lyapunov exponents of generic volume preserving and symplectic systems*, Ann. of Math. (2), to appear.

[38] J. Bochi and M. Viana, *Lyapunov exponents: how frequently are dynamical systems hyperbolic?*, in Modern Dynamical Systems and Applications, Cambridge Univ. Press, 2004, pp. 271–297.

[39] C. Bonatti, X. Gómez-Mont and M. Viana, *Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices*, Ann. Inst. H. Poincaré Anal. Non Linéarié **20** (2003), 579–624.

[40] C. Bonatti, C. Matheus, M. Viana and A. Wilkinson, *Abundance of stable ergodicity*, Comment. Math. Helv. **79** (2004), 753–757.

[41] C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting*, Israel J. Math. **115** (2000), 157–193.

[42] C. Bonatti and M. Viana, *Lyapunov exponents with multiplicity* 1 *for deterministic products of matrices*, Ergodic Theory Dynam. Systems **24** (2004), 1295–1330.

[43] F. Bonetto, G. Galavotti and P. Garrido, *Chaotic principle: an experimental test*, Phys. D **105** (1997), 226–252.

[44] J. Bourgain, *Positivity and continuity of the Lyapunov exponent for shifts on T^d with arbitrary frequency and real analytic potential*, Preprint, 2002.

[45] J. Bourgain and S. Jitomirskaya, *Continuity of the Lyapunov exponent for quasi-periodic operators with analytic potential*, J. Stat. Phys. **108** (2002), 1203–1218.

[46] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lect. Notes in Math. 470, Springer, 1975.

[47] M. Brin, *Bernoulli diffeomorphisms with nonzero exponents*, Ergodic Theory Dynam. Systems **1** (1981), 1–7.

[48] M. Brin, *Hölder continuity of invariant distributions*, in Smooth Ergodic Theory and Its Applications (A. Katok, R. de la Llave, Ya. Pesin and H. Weiss, eds.), Proc. Symp. Pure Math., Amer. Math. Soc., 2001, pp. 91–93.

[49] A. Broise-Alamichel and Y. Guivarc'h, *Exposants caractéristiques de l'algorithme de Jacobi–Perron et de la transformation associée*, Ann. Inst. Fourier (Grenoble) **51** (2001), 565–686.
[50] H. Bruin, S. Luzzatto and S. van Strien, *Rates of decay of correlation for one-dimensional dynamics*, Ann. Sci. École Norm. Sup. (4) **36** (2003), 621–646.
[51] L. Bunimovich, *Systems of hyperbolic type with singularities*, in Modern Problems of Mathematics 2 (Ya. Sinai, ed.), Springer, 1989.
[52] L. Bunimovich, *Billiards and other hyperbolic systems*, in Dynamical Systems, Ergodic Theory and Applications Series (Ya. Sinai, ed.), Encyclopaedia of Mathematical Sciences 100, Springer, 2000, pp. 192–233.
[53] K. Burns, D. Dolgopyat and Ya. Pesin, *Partial hyperbolicity, Lyapunov exponents and stable ergodicity*, J. Stat. Phys. **108** (2002), 927–942.
[54] K. Burns and M. Gerber, *Continuous invariant cone families and ergodicity of flows in dimension three*, Ergodic Theory Dynam. Systems **9** (1989), 19–25.
[55] K. Burns and R. Spatzier, *Manifolds of nonpositive curvature and their buildings*, Inst. Hautes Études Sci. Publ. Math. **65** (1987), 35–59.
[56] J. Buzzi and G. Keller, *Zeta functions and transfer operators for multidimensional piecewise affine and expanding maps*, Ergodic Theory Dynam. Systems **21** (2001), 689–716.
[57] J. Buzzi and V. Maume-Deschampes, *Decay of correlations for piecewise invertible maps in higher dimensions*, Israel J. Math. **131** (2003), 203–220.
[58] D. Bylov, R. Vinograd, D. Grobman and V. Nemyckii, *Theory of Lyapunov exponents and its application to problems of stability*, Izdat. "Nauka", Moscow, 1966, in Russian.
[59] A. Carverhill, *A nonrandom Lyapunov spectrum for nonlinear stochastic dynamical systems*, Stochastics **17** (1986), 253–287.
[60] C.-Q. Cheng and Y.-S. Sun, *Existence of invariant tori in three dimensional measure-preserving mappings*, Celestial Mech. Dyn. Astronom. **47**, (1989/90), no. 3, 275–292.
[61] N. Chernov, *Local ergodicity of hyperbolic systems with singularities*, Funct. Anal. Appl. **27** (1993) no. 1, 51–54.
[62] N. Chernov, *Markov approximations and decay of correlations for Anosov flows*, Ann. of Math. (2) **147** (1998), 269–324.
[63] N. Chernov, *Decay of correlations and dispersing billiards*, J. Stat. Phys. **94** (1999), 513–556.
[64] N. Chernov and L.-S. Young, *Decay of correlations for Lorenz gases and hard balls*, in *Hard Ball Systems and the Lorentz Gas*, Encyclopaedia of Mathematical Sciences 101, Springer, 2000, pp. 89–120.
[65] N. Chernov, *Invariant measures for hyperbolic dynamical systems*, in Handbook of Dynamical Systems 1A (B. Hasselblatt and A. Katok, eds.), Elsevier, 2002, pp. 321–407.
[66] N. Chernov and Ya. Sinai, *Ergodic properties of some systems of* 2*-dimensional discs and* 3*-dimensional spheres*, Russian Math. Surveys **42** (1987), 181–207.
[67] I. Cornfeld, S. Fomin and Ya. Sinai, *Ergodic Theory*, Springer, 1982.
[68] C. Dellago, H. Posch and W. Hoover, *Lyapunov instability in a system of hard disks in equilibrium and nonequilibrium steady states*, Phys. Rev. E **53** (1996), 1485–1501.
[69] C. Dettmann and G. Morriss, *Proof of Lyapunov exponent pairing for systems at constant kinetic energy*, Phys. Rev. E **53** (1996), 5541.
[70] C. Dettmann and G. Morriss, *Hamiltonian formulation of the Gaussian isokinetic thermostat*, Phys. Rev. E **54** (1996), 2495.
[71] W. Doeblin and R. Fortet, *Sur des chaînes à liaisons complétes*, Bull. Soc. Math. France **65** (1937), 132–148.
[72] D. Dolgopyat, *On decay of correlations in Anosov flows*, Ann. of Math. (2) **147** (1998), 357–390.
[73] D. Dolgopyat, *Prevalence of rapid mixing in hyperbolic flows*, Ergodic Theory Dynam. Systems **18** (1998), 1097–1114.
[74] D. Dolgopyat, *On dynamics of mostly contracting diffeomorphisms*, Comm. Math. Phys. **213** (2000), 181–201.
[75] D. Dolgopyat, *Prevalence of rapid mixing–II: topological prevalence*, Ergodic Theory Dynam. Systems **20** (2000), 1045–1059.
[76] D. Dolgopyat, *On mixing properties of compact group extensions of hyperbolic systems*, Israel J. Math. **130** (2002), 157–205.
[77] D. Dolgopyat, H. Hu and Ya. Pesin, *An example of a smooth hyperbolic measure with countably many ergodic components*, in Smooth Ergodic Theory and Its Applications (A. Katok,

R. de la Llave, Ya. Pesin and H. Weiss, eds.), Proc. Symp. Pure Math., Amer. Math. Soc., 2001, pp. 95–106.

[78] D. Dolgopyat and Ya. Pesin, *Every compact manifold carries a completely hyperbolic diffeomorphism*, Ergodic Theory Dynam. Systems **22** (2002), 409–435.

[79] P. Eberlein, *Geodesic flows on negatively curved manifolds. I*, Ann. of Math. (2) **95** (1972), 492–510.

[80] P. Eberlein, *When is a geodesic flow of Anosov type? I*, J. Differential Geometry **8** (1973), 437–463.

[81] P. Eberlein, *Geometry of Nonpositively Curved Manifolds*, Chicago Lectures in Mathematics, Chicago University Press, 1996.

[82] P. Eberlein, *Geodesic flows in manifolds of nonpositive curvature*, in Smooth Ergodic Theory and Its Applications (A. Katok, R. de la Llave, Ya. Pesin and H. Weiss, eds.), Proc. Symp. Pure Math., Amer. Math. Soc., 2001, pp. 525–571.

[83] L. Eliasson, *Reducibility and point spectrum for linear quasi-periodic skew-products*, Proceedings of the International Congress of Mathematicians II (Berlin, 1998), Doc. Math. **Extra Vol. II** (1998), 779–787.

[84] A. Fathi, M. Herman and J. Yoccoz, *A proof of Pesin's stable manifold theorem*, in Geometric Dynamics (Rio de Janeiro, 1981) (J. Palis, ed.), Lect. Notes in Math. 1007, Springer, 1983, pp. 177–215.

[85] A. Fathi, F. Laudenbach and V. Poénaru, *Travaux de Thurston sur les surfaces*, Séminaire Orsay, Astérisque 66–67, 1979, 284 pp.

[86] B. Fayad, *Polynomial decay of correlations for a class of smooth flows on the two torus*, Bull. Soc. Math. France **129** (2001), 487–503.

[87] P. Ferrero and B. Schmitt, *Produits aléatories d'opérateurs matrices de transfert*, Prob. Th. Relat. Fields **79** (1988), 227–248.

[88] M. Field, I. Melbourne and A. Török, *Decay of correlations, central limit theorems, and approximation by Brownian motion for compact Lie group extensions*, Ergodic Theory Dynam. Systems **23** (2003), 87–11.

[89] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Ann. of Math. (2) **155** (2002), 1–103.

[90] G. Forni, *On the Lyapunov exponents of the Kontsevich-Zorich cocycle*, in Handbook of Dynamical Systems 1B (B. Hasselblatt and A. Katok, eds.), Elsevier, to appear.

[91] A. Freire and R. Mañé, *On the entropy of the geodesic flow in manifolds without conjugate points*, Invent. Math. **69** (1982), 375–392.

[92] A. Furman, *On the multiplicative ergodic theorem for uniquely ergodic systems*, Ann. Inst. Henri Poincaré **33** (1997), 797–815.

[93] A. Furman, *Random walks on groups and random transformations*, in Handbook of Dynamical Systems 1A (B. Hasselblatt and A. Katok, eds.), Elsevier, 2002, pp. 931–1014.

[94] H. Furstenberg and Yu. Kifer, *Random matrix products and measures on projective spaces*, Israel J. Math. **46** (1983), 12–32.

[95] P. Garrido and G. Galavotti, *Billiards correlation functions*, J. Stat. Phys. **76** (1994), 549–585.

[96] M. Gerber, *Conditional stability and real analytic pseudo-Anosov maps*, Mem. Amer. Math. Soc. 321, 1985.

[97] M. Gerber and A. Katok, *Smooth models of Thurston's pseudo-Anosov maps*, Ann. Sci. École Norm. Sup. (4) **15** (1982), 173–204.

[98] I. Gol'dsheid and G. Margulis, *Lyapunov indices of a product of random matrices*, Russian Math. Surveys **44** (1989), 11–71.

[99] S. Gouëzel, *Sharp polynomial estimates for the decay of correlations*, Israel J. Math. **139** (2004), 29–65.

[100] S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems*, Preprint.

[101] R. Gunesch, *Precise volume estimates in nonpositive curvature*, Preprint, 2003.

[102] R. Gunesch, *Precise asymptotics for periodic orbits of the geodesic flow in nonpositive curvature*, Preprint, 2003.

[103] J. Hadamard, *Sur l'itération et les solutions asymptotiques des équations différentielles*, Bull. Soc. Math. France **29** (1901), 224–228.

[104] B. Hasselblatt, *Hyperbolic dynamical systems*, in Handbook of Dynamical Systems 1A (B. Hasselblatt and A. Katok, eds.), Elsevier, 2002, pp. 239–319.

[105] B. Hasselblatt and A. Katok *Principal structures*, in Handbook of Dynamical Systems 1A (B. Hasselblatt and A. Katok, eds.), Elsevier, 2002, pp. 1–203.

[106] B. Hasselblatt and Ya. Pesin, *Partially hyperbolic dynamical systems* , in Handbook of Dynamical Systems 1B (B. Hasselblatt and A. Katok, eds.), Elsevier, to appear.

[107] H. Hennion, *Sur un théorème spectral et son application aux noyaux lipschitziens*, Proc. Amer. Math. Soc. **118** (1993), 627–634.

[108] M. Hénon, *A two dimensional mapping with a strange attractor*, Comm. Math. Phys. **50** (1976), 69–77.

[109] M. Herman, *Construction d'un difféomorphisme minimal d'entropie topologique non nulle*, Ergodic Theory Dynam. Systems **1** (1981), 65–76.

[110] M. Herman, *Une méthode pour minorer les exposants de Lyapunov et quelques examples montrant le caractére local d'un théorèm d'Arnold et de Moser sur le tore de dimension* 2, Comment. Math. Helv. **58** (1983), 453–502.

[111] M. Herman, *Stabilité topologique des systémes dynamiques conservatifs*, Preprint, 1990.

[112] M. Hirayama and Ya. Pesin, *Non-absolutely continuous invariant foliations*, Mosc. Math. J., to appear.

[113] M. Hirsh, C. Pugh and M. Shub, *Invariant Manifolds*, Lect. Notes in Math. 583, Springer, 1977.

[114] F. Hofbauer and G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Math. Z. **180** (1982), 119–140.

[115] M. Holland, *Slowly mixing systems and intermittency maps*, Ergodic Theory Dynam. Systems, to appear.

[116] E. Hopf, *Statistik der geodätischen linien in mannigfaltigkeiten negativer krümmung*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **91** (1939), 261–304.

[117] L. Hörmander, *Complex Analysis in Several Variables*, Van Nostrand, 1966.

[118] H. Hu, *Conditions for the existence of SRB measures for "almost Anosov" diffeomorphisms*, Trans. Amer. Math. Soc. **352** (2000), 2331–2367.

[119] H. Hu, *Decay of correlations for piecewise smooth maps with indifferent fixed points*, Ergodic Theory Dynam. Systems **24** (2004), 495–524.

[120] H. Hu and L.-S. Young, *Nonexistence of SRB measures for some diffeomorphisms that are "almost Anosov"*, Ergodic Theory Dynam. Systems **15** (1995), 67–76.

[121] H. Hu, Ya. Pesin and A. Talitskaya, *Every compact manifold carries a hyperbolic ergodic flow*, in Modern Dynamical Systems and Applications, Cambridge Univ. Press, 2004.

[122] C. Ionescu-Tulcea and G. Marinescu, *Théorie ergodique pour des classes d'opérations non complétement continues*, Ann. of Math. (2) **52** (1950), 140–147.

[123] M. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps* Comm. Math. Phys. **81** (1981), 39–88.

[124] M. Jakobson and G. Światek, *One-dimensional maps*, in Handbook of Dynamical Systems 1A (B. Hasselblatt and A. Katok, eds.), Elsevier, 2002, pp. 599–664.

[125] R. Johnson, *On a Floquet theory for almost periodic, two-dimensional linear systems*, J. Differential Equations **37** (1980), 184–205.

[126] R. Johnson, K. Palmer and G. Sell, *Ergodic properties of linear dynamical systems*, SIAM J. Math. Anal. **18** (1987), 1–33.

[127] B. Kalinin and V. Sadovskaya, *On pointwise dimension of non-hyperbolic measures*, Ergodic Theory Dynam. Systems **22** (2002), 1783–1801.

[128] A. Karlsson and G. Margulis, *A multiplicative ergodic theorem and nonpositively curved spaces*, Comm. Math. Phys. **201** (1999), 107–123.

[129] A. Katok, *A conjecture about entropy*, in Smooth Dynamical Systems (D. Anosov, ed.), Mir, Moscow, 1977, pp. 181–203.

[130] A. Katok, *Bernoulli diffeomorphisms on surfaces*, Ann. of Math. (2) **110** (1979), 529–547.

[131] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. **51** (1980), 137–173.

[132] A. Katok in collaboration with K. Burns, *Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems*, Ergodic Theory Dynam. Systems **14** (1994), 757–785.

[133] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, 1995.

[134] A. Katok and J. Lewis, *Global rigidity results for lattice actions on tori and new examples of volume-preserving action*, Israel J. Math. **93** (1996), 253–280.

[135] A. Katok and L. Mendoza, *Dynamical systems with nonuniformly hyperbolic behavior*, in A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, 1995.

[136] A. Katok and R. Spatzier, *First cohomology of Anosov actions of higher rank Abelian groups and applications to rigidity*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 131–156.

[137] A. Katok and R. Spatzier, *Subelliptic estimates of polynomial differential operators and applications to cocycle rigidity*, Math. Research Letters **1** (1994), 193–202.

[138] A. Katok and J.-M. Strelcyn, *Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities*, with the collaboration of F. Ledrappier and F. Przytycki, Lect. Notes. in Math. 1222, Springer, 1986.

[139] G. Keller, *Interval maps with strictly contracting Perron–Frobenius operators*, Inter. J. Bifur. Chaos Appl. Sci. Engrg. **9** (1999), 1777–1783.

[140] K. Khanin and Ya. Sinai, *Mixing for some classes of special flows over rotations of the circle*, Funk. Anal. Prilozhen. **26** (1992), 155–169.

[141] Yu. Kifer, *A multiplicative ergodic theorem for random transformations*, J. Analyse Math. **45** (1985), 207–233.

[142] Yu. Kifer, *Ergodic Theory of Random Transformations*, Progress in Probability and Statistics 10. Birkhäuser, 1986.

[143] Yu. Kifer and P.-D. Liu, *Random dynamics*, in Handbook of Dynamical Systems 1B (B. Hasselblatt and A. Katok, eds.), Elsevier, 2002, to appear.

[144] G. Knieper, *Volume growth, entropy and the geodesic stretch*, Math. Res. Lett. **2** (1995), 39–58.

[145] G. Knieper, *On the asymptotic geometry of nonpositively curved manifolds*, Geom. Funct. Anal. **7** (1997), 755–782.

[146] G. Knieper, *The uniqueness of the measure of maximal entropy for the geodesic flows on rank* 1 *manifolds*, Ann. of Math. (2) **148** (1998), 291–314.

[147] G. Knieper, *Hyperbolic dynamics and Riemannian geometry*, in Handbook of Dynamical Systems 1A (B. Hasselblatt and A. Katok, eds.), Elsevier, 2002, pp. 453–545.

[148] A. Kocergin, *Mixing in special flows over a rearrangement of segments and in smooth flows on surfaces*, Mat. USSR Sbornik **25** (1975), 471–502.

[149] Z. Kowalski, *Ergodic properties of skew products with Lasota–Yorke type maps in the base*, Studia Math. **106** (1993), 45–57.

[150] A. Krámli, N. Simányi and D. Szász, *A "transversal" fundamental theorem for semi-dispersing billiards*, Comm. Math. Phys. **129** (1990), 535–560, erratum in Comm. Math. Phys. **138** (1991), 207–208.

[151] R. Krikorian, *Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts*, Astérisque 259, 1999.

[152] J. Lagarias, *The quality of the Diophantine approximations found by the Jacobi–Perron algorithm and related algorithms*, Monatsh. Math. **115** (1993), 299–328.

[153] A. Lasota and J. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488.

[154] F. Ledrappier, *Propriétes ergodiques des mesures de Sinaï*, Inst. Hautes Études Sci. Publ. Math. **59** (1984), 163–188.

[155] F. Ledrappier and M. Misiurewicz, *Dimension of invariant measures for maps with exponent zero*, Ergodic Theory Dynam. Systems **5** (1985), 595–610.

[156] F. Ledrappier and J.-M. Strelcyn, *A proof of the estimate from below in Pesin's entropy formula*, Ergodic Theory Dynam. Systems **2** (1982), 203–219.

[157] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula*, Ann. of Math. (2) **122** (1985), 509–539.

[158] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension*, Ann. of Math. (2) **122** (1985), 540–574.

[159] Y. Levy, *Ergodic properties of the Lozi map*, Lect. Notes in Math. 1109, Springer, 1985, pp. 103–116.

[160] J. Lewowicz, *Lyapunov functions and topological stability*, J. Differential Equations **38** (1980), 192–209.

[161] J. Lewowicz, *Lyapunov functions and stability of geodesic flows*, Geometric Dynamics (Rio de Janeiro, 1981) (J. Palis, ed.), Lect. Notes. in Math. 1007, Springer, 1983, pp. 463–479.

[162] J. Lewowicz and E. Lima de Sá, *Analytic models of pseudo-Anosov maps*, Ergodic Theory Dynam. Systems **6** (1986), 385–392.

[163] M. Lin, *Mixing for Markov operators*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **19** (1971), 231–242.

[164] P.-D. Liu and M. Qian, *Smooth Ergodic Theory of Random Dynamical Systems*, Lect. Notes in Math. 1606, Springer, 1995.

[165] C. Liverani, *Decay of correlations*, Ann. of Math. (2) **142** (1995), 239–301.

[166] C. Liverani, *On contact Anosov flows*, Ann. Math. (2) **159** (2004), 1275–1312.

[167] C. Liverani and M. Wojtkowski, *Ergodicity in Hamiltonian systems*, Dynam. Report. Expositions Dynam. Systems (N.S.) 4, Springer, 1995, pp. 130–202.

[168] S. Luzzatto, *Stochastic-like behaviour in nonuniformly expanding maps*, in Handbook of Dynamical Systems 1B (B. Hasselblatt and A. Katok, eds.), Elsevier, to appear.

[169] S. Luzzatto and M. Viana, *Parameter exclusions in Hénon-like systems*, Russian Math. Surveys **58** (2003), 1053–1092.

[170] A. Lyapunov, *The General Problem of the Stability of Motion*, Taylor & Francis, 1992.

[171] R. Mañé, *A proof of Pesin's formula*, Ergodic Theory Dynam. Systems **1** (1981), 95–102, errata in Ergodic Theory Dynam. Systems **3** (1983), 159–160.

[172] R. Mañé, *Lyapunov exponents and stable manifolds for compact transformations*, in Geometric Dynamics (Rio de Janeiro, 1981) (J. Palis, ed.), Lect. Notes. in Math. 1007, Springer, 1983, pp. 522–577.

[173] R. Mañé, *Oseledec's theorem from the generic viewpoint*, Proceedings of the International Congress of Mathematicians 1–2 (Warsaw, 1983), 1984, pp. 1269–1276.

[174] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge·Band 8, Springer, 1987.

[175] R. Mañé, *The Lyapunov exponents of generic area preserving diffeomorphisms*, in International Conference on Dynamical Systems (Montevideo, 1995) (F. Ledrappier, J. Lewowicz and S. Newhouse, eds.), Pitman Res. Notes Math. Ser. 362, Longman, 1996, pp. 110–119.

[176] A. Manning, *Topological entropy and the first homology group*, in Dynamical Systems (Warwick, 1974) (A. Manning, ed.), Lect. Notes in Math. 468, Springer, 1975, pp. 185–190.

[177] R. Markarian, *The fundamental theorem of Sinai–Chernov for dynamical systems with singularities*, Dynamical Systems (Santiago, 1990), Pitman Res. Notes Math. Ser. 285, Longman, 1993, pp. 131–158.

[178] R. Markarian, *Non-uniformly hyperbolic billiards*, Ann. Fac. Sci. Toulouse, VI. Sér., Math. **3** (1994), no. 5–6, 223-257.

[179] R. Markarian, *Billiards with polynomial decay of correlations*, Ergodic Theory Dynam. Systems **24** (2004), 177–197.

[180] V. Maume-Deschampes, *Projective metrics and mixing properties on towers*, Trans. Amer. Math. Soc. **353** (2001), 3371–3389.

[181] H. McCluskey and A. Manning, *Hausdorff dimension for horseshoes*, Ergodic Theory Dynam. Systems **3** (1983), 251–260, errata in Ergodic Theory Dynam. Systems **3** (1983), 319.

[182] V. Millionshchikov, *Les systèmes linéaires d'équations différentielles ordinaires*, Actes du Congrès International des Mathématiciens (Nice, 1970) 2, Gauthier-Villars, 1971, pp. 915–919.

[183] J. Milnor, *Fubini foiled: Katok's paradoxical example in measure theory*, Math. Intelligencer **19** (1997), 30–32.

[184] J. Milnor, *On the concept of attractor*, Comm. Math. Phys. **102** (1985), 517–519.

[185] M. Misiurewicz and F. Przytycki, *Entropy conjecture for tori*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **25** (1977), no. 6, 575–578.

[186] M. Misiurewicz and F. Przytycki, *Topological entropy and degree of smooth mappings*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **25** (1977), no. 6, 573–574.

[187] L. Mora and M. Viana, *Abundance of strange attractors*, Acta Math. **171** (1993), 1–71.

[188] S. Newhouse, *Entropy and volume*, Ergodic Theory Dynam. Systems **8*** (1988), 283–299.

[189] S. Newhouse, *Continuity properties of entropy*, Ann. of Math. (2) **129** (1989), 215–235.

[190] D. Ornstein and B. Weiss, *Geodesic flows are Bernoullian*, Israel J. Math. **14** (1973), 184–198.

[191] V. Oseledets, *A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–221.

[192] W. Parry and M. Pollicott, *The Chebotarov theorem for Galois coverings of Axiom A flows*, Ergodic Theory Dynam. Systems **6** (1986), 133–148.

[193] O. Perron, *Über stabilität und asymptotishes Verhalten der Lösungen eines Systemes endlicher Differenzengleichungen*, J. Reine Angew. Math. **161** (1929), 41–64.

[194] O. Perron, *Die stabilitätsfrage bei Differenzengleichungen*, Math. Zs. **32** (1930), 703–728.

[195] Ya. Pesin, *An example of a nonergodic flow with nonzero characteristic exponents*, Func. Anal. and its Appl. **8** (1974), 263–264.

[196] Ya. Pesin, *Families of invariant manifolds corresponding to nonzero characteristic exponents*, Math. USSR-Izv. **40** (1976), 1261–1305.

[197] Ya. Pesin, *Characteristic Ljapunov exponents, and smooth ergodic theory*, Russian Math. Surveys **32** (1977), 55–114.

[198] Ya. Pesin, *A description of the π-partition of a diffeomorphism with an invariant measure*, Math. Notes **22** (1977), 506–515.

[199] Ya. Pesin, *Geodesic flows on closed Riemannian manifolds without focal points*, Russ. Math. Surveys **11** (1977), 1195–1228.

[200] Ya. Pesin, *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties*, Ergodic Theory Dynam. Systems **12** (1992), 123–151.

[201] Ya. Pesin, *Formulas for the entropy of the geodesic flow on a compact Riemannian manifold without conjugate points*, Math. Notes **24:4** (1978), 796–805.

[202] Ya. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Mathematics, Chicago University Press, 1998.

[203] Ya. Pesin and Ya. Sinai, *Gibbs measures for partially hyperbolic attractors*, Ergodic Theory Dynam. Systems **2** (1982), 417–438.

[204] V. Pliss, *On a conjecture due to Smale*, Diff. Uravnenija **8** (1972), 262–282, in Russian.

[205] M. Pollicott, *On the rate of mixing of Axiom A flows*, Invent. Math. **81** (1985), 413–426.

[206] M. Pollicott, *On the mixing of Axiom A attracting flows and a conjecture of Ruelle*, Ergodic Theory Dynam. Systems **19** (1999), 535–548.

[207] C. Pugh, *The $C^{1+\alpha}$ hypothesis in Pesin theory*, Inst. Hautes Études Sci. Publ. Math. **59** (1984), 143–161.

[208] C. Pugh and M. Shub, *Ergodic attractors*, Trans. Amer. Math. Soc. **312** (1989), 1–54.

[209] E. Pujals and M. Sambarino, *Homoclinic bifurcations, dominated splitting and robust transitivity*, in Handbook of Dynamical Systems 1B (B. Hasselblatt and A. Katok, eds.), Elsevier, to appear.

[210] M. Qian and S. Zhu, *SRB-measures and Pesin's entropy formula for endomorphisms*, Trans. Amer. Math. Soc. **354** (2002), 1453–1471.

[211] M. Raghunathan, *A proof of Oseledec's multiplicative ergodic theorem*, Israel J. Math. **32** (1979), 356–362.

[212] D. Ruelle, *An inequality for the entropy of differentiable maps*, Bol. Soc. Brasil. Mat. **9** (1978), 83–87.

[213] D. Ruelle, *Thermodynamic Formalism. The Mathematical Structures of Classical Equilibrium Statistical Mechanics*, Encyclopedia of Mathematics and its Applications 5, Addison-Wesley, 1978.

[214] D. Ruelle, *Analyticity properties of the characteristic exponents of random matrix products*, Adv. Math. **32** (1979), 68–80.

[215] D. Ruelle, *Ergodic theory of differentiable dynamical systems*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 27–58.

[216] D. Ruelle, *Characteristic exponents and invariant manifolds in Hilbert space*, Ann. of Math. (2) **115** (1982), 243–290.

[217] D. Ruelle, *Flots qui ne mélangement pas exponentiellement*, C. R. Acad. Sci. Paris. **296** (1983), 191–193.

[218] D. Ruelle, *Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics*, J. Stat. Phys. **94** (1999), 393–468.

[219] D. Ruelle and A. Wilkinson, *Absolutely singular dynamical foliations*, Comm. Math. Phys. **219** (2001), 481–487.

[220] M. Rychlik, *Bounded variation and invariant measures*, Studia Math. **76** (1983), 69–80.

[221] R. Sacker and G. Sell, *Lifting properties in skew-product flows with applications to differential equations*, Mem. Amer. Math. Soc. 190, 1977.

[222] R. Sacker and G. Sell, *A spectral theory for linear differential systems*, J. Differential Equations **27** (1978), 320–358.

[223] O. Sarig, *Subexponential decay of correlations*, Invent. Math. **150** (2002), 629–653.

[224] B. Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps*, Israel J. Math. **116** (2000), 223–248.

[225] F. Schweiger, *The metrical theory of Jacobi–Perron algorithm*, Lect. Notes in Math. 334, Springer, 1973.

[226] G. Sell, *The structure of a flow in the vicinity of an almost periodic motion*, J. Differential Equations **27** (1978), 359–393.

[227] M. Shub, *Dynamical systems, filtrations, and entropy*, Bull. Amer. Math. Soc. (N.S.) **80** (1974), 27–41.

[228] M. Shub and D. Sullivan, *A remark on the Lefschetz fixed point formula for differentiable maps*, Topology **13** (1974), 189–191.

[229] M. Shub and A. Wilkinson, *Pathological foliations and removable zero exponents*, Invent. Math. (2000), 495–508.

[230] M. Shub and R. Williams, *Entropy and stability*, Topology **14** (1975), 329–338.

[231] B. Simon and M. Reed, *Methods of Modern Mathematical Physics. I: Functional Analysis*, Academic Press, 1980.

[232] Ya. Sinai, *Dynamical systems with countably-multiple Lebesgue spectrum II*, Amer. Math. Soc. Trans. (2) **68** (1966), 34–88.

[233] Ya. Sinai, *Dynamical systems with elastic reflections*, Russ. Math. Surveys **68** (1970), 137–189.

[234] Ya. Sinai and N. Chernov, *Entropy of hard spheres gas with respect to the group of space-time translations*, Tr. Semin. Im. I. G. Petrovskogo **8** (1982), 218–238, in Russian.

[235] L. Stoyanov, *Spectrum of the Ruelle operator and exponential decay of correlations for open billiard flows*, Amer. J. Math. **123** (2001), 715–759.

[236] D. Szász, ed., *Hard Ball Systems and the Lorentz Gas*, Encyclopaedia of Mathematical Sciences 101, Springer, 2000.

[237] A. Tahzibi, *C^1-generic Pesin's entropy formula*, C. R. Acad. Sci. Paris **335** (2002), 1057–1062.

[238] P. Thieullen, *Fibrés dynamiques asymptotiquement compacts. Exposants de Lyapunov. Entropie. Dimension*, Ann. Inst. H. Poincaré. Anal. Non Linéaire **4** (1987), 49–97.

[239] P. Thieullen, *Ergodic reduction of random products of two-by-two matrices*, J. Anal. Math. **73** (1997), 19–64.

[240] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Preprint.

[241] M. Tsujii, *Regular points for ergodic Sinai measures*, Trans. Amer. Math. Soc. **328** (1991), 747–777.

[242] I. Ugarcovici, *Hyperbolic measures and periodic orbits*, Preprint, 2003.

[243] S. Vaienti, *Ergodic properties of the discontinuous sawtooth map*, J. Stat. Phys. **67** (1992), 251–269.

[244] M. Viana, *Strange attractors in higher dimensions*, Bol. Soc. Brasil. Mat. (N.S.) **24** (1993), 13–62.

[245] P. Walters, *Unique ergodicity and matrix products*, in *Lyapunov exponents*, in Proc. Workshop (Bremen, Germany, 1984), Lect. Notes in Math. 1186, Springer, 1986, pp. 37–55.

[246] Q. Wang and L.-S.Young, *Strange attractors with one direction of instability*, Comm. Math. Phys. **218** (2001), 1–97.

[247] Q. Wang and L.-S.Young, *From invariant curves to strange attractors*, Comm. Math. Phys. **225** (2002), 275–304.

[248] M. Wojtkowski, *Invariant families of cones and Lyapunov exponents*, Ergodic Theory Dynam. Systems **5** (1985), 145–161.

[249] M. Wojtkowski, *Monotonicity, J-algebra of Potapov and Lyapunov exponents*, in Smooth Ergodic Theory and Its Applications (A. Katok, R. de la Llave, Ya. Pesin and H. Weiss, eds.), Proc. Symp. Pure Math., Amer. Math. Soc., 2001, pp. 499–521.

[250] M. Wojtkowski and C. Liverani, *Conformally symplectic dynamics and symmetry of the Lyapunov spectrum*, Comm. Math. Phys. **194** (1998), 47–60.

[251] Z. Xia, *Existence of invariant tori in volume-preserving diffeomorphisms*, Ergodic Theory Dynam. Systems **12** (1992), 621–631.
[252] J.-C. Yoccoz, *Travaux de Herman sur les tores invariants*, Séminaire Bourbaki, Astérisque 206, 1991/1992, exposé 754, pp. 311–344.
[253] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. **57** (1987), 285–300.
[254] Y. Yomdin, *C^k-resolution of semialgebraic mappings. Addendum to: "Volume growth and entropy"*, Israel J. Math. **57** (1987), 301–317.
[255] L.-S. Young, *Dimension, entropy and Lyapunov exponents*, Ergodic Theory Dynam. Systems **2** (1982), 109–124.
[256] L.-S. Young, *Bowen-Ruelle-Sinai measures for certain piecewise hyperbolic maps*, Trans. Amer. Math. Soc. **287** (1985), 41–48.
[257] L.-S. Young, *Some open sets of nonuniformly hyperbolic cocycles*, Ergodic Theory Dynam. Systems **13** (1993), 409–415.
[258] L.-S. Young, *Lyapunov exponents for some quasi-periodic cocycles*, Ergodic Theory Dynam. Systems **17** (1997), 483–504.
[259] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity* , Ann. of Math. (2) **147** (1998), 585–650.
[260] L.-S. Young, *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153–188.
[261] R. Zimmer, *Ergodic Theory and Semisimple Groups*, Monographs in Mathematics 81, Birkhäuser, 1984.

Index

Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal
E-mail address: barreira@math.ist.utl.pt
URL: http://www.math.ist.utl.pt/~barreira/

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
E-mail address: pesin@math.psu.edu
URL: http://www.math.psu.edu/pesin/

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
E-mail address: sarig@math.psu.edu
URL: http://www.math.psu.edu/sarig/