Aspects of Geometric Model Theory

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1 Introduction

In this paper (based on my tutorial in Utrecht) I want to discuss some themes from contemporary model theory, mainly originating in stability theory and classification theory, and point out some mathematical implications. Model theory has become largely the study of definable sets (or the category of definable sets and functions) in given structures, as well as the study of interpretability and bi-interpretability. These can either be specific, such as the field of p-adic numbers (as in applications), or can be arbitrary structures which satisy some model-theoretic hypotheses (stability, ω_1-categoricity, o-minimality). Among the themes or topics I will touch on are: dimension theory, how a structure is built up from "irreducible bits" (geometries), the fine structure of these "irreducible bits", modularity, orthogonality, equivariant model theory (definable groups and group actions), quotients and Galois theory. This paper is aimed at the non model-theorist logician. I want to explain a little of what is going on in model theory, but at the same time I do not want to simply repeat what has already been said in numerous surveys of this kind.

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I assume acquaintance with the basic concepts and results of first order logic and model theory: complete theories, compactness theorem, elementary substructure, saturation, Lowenheim-Skolem theorems. T will usually denote a complete theory in a language L. M will be an L-structure, usually a model of T. $x, y, ..$ denote finite sequences of variables, and $a, b, ..$ finite sequences of elements of a stucture M. If $\phi(x)$ is a formula, possibly with additional parameters from M, then $X = \{a \in M^n : M \models \phi(a)\}$ is called a definable set in M. We also write ϕ^M for X. If the parameters from ϕ belong to $A \subseteq M$ we say that X is A-definable. $tp_M(a/A)$ is the set of all formulas with parameters from A which are satisfied by a in M. A function $f : X \to Y$ is said to be definable (in M) if its graph is. So the category associated to M is the category of sets and functions definable in M. One may also want to (and should) include quotient objects X/E where E is a definable equivalence relation, as definable sets. This is unproblematic and will be discussed later. A rather basic notion is that of algebraic closure: suppose M is a structure $A \subseteq M$ and a a tuple from M. We will say that a is algebraic over A (in M) ($a \in acl_M(A)$) if there is a formula $\phi(x)$ over A (that is, with parameters from A) which is satisfied by a and which has only finitely many realizations in M. By compactness this is equivalent to there being some cardinal bound to the set of realisations of $tp_M(a/A)$ in any elementary extension of M. If we require in addition that a is the unique realization of $\phi(x)$ we say a is in the definable closure of A, $a \in dcl_M(A)$.

2 Morley's Theorem

Many of the elements of geometric model theory are either present in or naturally suggested by the Baldwin-Lachlan proof of Morley's Theorem [**?**], and I will introduce them in this way. Morley's theorem was the beginning of stability theory and classification theory. The context is a countable complete theory T. For κ an infinite cardinal, T is said to be κ-categorical if T has exactly one model of cardinality κ up to isomorphism. Morley's theorem states that if T is κ-categorical for some uncountable κ then it is κ-categorical for all uncountable a. This is a fundamental result on the expressive power of first order logic.

Proof of Morley's Theorem.
Assume T to be κ-categorical, where κ is an uncountable cardinal.

Step 1. Show that T is ω-stable (or totally transcendental). This means that for any countable model M of T the Stone space $S(M)$ of complete types over M in finitely many (or one) variable is countable. If not then there will be a model M' of T of cardinality κ realising uncountably many types over some countable elementary submodel M_0. On the other hand an Ehrenfeucht-Mostowski argument using Skolem functions and indiscernibles yields a model M' of T of cardinality κ such that over any countable subset A of M' only countably many types are realised in M'. Contradiction to κ-categoricity.

Consequences of ω-stability.
The first consequence is the existence of prime models over all sets: For any model M of T and subset A of M there is an elementary substructure $M(A)$ of M which contains A such that whenever M' is a model of T containing A such that $(M,a)_{a\in A} \equiv (M',a)_{a\in A}$ then there is an A-elementary embedding of $M(A)$ in M'. (Shelah subsequently proved that $M(A)$ is unique up to A-isomorphism.)
Another consequence (coming from the existence of Cantor-Bendixon rank on types) is that any formula $\phi(x)$ (with parameters from a model of T) has ordinal valued Morley rank.

Definition 2.1 *Let $M \models T$. Let $\phi(x)$ be a formula over M. We define $RM(\phi(x)) \geq 0$ if $\phi(x)$ is consistent (has a solution in M), and $RM(\phi(x)) \geq \alpha + 1$ if there is an elementary extension M' of M and formulas $\psi_i(x)$ over M' for $i < \omega$, pairwise inconsistent (in M'), each implying $\phi(x)$, and with $RM(\psi_i(x)) \geq \alpha$ for all $i < \omega$.*
(Also for δ limit, $RM(\phi(x)) \geq \delta$ if $RM(\phi(x)) \geq \alpha$ for all $\alpha < \delta$.)

The definition above makes sense for any theory T. In any case for the theory T currently under consideration, every formula has ordinal-valued Morley rank. In fact it was proved later (by Baldwin) that every formula has *finite* Morley rank.
If $RM(\phi(x)) = \alpha$ then $\phi(x)$ has an associated Morley degree (or multiplicity), the largest k such that there exist $\psi_i(x)$ for $i < k$ over some elementary extension M' of M, which imply $\phi(x)$, are pairwise inconsistent, and have Morley rank α.
In particular there is some formula $\phi(x)$ over a model M of T (where x can even be chosen to be a single variable) which has Morley rank 1 and

Morley degree 1. Such a formula is also called *strongly minimal*, and has the alternative characterization (which again makes sense in any theory):

Definition 2.2 *Let $\phi(x)$ be a formula over a model M of T. $\phi(x)$ is called strongly minimal if it has infinitely many realizations in M, and for any elementary extension M' of M and formula $\psi(x)$ over M', it is not the case that both $\phi(x) \wedge \psi(x)$ and $\phi(x) \wedge \psi(x)$ have infinitely many realizations.*

In the present context we will assume that some strongly minimal $\phi(x)$ can be found without parameters. (This is a delicate point. In general it is proved that $\phi(x)$ can be found with parameters in the prime model of T).
The set of realisations of $\phi(x)$ in a model M is what is called a strongly minimal set. Here are some basic properties and definitions.
(a) Algebraic closure on strongly minimal sets satisfies Steinitz exchange: if M is a model of T, $A \subset M$ and a, b satisfy $\phi(x)$ in M then, if $b \in acl(A, a) \setminus acl(A)$, then $a \in acl(A, b)$.
(b) A set $\{a_i : i \in I\}$ of realizations of $\phi(x)$ in a model M of T will be called (algebraically) independent, if $a_i \notin acl\{a_j : j \in I, j \neq i\}$ for each $i \in I$.
(c) If $(a_i : i < \alpha\}$ is an independent set of realisations of $\phi(x)$ in $M \models T$ and $(b_i : i < \alpha)$ is an independent set of realizations of $\phi(x)$ in $M' \models T$, then the map taking a_i to b_i $(i < \alpha)$ is elementary.
(d) A *basis* for $\phi(x)$ in M is by definition a maximal independent subset of ϕ^M. If $\{a_i : i < \alpha\}$ is such a basis then ϕ^M is contained in $acl_M(\{a_i : i < \alpha\}$.
(e) Any two bases for $\phi(x)$ in M have the same cardinality, which we call $dim(\phi, M)$.

Step 2. For any model M of T, M is prime and minimal over ϕ^M. Minimality means that there is no proper elementary substructure M' of M containing ϕ^M.
It is enough to show minimality. Suppose not. So there is a proper elementary substructure M' of M with $\phi^{M'} = \phi^M$, a so-called Vaughtian pair. We may assume that both M and M' are countable. An argument (using ω-stability) enables us to find an elementary extension M'' of M of cardinality κ with $\phi^{M''} = \phi^M$. This contradicts κ-categoricity of T, as there will exist another model N of T of cardinality κ in which ϕ^N has cardinality κ.

End of proof. Let M and N be two models of T of cardinality $\lambda > \omega$. As M is prime over ϕ^M the latter has cardinality ¿λ, and thus $dim(\phi, M) = \lambda$. Similarly $dim(\phi, N) = \lambda$. Let I be a basis for ϕ^M and J a basis for ϕ^N. So I

and J both have cardinality λ and there is an elementary map taking I to J (by (c)). This extends (by passing to the algebraic closures) to an elementary map taking ϕ^M to ϕ^N, and then also to an elementary embedding f of M in N (as M is prime over ϕ^M). As N is minimal over ϕ^N, f must be onto N, so M and N are isomorphic.

The proof shows that the isomorphism type of any model M of T is determined by $dim(\phi, M)$, and it follows fairly easily that the number of countable models of T must be either 1 or ω (the Baldwin-Lachlan Theorem).

3 Fine structure of uncountably categorical theories.

Step 1 above gave us in particular a "geometry" (the strongly minimal formula $\phi(x)$), and Step 2 told us that this formula controls every model (which we now call unidimensionality). It is technically convenient to identify the models of T (T any complete theory) with elementary submodels of a big saturated model $\bar{M}$ (or class if you wish), and work inside $\bar{M}$. We will follow this convention from now on. The geometric model-theoretic point of view (represented historically by Zilber) at this point asks: (I) exactly how does the geometry ϕ control the whole structure, or why and how is every model M prime over ϕ^M, and (II) what exactly are the possibilities for $\phi(x)$? Zilber especially needed answers to these questions in the case where T is also ω-categorical, in order to prove that totally categorical theories are not finitely axiomatizable.
Let us begin by looking at question (I). One possibility is of course is when $\phi(x)$ is "x=x", namely defines the whole structure. In this situation we say that T itself is strongly minimal, and we even call $M \models T$ a strongly minimal structure. Another possibility is the existence of a formula $\psi(x, y)$ defining a finite-to-one function from $\bar{M}$ onto X, where X is some $\emptyset$-definable subset of $(\phi^{\bar{M}})^n$. In this case, each model M is equal to $acl(\phi^M)$. Are there any other possibilities? Group actions and a kind of analogue of 'fibre bundles" (from differential geometry) turn out to give examples (and essentially the only other examples). I will describe this construction in full generality. The data will be a structure P (in some language), a definable subset X of P^n and a definable (in P) family $(G_a : a \in X)$ of definable groups (let's suppose

X and the family of groups to be $\emptyset$-definable). For each $a \in X$, let Y_a be a principal homogeneous space for G_a (namely a set on which G_a acts regularly, or equivalently strictly 1-transitively). From this data we manufacture a new structure M. The universe of M will be the disjoint union of P and the Y_a, where P is equipped with its original structure. The language for M will also contained a function symbol π, interpreted as the canonical surjection $\pi : \cup_a Y_a \to X$, as well as another function symbol $f(-,-,-)$ such that for each $a \in X$, $f(a,-,-)$ defines the action of G_a on Y_a. We will also allow arbitrary additional relations on M as long as no new structure is induced on P (namely every $\emptyset$-definable subset of P^m definable in the new structure M should be already $\emptyset$-definable in the original structure P). We call M a definable fibre bundle over P, with data $(G_a : a \in X)$. It is not hard to see that M is prime and minimal over P (and likewise for any N elementarily equivalent to M.) If $Th(P)$ is uncountably categorical, so is $Th(M)$. Note that if each group G_a is finite, then $M = acl_M(P)$. Also if X is a singleton $\{a\}$, and b any element of Y_a then $M = acl_M(P, b)$.
Zilber (see [**?**]) essentially proved that definable fibre bundles explain minimality over the strongly minimal set $\phi(x)$. (Shelah's semiregular types technology [**?**] together with an input from Hrushovski give a general account for superstable theories.)

Proposition 3.1 *If $Th(M)$ is uncountably categorical and $P \subseteq M$ is a strongly minimal set, then there are structures $P = P_0, P_1, .., P_k = M$ such that P_{i+1} is a definable fibre bundle over P_i. Moreover the relevant groups G_a can be taken to be living in P and to be elementary abelian, finite simple nonabelian, or infinite without definable proper infinite normal subgroups.*

One consequence of the proposition is that T has finite Morley rank. In any case we see in this result the strong relationship between the internal structure of a given model M and the problem of the classification all models of $Th(M)$.

A kind of restatement of the above proposition, which has a Galois-theoretic flavour is:

Remark 3.2 *Suppose T is uncountably categorical, $\phi(x)$ is strongly minimal (or actually any formula with infinitely many realizations), and $M \models T$. Then $M = \phi^M \cup \{a_i : i < \alpha\}$ where for each $\beta < \alpha$ either (i) $\beta \in acl(\phi^M \cup \{a_i : i < \beta\})$ or (ii) $tp(a_\beta / \phi^M \cup \{a_i : i < \beta\})$ is isolated by a formula $\chi(y)$ which* *defines a principal homogeneous space for a definable group G contained in ϕ. Moreover G can be chosen to be infinite, connected, and simple.*

The Galois-theoretic content of say case (ii) is that the group of elementary permutations of $dcl(\phi^M \cup \{a_i : i \leq \beta\})$ which fix $\phi^M \cup \{a_i : i < \beta\}$ pointwise is isomorphic to G.
The notion "almost strongly minimality" and variants, will be important later.

Definition 3.3 *T is said to be almost strongly minimal if there is some strongly minimal set P in $\bar{M}$ such that, after naming finitely many elements of $\bar{M}$ (namely adding finitely many constants to the language), $\bar{M} = acl_{\bar{M}}(P)$.*

Given a strongly minimal structure P containing an infinite definable group G, the basic fibre bundle constructed from P from the data P and $G_a = G$ for all $a \in P$, will be uncountably categorical and not strongly minimal. The reader may find it worthwhile looking in detail at the theory of $((Z/4Z)^\omega, +)$. This is a definable fibre bundle (with additional structure given by the group operation) over $(Z/2Z)^\omega$, and is not almost strongly minimal. Given a structure P and data $(G_a : a \in X)$, the issue of what the possible bundles over P with this data can be is essentially a cohomological question.

Now we pass to (II). The issue is to classify strongly minimal sets. What does this mean? The strongly minimal subset $P = \phi^{\bar{M}}$ of the model $\bar{M}$ can be viewed as a structure in its own right: for each formula $\psi(x_1, .., .x_n)$ of L let R_ψ be the set of realizations of ψ each of whose coordinates is in P. The resulting structure $(P, R_\psi)_\psi$ has quantifier-elimination. Zilber's hope was to classify such structures. Classification up to bi-interpretability is probably the most reasonable notion here, although the finer notion of identifying two structures if there is a bijection between their universes taking definable sets to definable sets is also useful.
There are three important examples of strongly minimal structures:
(i) An infinite set in the empty language (that is, equality the only relation).
(ii) An (infinite-dimensional) vector space V over a field F, in the language containing just $+, 0$ and f_a for each $a \in F$, representing scalar multiplication.
(iii) An algebraically closed field in the field language $(+, \cdot, 0, 1, -)$.
Let us fix a strongly minimal structure P, possibly living as above in $\bar{M}$. One

can study P by studying the behaviour of algebraic closure. Coming out of Steinitz exchange mentioned in the previous section, we assign a dimension or rank to any finite tuple from P: for any set A of parameters (maybe from $\bar{M}$) and finite tuple b from P, $dim(b/A)$ is defined to be the cardinality of any maximal A-algebraically independent subtuple of b (which is well-defined). (In fact $dim(a/A)$ coincides with the Morley rank of $tp(a/A)$ which is the smallest Morley rank of a formula in this type.)

Definition 3.4 *Let b, c be tuples from P and A and set of parameters. We say that b is independent from c over A, if $dim(b, c/A) = dim(b/A) + dim(c/A)$.*

Definition 3.5 *(i) We call P degenerate if for any subset B of P, $acl(B) \cap P = \bigcup_{b \in B} acl(b)$.*
(ii) We call P modular if after naming a small set of parameters, we have, for all tuples b, c from P, b is independent from c over $acl(b) \cap acl(c)$.

Degenerate implies modular. Example (i) above is degenerate, example (ii) modular and nondegenerate and example (iii) nonmodular.
Zilber conjectured, very early on:
Zilber conjecture. If P is nonmodular, then an infinite (necessarily algebraically closed) field is definable in the structure P.
In fact he tentatively conjectured that in the nonmodular case the structure P itself is essentially an algebraically closed field and all sets definable in P are defined in the field language.
The full conjecture was disproved by Hrushovski. However the conjecture was proved by Hrushovski and Zilber [**?**] under certain additional assumptions on P of a topological-geometric nature, namely assuming P to be a "Zariski geometry". I will not go into the definition. Zariski geometries are treated in detail in Marker's tutorial in Haifa [**?**]. Also in the concrete examples we'll be considering there are alternative proofs.
There are some rather more intuitive or geometric ways of seeing the above notions (modularity,..). P has Morley rank 1. P^2 (which one should think of as 2-space over P) has Morley rank 2, and contains various Morley rank 1 definable subsets, such as the diagonal $\{(x, x) : x \in P\}$, or for any $a \in P$, $\{(x, a) : x \in P\}$. Think of strongly minimal subsets of P^2 as curves over P. Modularity essentially says that if $X \subseteq P^m$ is definable and $(C_a : a \in X)$ is a definable family of curves (strongly minimal subsets of P^2) with pairwise

intersection finite, then $RM(X) \leq 1$: there are no 2 or higher dimensional families of curves. Degeneracy says that X is finite: there is *no* infinite family of curves.
Modularity (in this and more general contexts) is a central concept of geometric model theory. There is a well-developed general theory, and somewhat surprisingly, the degenerate/modular/nonmodular trichotomy has fundamential meaning in many mathematical contexts. In the next result, I summarise the structural consequences (due to Zilber [**?**]) of these notions in the uncountably categorical context. T will be uncountably categorical, $\bar{M}$ a (big) model of T and $P = \phi^{\bar{M}}$ a strongly minimal subset of $\bar{M}$.

Proposition 3.6 *(i) If P is degenerate, then there are no infinite definable groups in $\bar{M}$. In particular $\bar{M} = acl(P)$.*
(ii) If P is modular and nondegenerate then there is an infinite strongly minimal group definable in $\bar{M}$ (in fact essentially on P itself).
(iii) If P is modular, then for any definable group G in $\bar{M}$, G is abelian-by-finite, G has no infinite definable family of connected subgroups, and every definable (in $\bar{M}$) subset of G is a finite Boolean combination of cosets (translates of subgroups).

4 Quotients

I fudged over certain issues of definability in the last section, because certain of the objects mentioned (definable homogeneous spaces and definable groups for example) may only exists as quotients of definable sets by definable equivalence relations. So in this section I will discuss the status and model-theoretic treatment of such quotient objects as well as more general quotients (by type-definable equivalence relations). I will also discuss various Galois groups attached to theories. T is an arbitrary complete theory and we work in $\bar{M}$ a big saturated model of T. (I ignore set-theoretic complications.)
A quotient-definable set is something of the form X/E where X is a definable set and E a definable equivalence relation on X. It makes perfect sense to speak of a definable map from X/E to Y/F: it will be a map f induced by a definable relation R between X and Y. In various kinds of geometry the issue of quotient objects (for example the space of orbits of a manifold under the action of a Lie group), is a very delicate matter, because one wants

the quotient object to exist as a geometric object of the same kind. However, there is absolutely nothing at the general model-theoretic level which is problematic about such quotient objects (quotients of definable sets by definable equivalence relations), in the sense that they remain entirely within the framework of first order model theory. One formalism for seeing this is Shelah's formation of the many-sorted structure $\bar{M}^{eq}$. This is obtained by adding a new sort S_E for each $\emptyset$-definable equivalence relation E on $\bar{M}^n$, together with a function f_E from $\bar{M}^n \to S_E$. Then any Y/F (where Y, F may be defined with parameters) can be naturally identified with a definable subset of some sort S_E in $\bar{M}^{eq}$. Precisely for this reason, we should really understand the category of definable sets and functions in $\bar{M}^{eq}$. The structures P_i given in Lemma 3.1 are really definable in M^{eq}. This all suggests that we should from the start consider many-sorted structures, working in many-sorted logic. In fact there is no harm in even allowing a sort S_ϕ for each formula $\phi(x)$ of L (without parameters). To "understand" such a structure then amounts to identifying a certain family $(S_i)_i$ of sorts, classifing the definable subsets of these sorts (quantifier-elimination), and showing that for any sort S and $\emptyset$-definable equivalence relation E on S, there is a $\emptyset$-definable bijection between S/E and a definable subset of one of the S_i (elimination of imaginaries). Nevertheless, back in the one-sorted situation we make:

Definition 4.1 *T has elimination of imaginaries if for any $\emptyset$-definable set $X \subseteq \bar{M}^n$ and $\emptyset$-definable equivalence relation E on X there is a $\emptyset$-definable bijection f between X/E and some definable $Y \subseteq \bar{M}^m$.*

Remark 4.2 *(i) The definition above has a very minor discepancy with the usual notion of elimination of imaginaries, but agrees if there are two distinct constants in T.*
(ii) If T has elimination of imaginaries, then the right hand side of Definition 3.1 also holds where we no longer demand $\emptyset$-definability of X, E and f.

Maybe the only general model-theoretic result regarding elimination of imaginaries is:

Fact 4.3 *Suppose T is strongly minimal. Then, up to naming finitely many parameters T has weak elimination of imaginaries: for any X/E as in Definition 4.1, there there is a $\emptyset$-definable set Y and a $\emptyset$-definable surjective function $f : Y \to X/E$ with finite fibres.*

Fact 4.4 *The theory of algebraically closed fields of a given characteristic, the theory of real closed fields and the theory of differentially closed fields of characteristic* 0 *all eliminate imaginaries.*

In each of the theories above definable groups can be definably equipped with unique "geometric" structure: algebraic groups, Nash groups and differential algebraic groups respectively. Similarly for definable homogeneous spaces. So Fact 4.3 proves that quotients of such groups by definable subgroups exist as geometric objects.

Elements of the form a/E (a a finite tuple from $\bar{M}$ and E a $\emptyset$-definable equivalence relation) are usually called imaginaries. As we have explained there is no intrinsic model-theoretic problem with these objects. Suppose now that E is an equivalence relation on tuples of fixed, but possibly infinite (although small, relative to the saturation of $\bar{M}$) length, which is defined by a possibly infinite set of formulas (without parameters say). An equivalence class a/E is what we have recently called a *hyperimaginary*. A type-definable set of such objects: X/E where X is a set of tuples of the right length defined by a possibly infinite set of formulas, is a *hyperdefinable set*. The status of such objects (hyperimaginaries, and hyperdefinable sets) *is* a model-theoretic problem. If E happens to be an intersection of *definable* equivalence relations E_i (at least when restricted to $tp(a)$), then a/E can be identified naturally with the sequence $(a/E_i)_i$ of imaginaries, namely with a "pro-imaginary". We will say that T eliminates hyperimaginaries if this always happens: any type-definable equivalence relation is equivalent, on the set of realizations of any complete type, to the intersection of definable equivalence relations (any hyperimaginary is a pro-imaginary). Any stable theory (for example, an uncountably categorical theory) has elimination of hyperimaginaries. However, even if T does *not* eliminate hyperimaginaries, these objects still remain within first order model theory (namely subject to the compactness theorem). Although we cannot in a sensible way add new sorts S_E for each such E or even talk about definable sets of hyperimaginaries, we can make sense out of the complete type of a hyperimaginary (as a certain partial type), and this is enough to do model theory. Hyperdefinable, non pro-definable sets arise naturally in nonstandard analysis (where E might be the relation of being infinitely close). For example the so-called nonstandard hull $\hat{B}$ of a Banach space B is a hyperdefinable set in a saturated model ${}^*B$ of B. Henson and

Iovino have developed some stability theory for Banach spaces using Henson's Banach space logic. An alternative treatment would be to use standard stability theory directly on the hyperdefinable set $\hat{B}$.

By $acl^{eq}(A)$ we mean the set of elements of $\bar{M}^{eq}$ in the algebraic closure of A. For T the theory of algebraically closed fields of characteristic 0, $acl^{eq}(\emptyset)$ is precisely $\bar{\mathbf{Q}}$. The automorphism group of $\bar{M}$, $Aut(\bar{M})$ acts naturally on $acl^{eq}(\emptyset)$, and the corresponding group $Aut(\bar{M})$ quotiented by the normal subgroup consisting of those σ which fix $acl^{eq}(\emptyset)$ pointwise, has the natural structure of a profinite group (which is precisely the absolute Galois group of $\mathbf{Q}$ in the characteristic 0 algebraically closed fields case). We denote this group by $Gal_{pf}(T)$. It is an invariant of the bi-interpretability type of T. The structure of $Gal_{pf}(T)$ has various implications. For example, if T is ω-categorical and $Gal_{pf}(T)$ is finite (even after naming any finite set of parameters), then Lascar [**?**] proved that the bi-interpretability type of T can be recovered from the abstract group $Aut(\bar{M})$. Also Hrushovski [**?**] remarks that if T is uncountably categorical and finitely axiomatizable, then only finitely many finite simple groups occur as quotients of $Gal_{pf}(T)$ (even after naming finitely many parameters). Also there is a Galois theory. For any T there is a Galois correspondence between closed subgroups of $Gal_{pf}(T)$ and definably closed subsets of $acl^{eq}(\emptyset)$.

The analogous construction can be made for hyperimaginaries. Let $bdd^{heq}(A)$ be the set of hyperimaginaries which have small orbit under $Aut_A(\bar{M})$. Then $Aut(\bar{M})$ acts on $bdd^{heq}(\emptyset)$. As above we obtain a group which we denote $Gal_c(T)$, the closed Galois group. This has naturally the structure of a compact (Hausdorff) topological group, and is again an invariant of the bi-interpretability type of T. $Gal_{pf}(T)$ is naturally a quotient of $Gal_c(T)$. The two groups are equal just if there is no hyperdefinable (over an element $c \in acl^{eq}(\emptyset)$) connected compact Lie group acting as automorphisms of some hyperdefinable (over c) set X. Again there is a Galois correspondence between closed subgroups of $Gal_c(T)$ and definably closed subsets of $bdd^{heq}(\emptyset)$. (See [**?**].)

There is a third, natural but rather mysterious group $Gal(T)$, which lives rather in the "descriptive set theory" of $\bar{M}$. Consider equivalence relations E on possibly infinite tuples, which are invariant under $Aut(\bar{M}$. Let $bdd^{inveq}(A)$ be the set of classes of such E which have small orbit under $Aut(\bar{M})$. Again

$Aut(\bar{M})$ acts on $bdd^{inveq}(\emptyset)$ and $Gal(T)$ is the resulting group. $Gal_c(T)$ is a quotient of $Gal(T)$, and gives $Gal(T)$ the structure of a quasicompact (not necessarily Hausdorff) topological group. Ziegler recently gave an example where $Gal(T) \neq Gal_c(T)$. We know very little about $Gal(T)$. For example what can be the cardinality of the quotient $Gal(T)/Gal_c(T)$?

5 Finite rank structures

The basic ingredients in the proof of Morley's theorem were generalized by Shelah [**?**] to build the enormous machinery of stability theory which he used to solve the spectrum problem for countable theories. Analogously, the geometric ideas from section 3 have much wider applicability. The key points are the identification of certain "geometries", the fine structure of these geometries, and structural consequences for parts of the ambient structure controlled by these geometries. A new feature, absent in the ω_1-categorical context, will be orthogonality. For example, there may be several strongly minimal formulas ϕ_i which together control the whole structure (every model M of T is prime over $\bigcup_i \phi_i^M$) but the ϕ_i may have no mutual interaction. Also in the general superstable case, the geometries that have to be considered may have infinite Morley rank. Here we will restrict our attention to finite rank structures where it is the strongly minimal geometries that are relevant. This is a small generalization of the situation considered in section 3. The possible presence of orthogonal strongly minimal sets is the new feature. In subsequent applications, these finite rank structures may arise as definable sets in some ambient (infinite rank or even unranked) structures. Also at this point, the reader may wish to think of our theories and structures being many sorted.

Definition 5.1 *We will say that a structure M (equivalently $Th(M)$) has finite Morley rank, if in $Th(M)$ every formula has finite Morley rank. We will call M finite-dimensional, if in addition, there are a finite set $\{\phi_i : i = 1, .., k\}$ of strongly minimal formulas in T^{eq} such that every model N of $Th(M)$ is prime and minimal over $\bigcup_i \phi_i^N$.*

We assume for the remainder of this section that T is a theory of finite Morley rank, $M \models T$ and $\bar{M}$ a saturated model of T. We will work in $\bar{M}$ unless we say otherwise. Recall that the Morley rank of a complete type $p(x)$

over a set A is the minimum of the Morley ranks of the formulas in p. We write $RM(a/A)$ for $RM(tp(a/A))$. If a happens to be a tuple from a strongly minimal set, this coincides with $dim(a/A)$ as defined in section 3.

Definition 5.2 *Let a be a finite tuple, $A \subset B$. We say a is independent from B over A if $RM(a/B) = RM(a/A)$.*

Remark 5.3 *This coincides with independence in the sense of nonforking. We have the basic properties: a is independent from B over A if a is independent from $A \cup \{b\}$ for all finite tuples b from B iff b is independent from $A \cup \{a\}$ over A for all finite tuples b from B. Given a and $A \subseteq B$, there is a' realising $tp(a/A)$ such that a' is independent from B over A. $tp(a/A)$ is stationary if for all $B \supset A$, and a', a'' realising $tp(a/A)$ independent from B over A, $tp(a'/B) = tp(a''/B)$. $tp(a/A)$ is stationary iff it has Morley degree 1.*

Definition 5.4 *T is modular, if for all finite tuples a, b, a is independent from b over $acl^{eq}(a) \cap acl^{eq}(b)$.*

This agrees with Definition 3.5 (ii) if T happens to be strongly minimal. Here are some generalizations of Proposition 3.6.

Fact 5.5 *(i) T is modular iff each strongly minimal formula in $\bar{M}^{eq}$ is modular.*
(ii) If T is modular and G is a definable group in $\bar{M}^{eq}$ then G is abelian-by-finite, G has no infinite definable family of infinite connected subgroups, and all definable subsets of G are Boolean combinations of cosets.

Definition 5.6 *Let X, Y be definable sets in $\bar{M}^{eq}$, both defined over A say. We say that X and Y are fully orthogonal if whenever b is a finite tuple from X and c a finite tuple from Y then b is independent from c over A.*

Remark 5.7 *(i) The content of full orthogonality is that any definable subset of $X^n \times Y^m$ is a finite union of products of definable subsets of X^n and definable subsets of Y^n.*
(ii) Restricted to strongly minimal sets, full orthogonality is usually just called orthogonality. Nonorthogonality is an equivalence relation on strongly minimal sets.

If T is also finite-dimensional, then Proposition 3.1 holds, in the sense that there are (given M) $P_0,..,P_n = M$ such that $P_0 = \phi_1^M \cup .. \cup \phi_k^M$ and P_{i+1} is a definable fibre bundle over P_i.

Groups of finite Morley rank. We mean a group G with possibly additional structure such that $Th(G)$ has finite Morley rank. In many applications such groups will arise as definable groups in some ambient structure. A result of Lascar (see [**?**]) says that any finite Morley rank group is finite-dimensional (in the sense of 5.1). From Proposition 3.1, we see that the structure of simple groups of finite Morley rank is relevant to the fine structure of uncountably categorical theories. An underlying and fundamental result due to Macintyre is that any infinite field of finite Morley rank is algebraically closed. G is said to be connected if G has no proper definable subgroup of finite index. All the above notions have various equivariant implications and interpretations. For example

Fact 5.8 *Suppose H_1, H_2 are definable connected fully orthogonal subgroups of G then H_1 and H_2 commute. Moreover any definable subset X of $H_1.H_2$ is essentially the product of a definable subset X_1 of H_1 with a definable subset X_2 of H_2.*

Let X be a definable subset of G of Morley multiplicity 1. The *stabilizer* of X in G, $Stab_G(X)$ is by definition the set of $g \in G$ such that $RM(X) = RM(X \cap g \cdot X)$. This is a definable subgroup of G. A kind of stability-theoretic analogue of the *socle* of a group (subgroup generated by minimal normal subgroups) is the largest connected definable subgroup of G contained in the algebraic closure of some finite set $(\phi_i(x))_i$ of strongly minimal formulas. (This should be read in a saturated model.) We will call this object $s(G)$, hopefully without ambiguity. If G happened to be uncountably categorical, this is precisely the maximal connected almost strongly minimal subgroup of G. As remarked in section 3, if G is $(Z/4Z)^\omega$, this is $2G$. A useful and relatively elementary result relating the structure of definable sets in a commutative group G to $s(G)$ is the following [**?**]:

Lemma 5.9 *Let G be a commutative connected group of finite Morley rank, defined in an ambient structure $\bar{M}$ over a set A. Assume that every connected definable subgroup of G is $acl^{eq}(A)$ definable. Let $H = s(G)$. Let X be any definable set of Morley degree 1 which has finite stabilizer. Then, up to a set of smaller Morley rank, X is contained in a single translate of H.*

6 Examples.

Algebraically closed fields. The theory of algebraically closed fields of a fixed characteristic say 0 (ACF_0) is strongly minimal, with quantifier-elimination in the language $(+, \cdot, 0, 1, -)$. The dichotomies and results from sections 3 and 5 are largely vacuous here. All definable sets are mutually nonorthogonal, all definable groups are almost strongly minimal, nothing is modular. The basic objects of algebraic geometry are certain definable sets, varieties, defined by finite systems of polynomial equations. Morley rank and algebraic-geometric dimension coincide for such objects. Any definable set is a finite Boolean combination of such things, even a finite Boolean combination of smooth projective varieties. Definable functions are piecewise rational. One of the aims of algebraic geometry is the classification of varieties up to birational isomorphism. This is pretty close to classifying definable sets up to definable isomorphism. The general model-theory of ACF_0 says very little about this problem. We will see later however that the model theory of certain enriched structures (such as differentially closed fields) is meaningful for issues such as the deformation theory of algebraic varieties.

A definable group can be uniquely equipped with the geometric structure of a variety (pieced together from finitely many affine varieties with rational transition maps) such that multiplication becomes a morphism. A class of such groups which is very important for geometry and arithmetic is the class of abelian varieties, connected algebraic groups whose underlying variety is a closed subvariety of some projective space $\mathbf{P}^n$. These are commutative groups, and any smooth projective curve which is not isomorphic to $\mathbf{P}^1$ will embed in a unique smallest such abelian variety (its Jacobian variety). Hence their importance for the study of curves, at least. Any strongly minimal definable set X in $(K, +, \cdot)$ is, up to finite, a smooth projective curve C. Assume $K = \mathbf{C}$. A smooth projective curve C is a compact Riemann surface (so a compact 2-dimensional manifold), and has a finite number of "handles" g, the genus of the curve. (g is the dimension of its Jacobian variety.) This is a fundamental invariant of the curve, and also of the strongly minimal set X. We will use below the following fact: the strongly minimal set (or curve if you wish) has genus ≥ 2 if and only if there is no definable group structure on X, even after adding or subtracting finitely many points. See [**?**] for more background.

Compact complex manifolds. A complex manifold M is a topological space with a covering by open sets homeomorphic to open subsets of some $\mathbf{C}^n$ such that the transition maps are holomorphic (complex analytic). If M is such, so are $M \times M$, $M \times M \times M$ etc. By an analytic subset or subvariety) of M we mean a subset X of M such that for every $a \in M$, there is an open neighbourhood U of a in M and holomorpic functions $f_1, .., f_n$ on U such that $U \cap X = \{x \in U : f_1(x) = ... = f_n(x) = 0\}$. Let M be a compact complex manifold and consider M as a relational structure by adding a predicate R_X for each analytic subvariety of M^n. Zilber pointed out (using a theorem of Remmert)) that M is a structure of finite Morley rank with quantifier-elimination (but clearly not saturated as every element of M is essentially named by a constant), in fact even a "Zariski structure" in a generalized sense. We can actually consider the whole category of compact complex manifolds as a many sorted structure; the relations on $M_1 \times ... \times M_n$ being again the analytic subvarieties. This category is again a structure of finite Morley rank (every sort has finite Morley rank). It turns out that the machinery developed earlier *is* meaningful for complex compact manifolds (either one at a time, or the whole category). We let $\mathcal{A}$ denote the many-sorted structure. Among the sorts is $\mathbf{P}^1(\mathbf{C})$ (we will just say $\mathbf{P}^1$), which is, by Chow's theorem, essentially just the structure $(\mathbf{C}, +, \cdot)$ considered above. Basically all the general theory we have discussed has meaning in the structure $\mathcal{A}$. A complex torus T is a complex Lie group of the form $\mathbf{C}^n/\Lambda$ where Λ is a lattice of real rank $2n$. It is precisely a compact complex Lie group so is already a sort in $\mathcal{A}$ (and the group operation is definable). For suitably general Λ and for $n > 1$, T will be fully orthogonal to $\mathbf{P}^1$. The complex analytic literature ([**?**]) already contains the result that any complex torus which is fully orthogonal to $\mathbf{P}^1$ must be modular. This implies:

Proposition 6.1 *Suppose T is a complex torus with no proper subtori. Then ether T is modular or T is (definably) isomorphic to an abelian variety.*

The Zilber conjecture is true in $\mathcal{A}$ (via Zariski geometries). This may have interesting implications for the classification of compact complex manifolds up to bimeromorphic equivalence. Some recent results ([**?**]) are:

Proposition 6.2 *Let G be a strongly minimal modular group definable in $\mathcal{A}$. Then G is definably isomorphic to a complex torus.*

Proposition 6.3 *Suppose X is a strongly minimal set definable in $\mathcal{A}$. Suppose $\bar{X}$ is a connected compact complex manifold such that X is a definable open subset of $\bar{X}$. Then X is degenerate if and only if (i) there are no nonconstant meromorphic functions (to $\mathbf{P}^1$) on $\bar{X}$, and (ii) there is no generically surjective meromorphic map from $\bar{X}$ to any space of the form T/G where T is a complex torus and G a finite group of (holomorphic) automorphisms of T.*

Definable groups and homogeneous spaces come into the picture much as in Proposition 3.1. For example, it follows from the general theory that if M is a compact Kahler manifold, and $f : M \to X$ is the algebraic reduction of M (f definable map onto an algebraic variety X of maximal possible dimension) and the general fibre of f is isomorphic to an algebraic variety Y, then Y is a homogeneous space for a complex algebraic group.

Differential equations. One algebraic route to the study of (algebraic) differential equations, is via differential rings and fields. A differential field is a field F equipped with a derivation D. The theory of differential fields of characteristic 0 has a model companion, the theory of differentially closed fields DCF_0. This theory is ω-stable, but of infinite Morley rank. The definable sets of finite rank turn out to witness very nicely the geometric-model-theoretic themes discussed above. In fact, via model theory we see quite amazing analogies between the category of definable sets of finite Morley rank in a model of DCF_0 and the category $\mathcal{A}$ discussed in the previous section. (See [**?**] for more on this).
We fix a large model $(K, +, ., D)$ of DCF_0. k denotes the field of constants (which is a strongly minimal set). Again the Zilber conjecture is true for strongly minimal sets definable in this structure, via the Zariski geometries theorem. There exists a direct differential algebraic-geometric proof in the case where the strongly minimal set is already a subset of a finite Morley rank group [**?**], yielding the following (first proved in [**?**]).
Suppose A to be an abelian variety (defined in the algebraically closed field $(K, +, \cdot)$). By $A^\sharp$ we mean the smallest definable (in $(K, +, \cdot, D)$) subgroup of A containing the group of torsion points of A.

Proposition 6.4 *(i) $A^\sharp$ has finite Morley rank, and for any finite Morley rank G with $A^\sharp < G < A$, $A^\sharp = s(G)$.*
(ii) If A is defined over the field k of constants, then $A^\sharp$ is precisely $A(k)$,

the group of points of A with coefficients in k.
(iii) $A^\sharp$ is modular if and only if A has no abelian subvariety isomorphic (as an algebraic group) to an abelian variety defined over k (A has k-trace 0).
(iv) Let A_1, A_2 be simple abelian varieties, each nonisomorphic to any abelian variety defined over k. Then there is a rational isogeny from A_1 to A_2 iff $A_1^\sharp$ and $A_2^\sharp$ are nonorthogonal.

The analogue of 6.3 is:

Proposition 6.5 *Let X be a strongly minimal set in $(K, +, ., D)$. Then exactly one of the following holds:*
(i) X is degenerate,
(ii) X is definably isomorphic to a definable subset of k^n (for some n),
(iii) there is a simple abelian variety A with k-trace 0 and a generically surjective definable map from X to $A^\sharp/G$ for some finite group G of definable automorphisms of $A^\sharp$.

Fundamental questions remain concerning degenerate strongly minimal sets. An important invariant of a definable set of finite Morley rank is its "order" (more or less the order of the differential polyynomial defining it). In the order 1 case, the situation is rather clear (by a finiteness theorem of Jouanolou). But for higher orders nothing is known.
Again the definable homogeneous space technology from section 3 is relevant. It both explains and generalizes the classical Picard-Vessiot Galois theory of linear differential equations.

Finiteness theorems. Faltings proved [**?**] that a curve X of genus ≥ 2 defined over $\mathbf{Q}$ has only finitely many points with coordinates in $\mathbf{Q}$. A version over function fields (conjectured by Lang) was proved earlier by Manin:

Proposition 6.6 *Let X be a curve defined over F where F is a function field over an algebraically closed k (characteristic 0). Then either (i) X is not isomorphic to a curve defined over k and X(F) is finite, or (ii) X is isomorphic to a curve X_0 defined over k and all but finitely many points of X(F) come from points of $X_0(k)$ via this isomorphism.*

As is well-known, model theoretic methods give a proof of the first part (using especially 6.4(iii)). Rather easier aspects of the theory (a small part of (i) and (ii) of 6.4) yield the second part, which I will outline. This second part is called the Theorem of di Franchis and can be restated:

Proposition 6.7 *(characteristic 0.) Let X be a curve of genus ≥ 2 and W be any variety. Then there are only finitely many generically surjective rational maps from W to X.*

We sketch a proof, coming essentially from [**?**]. Let k be an algebraically closed field over which X and W are defined. Let F be the function field of W. A generically surjective rational map from W to X corresponds to a point of $X(F) \setminus X(k)$, so we must show there are finitely many such things. Let D be a derivation on F whose field of constants is k. Extend to a derivation D on a differentially closed field K containing F (still with constants k). Let A be an abelian variety containing X and defined over k (the Jacobian variety of X). The only use of $genus(X) \geq 2$ will be that the algebraic-geometric stabilizer $Stab(X) = \{a \in A : a + X = X\}$ is finite (otherwise X would be a translate of its stabilizer). Now there is a certain definable (in the differentially closed field) homomorphism, the logarithmic derivative, from $A(K)$ to some K^n whose kernel is $A(k)$. As $A(F)/A(k)$ is finitely generated (by the Lang-Neron theorem) there is a finite Morley rank definable subgroup G of A containing $A(F)$. Let $Y = X \cap G$. We must show Y to be finite. Suppose otherwise. As X has genus ≥ 2, $Stab_G(Y)$ is finite, so by 6.4 (i), (ii), and 5.9, Y is contained in a single coset of $A(k)$. Let $a \in Y$ (so $a \notin A(k)$. As Y is infinite, there are infinitely many $b \in A(k)$ such that $b + a \in Y$. Let $k_0 < k$ be such that the curve X is defined over k_0. So we can find $b \notin acl(k_0)$ such that $b + a \in Y$. In particular, working in the algebraically closed field K, a is a generic point of X over k_0, a is independent from b over k_0 and $b + a$ is also a generic point of X over k_0. So $b \in Stab(X)$ and as $b \notin acl(k_0)$ this shows $Stab(X)$ to be infinite, a contradiction.

It should be remarked that proofs like the above (using auxiliary model-theoretic structures) automatically yield good bounds, in the above case doubly exponential bounds, as a function of the shape of the equations defining X and the rank of the finitely generated group $A(F)/A(k)$. (See [**?**].)

Faltings [**?**] subsequently proved a generalization of Mordell's conjecure, again conjectured by Lang: If A is a complex abelian variety, X a subvariety and Γ a finitely generated subgroup of A then $X \cap \Gamma$ is a finite union of cosets. The same holds for semiabelian varieties, algebraic groups which are extensions of an abelian variety by $(\mathbf{C}^*)^n$. (As is well-known, methods using above machinery yield a proof of this in the function field case, also in positive

characteristic [**?**].)
One can ask whether an analogous (absolute) statement holds in the category $\mathcal{A}$. Here the analogue of a semiabelian variety is a extension A of a complex torus by some $(\mathbf{C}^*)^n$. Such a group A is commutative and has a compactification $\bar{A}$ living in $\mathcal{A}$ such that A and its group structure are definable (in $\mathcal{A}$). The analogue of a subvariety is an analytic subvariety of A definable in $\mathcal{A}$. With then have:

Proposition 6.8 *Suppose A is an extension of a complex torus by $(\mathbf{C}^*)^n$. Suppose X is a "subvariety" of A and Γ a finitely generated subgroup of A. Them $X \cap \Gamma$ is a finite union of cosets.*

Sketch of proof. This is a straightforward reduction to the semiabelian variety case using methods from [**?**]. We write the group operation additively. We may assume that X is irreducible, and $X \cap \Gamma$ is "Zariski dense" in X, namely X is the smallest "analytic" subvariety of A containing $X \cap \Gamma$. We want to show that X is a translate of a definable subgroup of A. Let $S = \{a \in A : a + X = X\}$. Let $\pi : A \to A/S$ be the natural homomorphism. Then $\pi(\Gamma)$ is finitely generated and $\pi(\Gamma) \cap \pi(X)$ is Zariski-dense in X. Also X is a union of translates of S. So (replacing A, X, Γ, by $\pi(A), \pi(X), \pi(\Gamma)$ we may assume that S is finite (or even trivial). Let $s(A)$ be the definable socle of A. By 5.9 we may assume that X is contained in $s(A)$. By the results in section 6, $s(A)$ is the almost direct sum of A_1 a semiabelian variety, and A_2 a modular complex torus. A_1 and A_2 are fully orthogonal, so by 5.7, $X = X \cap A_1 + X \cap A_2$. As A_2 is modular $X \cap A_2$ is a translate of a subgroup A_3 of A_2. But then A_3 is contained in S, so A_3 is a point. So up to translation X is contained in A_1, a semiabelian variety. By [**?**] (the generalization of Faltings theorem to semiabelian varieties), X is a translate of a semiabelian subvariety. We either both a proof of the theorem and a contradiction.

7 Variants

The main problem is the development of geometric model theory outside the finite Morley rank context as well as finding applications. The theory is fully in place ([**?**],[**?**]) for superstable theories (T is superstable if for all sufficiently large models M of T there are at most $|M|$-many complete types over M).

There is a rank on types, the U-rank which is ordinal valued but maybe infinite. Strongly minimal formulas are replaced by regular types, types of U-rank ω^α. There is as yet no formulation of a "Zariski geometry" on regular types, and also no general theorems regarding the Zilber conjecture in this context.

In fact the general theory is in place for stable theories, except that regular types no longer coordinatize the structure. (T is stable if for any model M of T there are at most $|M|^{|T|}$ complete types over M.) See [**?**] for a reasonably comprehensive acount.

There are not many stable theories. In the past five years there has been an enormous amount of work done generalizing the machinery of stability theory to a larger class of theories, the *simple* theories. The model companion of fields equipped with an automorphism, $ACFA$, is simple but unstable. The validity of the Zilber conjecture here ([**?**]) has led to more model-theoretic applications to diophantine geometry [**?**].

The study of o-minimal structures (neither stable, nor simple) is another thriving area [**?**]. However a lot of the general geometric theory is vacuous here, as 1-dimensionality is built in to the situation. One would like to see a theory of finite-dimensional structures which are coordinatized by o-minimal structures, in which the principal homogeneous spaces from section 3 play a role.

In some sense the various kinds of theories for which there is a developed model theory correspond to structures surrounding number theory: the complex field (strongly minimal), the real field (o-minimal), ultraproducts of finite fields (simple of SU-rank 1). There is however no really "general" theory in place corresponding to the field of p-adics, and one would hope to see some progress here.

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