GIBBS MEASURES OF CONTINUOUS SYSTEMS: AN ANALYTIC APPROACH

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ABSTRACT. We present a new method to prove existence and uniform *à-priori* estimates for Gibbs measures associated with classical particle systems in continuum. The method is based on the choice of appropriate Lyapunov functionals and on the corresponding exponential bounds for the local Gibbs specification.

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1. Introduction

The paper is devoted to mathematical aspects of the equilibrium statistical mechanics of classical particle systems in continuum. Our goal is to establish a *new* analytic method for proving *existence* and *à-priori* bounds for the corresponding Gibbs measures $\mu \in \mathcal{G}$. The main advantage of this method is its *simplicity*. Besides some elementary constructions from (infinite dimensional) analysis and probability, no advanced tools are needed. To illustrate the key ideas, we first concentrate on the standard case of a (non-translation invariant, possibly discontinuous) pair interaction $V(x, y)$ assigned to particles in Euclidean space $\mathbb{R}^d$, $d \geq 1$. Afterwards, we demonstrate how to remove technical assumptions such as finite range of the interaction or spatial regularity of the intensity measure and how to handle the case of multibody interactions. As will be pointed out below, some of these generalizations are far beyond the reach of previously available techniques. Furthermore, our method straightforwardly extends to marked Gibbs measures or to the configuration spaces over manifolds, which will be elaborated in more detail in subsequent papers.

An initial step in studying Gibbs measures is always the existence problem. However, for most of infinite particle systems with non-compact phase spaces, existence turns out to be far from evident. Our situation becomes even more complicated, since the Gibbs measures to be constructed are supported by the space $\mathit{\Gamma}$ of locally finite sets (i.e., *configurations*) γ over $\mathbb{R}^d$, which by itself is an *infinite dimensional* manifold. Note that we are only considering simple configurations $\gamma \in \mathit{\Gamma}$ (i.e., those with at most one particle at each point $x \in \mathbb{R}^d$), which is justified by the applications in physics. Let us recall that for classical continuous systems there are two commonly accepted, but essentially different approaches employing respectively *Ruelle's superstability estimates* [42, 43, 26] and *Dobrushin's existence criterion* [7, 8, 3, 24] (see the comments in Subsection 2.4). Instead of inductive or combinatorial techniques typical in those approaches, we shall basically use the analysis and geometry on the configuration space $\mathit{\Gamma}$ that were developed in [1, 2]. This yields not only a direct way of constructing the corresponding Gibbs states, but also a lot of easily computable bounds for them in terms of the interaction parameters.

Before going into further details, let us mention the important notion of *temperedness* that naturally arises in all systems of unbounded spins. It is common knowledge that any detailed study of Gibbs measures is impossible without assuming any *prior* information about their properties. A practicable compromise is to confine ourselves to a proper subset $\mathcal{G}^{\mathrm{t}}$ of so-called *tempered* Gibbs measures with controlled growth. The definition of temperedness we suggest below (cf. (2.65), (2.66)) is more general than those in the existing literature.

Our key idea is to prove exponential integrability of a certain *Lyapunov functional*, which is given by the energy $H(\gamma_{\Lambda_k})$ of a configuration γ restricted to a small cube $\Lambda_k := Q_{gk} \subset \mathbb{R}^d$ of edge length $g > 0$, see Lemma 3.1. An important issue here is the *weak* dependence of the corresponding bound (3.1) on the boundary values $\xi \in \mathit{\Gamma}$, fixed outside Q_{gk}. Using the consistency property of the local Gibbs specification, in Lemma 3.3 we then extend the above estimate to large volumes $\Lambda \subset \mathbb{R}^d$ constructed by means of the partition $\mathbb{R}^d = \coprod_{k \in \mathbb{Z}^d} Q_{gk}$. For the kernels of the Gibbs specifications $\pi_\Lambda(\gamma|\xi)$, this implies the necessary tightness to prove that $\mathcal{G}^{\mathrm{t}} \neq \varnothing$ (cf. Theorem 2.8). Since the bounds obtained for $\pi_\Lambda(\gamma|\xi)$ are asymptotically uniform as $\Lambda \nearrow \mathbb{R}^d$ and hold for μ-almost all $\xi \in \Gamma$, by the *DLR* equation we immediately get a similar bound for *all* $\mu \in \mathcal{G}^{\mathrm{t}}$ (cf. Theorem 2.9). At this point there is a principal difference in comparison with Dobrushin's approach, which just ensures the existence of *some* tempered Gibbs states μ without any information about the whole set $\mathcal{G}^{\mathrm{t}}$. The same scheme works for interactions of infinite range, for which we should control the weak dependence, contractivity, and compactness properties of the specification π_Λ in suitable weighted seminorms over Γ (see Subsection 4.3). In contrast to Ruelle's approach, there is no reason to decompose into n-particle configurations located in bounded domains Λ and to analyze the associated correlation functions. Actually, the class of measures $\mathcal{G}^{\mathrm{t}}$ we construct and study below includes all μ having Ruelle's support, cf. Remark 4.9. For the precise links to Dobrushin's and Ruelle's techniques see Subsections 2.4 and 4.5, 4.6, respectively.

The paper is organized as follows. In Section 2 we fix the standard framework for Gibbs measures on configuration spaces and present our main results. In Subsection 2.1 we shortly recall a few basic facts about the Poisson measure $\pi_{z\sigma}$ on Γ to be used later on. The local Gibbs specification $\Pi = \{\pi_\Lambda(\mathrm{d}\gamma|\xi)\}$ and the corresponding Gibbs measures $\mu \in \mathcal{G}$ as solutions to the *DLR* equation are defined in Subsection 2.2. In Subsection 2.3 we give precise conditions on the interaction and relate them to Ruelle's superstability. Next, in Subsection 2.4 we introduce the set of tempered configurations $\gamma \in \Gamma^{\mathrm{t}}$ and the set of tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ obeying $\mu(\Gamma^{\mathrm{t}}) = 1$. Then, we formulate our main Theorems 2.8 and 2.9 on existence and *à-priori* estimates for $\mu \in \mathcal{G}^{\mathrm{t}}$. Comments, which in particular compare these results with previous ones obtained by other methods, conclude this section.

In Section 3 we prove our main Theorems 2.8 (existence) and 2.9 (*à-priori bounds*). In Subsection 3.1 we prepare the key technical Lemmas 3.1 and 3.3 about the integrability properties of the specification kernels $\pi_\Lambda(\mathrm{d}\gamma|\xi)$. Thereafter, in Subsection 3.2 we give the complete proofs of Theorems 2.8 and 2.9, which as mentioned before turn out to be quite elementary.

In Section 4 we extend our initial model in several core directions. A natural question how *singularity* of the potential $V(x,y)$ on the diagonal may improve the regularity properties of the corresponding $\mu \in \mathcal{G}^{\mathrm{t}}$, is addressed in Subsection 4.1. Possible refinements of Theorems 2.8 and 2.9 in the case of *strong superstable* interactions are outlined in Subsection 4.2. It is worth noting that in most examples of physical interest, the interaction is usually strongly superstable. In Subsection 4.3 we modify our method to the particle systems with *multibody* interactions. The new situation when the intensity measure σ is no longer translation invariant on $\mathbb{R}^d$ is treated in Subsection 4.4. Subsequently, in Subsection 4.5 we obtain precise information on support properties of all tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$. In Subsection 4.6 we take a closer look at the correlation functional k_μ and point out further relations to Ruelle's approach.

Finally, in Section 5 we demonstrate how to handle interactions of infinite range. A principal difficulty here is to identify the limit points $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ as $\Lambda \nearrow \mathbb{R}^d$ as solutions of the *DLR* equation. To this end we implement an alternative way, based on the (almost) *continuity* of the Gibbs specification in certain spaces of tempered configurations, see Proposition 5.2. The proof works for physically relevant potentials (e.g., for the *Lennard-Jones* potential) with a singularity at the diagonal.

Finally, we emphasize that our methods appear to be quite universal and applicable to different classes of models: classical or quantum, on a lattice, on an infinite graph, or in the continuum. In this respect let us mention here some recent papers, where appropriate modifications of our technique were firstly implemented to quantum anharmonic crystals [25, 34, 35] and to classical spin systems on graphs [23]. Therefore, we hope that the present work will contribute to make the theory of Gibbs measures more accessible to a wider audience, in particular, for all specialists being interested in applications of infinite dimensional analysis to problems of mathematical physics.

2. Description of the Model and Main Result

In this section we recall the standard setting of classical statistical mechanics and present the main theorems obtained in the paper.

2.1. **Preliminaries on configuration spaces.** We consider a mechanical system of identical particles that interact via a pair potential $V(x,y)$ obeying certain stability properties to be specified below. Any such particle is entirely described by a point (its position) $x=(x^{(i)})_{i=1}^d$ in the phase space $\mathbb{R}^d$, $d\in\mathbb{N}$. This space is equipped with the Euclidean distance $|\cdot|$ and the *Borel* σ-algebra $\mathcal{B}(\mathbb{R}^d)$ generated by the family $\mathcal{O}(\mathbb{R}^d)$ of its open subsets. By $\mathcal{O}_c(\mathbb{R}^d)$ and $\mathcal{B}_c(\mathbb{R}^d)$ we denote the systems of all *bounded* elements (i.e., those with compact closures) in $\mathcal{O}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d)$, respectively. So, $\mathcal{O}_c(\mathbb{R}^d)$ contains all open balls $B_r(y):=\{x\in\mathbb{R}^d \mid |x-y|<r\}$ with center $y\in\mathbb{R}^d$ and radius $r>0$.

For a subset $\Lambda\subseteq\mathbb{R}^d$, let Λ^c, $\mathring{\Lambda}$, $\bar{\Lambda}$, and $\partial\Lambda:=\bar{\Lambda}\backslash\mathring{\Lambda}$ denote its complement, interior, closure, and topological boundary, respectively. Subsets constituted by points $k=(k^{(i)})_{i=1}^d$ of the integer lattice $\mathbb{Z}^d\subseteq\mathbb{R}^d$ will be denoted by $\mathcal{K}$. For shorthand we write $\mathcal{K}\Subset\mathbb{Z}^d$ if the set $\mathcal{K}$ is nonvoid and finite, i.e., its cardinality obeys $0<|\mathcal{K}|<\infty$. A sequence of $\Lambda_N\in\mathcal{B}_c(\mathbb{R}^d)$ (or $\mathcal{K}_N\Subset\mathbb{Z}^d$) indexed by $N\in\mathbb{N}$ is called *order generating* if it is ordered by inclusion and exhausts the whole $\mathbb{R}^d$ (respectively, $\mathbb{Z}^d$). Furthermore, $\Lambda\nearrow\mathbb{R}^d$ (or $\mathcal{K}\nearrow\mathbb{Z}^d$) means the limit taken along any unspecified sequence of this type. Finally, we abbreviate $\mathbb{R}_+:=[0,\infty)$, $\bar{\mathbb{R}}:=\mathbb{R}\cup\{+\infty\}$, $\mathbb{Z}_+:=\{0,1,2,\ldots\}$, and $\bar{\mathbb{Z}}_+:=\mathbb{Z}_+\cup\{+\infty\}$.

The *configuration space* $\Gamma:=\Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$ consists of all locally finite subsets, i.e.,

$$\Gamma:=\left\{\gamma\subset\mathbb{R}^d \;\middle|\; |\gamma_\Lambda|<\infty,\quad \forall\Lambda\in\mathcal{B}_c(\mathbb{R}^d)\right\}, \tag{2.1}$$

where $|\gamma_\Lambda|$ is the number of points in the restriction $\gamma_\Lambda:=\gamma\cap\Lambda$. The elements $\gamma\in\Gamma$ are called *simple configurations.* For technical reason we shall also need the larger space of *multiple configurations*

$$\ddot{\Gamma}:=\{(\gamma,n)\mid \gamma\in\Gamma,\ n:\gamma\to\mathbb{N}\}\,. \tag{2.2}$$

A multiple configuration (γ,n) can be interpreted as follows: $\gamma\in\Gamma$ is the support set describing positions of particles in $\mathbb{R}^d$, whereas $n(x)$ is the number of particles at each point $x\in\gamma$. If there is no confusion, the notation $\gamma\in\ddot{\Gamma}$ will be understood as $(\gamma,n)\in\ddot{\Gamma}$. The total number of particles in $\gamma_\Lambda:=\gamma\cap\Lambda$ is then given by $|\gamma_\Lambda|:=\sum_{x\in\gamma_\Lambda}n(x)$. Each $\gamma\in\Gamma$ (respectively, $\gamma\in\ddot{\Gamma}$) can be identified with the $\bar{\mathbb{Z}}_+$-valued counting measure $\sum_{x\in\gamma}\delta_x$ (respectively, $\sum_{x\in\gamma}n(x)\delta_x$), where δ_x is the Dirac distribution with mass at point x. Here and elsewhere, all sums over the empty index set are set to be zero. In particular, $\gamma = \varnothing$ corresponds to the zero measure on $\mathbb{R}^d$. In this sense we have a natural embedding

$$\Gamma \ \subset \ \ddot{\Gamma} \ \subset \ \mathcal{M}(\mathbb{R}^d),$$

where $\mathcal{M}(\mathbb{R}^d)$ denotes the linear space of all signed Radon measures on $\mathbb{R}^d$. The spaces Γ and $\ddot{\Gamma}$ will be equipped with the *vague topology* inherited from $\mathcal{M}(\mathbb{R}^d)$. Below we summarize the basic properties of this topology which will be essential for our considerations.

By definition, $\mathcal{O}_v(\mathcal{M})$ is the coarsest topology on $\mathcal{M}(\mathbb{R}^d)$ with respect to which each of the following maps is continuous

$$\mathcal{M}(\mathbb{R}^d) \ni \nu \to \langle f, \nu \rangle := \int_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x), \quad f \in C_0(\mathbb{R}^d), \tag{2.3}$$

where $C_0(\mathbb{R}^d)$ denotes the set of all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support. It is well known that $\mathcal{M}(\mathbb{R}^d)$ is a *Polish* (i.e., a separable completely metrizable) space in the vague topology $\mathcal{O}_v(\mathcal{M})$. An example of such metrization to be consistent with $\mathcal{O}_v(\mathcal{M})$ is given by

$$\rho_v(\nu, \nu') := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\langle f_k, \nu \rangle - \langle f_k, \nu' \rangle|}{1 + |\langle f_k, \nu \rangle - \langle f_k, \nu' \rangle|}, \quad \nu, \nu' \in \mathcal{M}(\mathbb{R}^d), \tag{2.4}$$

where $(f_k)_{k\in\mathbb{N}} \subset C_0(\mathbb{R}^d)$ is a proper measure determining class for $\mathcal{M}(\mathbb{R}^d)$, see 15.7.7 in [16]. By $\mathcal{O}_v(\Gamma)$ and $\mathcal{O}_v(\ddot{\Gamma})$ we denote the induced topologies on Γ and $\ddot{\Gamma}$, respectively. Since $\ddot{\Gamma}$ is a vaguely closed subset of $\mathcal{M}(\mathbb{R}^d)$, the metric space $(\ddot{\Gamma}, \rho_v)$ will be Polish as well, see 15.7.4 in [16] or 3.2.4 in [17]. The latter property obviously fails for (Γ, ρ_v). Nevertheless, it can be shown that Γ is a dense G_δ-set in $(\ddot{\Gamma}, \rho_v)$, and hence by the Alexandrov–Hausdorff theorem there exists some other metrization of $(\Gamma, \mathcal{O}_v(\Gamma))$ making it a *Polish* space, see [22, 47]. A large class of the appropriate metrics (indeed stronger than ρ_v) was constructed in [22] (whose explicit form, however, is not relevant for our purposes).

An intrinsic description of the topology $\mathcal{O}_v(\Gamma)$ can be given by a subbase of open sets

$$\{\gamma \in \Gamma \mid |\gamma_\Lambda| = n, \ \gamma_{\partial\Lambda} = \varnothing\}, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d), \ n \in \mathbb{Z}_+. \tag{2.5}$$

The sequential convergence in $(\Gamma, \mathcal{O}_v(\Gamma))$ is then characterized as follows:

$$\gamma^{(N)} \xrightarrow{v} \gamma \ \text{ as } N \to \infty, \ \text{ iff}$$

$$|\gamma_\Lambda^{(N)}| \to |\gamma_\Lambda| \ \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \text{ with } |\gamma_{\partial\Lambda}| = 0. \tag{2.6}$$

Note that (2.6) implies the usual convergence of points in the configurations being restricted to any domain $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ such that $\gamma \cap \partial\Lambda = \varnothing$. More precisely, the number of particles in Λ stabilizes, i.e., $|\gamma_\Lambda^{(N)}| = |\gamma_\Lambda|$ for large enough $N \geq N(\Lambda)$, and there are appropriate enumerations $\{x_j\}_{j=1}^{|\gamma_\Lambda|} = \gamma_\Lambda$ and $\{x_j^{(N)}\}_{j=1}^{|\gamma_\Lambda|} = \gamma^{(N)}$ such that

$$\lim_{N\to\infty} x_j^{(N)} = x_j \quad \text{for all } 1 \leq j \leq |\gamma_\Lambda|. \tag{2.7}$$

This can easily be seen by analyzing the limits $\langle f_j^{(M)}, \gamma^{(N)} \rangle \to x_j$ as $M, N \to \infty$, where $\{f_j^{(M)}\}$ is a system of functions from $C_0(\mathbb{R}^d)$ such that $\mathbf{1}_{B_{1/2M}(x_j)} \leq f_j^{(M)} \leq \mathbf{1}_{B_{1/M}(x_j)}$. In the case $\gamma \cap \partial\Lambda \neq \varnothing$ one should be more careful. So, for an open

$\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ we can only say that $|\gamma_\Lambda^{(N)}| \geq |\gamma_\Lambda|$ as $N \to \infty$, whereas for each $j > |\gamma_\Lambda|$ there may occur sequences $x_j^{(N)} \in \gamma_\Lambda^{(N)}$ having accumulation points outside Λ, see e.g. page 212 of [45]. In particular, the *cut-off* operator $\gamma \mapsto \gamma_\Lambda$ and the *counting* map $\gamma \mapsto |\gamma_\Lambda|$ are vaguely *continuous* at some $\gamma \in \Gamma$ iff $\gamma_{\partial\Lambda} = \varnothing$, cf. Subsection 5.4.

By $\mathcal{B}(\Gamma)$ we denote the associated *Borel* σ-algebra, which also coincides with the smallest σ-algebra generated by the cylinder sets

$$\{\gamma \in \Gamma \mid |\gamma_\Lambda| = n\}, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d),\ n \in \mathbb{Z}_+. \tag{2.8}$$

Similarly, for a fixed $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, we define $\mathcal{B}_\Lambda(\Gamma)$ $(\subset \mathcal{B}(\Gamma))$ as the smallest σ-algebra generated by the sets

$$\{\gamma \in \Gamma \mid |\gamma_{\Lambda'}| = n\}, \quad \Lambda' \in \mathcal{B}_c(\mathbb{R}^d),\ \Lambda' \subset \Lambda,\ n \in \mathbb{Z}_+. \tag{2.9}$$

Obviously, $\mathcal{B}(\Gamma) = \mathcal{B}(\ddot{\Gamma}) \cap \Gamma$, where $\mathcal{B}(\ddot{\Gamma})$ is the σ-algebra of Borel sets in $\ddot{\Gamma}$. The family of all probability measures μ on $(\Gamma, \mathcal{B}(\Gamma))$ (also called *simple point processes* [16, 17]) will be denoted by $\mathcal{P}(\Gamma)$.

Let $(\tau_s)_{s\in\mathbb{R}^d}$ be the group of translations in $\mathbb{R}^d$, i.e., $\tau_s x := s + x$ for all $x \in \mathbb{R}^d$. The corresponding transformation on the configurations $\gamma \in \Gamma$ is given by $\tau_s\gamma := \sum_{x\in\gamma} \delta_{s+x}$; it is vaguely continuous and hence measurable. A measure $\mu \in \mathcal{P}(\Gamma)$ is called *translation invariant* if $\mu \circ \tau_s^{-1} = \mu$ for all $s \in \mathbb{R}^d$.

Next, we introduce some commonly used spaces of *finite* configurations. The space of configurations located in $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ is defined as the disjoint union

$$\Gamma_\Lambda := \left\{\gamma \in \Gamma \,\middle|\, \gamma_{\mathbb{R}^d\setminus\Lambda} = \varnothing\right\} = \bigsqcup\nolimits_{n\in\mathbb{Z}_+} \Gamma_\Lambda^{(n)} \tag{2.10}$$

of the n-particle subsets

$$\Gamma_\Lambda^{(n)} := \{\gamma \in \Gamma_\Lambda \mid |\gamma| = n\}. \tag{2.11}$$

In a standard way, each Γ_Λ is equipped with the topology $\mathcal{O}_v(\Gamma_\Lambda)$ induced from $\mathcal{O}_v(\Gamma)$ under the natural projection

$$\Gamma \ni \gamma \longmapsto \mathbb{P}_\Lambda\gamma := \gamma_\Lambda \in \Gamma_\Lambda \tag{2.12}$$

and with the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_\Lambda) := \mathcal{B}(\Gamma) \cap \Gamma_\Lambda$. A subbase of the topology $\mathcal{O}_v(\Gamma_\Lambda)$ consists of the open sets

$$\{\gamma \in \Gamma \mid |\gamma_{\Lambda'}| = n,\ \gamma_{\partial\Lambda'} = \varnothing\}, \tag{2.13}$$

indexed by all possible $n \in \mathbb{Z}_+$ and $\Lambda' \in \mathcal{B}_c(\mathbb{R}^d)$ with $\bar{\Lambda}' \subset \Lambda$. There is obvious relations

$$\mathcal{B}(\Gamma_\Lambda) = \mathbb{P}_\Lambda(\mathcal{B}_\Lambda(\Gamma)) = \mathcal{B}(\Gamma) \cap \Gamma_\Lambda,$$

which shows that $\mathcal{B}_\Lambda(\Gamma)$ and $\mathcal{B}(\Gamma_\Lambda)$ are σ-isomorphic. An important observation is that each $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ will be a *standard Borel* space (see Section A.5 of [26]). In other words, $\mathcal{B}(\Gamma_\Lambda)$ coincides with the Borel σ-algebra generated by some separable and complete metric on Γ_Λ. An example of such metrization of Γ_Λ (inducing a topology stronger than $\mathcal{O}_v(\Gamma_\Lambda)$) can be found in Section 3.1 of [27].

For all $\Lambda, \Lambda' \in \mathcal{B}_c(\mathbb{R}^d)$ with $\Lambda' \subset \Lambda$, the maps

$$\Gamma_\Lambda \ni \gamma_\Lambda \longmapsto \mathbb{P}_{\Lambda',\Lambda}\gamma_\Lambda := \gamma_{\Lambda'} \in \Gamma_{\Lambda'}$$

are continuous as well. This yields a '*localized*' description of Γ as the *projective* limit of the spaces $(\Gamma_\Lambda)_{\Lambda\in\mathcal{B}_c(\mathbb{R}^d)}$ as $\Lambda \nearrow \mathbb{R}^d$. Below we shall use a corresponding version of *Kolmogorov's theorem* for projective limit spaces (cf. Theorem V.3.2 in [33]), according to which any probability measure $\mu \in \mathcal{P}(\Gamma)$ is *uniquely* determined by the family of its finite volume projections $\mu_\Lambda := \mu \circ \mathbb{P}_\Lambda^{-1} \in \mathcal{P}(\Gamma_\Lambda)$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. As a measure determining class for $\mathcal{P}(\Gamma)$ one can take the family $\mathcal{F}C_{\mathrm{b}}(\Gamma)$ of all *cylinder* functions $F : \Gamma \to \mathbb{R}$ allowing the representation

$$F(\gamma) := g(< f_1, \gamma > \ldots < f_M, \gamma >), \quad \gamma \in \Gamma, \tag{2.14}$$

with some $f_1, \ldots, f_M \in C_0(\mathbb{R}^d)$, $g \in C_{\mathrm{b}}(\mathbb{R}^M)$, and $M \in \mathbb{N}$. Such functions are $\mathcal{B}_0(\Gamma)$-measurable, where

$$\mathcal{B}_0(\Gamma) := \bigcup\nolimits_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} \mathcal{B}(\Gamma_\Lambda)$$

denotes the algebra of all *local* Borel sets. We also shall need the subset of configurations *finite* in all of $\mathbb{R}^d$

$$\Gamma_0 := \bigcup\nolimits_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} \Gamma_\Lambda = \bigsqcup\nolimits_{n \in \mathbb{Z}_+} \Gamma_0^{(n)}, \tag{2.15}$$

where, similarly to (2.11),

$$\Gamma_0^{(n)} := \{\gamma \in \Gamma_0 \mid |\gamma| = n\} = \bigcup\nolimits_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} \Gamma_\Lambda^{(n)}, \quad \Gamma_0^{(0)} := \varnothing.$$

For more details on topological structure of the above configuration spaces see e.g. [1, 20, 22, 26, 30, 31].

In statistical physics, the state of an ideal gas is described by a *Poisson* random point field $\pi_{z\sigma}$ on Γ. We next recall the well-known explicit construction of $\pi_{z\sigma}$ (see e.g. Section 2.1 in [1] or Section 2.4 in [10]). To this end we fix a chemical *activity* parameter $z > 0$ and an *intensity* measure $\sigma \geq 0$ on the underlying phase space $\mathbb{R}^d$. As usual, one assumes that σ is a *nonatomic Radon* measure on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$, i.e.,

$$\sigma(\mathbb{R}^d) = \infty, \quad \sigma(\Lambda) < \infty \ \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d),$$
$$\text{and } \ \sigma(\{x\}) = 0 \ \text{ for all } x \in \mathbb{R}^d.$$

For convenience, we also suppose a kind of spatial *regularity*

$$\sup_{y \in \mathbb{R}^d} \sigma\,[B_r(y)] < \infty \tag{2.16}$$

to be hold for some (and hence for all) finite $r > 0$. Obviously, (2.16) is fulfilled for any translation invariant measure on $\mathbb{R}^d$ (in particular, for the Lebesgue measure $\mathrm{d}x$). A possible way to omit this technical condition will be discussed in Subsection 4.4. For each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the corresponding σ-*Poisson* measure $\lambda_{z\sigma} := \lambda_{z\sigma}^\Lambda$ (or the *Lebesgue-Poisson* measure λ_z if $\sigma(\mathrm{d}x) = \mathrm{d}x$) is defined on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ by the identity

$$\int_{\Gamma_\Lambda} F(\gamma_\Lambda) \mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda)$$
$$= \quad F(\varnothing) + \sum_{n \in \mathbb{N}} \frac{z^n}{n!} \int_{\Lambda^n} F(\{x_1, \ldots, x_n\}) \mathrm{d}\sigma(x_1) \ldots \mathrm{d}\sigma(x_n), \tag{2.17}$$

which holds for all bounded measurable functions $F \in L^\infty(\Gamma_\Lambda)$. Taking into account that $\lambda_{z\sigma}^\Lambda(\Gamma_\Lambda) = e^{z\sigma(\Lambda)}$, on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ we then introduce the probability measures

$$\pi_{z\sigma}^\Lambda := e^{-z\sigma(\Lambda)} \lambda_{z\sigma}^\Lambda. \tag{2.18}$$

Note that the family $\{\pi^{\Lambda}_{z\sigma} \,|\, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\}$ is consistent, which means

$$\pi^{\Lambda'}_{z\sigma} = \pi^{\Lambda}_{z\sigma} \circ \mathbb{P}^{-1}_{\Lambda',\Lambda} \quad \text{whenever } \Lambda' \subset \Lambda.$$

By Kolmogorov's theorem (cf. Theorem V.3.2 in [33]), the *Poisson* measure $\pi_{z\sigma}$ is the unique probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ such that

$$\pi^{\Lambda}_{z\sigma} = \pi_{z\sigma} \circ \mathbb{P}^{-1}_{\Lambda} \quad \text{for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d). \tag{2.19}$$

Furthermore, $\pi_{z\sigma}$ is equivalently defined as the unique probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ such that, for any collection of disjoint domains $(\Lambda_j)_{j=1}^N \subset \mathcal{B}_c(\mathbb{R}^d)$, the random variables $|\gamma_{\Lambda_j}|$ are mutually independent and distributed by the Poissonian law with parameters $z\sigma(\Lambda_j)$, i.e.,

$$\pi_{z\sigma}\left(\left\{\gamma \in \Gamma \,\Big|\, |\gamma_{\Lambda_j}| = n\right\}\right) = \frac{z^n \sigma^n(\Lambda_j)}{n!} e^{-z\sigma(\Lambda_j)}, \quad n \in \mathbb{Z}_+. \tag{2.20}$$

Another analytic characterization of $\pi_{z\sigma}$ is through its Laplace transform, see e.g. [11],

$$\int_{\Gamma} \exp\langle f, \gamma\rangle \mathrm{d}\pi_{z\sigma}(\gamma) := \exp\left\{\int_{\mathbb{R}^d} \left(e^{f(x)} - 1\right) z \mathrm{d}\sigma(x)\right\}, \quad f \in C_0(\mathbb{R}^d).$$

2.2. **Gibbsian formalism.** Now we define Gibbs reconstructions of the '*free*' measure $\pi_{z\sigma}$. Let there be given a *pair interaction potential* which is a symmetric measurable function

$$V : \mathbb{R}^d \times \mathbb{R}^d \to \bar{\mathbb{R}}, \quad V(x,y) = V(y,x) \quad \text{for all } x, y \in \mathbb{R}^d. \tag{2.21}$$

We stress that neither *translation invariance* nor *continuity* of V need to be assumed (in particular, V may be a hard-core or a potential singular everywhere). For simplicity, we here impose the following technical restriction on the potential.

(**FR**): ***Finite range*: *There exists*** $R \in (0, \infty)$ *such that*

$$V(x,y) = 0 \quad \textit{if} \quad |x - y| \geq R. \tag{2.22}$$

The case of *long-range* interactions with $R = \infty$ requires a more detailed analysis, which will be performed in Section 5.

The *Hamiltonian* (or *energy functional*) $H : \Gamma_0 \to \bar{\mathbb{R}}$ is given on finite configurations $\gamma \in \Gamma_0$ by

$$H(\gamma) := \sum_{\{x,y\} \subset \gamma} V(x,y) \in \bar{\mathbb{R}}, \tag{2.23}$$

where the sum runs over all (unordered) pairs of distinct points $x, y \in \gamma$. By convention, this functional vanishes at the empty and one-point configurations, i.e.,

$$H(\varnothing) = 0, \quad H(\{x\}) = 0 \quad \text{for all } x \in \mathbb{R}^d. \tag{2.24}$$

Respectively, for each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\gamma, \xi \in \Gamma$,

$$W_{\Lambda}(\gamma_{\Lambda}|\xi) := \sum_{x \in \gamma_{\Lambda},\, y \in \xi_{\Lambda^c}} V(x,y) \tag{2.25}$$

is the *interaction energy* between $\gamma_{\Lambda} \in \Gamma_{\Lambda}$ and $\xi_{\Lambda^c} := \xi \cap \Lambda^c$, which is well-defined because of Assumption (**FR**) Combining (2.23) and (2.25), we introduce the *conditional Hamiltonians* $H_{\Lambda}(\cdot|\xi) : \Gamma_{\Lambda} \to \bar{\mathbb{R}}$ by

$$H_{\Lambda}(\gamma_{\Lambda}|\xi) := H(\gamma_{\Lambda}) + W_{\Lambda}(\gamma_{\Lambda}|\xi). \tag{2.26}$$

For fixed inverse temperature $\beta := 1/T > 0$, the *local Gibbs state* with boundary condition ξ is defined by

$$\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi) := [Z_\Lambda(\xi)]^{-1} \exp\{-\beta H_\Lambda(\gamma_\Lambda|\xi)\}\, \lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda) \tag{2.27}$$

provided the corresponding *partition function* (cf. (2.17) and (2.24))

$$\begin{aligned} Z_\Lambda(\xi) \quad : &= \int_{\Gamma_\Lambda} \exp\{-\beta H_\Lambda(\gamma_\Lambda|\xi)\}\, \mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) \\ &= 1 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\Lambda^n} \exp\{-\beta H_\Lambda(\{x_1,\ldots,x_n\}|\xi)\}\, \mathrm{d}\sigma(x_1)\ldots\mathrm{d}\sigma(x_n) \end{aligned} \tag{2.28}$$

is finite. In the case $Z_\Lambda(\xi) = +\infty$ we respectively set $\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi) = 0$. From (2.28) we see that we *always* have

$$Z_\Lambda(\xi) = [\mu_\Lambda(\{\varnothing_\Lambda\}|\xi)]^{-1} \geq 1, \tag{2.29}$$

which will be repeatedly used in the subsequent estimates.

The *local specification* $\Pi = \{\pi_\Lambda\}_{\Lambda\in\mathcal{B}_c(\mathbb{R}^d)}$ is a family of stochastic kernels

$$\mathcal{B}(\Gamma) \times \Gamma \ni (B,\xi) \mapsto \pi_\Lambda(B|\xi) \in [0,1] \tag{2.30}$$

given by

$$\pi_\Lambda(B|\xi) := \mu_\Lambda(B_{\Lambda,\xi}|\xi), \quad B_{\Lambda,\xi} := \{\gamma_\Lambda \in \Gamma_\Lambda \mid \gamma_\Lambda \cup \xi_{\Lambda^c} \in B\} \in \mathcal{B}(\Gamma_\Lambda). \tag{2.31}$$

By construction (cf. Proposition 6.3 in [38] or Proposition 2.6 in [39]), the family (2.31) obeys the *consistency property*, which means that for all $B \in \mathcal{B}(\Gamma)$ and $\xi \in \Gamma$

$$\int_\Gamma \pi_\Delta(B|\gamma)\pi_\Lambda(\mathrm{d}\gamma|\xi) = \pi_\Lambda(B|\xi), \quad \Delta \subseteq \Lambda. \tag{2.32}$$

In Subsection 3 we shall impose some natural conditions on V to guarantee that $Z_\Lambda(\xi) < \infty$ (cf. (2.60)); then each specification kernel $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ will be a *probability* distribution on $(\Gamma, \mathcal{B}(\Gamma))$.

Given $F \in L^\infty(\Gamma)$ and $\mu \in \mathcal{P}(\Gamma)$, let us define $\pi_\Lambda F \in L^\infty(\Gamma)$ and $\pi_\Lambda\mu \in \mathcal{P}(\Gamma)$ by

$$(\pi_\Lambda F)(\xi) \quad : = \int_\Gamma F(\gamma)\pi_\Lambda(\mathrm{d}\gamma|\xi), \quad \xi \in \Gamma, \tag{2.33}$$

$$(\pi_\Lambda \mu)(B) \quad : = \int_\Gamma \pi_\Lambda(B|\gamma)\mu(\mathrm{d}\gamma), \quad B \in \mathcal{B}(\Omega), \tag{2.34}$$

which are obviously related by the duality $\langle \pi_\Lambda F, \mu\rangle = \langle F, \pi_\Lambda \mu\rangle$. Here and elsewhere, we use the following shorthand for expectations

$$\langle F, \mu\rangle := \mu(F) := \int_\Omega F \mathrm{d}\mu. \tag{2.35}$$

Definition 2.1. *A probability measure $\mu \in \mathcal{P}(\Gamma)$ is called a grand canonical **Gibbs measure** (or **state**) with pair potential V, activity z, and intensity σ if it satisfies the Dobrushin-Lanford-Ruelle (**DLR**) equilibrium equation*

$$(\pi_\Lambda \mu)(B) := \int_\Gamma \pi_\Lambda(B|\gamma)\mu(\mathrm{d}\gamma) = \mu(B) \tag{2.36}$$

for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $B \in \mathcal{B}(\Omega)$. For fixed inverse temperature β, the associated set of all Gibbs states will be denoted by $\mathcal{G}$.

Remark 2.2. By (2.32) a measure μ is Gibbs iff it solves the *DLR* equation (2.36) just for *some* order generating sequence $\Lambda_N \nearrow \mathbb{R}^d$ (e.g., for the cubes $[-N,N]^d$ or balls $B_N(0)$). Equation (2.36) can be rewritten in the dual form

$$\langle \pi_\Lambda F, \mu\rangle := \int_\Gamma \int_\Gamma F(\gamma)\pi_\Lambda(\mathrm{d}\gamma|\xi)\mu(\mathrm{d}\xi) = \int_\Omega F(\gamma)\mathrm{d}\mu(\gamma) = \langle F, \mu\rangle, \tag{2.37}$$

which has to hold for all $F \in L^\infty(\Gamma)$. Furthermore, it would suffice to check (2.37) on all cylinder functions $F \in \mathcal{F}C_b(\Gamma)$ as defined in (2.14), which constitute a measure determining class for $\mathcal{P}(\Gamma)$.

We recall that the standard sources on the *DLR* approach in statistical mechanics are the monographs [12, 38].

2.3. **Assumptions on the interaction and superstability.** It is instructive to get started with the following simplest but physically realistic model, which then will be enriched step by step. An advantage of this model is that it allows a direct control of attraction-repulsion effects. Namely, throughout this section we impose the next two conditions on the interaction potential.

(**LB**)**:** ***Lower boundedness***: *There exist $M \geq 0$ and $r_1, r_2 \in [0,R]$, $r_1 \leq r_2$, such that*

$$\inf_{x,y\in\mathbb{R}^d} V(x,y) \geq -M \quad \textit{and} \tag{2.38}$$

$$V(x,y) \geq 0 \quad \textit{if} \quad |x-y| \leq r_1 \quad \textit{or} \quad |x-y| \geq r_2. \tag{2.39}$$

(**RC**)**:** ***Repulsion condition***: *There exists $\delta > 0$ such that*

$$\inf_{x,y:\,|x-y|\leq\delta} V(x,y) =: A_\delta > 2Mm_\delta, \tag{2.40}$$

where

$$m_\delta := m_\delta(d, r_1, r_2) := v_d d^{d/2}\left[(r_2/\delta + 3/2)^d - (r_1/\delta - 3/2)^d\right] \tag{2.41}$$

and $v_d := \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ is the volume of a unit ball in $\mathbb{R}^d$.

Remark 2.3. The role of the parameter m_δ will become clear from (2.49) below; note from (2.41) that $m_\delta = \mathcal{O}\left(\delta^{-d}\right)$ as $\delta \to 0$. The relation $A_\delta > 2Mm_\delta$ is essential for proving Lemma 3.3, where it enters via (3.9). An obvious example that fulfills (2.40), with an arbitrary large but fixed $M > 0$, is any potential V obeying the following asymptotic behavior

$$\lim_{|x-y|\to 0} \frac{V(x,y)}{|x-y|^d} \to +\infty, \quad \text{and hence} \quad \lim_{\delta\to 0} \frac{A_\delta}{\delta^d} \to +\infty. \tag{2.42}$$

In particular, this includes the class of so-called *Dobrushin-Fisher-Ruelle* (*DFR*) potentials which are characterized by the following growth at the diagonal: for some $\varkappa, C > 0$

$$V(x,y) \geq C|x-y|^{-(d+\varkappa)} \quad \text{as } |x-y| \to 0. \tag{2.43}$$

The trivial situation when $V(x,y) = 0$ for all $x, y \in \mathbb{R}^d$ is described by the choice of $r_1 = r_2$ and $A = M = 0$ and thus formally does not fit (2.40); the existence of $\mu \in \mathcal{G}$ for any nonnegative V is shown in Remark 3.4. Merely speaking, Condition (**RC**) means that the *repulsive* part $V^+ := \max\{V, 0\}$ of the pair interaction dominates the *attractive* one $V^- := \min\{V, 0\}$. In the case of translation invariant potentials (i.e., when $V(\tau_s x, \tau_s y) = V(x,y)$ for all $x, y, s \in \mathbb{R}^d$), a similar assumption was first employed in [37]. For a better control of V in (2.39), one may use a finite system of intervals $(r_1^{(n)}, r_2^{(n)})$ such that $V(x,y)$ is nonnegative for $|x-y|$ outside each of them. Furthermore, proceeding in the spirit of [8], it is possible to refine the global bounds (2.38) and (2.40) by certain integral conditions on V^+ and V^-.

To analyze the consequences of the above assumptions, let us consider a *partition* of the phase space $\mathbb{R}^d = \bigsqcup_{k\in\mathbb{Z}^d} Q_{gk}$ by the cubes

$$Q_{gk} := \left\{ x = (x^{(i)})_{i=1}^d \in \mathbb{R}^d \,\middle|\, g\left(k^{(i)} - 1/2\right) \le x^{(i)} < g\left(k^{(i)} + 1/2\right) \right\}. \tag{2.44}$$

These cubes have edge length $g > 0$ and are centered at the points gk, $k \in \mathbb{Z}^d$. Recall that $\mathring{Q}_{gk}$ and $\bar{Q}_{gk}$ denote respectively the interior and closure of Q_{gk} in $\big(\mathbb{R}^d, |\cdot|\big)$. For $k \in \mathbb{Z}^d$ and $\gamma \in \Gamma$, we then define

$$\Gamma_k := \Gamma_{Q_{gk}}, \quad \gamma_k := \gamma \cap Q_{gk}, \quad \bar{\gamma}_k := \gamma \cap \bar{Q}_{gk}. \tag{2.45}$$

In what follows, we pick the parameter $g := \delta/\sqrt{d}$ with some $\delta > 0$ satisfying Assumption (**RC**)**.** By construction

$$\operatorname{diam}(Q_{gk}) := \sup_{x,y\in Q_{gk}} |x-y| = \delta, \tag{2.46}$$

which implies that $V(x,y) \ge A$ for all $x, y \in \gamma_k$. Here and below we shall often drop δ in the notation for the corresponding constants A, m in (2.40).

Technically we need to control only those pairs $\{x,y\} \subset \gamma$ for which $V(x,y) < 0$. It is clear that $V(x,y)$ may be negative for some $x \in \gamma_k$ and $y \in \gamma_j$ whenever

$$j \in \partial_g^- k := \{k' \neq k \mid \vartheta_1 < |k'-k| < \vartheta_2\}, \tag{2.47}$$

$$\vartheta_1 := (r_1/\delta - 1)\sqrt{d}, \quad \vartheta_2 := (r_2/\delta + 1)\sqrt{d}. \tag{2.48}$$

The total number of such '*neighbor*' cubes Q_{gj} can be roughly estimated by

$$|\partial_g^- k| \le m_\delta(r_1, r_2), \tag{2.49}$$

which is the same constant as in (2.41). Note that to each index set $\mathcal{K} \Subset \mathbb{Z}^d$ there corresponds the '*cubic*' domain

$$\Lambda_{\mathcal{K}} := \bigsqcup_{k\in\mathcal{K}} Q_{gk} \in \mathcal{B}_c(\mathbb{R}^d); \tag{2.50}$$

the family of all such domains will be denoted by $\mathcal{Q}_c(\mathbb{R}^d)$. On the other hand, for a given volume $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, we can construct its *'minimal'* covering

$$\Lambda_g := \bigsqcup_{k\in\mathcal{K}_\Lambda} Q_{gk} \in \mathcal{Q}_c(\mathbb{R}^d) \;\text{ with }\; \mathcal{K}_\Lambda := \left\{k \in \mathbb{Z}^d | \; \Lambda \cap Q_{gk} \neq \varnothing \right\}, \tag{2.51}$$

where $|\mathcal{K}_\Lambda|$ is the number of cubes Q_{gk} having nonvoid intersection with Λ.

The first claim of Lemma 2.4 says that under the above assumptions the interaction is *superstable* in the usual sense of *Ruelle* [43] (this gives a positive answer to a question posed on page 146 of [37]). The second claim (playing a crucial role in our approach) establishes a *lower bound* on the local Hamiltonians $H_\Lambda(\gamma|\xi)$ in terms of the boundary condition ξ_{Λ^c}, which has to be valid in *small volumes* Λ. For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ we denote its *'interaction'* neighborhood

$$\partial_R \Lambda := \left\{ y \in \mathbb{R}^d \;\mid \operatorname{dist}(y, \Lambda) \le R \right\} \in \mathcal{B}_c(\mathbb{R}^d), \tag{2.52}$$

so that for all $x \in \Lambda$ and $y \notin \mathcal{U}_R(\Lambda)$ we have $V(x,y) = 0$.

Lemma 2.4. *(i) For any partition of $\mathbb{R}^d$ by the cubes (2.44) with edge length $g > 0$, there exist $D_g, E_g > 0$ such that for all $\gamma \in \Gamma_0$ the following holds:*

$$\text{(2.53)} \qquad (\mathbf{SS})\text{: } \textit{Ruelle's Superstability:} \quad H(\gamma) \geq D_g \sum_{k \in \mathbb{Z}^d} |\gamma_k|^2 - E_g |\gamma|.$$

(ii) Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ be such that $\operatorname{diam}(\Lambda) \leq \delta$, *then for all $\gamma, \xi \in \Gamma$*

$$\text{(2.54)} \qquad H_\Lambda(\gamma_\Lambda|\xi) \geq \frac{A}{2}\left(|\gamma_\Lambda|^2 - |\gamma_\Lambda|\right) - M|\gamma_\Lambda| \cdot |\xi_{\Lambda^c \cap \partial_R \Lambda}|.$$

In particular, for any $\varepsilon \in (0,1]$

$$\text{(2.55)} \qquad H_\Lambda(\gamma_\Lambda) \geq \frac{A}{2}(1-\varepsilon)|\gamma_\Lambda|^2 - \frac{A}{8\varepsilon}.$$

Proof. (i) Note that it suffices to check (2.53) just for $g := \delta/\sqrt{d}$. Because of the elementary inequality

$$\text{(2.56)} \qquad \sum_{k=1}^{K} |\gamma_k|^2 \leq \left(\sum_{k=1}^{K} |\gamma_k|\right)^2 \leq K \sum_{k=1}^{K} |\gamma_k|^2, \quad K \in \mathbb{N},$$

this would readily imply (**SS**) for *all* $g > 0$, but with their own constants D_g and E_g. By (2.38), (2.40), (2.49), and (2.51) we see that for each $\gamma \in \Gamma_\Lambda$ with $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$

$$\begin{aligned} H(\gamma) &= \sum_{k \in \mathcal{K}_\Lambda} \sum_{\{x,y\} \subset \gamma_k} V(x,y) + \sum_{\{k,j\} \subset \mathcal{K}_\Lambda} \sum_{x \in \gamma_k,\ y \in \gamma_j} V(x,y) \\ &\geq A \sum_{k \in \mathcal{K}_\Lambda} \binom{|\gamma_k|}{2} - \frac{M}{2} \sum_{k \in \mathcal{K}_\Lambda} \sum_{j \in \partial_g^- k \cap \mathcal{K}_\Lambda} |\gamma_k| \cdot |\gamma_j| \\ &\geq \frac{1}{2}(A - Mm) \sum_{k \in \mathcal{K}_\Lambda} |\gamma_k|^2 - \frac{A}{2}|\gamma|. \qquad \text{(2.57)} \end{aligned}$$

For the value of g chosen as above, this yields the claim (i) with $D_g := (A - Mm)/2$ and $E_g := -A/2$.

(ii) The proof of (2.54) is similar, see also Lemma 1 in [37]. Indeed, by (2.38), (2.40), and (2.52) we have for any $\Lambda \subset \mathbb{R}^d$ with $\operatorname{diam}(\Lambda) \leq \delta$

$$\begin{aligned} H_\Lambda(\gamma_\Lambda|\xi) &\geq A\binom{|\gamma_\Lambda|}{2} + \sum_{\substack{x \in \gamma_\Lambda \\ y \in \xi_{\Lambda^c}}} V(x,y) \\ &\geq \frac{A}{2}\left(|\gamma_\Lambda|^2 - |\gamma_\Lambda|\right) - M|\gamma_\Lambda| \cdot |\xi_{\Lambda^c \cap \partial_R \Lambda}|. \end{aligned}$$

Finally, (2.55) follows from (2.54) by Young's inequality. □

Remark 2.5. (i) Let us consider $\Lambda := \bigsqcup_{k \in \mathcal{K}} Q_{gk} \in \mathcal{Q}_c(\mathbb{R}^d)$ being a finite union of partition cubes, cf. (2.50). Note that $\operatorname{diam}(\Lambda) \geq g|\mathcal{K}|^{1/d}$. Using (2.56) with $K := |\mathcal{K}|$, we can continue (2.53) with $B_g := g^d D_g$ as

$$\text{(2.58)} \qquad (\mathbf{GSS})\text{:} \quad H(\gamma_\Lambda) \geq B_g \frac{1}{[\operatorname{diam}(\Lambda)]^d} |\gamma_\Lambda|^2 - E_g |\gamma_\Lambda|,$$

which means we have superstability in the sense of *Ginibre* (see page 29 of [14]).

(ii) The superstability (**SS**) implies the usual *stability* property,

$$\textbf{(S)}\textbf{:}\quad H(\gamma) \geq -E|\gamma| \quad \text{for all } \gamma \in \mathit{\Gamma}_0, \tag{2.59}$$

which is necessary for a rigorous description of thermodynamics of infinite particle systems. As was calculated in (2.57), a possible choice here is $E = A/2$.

It is well known (see e.g. Section 3.2 of [42]) that the stability of the interaction (2.59) implies that the partition function (2.28) is finite for all $\xi \in \mathit{\Gamma}$. More explicitly, we have the following bound in terms of the interaction parameters

$$\begin{aligned} Z_\Lambda(\xi) &\leq \sum_{n=0}^{\infty} \frac{z^n}{n!}\sigma(\Lambda)^n \exp\left[n\beta\left(\frac{A}{2} + M|\xi_{\Lambda^c\cap\partial_R\Lambda}|\right)\right] \\ &= \exp\left\{z\sigma(\Lambda)\exp\left[\beta\left(\frac{A}{2} + M|\xi_{\Lambda^c\cap\partial_R\Lambda}|\right)\right]\right\}. \end{aligned} \tag{2.60}$$

Therefore, each $\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi)$ (and hence, $\pi_\Lambda(\mathrm{d}\gamma|\xi)$) is well defined as a probability measure on $\mathit{\Gamma}_\Lambda$ (respectively, on $\mathit{\Gamma}$); see the discussion in Subsection 2.2. Furthermore, the following *exponential integrability* property of $\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi)$ will be strongly used below. For $\kappa, \lambda \geq 0$, let us define

$$\mathit{\Gamma}_0 \ni \gamma \mapsto \Phi(\gamma) := \kappa H(\gamma) + \lambda|\gamma|^2, \tag{2.61}$$

which will play the role of *Lyapunov functional* in establishing stability properties of our model. According to Hypotheses (2.40) and (2.46),

$$\Phi(\gamma) \geq 0 \quad \text{for any } \gamma \in \mathit{\Gamma}_k,\ k \in \mathbb{Z}^d.$$

Lemma 2.6. *Let the parameters $\kappa \in [0, \beta]$ and $\lambda \geq 0$ obey the relation*

$$\kappa A + 2\lambda \leq \beta(A - Mm). \tag{2.62}$$

In particular, one may choose here either $\kappa = 0$ and $\lambda = \beta(A - Mm)/2$ or $\lambda = 0$ and $\kappa = \beta\,(1 - Mm/A)$. Then, for any $k \in \mathbb{Z}^d$, $\xi \in \mathit{\Gamma}$ and for all sets $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ containing Q_{gk},it holds

$$\int_{\mathit{\Gamma}_\Lambda} \exp\left\{\Phi(\gamma_k)\right\}\mu_\Lambda(\mathrm{d}\gamma|\xi) \leq \mathit{\Psi}(\Lambda, \xi) < \infty. \tag{2.63}$$

Proof. Calculations similar to those in (2.57) and (2.60) show that

$$\begin{aligned}
&\int_{\Gamma_\Lambda} \exp\{\Phi(\gamma_k)\}\,\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi) \le \int_{\Gamma_\Lambda} \exp\{\Phi(\gamma_k) - \beta H_\Lambda(\gamma_\Lambda|\xi)\}\,\mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) \\
\le\ &\int_{\Gamma_\Lambda} \exp\Bigg\{\kappa H(\gamma_k) + \lambda|\gamma_k|^2 - \beta\sum_{j\in\mathcal{K}_\Lambda} H(\gamma_j) \\
&+\frac{1}{2}\beta M \sum_{k\in\mathcal{K}_\Lambda}\ \sum_{j\in\partial_g^- k\cap\mathcal{K}_\Lambda} |\gamma_i|\cdot|\gamma_j| + \beta M|\gamma_\Lambda|\cdot|\xi_{\Lambda^c\cap\partial_R\Lambda}|\Bigg\}\,\mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) \\
\le\ &\int_{\Gamma_\Lambda} \exp\Bigg\{\frac{1}{2}\left[-(\beta-\kappa)A + 2\lambda + \beta M m\right]\cdot|\gamma_k|^2 \\
&+\frac{1}{2}\beta(-A+Mm)\sum_{j\in\mathcal{K}_\Lambda}|\gamma_j|^2 + \left[(\beta-\kappa)\frac{A}{2} + \beta M|\xi_{\Lambda^c\cap\partial_R\Lambda}|\right]\cdot|\gamma_\Lambda|\Bigg\}\,\mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda) \\
\le\ &\exp\left\{z\sigma(\Lambda)\exp\left[\beta\left(\frac{A}{2} + M|\xi_{\Lambda^c\cap\partial_R\Lambda}|\right)\right]\right\}. \qquad (2.64)
\end{aligned}$$

□

Remark 2.7. As is seen from the proof of Lemma 2.6, straightforward arguments lead to the upper bound $C(\Lambda,\xi)$, which is exponentially growing as $\Lambda \nearrow \mathbb{R}^d$. This is surely not optimal for our purposes. Proceeding more subtle, in Lemma 3.3 we shall prove that the integral in (2.63) can be estimated *uniformly* in Λ.

2.4. **Main theorems and comments.** The paper contains two main results, Theorems 2.8 and 2.9 below, describing the set $\mathcal{G}^{\mathrm{t}}$ of *tempered* (or *physically relevant*) Gibbs measures. Possible improvements of these theorems will be discussed in Sections 4 and 5 below.

A starting point in any theory of unbounded spin systems is the proper notion of *temperedness*. In the present context, it is natural to introduce the following subsets of *tempered configurations*

$$\Gamma^{\mathrm{t}} := \bigcap_{\alpha>0}\Gamma_\alpha = \bigcap_{\alpha>0}\tilde{\Gamma}_\alpha,$$

$$\Gamma_\alpha := \left\{\gamma\in\Gamma \;\middle|\; |\gamma|_\alpha := \sup_{k\in\mathbb{Z}^d}\left[|\gamma_k|^2\exp\{-\alpha|k|\}\right]^{1/2} < \infty\right\},$$

$$\tilde{\Gamma}_\alpha := \left\{\gamma\in\Gamma \;\middle|\; ||\gamma||_\alpha := \left[\sum_{k\in\mathbb{Z}^d}|\gamma_k|^2\exp\{-\alpha|k|\}\right]^{1/2} < \infty\right\}. \qquad (2.65)$$

From (2.8) it is clear that all these sets are measurable, i.e., $\Gamma_\alpha, \tilde{\Gamma}_\alpha, \Gamma^{\mathrm{t}} \in \mathcal{B}(\Gamma)$. Furthermore, $\tilde{\Gamma}_\alpha \subset \Gamma_\alpha \subset \tilde{\Gamma}_{\alpha'}$ whenever $0<\alpha<\alpha'$. Respectively, the subset $\mathcal{G}^{\mathrm{t}}$ of *tempered Gibbs states* is defined to consist of those $\mu\in\mathcal{G}$ which are supported by Γ^{t}, i.e.,

$$\mathcal{G}^{\mathrm{t}} := \mathcal{G}\cap\mathcal{P}^{\mathrm{t}}(\Gamma), \quad \text{where} \quad \mathcal{P}^{\mathrm{t}}(\Gamma) := \left\{\mu\in\mathcal{P}(\Gamma) \;\middle|\; \mu(\Gamma^{\mathrm{t}}) = 1\right\}. \qquad (2.66)$$

Note that $|\gamma|_\alpha$ and $||\gamma||_\alpha$ extend to the seminorms on the linear space $\mathcal{M}(\mathbb{R}^d)$ of all signed Radon measures on $\mathbb{R}^d$. Furthermore, the sets Γ_α and Γ^{t} themselves do not depend on the choice of the parameter $g > 0$, which is the edge length of the partition cubes Q_{gk}.

As is commonly accepted in mathematical physics, configurations (or respectively, measures) with controlled growth are called *tempered*. It is worth noting that our notion of temperedness is more general than those used in earlier papers. In particular, $\mathcal{G}^{\mathrm{t}}$ contains the class of so-called *Ruelle* type *'superstable'* Gibbs states $\mu \in \mathcal{G}^{\mathrm{st}}$ which are characterized by the support condition (see Equation 5.10 in [43])

$$\sup_{K\in\mathbb{N}} \left\{ K^{-d} \sum_{|k|\le K} |\gamma_k|^2 \right\} =: C(\gamma) < \infty \quad \text{for all } \gamma \in \Gamma \quad (\mu - \text{a.e.}). \tag{2.67}$$

We begin by proving the *existence* of tempered Gibbs measures.

Theorem 2.8. *Under Hypotheses* (**FR**), (**LB**) *and* (**RC**)*, the set of tempered Gibbs measures is not empty, i.e.,* $\mathcal{G}^{\mathrm{t}} \neq \varnothing$*, at all positive values of the temperature* β *and activity* z*.*

Next we establish an exponential moment estimate similar to (2.63), which holds *for all* tempered Gibbs measures.

Theorem 2.9. *There exists a positive constant* Ψ *such that uniformly for all* $\mu \in \mathcal{G}^{\mathrm{t}}$ *and all* $\kappa, \lambda \ge 0$ *related by (2.62)*

$$\sup_{k\in\mathbb{Z}^d} \int_{\Omega} \exp\left\{\kappa H(\gamma_k) + \lambda|\gamma_k|^2\right\} \mu(\mathrm{d}\gamma) \le \Psi. \tag{2.68}$$

An explicitly computable value of Ψ, which depends on the inverse temperature β and parameters of the model only, will be given by (3.18). The proof of the above assertions strongly exploits the regularity properties of the specification $\Pi = \{\pi_\Lambda\}$. Because of independent interest, we separately state them as Propositions 2.10 and 2.11 below.

On the space of all probability measures $\mathcal{P}(\Gamma)$ let us introduce the topology of *local setwise convergence* (see e.g. Section 4.1 of [12]). This topology, which we denote by $\mathcal{T}_{\mathrm{loc}}$, is defined as the coarsest topology making the maps $\mu \mapsto \mu(B)$ continuous for all sets B from the algebra $\mathcal{B}_0(\Gamma) := \cup_{|\Lambda|<\infty}\mathcal{B}(\Gamma_\Lambda)$. Equivalently, $\mathcal{T}_{\mathrm{loc}}$ is the coarsest topology such that $\mu \mapsto \mu(F)$ is continuous for all bounded $\mathcal{B}_0(\Gamma)$-measurable functions $F : \Gamma \to \mathbb{R}$. The reader is, however, warned that the topology $\mathcal{T}_{\mathrm{loc}}$ is not metrizable (see page 57 in [12]), so that the notions of convergence and sequential convergence in $\mathcal{T}_{\mathrm{loc}}$ do not coincide.

Proposition 2.10. $\mathcal{G}^{\mathrm{t}}$ *is a compact set in the topology* $\mathcal{T}_{\mathrm{loc}}$*.*

Proposition 2.11. *The specification* $\Pi = \{\pi_\Lambda\}$ *is compact in the following sense: for each* $\xi \in \Gamma^{\mathrm{t}}$*, the family* $\{\pi_{\Lambda_{\mathcal{K}}}(\mathrm{d}\gamma_{\Lambda_{\mathcal{K}}}|\xi)\}_{\mathcal{K}\Subset\mathbb{Z}^d}$ *defined by (2.36), (2.50) is relatively* $\mathcal{T}_{\mathrm{loc}}$*-compact. Furthermore, all its limit points for* $\mathcal{K} \nearrow \mathbb{Z}^d$ *belong to* $\mathcal{G}^{\mathrm{t}}$*.*

Remark 2.12. (i) Let us give some historical comments. The existence problem for Gibbs measures goes back to the pioneering works of *R. Dobrushin* [7, 8] and *D. Ruelle* [42, 43]. It is well known that Stability Condition (**S**) allows to construct

$\mu \in \mathcal{G}^{\mathrm{t}}$ at *small* values of the inverse temperature β and activity z. This can be done either by the method of *cluster expansions* or by the contraction method for the *Kirkwood-Salsburg equation* (see respectively Sections 4.4 and 4.2 of [42]). In order to solve the DLR equation at *all* values of $\beta, z > 0$, one has to impose much stronger assumptions, typically given by Ruelle's Superstability Condition (**SS**) (see [14, 40] and Theorem 5.5 of [43]). The famous *Ruelle approach* then applies, which is based on certain *à-priori* bounds on correlation functions of the Gibbs measures. In turn, these fundamental bounds are derived from the superstability properties of the interaction. An alternative way is given by *Dobrushin's approach* and relies on the reduction to the associated lattice system and the subsequent use of the general Dobrushin existence criterion for Gibbs fields on $\mathbb{Z}^d$. Since the original papers [7, 8], the latter approach has been further developed in [3, 24, 37] (see Section 3 for more details). The drawback to this method is that it does not give enough information about the support of $\mu \in \mathcal{G}$, see Remark 3.2 (i).

(ii) For the simplest model, that is, the one we deal with in this section, the existence result of Theorem 2.8 can be obtained (at least for translation invariant V's of the *DFR* type) within either of the Ruelle or Dobrushin approaches. Our main aim here is thus not to improve the known existence results (which especially will be done in Sections 4 and 5 as well as in the forthcoming paper [23], but rather to present a *new* elementary technique of getting both the existence and *à-priori* bounds for $\mu \in \mathcal{G}^{\mathrm{t}}$. As an immediate outcome of our approach, in Theorem 2.9 (cf. also Theorem 4.8) we obtain the description of the set of *all* tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$, which seems to be entirely *new* in the literature. Among others, such integrability properties of $\mu \in \mathcal{G}^{\mathrm{t}}$ are important in studying the associated equilibrium stochastic dynamics via the theory of Dirichlet forms, see [2]. Further equivalent characterizations of the set $\mathcal{G}^{\mathrm{t}}$ are given by Theorem 4.8.

(iii) To the best of our knowledge, the statement of Propositions 2.10 and 2.11 about the compactness of the set $\mathcal{G}^{\mathrm{t}}$ and the specification $\Pi = \{\pi_\Lambda\}$, respectively, were not yet available for continuous systems. This opens a direct way of constructing $\mu \in \mathcal{G}^{\mathrm{t}}$ as limit points of the specification kernels $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ for $\Lambda \nearrow \mathbb{R}^d$, and thus allows to avoid the highly nontrivial analysis of their correlations functions, cf. (4.57). Furthermore, Proposition 2.11 says that such limit points exist for *each* boundary condition $\xi \in \Gamma^{\mathrm{t}}$, while the previous methods mainly dealt with $\xi = \varnothing$. We call (2.68) the *à-priori* bound, since it can be proven simultaneously with the fact of existence of $\mu \in \mathcal{G}^{\mathrm{t}}$ and independently from their (non-)uniqueness. In doing so, we are interested not just in the mere finiteness of the constants appearing in the estimates, but in handy expressions that can be *readily evaluated* in terms of the model parameters. For possible generalizations of these results see Subsection 4.2.

(iv) In the paper we do not touch the uniqueness problem for $\mu \in \mathcal{G}^{\mathrm{t}}$. In the appropriated classes of tempered Gibbs measures, the uniqueness can be studied by means of Ruelle's as well as Dobrushin's approaches mentioned above (see [43] resp. [3, 37]). The latter approach relies on the Dobrushin-Pechersky uniqueness criterion for unbounded spin systems (cf. Theorem 1 in [9]); its refinements and consequences (including the decay of correlations) will be discussed in the forthcoming paper [19].

3. Existence and *à-priori* Bounds for $\mu \in \mathcal{G}^{\mathrm{t}}$

3.1. **Moment estimates for local Gibbs measures.** This subsection plays a key role in carrying out our strategy. Here we establish the integrability properties of the kernels $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ needed later for proving Theorems 2.8 and 2.9. For the matter of clarity, all the constants appearing in Lemmas 3.1 and 3.3 below will be given explicitly.

Recall that the Lyapunov function $\Phi(\gamma)$ was introduced in (2.61). Our aim is to show that the left-hand side in (2.64) can be estimated *uniformly* in volume (which cannot be seen directly from the definition of $\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi)$, cf. Remark 2.7). To this end we shall perform an inductive scheme based on the consistency property (2.32). A starting point is the following trivial bound for the '*one-point*' kernels $\mu_k(\mathrm{d}\gamma_k|\xi) := \mu_{Q_{gk}}(\mathrm{d}\gamma_{Q_{gk}}|\xi)$ subject to the fixed boundary condition $\xi \in \Gamma$.

Lemma 3.1. *There exists a universal constant $\Upsilon > 0$ such that for all $k \in \mathbb{Z}^d$, $\xi \in \Gamma$ and $\kappa, \lambda \geq 0$ being the same as in Lemma 2.6*

$$\int_{\Gamma_k} \exp\left\{\kappa H(\gamma_k) + \lambda|\gamma_k|^2\right\} \mu_k(\mathrm{d}\gamma_k|\xi) \leq \exp\left\{\Upsilon + \frac{1}{2}\beta M \sum_{j\in\partial_g^- k} |\xi_j|^2\right\}. \tag{3.1}$$

Proof. Repeating the estimate (2.64) for $\Lambda := Q_{gk}$, we get by means of the Cauchy inequality that

$$\begin{aligned}
&\int_{\Gamma_k} \exp\left\{\Phi(\gamma_k)\right\} \mu_k(\mathrm{d}\gamma_k|\xi) \leq \int_{\Gamma_k} \exp\left\{\Phi(\gamma_k) - \beta H_k(\gamma_k|\xi)\right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\
&\leq \int_{\Gamma_k} \exp\left\{\left[\lambda - \frac{A}{2}(\beta-\kappa)\right]|\gamma_k|^2 + \left[\frac{A}{2}(\beta-\kappa) + \beta M \sum_{j\in\partial_g^- k}|\xi_j|\right]|\gamma_k|\right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\
&\leq \int_{\Gamma_k} \exp\left\{\left[\lambda - \frac{A}{2}(\beta-\kappa) + \frac{1}{2}\beta M m\right]|\gamma_k|^2 + \frac{A}{2}(\beta-\kappa)|\gamma_k|\right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\
&\qquad \times \quad \exp\left\{\frac{1}{2}\beta M \sum_{j\in\partial_g^- k}|\xi_j|^2\right\}.
\end{aligned} \tag{3.2}$$

In view of (2.16), (2.62), and (2.64) the claim holds with

$$\begin{aligned}
\Upsilon :=& \sup_k \log \int_{\Gamma_k} \exp\left\{\frac{A}{2}\beta|\gamma_k|\right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\
=& \; z\exp\left\{\frac{A}{2}\beta\right\} \sup_k \sigma(Q_{gk}) < \infty.
\end{aligned} \tag{3.3}$$

□

Remark 3.2. (i) A subsequent application of Jensen's inequality to both sides in (3.1) yields the following *'Dobrushin-type'* estimate (cf. Equation 4.9 in [8])

$$\int_{\Gamma_k} \Phi(\gamma_k)\mu_k(\mathrm{d}\gamma_k|\xi) \leq \Upsilon + \frac{\beta M}{2}\sum_{j\in\partial_g^- k}|\xi_j|^2 \leq \Upsilon + \frac{\beta M}{2\lambda}\sum_{j\in\partial_g^- k}\Phi(\xi_j). \tag{3.4}$$

Crucial in Dobrushin's method is the *weak* dependence on boundary conditions $\xi \in \Gamma$, which analytically means that $\beta M m/2\lambda < 1$. This can be always achieved by a proper choice of λ, as is ensured by Assumption (**RC**). By considering a lattice counterpart of our continuous model (as it was done in [3, 8, 24, 37]), we then have the possibility to employ here the general *Dobrushin existence criterion* for Gibbs random fields (Theorem 1 in [7]). However, the later scheme is rather cumbersome and leads to the theory of Gibbs measures on a larger space of *multiple* configurations $\ddot{\Gamma} \supset \Gamma$, see [3, 24, 27, 37]. Such technical extension to $\ddot{\Gamma}$ contradicts the physical intuition and leaves open the initial question about the existence of $\mu \in \mathcal{G}^{\mathrm{t}}$ supported by Γ^{t}.

(ii) Although the exponential bound (3.1) is stronger than Dobrushin's bound (3.4), actually it is derived much more *easily* in view of the additive structure of the Hamiltonian $H(\gamma)$ and the multiplication rule for exponents (see the proof of Lemma 3.1). Direct verification of Dobrushin bounds like (3.4) for particular models is a delicate technical problem, which usually requires additional assumptions on the interaction (ferromagneticity, translation invariance etc.).

(iii) Another principle difference from the previous schemes is that the function $\Phi(\gamma_k)$ constructed in (2.68) is a combination of the local *energy* $H(\gamma_k)$ and the *stabilizing* term $|\gamma_k|^2$. This not only greatly simplifies all calculations, but also provides optimal estimates on $\mu \in \mathcal{G}^{\mathrm{t}}$ as well. Unlike Dobrushin's criterion, compactness of the function $\Phi(\gamma_k)$ is *not* relevant for our existence result; actually in the proof of Theorem 2.8 we exploit only that $\Phi(\gamma_k) \nearrow \infty$ as $|\gamma_k| \nearrow \infty$. For translation invariant ferromagnets on a lattice, the exponential bound (3.1) with a standard choice of $\Phi(x_k) := \lambda|x_k|$, $\lambda > 0$, was first proved in [4], whereas the validity of Dobrushin's criterion was directly checked in [44]. For continuous particle systems in $\mathbb{R}^d$, Dobrushin bounds like (3.4) with $\Phi(\gamma_k) := \exp\{\lambda|\gamma_k|^\alpha\}$ or $\Phi(\gamma_k) := |\gamma_k|^\alpha$, $\alpha \geq 1$, were established in [3, 8, 13, 24, 37]. First attempts to consider compact functions like $\Phi(\gamma_k) := \exp\{\lambda H(\gamma_k)\}$ can be found in Section 5.2 of [27]. To some extent, one can see here a certain analogy with the lattice case, whereby instead of the single spins $x_k \in \mathbb{R}^d$ one has to control '*collective*' variables $|\gamma_k|$, i.e., the number of particles in the partition cubes Q_{gk}, $k \in \mathbb{Z}^d$.

Consider now arbitrary large cubic domains $\Lambda_{\mathcal{K}} := \bigsqcup_{k\in\mathcal{K}} Q_{gk} \in \mathcal{Q}_c(\mathbb{R}^d)$ indexed by $\mathcal{K} \Subset \mathbb{Z}^d$. Note that $\Lambda_{\mathcal{K}} \nearrow \mathbb{R}^d$ as $\mathcal{K} \nearrow \mathbb{Z}^d$. Using the '*one-point*' estimate (3.1) and the consistency property (2.32), our next step will be to get similar moment estimates for all specification kernels $\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) := \pi_{\Lambda_{\mathcal{K}}}(\mathrm{d}\gamma|\xi)$.

Lemma 3.3. *Under the assumptions of Lemma 3.1, there exists a finite $\Psi > 0$ such that uniformly for all $k \in \mathbb{Z}^d$, $\xi \in \Gamma^{\mathrm{t}}$, and $\kappa, \lambda \geq 0$ related by (2.62)*

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_\Gamma \exp\left\{\kappa H(\gamma_k) + \lambda|\gamma_k|^2\right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \leq \Psi. \tag{3.5}$$

Therefore, for each $\alpha > 0$ one finds a certain $\nu_\alpha > 0$ such that

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_\Gamma \exp\left\{\nu_\alpha ||\gamma||_\alpha^2\right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \leq \Psi, \tag{3.6}$$

where the seminorm $||\gamma||_\alpha$ was defined in (2.65).

Proof. For a fixed $\xi \in \Gamma^{\mathrm{t}}$ let us consider the quantities

$$n_k(\mathcal{K}|\xi) := \log\left\{\int_\Gamma \exp\left\{\Phi(\gamma_k)\right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)\right\}, \quad k \in \mathbb{Z}^d, \tag{3.7}$$

which are nonnegative and finite by Lemma 3.1. In particular,

$$n_k(\mathcal{K}|\xi) := \Phi(\xi_k) \quad \text{if} \quad k \notin \mathcal{K}.$$

A natural idea is to establish global bounds on the whole sequence $(n_k(\mathcal{K}|\xi))_{k\in\mathbb{Z}^d}$, which then imply the required estimates on each of its components.

Let us set

$$\vartheta := (r_2/\delta + 1)\sqrt{d} \tag{3.8}$$

so that $|k-j| \leq \vartheta$ for all $j \in \partial_g^- k$, cf. (2.47) and (2.48). Without loss of generality, we may assume by (2.40) and (2.62) that

$$\beta M m e^{\alpha\vartheta} + \kappa\varepsilon A < 2\lambda + \kappa A = \beta\,(A - Mm)\,, \tag{3.9}$$

which can be always achieved by choosing small enough $\alpha, \varepsilon > 0$ such that

$$e^{\alpha\vartheta} < (1-\varepsilon)\left(\frac{A}{Mm} - 1\right).$$

Let us start from (3.1) written for all specification kernels $\pi_k(\mathrm{d}\gamma_k|\gamma)$ with boundary conditions $\gamma \in \Gamma$

$$\int_\Gamma \exp\{\kappa\Phi(\eta_k)\}\,\pi_k(\mathrm{d}\eta_k|\gamma) \leq \exp\left\{\Upsilon + \frac{1}{2}\beta M \sum_{j\in\partial_g^- k} |\gamma_j|^2\right\}. \tag{3.10}$$

Integrating both sides of (3.10) with respect to $\pi_\mathcal{K}(\mathrm{d}\gamma|\xi)$ and taking into account the consistency property (2.32), we arrive at the following estimate for each $k \in \mathcal{K}$

$$\begin{aligned} n_k(\mathcal{K}|\xi) \quad &\leq \quad \Upsilon + \frac{\beta M}{2} \sum_{j\in\mathcal{K}^c\cap\partial_g^- k} |\xi_j|^2 \\ + \quad &\log\left\{\int_\Gamma \exp\left(\frac{\beta M}{2}\sum_{j\in\mathcal{K}\cap\partial_g^- k}|\gamma_j|^2\right)\pi_\mathcal{K}(\mathrm{d}\gamma|\xi)\right\} \\ &\leq \quad \Upsilon_\varepsilon + \frac{\beta M}{2}\sum_{j\in\mathcal{K}^c\cap\partial_g^- k}|\xi_j|^2 \\ + \quad &\frac{\beta M}{2\lambda + \kappa(1-\varepsilon)A}\sum_{j\in\mathcal{K}\cap\partial_g^- k} n_j(\mathcal{K}|\xi), \end{aligned} \tag{3.11}$$

where Υ is the same as in (3.3) and

$$\Upsilon_\varepsilon := \Upsilon + \frac{\beta\kappa A M m}{8\varepsilon\,[2\lambda + \kappa(1-\varepsilon)A]}. \tag{3.12}$$

Note that the terms $n_j(\mathcal{K}|\xi)$ enter into the right-hand side of (3.12) via the bound, cf. (2.55),

$$|\gamma_j|^2 \leq \frac{1}{2\lambda + \kappa(1-\varepsilon)A}\left[2\Phi(\gamma_j) + \frac{\kappa A}{4\varepsilon}\right], \quad \varepsilon \in (0,1]. \tag{3.13}$$

Here we have used the multiple Hölder inequality

$$\mu\left(\prod_{j=1}^K f_j^{s_j}\right) \leq \prod_{j=1}^K \mu^{s_j}(f_j), \qquad \mu(f_j) := \int f_j \mathrm{d}\mu, \tag{3.14}$$

valid for any probability measure μ, measurable functions $f_j \geq 0$, and numbers $s_j \geq 0$ such that $\sum_{j=1}^K s_j \leq 1$. In our context, $f_j := \exp\left\{\Phi(\gamma_j)\right\}$, $s_j := \beta M \left[2\lambda + \kappa(1-\varepsilon)A\right]^{-1} < 1/m$, and $K := |\mathcal{K} \cap \partial_g^- k| \leq m$.

Now let us consider any domain $\mathcal{K} \Subset \mathbb{Z}^d$ containing a fixed point $k_0 \in \mathbb{Z}^d$. After taking the upper bound in (3.11) with the weights $\exp\{-\alpha|k-k_0|\}$, we get

$$\begin{aligned} &\sup_{k\in\mathcal{K}} \left[n_k(\mathcal{K}|\xi)\exp\{-\alpha|k_0-k|\}\right] \\ \leq\ & \Upsilon_\varepsilon + \frac{\beta M}{2} \sup_{k\in\mathcal{K}} \sum_{j\in\mathcal{K}^c\cap\partial_g^- k} |\xi_j|^2 \exp\{\alpha\left[|j-k| - |j-k_0|\right]\} \\ & + \frac{\beta M}{2\lambda+\kappa(1-\varepsilon)A} \sup_{k\in\mathcal{K}} \sum_{j\in\mathcal{K}\cap\partial_g^- k} n_j(\mathcal{K}|\xi)\exp\{\alpha\left[|j-k|-|j-k_0|\right]\} \end{aligned}$$

and hence

$$\begin{aligned} n_{k_0}(\mathcal{K}|\xi) &\leq \sup_{k\in\mathcal{K}}\left[n_k(\mathcal{K}|\xi)\exp\{-\alpha|k_0-k|\}\right] \\ &\leq \left[1 - \frac{\beta M m}{2\lambda+\kappa(1-\varepsilon)A}e^{\alpha\vartheta}\right]^{-1}\left[\Upsilon_\varepsilon + \frac{\beta M}{2} m e^{\alpha(\vartheta+|k_0|)}||\xi_{\mathcal{K}^c}||_\alpha^2\right]. \end{aligned} \tag{3.15}$$

Since for $\xi \in \Gamma^{\mathrm{t}}$ the seminorm $||\xi_{\mathcal{K}^c}||_\alpha$ tends to zero as $\mathcal{K} \nearrow \mathbb{Z}^d$, we obtain for each $k_0 \in \mathbb{Z}^d$

$$\begin{aligned} &\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \sup_{k\in\mathcal{K}}\left[n_k(\mathcal{K}|\xi)\exp\{-\alpha|k_0-k|\}\right] \\ \leq\ & \Upsilon_\varepsilon \left[1 - \frac{\beta M m}{2\lambda+\kappa(1-\varepsilon)A}e^{\alpha\vartheta}\right]^{-1} \end{aligned} \tag{3.16}$$

and thus, by letting $\alpha \to 0$,

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} n_{k_0}(\mathcal{K}|\xi) \leq \Upsilon_\varepsilon \left[1 - \frac{\beta M m}{2\lambda+\kappa(1-\varepsilon)A}\right]^{-1}. \tag{3.17}$$

Substituting $\varepsilon = \beta(A-2Mm)/(2\kappa A)$ (which is just one of possible choices for ε fitting (3.9)) into the right-hand side in (3.17) under the constraint (3.9), we get the following bound

$$\begin{aligned} &\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_\Gamma \exp\left\{\Phi(\gamma_k)\right\}\pi_\mathcal{K}(\mathrm{d}\gamma|\xi) \\ \leq\ & \exp\left\{\frac{A}{A-2Mm}\left(\Upsilon + \frac{\beta A^2}{4(A-2Mm)}\right)\right\} =: \Psi, \end{aligned} \tag{3.18}$$

which completes the proof of (3.5). In the particular case $\kappa = 0$ we have from (3.17) that for each $\lambda \leq \lambda_0 := \beta(A - Mm)/2$

$$\begin{aligned} &\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_\Gamma \exp\left\{\lambda|\gamma_k|^2\right\}\pi_\mathcal{K}(\mathrm{d}\gamma|\xi) \\ \leq\ & \exp\left\{\Upsilon\frac{A-Mm}{A-2Mm}\right\} = \exp\left\{ze^{\beta A/2}\frac{A-Mm}{A-2Mm}\sup_k \sigma(Q_{gk})\right\} =: \Psi_0. \end{aligned} \tag{3.19}$$

Finally, setting

$$\nu_\alpha := \beta(A - Mm)\left[2\sum_{k\in\mathbb{Z}^d}\exp\{-\alpha|k|\}\right]^{-1},$$

by the Hölder inequality (3.14) we see that the left-hand side of (3.6) also does not exceed the same Ψ_0 as in (3.19). □

3.2. **Proof of the main theorems.** Here we prove our main Theorems 2.8 and 2.9. Rather than reducing to the lattice case and then applying the general Dobrushin criterion (see Remark 3.2 (i)), we give a direct construction of the corresponding Gibbs states using the exponential moment bounds established in Lemma 3.3. To some extent our strategy is inspired by the paper of J. Bellissard and R. Høegh-Krohn [4] which, however, was dealing with a much different (and simpler) model of classical ferromagnets on $\mathbb{Z}^d$. The main idea is to show that the uniform bounds (3.5) and (3.6) for $\pi_\Lambda(\mathrm{d}\gamma|\xi)$, along with the compactness argument in the $\mathcal{T}_{\mathrm{loc}}$-topology on $\mathcal{P}(\Gamma)$, readily imply the existence of certain $\mu \in \mathcal{G}^{\mathrm{t}}$ being the limit points of $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ as $\Lambda \nearrow \mathbb{R}^d$. As compared with the weak convergence topology on $\mathcal{P}(\Gamma)$ standardly used in Dobrushin's approach, an advantage of the local setwise convergence topology $\mathcal{T}_{\mathrm{loc}}$ is that no continuity assumptions on V are needed at all. Furthermore, this way we also obtain the *à-priori* estimates (2.63) to be valid uniformly *for all* tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$. Note that the latter information cannot be extracted from Dobrushin's criterion alone, which just provides existence of *at least one* $\mu \in \mathcal{G}^{\mathrm{t}}$.

Proof of Theorem 2.8 (also including **Proposition 2.11**)**.** Let us fix a boundary condition $\xi \in \Gamma^{\mathrm{t}}$ and an order generating sequence $\mathcal{K}_N \nearrow \mathbb{Z}^d$ as $N \to \infty$. Taking into account (3.6), for each $\alpha > 0$ one finds a corresponding $\mathcal{C}_\alpha(\xi)$ such that uniformly for all domains $\Lambda_N := \Lambda_{\mathcal{K}_N} \in \mathcal{Q}_c(\mathbb{R}^d)$

$$\int_\Gamma ||\gamma||_\alpha^2 \pi_{\Lambda_N}(\mathrm{d}\gamma|\xi) \le \mathcal{C}_\alpha(\xi). \tag{3.20}$$

It would suffice to show that the family of specification kernels $\{\pi_{\Lambda_N}(\mathrm{d}\gamma|\xi)\}_{N\in\mathbb{N}}$ is *locally equicontinuous* in the following sense: for all $\Delta \in \mathcal{Q}_c(\mathbb{R}^d)$ and each sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_\Delta(\Gamma)$ with $B_n \downarrow \varnothing$

$$\lim_{n\to\infty}\limsup_{N\to\infty}\pi_{\Lambda_N}(B_n|\xi) = 0. \tag{3.21}$$

In other words, for any $\epsilon > 0$ one finds $N(\varepsilon), n(\varepsilon) \in \mathbb{N}$ such that $\pi_{\Lambda_N}(B_{n_\epsilon}|\xi) < \epsilon$ for all $N > N(\varepsilon)$. By Propositions 4.9 and 4.15 in [12] combined with Theorem V.3.2 in [33], any locally equicontinuous net in $\mathcal{P}(\Gamma)$ has at least one $\mathcal{T}_{\mathrm{loc}}$-cluster point; furthermore, each of the cluster points can be obtained as a limit of a certain subsequence. Thus, there exists a limit point $\mu := \lim_{M\to\infty}\pi_{\Lambda_{N_M}}(\mathrm{d}\gamma|\xi) \in \mathcal{P}(\Gamma)$ taken along some subsequence $\Lambda_{N_M} \nearrow \mathbb{R}^d$, so that for all local sets $B \in \mathcal{B}_0(\Gamma)$

$$\int_\Gamma \pi_\Delta(B|\gamma)\pi_{\Lambda_{N_M}}(\mathrm{d}\gamma|\xi) = \pi_{\Lambda_{N_M}}(B|\xi) \to \mu(B) \quad \text{as } M \to \infty. \tag{3.22}$$

Note that in the left hand-side in (3.22) we have used the consistency property (2.32) valid for all $\Lambda_{N_M} \supseteq \Delta$. As the interaction has finite range, the function $\gamma \mapsto \pi_\Delta(B|\gamma)$ is bounded and $\mathcal{B}_0(\Gamma)$-measurable. Thus, for each $\Delta \in \mathcal{B}_c(\mathbb{R}^d)$ we can pass to the limit in the left hand-side in (3.22) and check that this μ solves the

DLR equation (2.36). By B. Levi's monotone convergence theorem we conclude from (3.20) that for all $\alpha > 0$

$$\int_{\Gamma} ||\gamma||_{\alpha}^2 \mu(\mathrm{d}\gamma) = \lim_{K\to\infty} \lim_{L\to\infty} \sum_{|k|\le K} \exp\{-\alpha|k|\} \int_{\Gamma} \left\{|\gamma_k|^2 \wedge L\right\} \mu(\mathrm{d}\gamma)$$

$$= \lim_{K\to\infty} \lim_{L\to\infty} \lim_{M\to\infty} \sum_{|k|\le K} \exp\{-\alpha|k|\} \int_{\Gamma} \left\{|\gamma_k|^2 \wedge L\right\} \pi_{\Lambda_{N_M}}(\mathrm{d}\gamma|\xi) \le \mathcal{C}_\alpha(\xi),$$

which implies that μ is supported by Γ^{t}.

So, it remains to check the local equicontinuity (3.21). We adapt the arguments used for proving Theorem 4.12 and Corollary 4.13 in [12] to the configuration space Γ. Fix $\Delta \in \mathcal{Q}_c(\mathbb{R}^d)$, and let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of sets from $\mathcal{B}_\Delta(\Gamma)$ such that $B_n \downarrow \varnothing$ as $n \to \infty$. Consider the following subsets from $\mathcal{B}_0(\Gamma)$

$$\Gamma_{R,T} := \left\{\gamma \in \Gamma \;\middle|\; |\gamma_{\partial_R(\Delta)}| \le T\right\}, \quad T > 0,$$

where, cf. (2.52),

$$\partial_R(\Delta) := \left\{x \in \mathbb{R}^d \;\mid \mathrm{dist}(x,\Delta) \le R\right\} \in \mathcal{B}_c(\mathbb{R}^d)$$

and $R > 0$ is the interaction radius in Assumption (**FR**). For each $\xi \in \Gamma^{\mathrm{t}}$ and $\Lambda_N \supseteq \Delta$ we have by (2.27), (2.31), and (2.32)

$$\begin{aligned}&\pi_{\Lambda_N}(B_n|\xi) \le \pi_{\Lambda_N}(B_n \cap [\Gamma_{R,T}]^c|\xi)\\ &+ \int_{\Gamma}\int_{\Gamma_\Delta} \chi_{B_n\cap\Gamma_{R,T}}(\eta_\Delta \cup \gamma_{\Delta^c}) \exp\left\{-\beta H_\Delta(\eta_\Delta|\gamma)\right\} \lambda_{z\sigma}(\mathrm{d}\eta_\Delta)\pi_{\Lambda_N}(\mathrm{d}\gamma|\xi).\end{aligned} \tag{3.23}$$

Chebyshev's inequality applied to (3.20) ensures that for each $\epsilon > 0$ there exists $T(\epsilon,\xi) > 0$ such that

$$\pi_{\Lambda_N}([\Gamma_{R,T}]^c|\xi) \le \epsilon \quad \text{for all } T \ge T(\epsilon,\xi) \text{ and } N \in \mathbb{N}. \tag{3.24}$$

On the other hand, for all $\gamma \in \Gamma$

$$\begin{aligned}&\int_{\Gamma_\Delta} \chi_{B_n\cap\Gamma_{R,T}}(\eta_\Delta \cup \gamma_{\Delta^c}) \exp\left\{-\beta H_\Delta(\eta_\Delta|\gamma)\right\} \lambda_{z\sigma}(\mathrm{d}\eta_\Delta)\\ \le\;& \int_{\Gamma_\Delta} \chi_{B_n\cap\Gamma_{R,T}}(\eta_\Delta \cup \gamma_{\Delta^c}) \exp\left\{\beta M|\eta_\Delta \cup \gamma_{\Delta^c}|^2\right\} \lambda_{z\sigma}(\mathrm{d}\eta_\Delta)\\ \le\;& \exp\left\{\beta M T^2\right\} \lambda_{z\sigma}(B_n) \le \epsilon \quad \text{as soon as } n \ge n(\epsilon).\end{aligned} \tag{3.25}$$

Plugging (3.24) and (3.25) back into (3.23), we get the required equicontinuity of the family $\{\pi_{\Lambda_N}(\mathrm{d}\gamma|\xi)\}_{N\in\mathbb{N}}$. □

Proof of Theorem 2.9. Consider an arbitrary $\mu \in \mathcal{G}^{\mathrm{t}}$. With the help of (2.36), (3.20), and Fatou's lemma we see that for any $k \in \mathbb{Z}^d$ and $K > 0$

$$\begin{aligned}&\int_{\Gamma^{\mathrm{t}}} \exp\left\{\Phi(\gamma_k) \wedge K\right\} \mu(\mathrm{d}\gamma)\\ =\;& \lim_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_{\Gamma^{\mathrm{t}}}\int_{\Gamma^{\mathrm{t}}} \left\{\exp \Phi(\gamma_k) \wedge K\right\} \pi_{\Lambda_{\mathcal{K}}}(\mathrm{d}\gamma|\xi)\mu(\mathrm{d}\xi)\\ \le\;& \int_{\Gamma^{\mathrm{t}}} \left[\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_{\Gamma^{\mathrm{t}}} \exp\left\{\Phi(\gamma_k) \wedge K\right\} \pi_{\Lambda_{\mathcal{K}}}(\mathrm{d}\gamma|\xi)\right] \mu(\mathrm{d}\xi) \le \Psi,\end{aligned} \tag{3.26}$$

where the constant Ψ was computed in (3.5) and (3.18). By B. Levi's theorem we further conclude from (3.26) that

$$\begin{aligned} &\int_{\Gamma_\alpha} \exp\{\Phi(\gamma_k)\}\,\mu(\mathrm{d}\gamma) \\ &= \limsup_{K\to\infty}\int_{\Gamma_\alpha}\exp\{\Phi(\gamma_k)\wedge K\}\,\mu(\mathrm{d}\gamma) \le \Psi, \end{aligned} \tag{3.27}$$

which yields the desired estimate (2.68). □

Proof of Proposition 2.10. The line of reasoning is similar to that implemented in the proof of Theorem 2.8. More precisely, one can show that the uniform bound

$$\sup_{\mu\in\mathcal{G}^{\mathrm{t}}}\int_{\Gamma}||\gamma||_\alpha^2\mu(\mathrm{d}\gamma)<\infty,$$

implies the local equicontinuity of the family $\mathcal{G}^{\mathrm{t}}$ in $\mathcal{P}(\Gamma)$. To this end we repeat the estimates (3.23)–(3.25) with $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ replaced by any $\mu\in\mathcal{G}^{\mathrm{t}}$. Thus, every net in $\mathcal{G}^{\mathrm{t}}$ has a subnet with a limit in $\mathcal{G}^{\mathrm{t}}$, which is equivalent to the compactness of $\mathcal{G}^{\mathrm{t}}$ in the topology $\mathcal{T}_{\mathrm{loc}}$. □

Remark 3.4. (i) Suppose we know nothing else about the interaction potential but that $V(x,y)\ge 0$ for all $x,y\in\mathbb{R}^d$. Without assuming (**RC**) we cannot guarantee that the exponential moments in (3.4) and (3.7) are finite, so that the previous proof of the existence result formally does not work. To show how to overcome this problem let us start from the obvious estimate

$$\int_{\Gamma_k}\exp\{\lambda|\gamma_k|\}\,\mu_k(\mathrm{d}\gamma_k|\xi)\le\int_{\Gamma_k}\exp\{\lambda|\gamma_k|\}\,|\mathrm{d}\lambda_{z\sigma}(\gamma_k)\le\Psi(\lambda),$$

which holds for any $\lambda>0$ with constant $\mathcal{C}(\lambda):=\exp\left\{ze^\lambda\sup_k\sigma(Q_{gk})\right\}$ being the same for all $\xi\in\Gamma$ and $k\in\mathbb{Z}^d$. Repeating the proof of Lemma 3.3 with $\Phi(\gamma_k):=\lambda|\gamma_k|$ and $M=0$, we get that

$$\sup_{k\in\mathcal{K}\Subset\mathbb{Z}^d}\int_\Gamma\exp\{\lambda|\gamma_k|\}\,\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)\le\Psi(\lambda), \tag{3.28}$$

and hence by Hölder's inequality

$$\sup_{\xi\in\Gamma^{\mathrm{t}}}\ \limsup_{\mathcal{K}\nearrow\mathbb{Z}^d}\int_\Gamma||\gamma||_\alpha^{2n}\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)=:\mathcal{C}_\alpha(n)<\infty,\quad n\in\mathbb{N}. \tag{3.29}$$

The last bound enables us to mimic the proof of Theorem 2.8 and so check that the family $\{\pi_\Lambda(\mathrm{d}\gamma|\xi)|\ \Lambda\in\mathcal{Q}_c(\mathbb{R}^d)\}$ has a $\mathcal{T}_{\mathrm{loc}}$-limit point $\mu\in\mathcal{G}^{\mathrm{t}}$. Furthermore, replacing (3.5) by (3.28) in the proof of Theorem 2.9, we show that all $\mu\in\mathcal{G}^{\mathrm{t}}$ must obey the à-priori bound

$$\sup_{k\in\mathbb{Z}^d}\int_\Omega\exp\{\lambda|\gamma_k|\}\,\mu(\mathrm{d}x)\le\mathcal{C}(\lambda). \tag{3.30}$$

In contrast to the superstable case, here one cannot expect the similar estimate with $\exp\left\{\lambda|\gamma_k|^2\right\}$ because this function is not integrable with respect to the Poisson measure $\pi_{z\sigma}$.

(ii) Instead of the consistency property (2.32), in proving Lemmas 3.1 and 3.3 one can use the so-called Ruelle equation (cf. (5.12) in [43])

$$\begin{aligned}(3.31)\quad &\int_{\Omega} F(\gamma)\pi_{\Delta}(\mathrm{d}\gamma|\xi)\\ &= \int_{\Gamma_{\Lambda^c}}\int_{\Gamma_{\Lambda}} F(\gamma_{\Delta}\cup\eta_{\Lambda\setminus\Delta}\cup\xi_{\Lambda^c})\exp\left\{-\beta H_{\Delta}(\gamma_{\Delta}|\eta_{\Lambda\setminus\Delta}\cup\xi_{\Lambda^c})\right\}\lambda_{z\sigma}(\mathrm{d}\gamma_{\Delta})\mu(\mathrm{d}\eta),\end{aligned}$$

valid for all $\Delta\subseteq\Lambda\Subset\mathbb{R}^d$, $\xi\in\Gamma$, and $F\in L^{\infty}(\Gamma)$. Since (2.32) and (3.31) are known to be equivalent, they lead to the same estimates (3.5) and (3.27)

4. Further Extensions

Here we outline possible improvements of Theorems 2.8 and 2.9. So, we extend our initial model in several directions including: essentially *singular* potentials (Subsection 4.1), *strong superstable* interactions (Subsection 4.2), *multibody* Hamiltonians (Subsection 4.3), and *general intensities* (4.4). The later case, when the underlying measure σ is no longer translation invariant in $\mathbb{R}^d$, has not yet been treated in the literature. In Subsection 4.5, for the corresponding Gibbs measures we discuss equivalent characterizations of temperedness in terms of their support sets, local densities, and integrability properties. Finally, in Subsection 4.6 we point out some relations to Ruelle's approach.

4.1. **Regularity of $\mu\in\mathcal{G}^{\mathrm{t}}$ due to singular potentials.** Of special interest in physical applications is the case when

$$(4.1)\qquad V(x,y)\geq v(|x-y|)\quad\text{for all}\quad x,y\in\mathbb{R}^d,$$

with a majorizing function $v:\mathbb{R}_+\to\mathbb{R}$ such that $\lim_{t\to+0}v(t)=+\infty$. Theorem 2.9 then says that all Gibbs measures $\mu\in\mathcal{G}^{\mathrm{t}}$, which are initially supported by the set Γ^{t} defined by (2.65), *à-posteori* must obey the much stronger integrability property

$$(4.2)\qquad \sup_{k\in\mathbb{Z}^d}\int_{\Gamma}\exp\left\{\frac{1}{2}\beta\sum_{\{x,y\}\subset\gamma_k}v(|x-y|)\right\}\mu(\mathrm{d}\gamma)\leq\Psi.$$

From a technical point of view the possibility of such a regularization relies on the following fact: for any boundary condition $\xi\in\Gamma$, the right-hand sides in the basic estimates (3.1) and (3.15) do *not* depend on the values of $V(x,y)$ taken at points $x,y\in\xi$. A typical assumption here is that (similarly to (2.43)) for some $C,\varkappa>0$

$$(4.3)\qquad v(x)\geq C|x|^{-(d+\varkappa)}\quad\text{as}\quad |x|\to 0.$$

Then according to (4.2) one finds a certain $\vartheta>0$ such that

$$(4.4)\qquad \sup_{\mu\in\mathcal{G}^{\mathrm{t}}}\sup_{k\in\mathbb{Z}^d}\int_{\Gamma}\exp\left\{\vartheta\sum_{\{x,y\}\subset\gamma_k}|x-y|^{-(d+\varkappa)}\right\}\mu(\mathrm{d}\gamma)<\infty.$$

4.2. **Strong superstable interactions.** The so-called *strong superstable* interactions may be viewed as a refined version of the classical superstability property introduced by D. Ruelle. All such stability conditions are formulated directly in terms of the Hamiltonian $H(\gamma)$, $\gamma \in \Gamma_0$, so that the particular structure of the potential $V(x,y)$ becomes no longer relevant. Below we demonstrate that an appropriate abstract setting (including the so-called ϕ-superstability defined by (4.21), (4.22)) is best suited to carry on the general strategy of getting the moment estimates described in Section 3. Furthermore, we indicate typical classes of the pair potentials $V(x,y)$ to which all this apply.

Let us replace the both Assumptions (**LB**) and (**RC**) on the behavior of $V(x,y)$ from Subsection 2.3 by the following one:

(**SSS**)**:** ***Strong Superstability***: *For a given $P > 2$ and a certain partition $\mathbb{R}^d = \bigsqcup_{k\in\mathbb{Z}^d} Q_{gk}$ with $g > 0$, cf. (2.44), there exist positive D, E such that*

$$H(\gamma) \;\geq\; D \sum_{k\in\mathbb{Z}^d} |\gamma_k|^P - E|\gamma| \quad \text{for all } \gamma \in \Gamma_0. \tag{4.5}$$

Since $|\gamma_k|^P \geq |\gamma_k|^2$ for any $\gamma_k \in \Gamma_k$, it is obvious that (**SSS**) is stronger than Ruelle's superstability (**SS**) resulting in particular from Assumption (**RC**), see Lemma 2.4. Actually, (4.5) implies (2.53) with an arbitrary large constant in the front of $|\gamma_k|^2$. From the proof of the same lemma, it is clear that (4.5) should hold at once for *all* $g > 0$, but with proper $D_g, E_g > 0$. According to (4.5), the pair potential V is semibounded below, i.e.,

$$\inf_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}} V(x,y) := \inf_{\{x,y\}\subset\mathbb{R}^d} H(\{x,y\}) =: -M \geq 2\left(2^{P-1}D - E\right), \tag{4.6}$$

which agrees with the initial Assumption (**LB**). The same is true for the energy in every partition cube Q_{gk}, i.e.,

$$\inf_{\gamma_k\in\Gamma_k} H(\gamma_k) =: -C \geq -D^{\frac{1}{1-P}}\, E^{\frac{P}{P-1}}. \tag{4.7}$$

We still keep the finite range Assumption (**FR**) from Subsection 2.2. Then, for $x \in Q_{gk}$ and $y \in Q_{gj}$, the interaction $V(x,y)$ is zero unless

$$j \in \partial_g k := \left\{ k' \in \mathbb{Z}^d \;\middle|\; |k - k'| < \sqrt{d}\,(1 + R/\delta) \right\}. \tag{4.8}$$

Similarly to (2.47) and (2.49), the number of such neighbor cubes Q_{gj} (having the diagonal $\delta := g\sqrt{d}$) does not exceed

$$|\partial_g k| \leq m := v_d d^{d/2}(R/\delta + 3/2)^d. \tag{4.9}$$

Remark 4.1. In its explicit form the notion of strong superstability was first introduced in [32], although some arguments leading to better analysis of stability could be already found in the earlier contributions [6, 8, 36, 42]. The most recent reference is [41], which contains a historical survey and account of sufficient conditions for (**SSS**) in terms of the potentials V. So, Theorem 2.3 there presents one of the best-understood examples of the strong superstable interactions. Namely, let V behave like $V(x,y) \geq c|x-y|^{-d(1+\varkappa)}$ as $|x-y| \to 0$ (cf. (2.43) and (4.3)), so that the classical *Dobrushin–Fisher–Ruelle* criterion (Proposition 1.4 in [43]) applies. Then V is not only superstable with an arbitrary large $D_g > 0$, but also fulfills (**SSS**) with $P = 2 + \varkappa$. This result naturally extends to the interactions of infinite range, see Section 5. Actually, most of the stable potentials used in statistical physics turn out to be strong superstable.

Supposing (**FR**) and (**SSS**), we can get a substantial refinement of the previous results. So, it is natural to consider all $\mu \in \mathcal{G}^{\mathrm{t}}$ supported by

$$\begin{aligned} \varGamma^{\mathrm{t}} &:= \left\{ \gamma \in \varGamma \;\middle|\; \forall \alpha > 0: \; |\gamma|_\alpha^P := \sup_{k \in \mathbb{Z}^d} \left[|\gamma_k|^P \exp\{-\alpha |k|\} \right] < \infty \right\} \\ &= \left\{ \gamma \in \varGamma \;\middle|\; \forall \alpha > 0: \; ||\gamma||_\alpha^P := \sum_{k \in \mathbb{Z}^d} |\gamma_k|^P \exp\{-\alpha |k|\} < \infty \right\}. \end{aligned} \tag{4.10}$$

So far, it is clear that either of the representations (2.65) and (4.10) describes the same subset of tempered configurations $\varGamma^{\mathrm{t}}$. Fixing the parameters

$$0 \le \kappa < \beta \quad \text{and} \quad 0 \le \lambda < (\beta - \kappa)\, D, \tag{4.11}$$

let us define the Lyapunov functional

$$\varGamma_k \ni \gamma_k \to \Phi(\gamma_k) := \kappa H(\gamma_k) + \lambda |\gamma_k|^P \ge -\beta C, \tag{4.12}$$

where $C \in \mathbb{R}$ is the same as in (4.7). A starting point is the following modification of the exponential bound (3.1)–(3.3) in Lemma 3.1

$$\begin{aligned} &\int_{\varGamma_k} \exp\{\Phi(\gamma_k)\}\, \mu_k(\mathrm{d}\gamma_k|\xi) \\ &\le \int_{\varGamma_k} \exp\left\{ [\lambda - (\beta-\kappa)\, D]\, |\gamma_k|^P + \left[(\beta - \kappa)\, E + \beta M \sum_{j \in \partial_g k} |\xi_j| \right] |\gamma_k| \right\} \mathrm{d}\lambda_{z\sigma}(\gamma_k) \\ &\le \exp\left\{ \varUpsilon_\epsilon + \epsilon \sum_{j \in \partial_g k} |\xi_j|^P \right\}, \end{aligned} \tag{4.13}$$

holding for any

$$0 < \epsilon < \frac{1}{m} \min\{\kappa D + \lambda; \;\; (\beta - \kappa)\, D - \lambda\}$$

and the corresponding

$$\varUpsilon_\epsilon := m \epsilon^{\frac{2}{2-P}} (\beta M)^{\frac{P}{P-2}} + z \exp\{\beta E\} \sup_k \sigma(Q_{gk}).$$

In deriving (4.13) we have used Young's inequality in the form

$$st \le \varkappa \left(s^P + t^P \right) + \varkappa^{\frac{2}{2-P}} \quad \text{for any } \varkappa, s, t > 0. \tag{4.14}$$

Picking next any $0 < \delta < \kappa D + \lambda - \epsilon m$ and noting that by (4.5) and (4.12)

$$|\xi_j|^P \le \frac{1}{\lambda + \kappa D - \delta} \left[\Phi(\xi_j) + \delta^{\frac{1}{1-P}} (\kappa E)^{\frac{P}{P-1}} \right], \tag{4.15}$$

we then follow step by step the arguments used in proving Lemma 3.3 and Theorems 2.8, 2.9. As $\Phi(\gamma_k)$ may take negative values, instead of $n_k(\mathcal{K}|\xi)$ defined by (3.7) we have to consider the quantities

$$\begin{aligned} 0 \le \tilde{n}_k(\mathcal{K}|\xi) &= n_k(\mathcal{K}|\xi) + \beta C : \\ &= \log\left\{ \int_{\varGamma} \exp\{\Phi(\gamma_k) + \beta C\}\, \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \right\}, \quad k \in \mathbb{Z}^d. \end{aligned}$$

By (4.13), (4.15) and the consistency property (2.32) we have the following relation

$$\begin{aligned}\tilde{n}_k(\mathcal{K}|\xi) \quad &\leq \quad \Upsilon_\epsilon + \beta C + \frac{\epsilon}{\lambda + \kappa D - \delta} \sum_{j \in \mathcal{K} \cap \partial_g k} \tilde{n}_j(\mathcal{K}|\xi) \\ &+ \epsilon \sum_{j \in \mathcal{K}^c \cap \partial_g k} |\xi_j|^P + \frac{\epsilon m}{\lambda + \kappa D - \delta} \left[\delta^{\frac{1}{1-P}} (\kappa E)^{\frac{P}{P-1}} - \beta C \right], \quad k \in \mathcal{K}.\end{aligned}$$

Herefrom, continuing similarly to (3.15)–(3.17), we conclude that for each $k_0 \in \mathcal{K}$ and a fixed but small enough $\alpha > 0$

$$\begin{aligned}\tilde{n}_{k_0}(\mathcal{K}|\xi) \leq &\left[1 - \frac{\epsilon m}{\lambda + \kappa D - \delta} e^{\alpha \vartheta} \right]^{-1} \\ &\times \left[\Upsilon_\varepsilon + \delta^{\frac{1}{1-P}} (\kappa E)^{\frac{P}{P-1}} \frac{\epsilon m}{\lambda + \kappa D - \delta} + \beta C \left(1 - \frac{\epsilon m}{\lambda + \kappa D - \delta} \right) + \epsilon m e^{\alpha(\vartheta + |k_0|)} ||\xi_{\mathcal{K}^c}||_\alpha^P \right].\end{aligned}$$

Finally, by letting $\alpha \searrow 0$, we get the estimate

$$\begin{aligned}&\limsup_{|\mathcal{K}| \nearrow \mathbb{Z}^d} \int_\Gamma \exp\left\{ \kappa H(\gamma_k) + \lambda |\gamma_k|^P \right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \\ &\leq \quad \exp\left\{ \left[1 - \frac{\epsilon m}{\lambda + \kappa D - \delta} \right]^{-1} \left[\Upsilon_\varepsilon + \delta^{\frac{1}{1-P}} (\beta E)^{\frac{P}{P-1}} \right] \right\} =: \Psi_{\epsilon,\delta}, \qquad (4.16)\end{aligned}$$

which is uniform for all $k \in \mathbb{Z}^d$ and $\xi \in \Gamma^{\mathrm{t}}$. The latter allows us to construct $\mu \in \mathcal{G}^{\mathrm{t}}$ by applying the compactness argument in $(\mathcal{P}(\Gamma), \mathcal{T}_{\mathrm{loc}})$. A further sequel of (4.16) is the *à-priori* bound

$$\sup_{k \in \mathbb{Z}^d} \int_\Omega \exp\left\{ \kappa H(\gamma_k) + \lambda |\gamma_k|^P \right\} \mu(\mathrm{d}\gamma) \leq \inf_{\epsilon,\delta} \Psi_{\epsilon,\delta} := \Psi \qquad (4.17)$$

to be fulfilled by all $\mu \in \mathcal{G}^{\mathrm{t}}$. This leads to the following conclusion.

Theorem 4.2. *Under Assumptions* (**FR**) *and* (**SSS**)*, the set* $\mathcal{G}^{\mathrm{t}}$ *is nonvoid and all its elements obey the bound (4.17).*

Remark 4.3. (i) In the proof of (4.13) we used only the '*local*' version of Assumption (**SSS**) in the elementary cubes Q_{gk}, that is

$$(\mathbf{LSS}) \qquad H(\gamma_k) \ \geq \ D|\gamma_k|^P - E|\gamma_k| \quad \text{for all } \gamma \in \Gamma_k, \ k \in \mathbb{Z}^d. \qquad (4.18)$$

For $P > 2$, the concrete values of $D, E > 0$ are not relevant for the existence result. Nevertheless, it can be easily shown that (2.22) and (4.18) imply the '*global*' condition (4.5) with new constants $\tilde{D} \in (0, D)$ and $\tilde{E} > 0$, where $\tilde{D}$ can be chosen arbitrarily close to D.

(ii) To run the above scheme for $P = 2$ we have to assume that $D > Mm$, where $D > 0$ is the maximal possible constant in (4.18) and $M := -\inf_{\mathbb{R}^{2d}} V$. For the same $g > 0$, this yields Ruelle's superstability (2.53) with any $D_g \in (0, D - Mm)$. Obviously,

$$\inf_{\gamma_k \in \Gamma_k} H(\gamma_k) =: -C \geq -\frac{E^2}{4D}.$$

Respectively, we define

$$\Phi(\gamma_k) := \kappa H(\gamma_k) + \lambda |\gamma_k|^2 \geq -\beta \frac{E^2}{4D},$$

where the parameters $\kappa, \lambda > 0$ meet the constraint

$$\lambda + \kappa D \leq \beta \left(D - Mm/2\right). \tag{4.19}$$

The basic estimates (3.1) and (3.5) then hold with

$$\Upsilon := z \exp\{\beta E\} \sup_k \sigma(Q_{gk}), \quad \Psi := \exp\left\{ \frac{D}{D - Mm} \left(\Upsilon + \frac{\beta E^2}{2(D - Mm)} \right) \right\}. \tag{4.20}$$

The *most general* setup that completely includes all previous considerations can be given as follows. Suppose there exist constants $C, D, E, M \geq 0$ and a function

$$\phi : \Gamma_0 \to \mathbb{R}_+ \quad \text{with} \quad \liminf_{|\gamma| \to \infty} \frac{\phi(\gamma)}{|\gamma|} = +\infty,$$

such that, for all $\gamma_k \in \Gamma_k$ and $\xi_j \in \Gamma_j$ with $k \neq j$,

$$H(\gamma_k) \ \geq \ D\phi(\gamma_k) - E|\gamma_k| \ \geq \ -C, \tag{4.21}$$

$$W(\gamma_k|\xi_j) := \sum_{x \in \gamma_k,\ y \in \xi_j} W(x, y) \geq -\frac{M}{2} \left[\phi(\gamma_k) + \phi(\xi_j)\right]. \tag{4.22}$$

In addition, let $D > Mm$ with the parameter m as defined in (4.9). Note that the standard *superstability* (or *strong superstability*) analyzed before is reduced to the *particular* choice of $\phi(\gamma_k) = |\gamma_k|^2$ (or $\phi(\gamma_k) = |\gamma_k|^P$ with $P > 2$). In this regard the interactions obeying (4.21), (4.22) may be called *ϕ-superstable*. The definition of temperedness is naturally modified as follows: the set $\mathcal{G}^{\mathrm{t}}$ now consists of those $\mu \in \mathcal{G}$ which are carried by

$$\begin{aligned} \Gamma^{\mathrm{t}} := & \left\{ \gamma \in \Gamma \,\middle|\, \forall \alpha > 0 : \ \sup_{k \in \mathbb{Z}^d} \left[\phi(\gamma_k) \exp\{-\alpha|k|\}\right] < \infty \right\} \\ = & \left\{ \gamma \in \Gamma \,\middle|\, \forall \alpha > 0 : \ \sum_{k \in \mathbb{Z}^d} \phi(\gamma_k) \exp\{-\alpha|k|\} < \infty \right\}. \end{aligned} \tag{4.23}$$

Respectively, our main statements will concern the Lyapunov functional

$$\Gamma_0 \ni \gamma \to \Phi(\gamma) := \kappa|H(\gamma)| + \lambda\phi(\gamma), \tag{4.24}$$

where $\kappa \in [0, \beta]$ and $\lambda \geq 0$ are related by (4.19), i.e.,

$$\lambda + \kappa D \leq \beta \left(D - Mm/2\right).$$

Going through the proof of Lemma 3.1 and making use of (4.21)–(4.24), we obtain the exponential bound

$$\int_{\Gamma_k} \exp \Phi(\gamma_k) \mu_k(\mathrm{d}\gamma_k|\xi) \leq \exp\left\{ \Upsilon + \frac{1}{2}\beta M \sum_{j \in \partial_g k} \phi(\xi_j) \right\}, \tag{4.25}$$

where the constant Υ is the same as in (4.20). In turn, (4.25) implies that the set $\mathcal{G}^{\mathrm{t}}$ is nonvoid and that all its elements obey

$$\sup_{\mu \in \mathcal{G}^{\mathrm{t}}} \sup_{k \in \mathbb{Z}^d} \int_{\Omega} \exp\left\{\kappa H(\gamma_k) + \lambda\phi(\gamma_k)\right\} \mu(\mathrm{d}x) < \infty. \tag{4.26}$$

4.3. **Multibody interactions.** This case is technically more difficult and thus less studied in the literature. Here we briefly outline how to modify our method; for details see [23] in preparation.

Let us be given a finite family of potential functions $V_n(x_1, \dots x_n)$, $2 \le n \le N \in \mathbb{N}$, describing the interaction between each n-tuple of the particles $\{x_1, \dots x_n\} \subset \mathbb{R}^d$. Every potential $V_n : \mathbb{R}^{nd} \to \bar{\mathbb{R}}$ is a bounded below, symmetric function of finite range, i.e., there exist some $M, R \ge 0$ such that for all $n \le N$

$$\inf_{(x_1,\dots x_n)\in\mathbb{R}^{nd}} V_n(x_1, \dots x_n) \ge -M,$$

$$V_n(x_1, \dots x_n) = 0 \quad \text{if} \quad \operatorname{diam}\{x_1, \dots x_n\} \ge R. \tag{4.27}$$

For $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\xi \in \Gamma$, the local Hamiltonians $H_\Lambda(\cdot|\xi) : \Gamma_\Lambda \to \bar{\mathbb{R}}$ are defined by

$$H_\Lambda(\gamma_\Lambda|\xi) := H(\gamma_\Lambda) + W_\Lambda(\gamma_\Lambda|\xi),$$

where respectively

$$\begin{aligned} H(\gamma_\Lambda) \quad &: \quad = \sum_{n=2}^{N} \sum_{\{x_1,\dots x_n\}\subset\gamma_\Lambda} V_n(x_1, \dots x_n), \\ W_\Lambda(\gamma_\Lambda|\xi) \quad &: \quad = \sum_{n=2}^{N} \sum_{p=1}^{n-1} \sum_{\substack{\{x_1,\dots,x_p\}\subset\gamma_\Lambda \\ \{y_{p+1},\dots,y_n\}\subset\xi_{\Lambda^c}}} V_n(x_1, \dots x_p, y_{p+1}, \dots, , y_n). \end{aligned}$$

Not trying to be optimal, for all $N \ge 2$ the energies in the elementary cubes $\Lambda := Q_{gk}$, $k \in \mathbb{Z}^d$, can be roughly estimated by

$$H(\gamma_k) \ge -M \sum_{n=2}^{N} \binom{|\gamma_k|}{n} \ge -M|\gamma_k|^N \sum_{n=2}^{N|} \frac{1}{n!} \ge -M|\gamma_k|^N, \tag{4.28}$$

$$\begin{aligned} W_k(\gamma_k|\xi) &\ge -M \sum_{n=2}^{N} \sum_{p=1}^{n-1} \binom{|\gamma_k|}{p} \binom{|\xi_{\partial_g k}|}{n-p} \\ &\ge -M \sum_{n=2}^{N} \frac{1}{n!} \sum_{p=1}^{n-1} \frac{n!}{p!(n-p)!} |\gamma_k|^p |\xi_{\partial_g k}|^{n-p} \\ &\ge -M \sum_{n=2}^{N} \frac{1}{n!} \left(|\gamma_k| + |\xi_{\partial_g k}| \right)^n \ge -M(m+1)^{N-1} \left[|\gamma_k|^N + \sum_{j\in\partial_g k} |\xi_j|^N \right], \end{aligned} \tag{4.29}$$

where m is the same as in (4.9). To make our method work it would suffice to impose the (local) strong superstability (4.18) of order equal to the interaction rank (i.e., $P = N$) and with a large enough $D > D_N$, that is

$$H(\gamma_k) \ \ge \ D|\gamma_k|^N - E|\gamma_k| \quad \text{for all } \gamma \in \Gamma_k, \ k \in \mathbb{Z}^d. \tag{4.30}$$

As is seen from (4.29) and Remark 4.3 (ii), one may choose $D_N := 6Mm^{N-1}$ for $N > 2$ and respectively $D_N := Mm$ for $N = 2$. This yields existence of $\mu \in \mathcal{G}^{\mathrm{t}}$ and the *à-priori* bound (4.17) for all κ, λ obeying

$$\kappa D + \lambda \le \beta \left(D - D_N/2 \right).$$

In view of (4.28), a possible way of getting (4.30) is to assume (**SSS**) of order $P_{n_0} > N$ just for one '*stabilizing*' potential V_{n_0} with some index $n_0 \le N$. Sufficient conditions for such superstability can be found e.g. in Lemma 4 of [3], Remark 2.1 of [28] or Theorem 2.1 of [46]. To this end, a typical behavior of the stabilizing potential should be

$$V_{n_0}(x_1, \dots x_n) \geq C \left[\mathrm{diam}\{x_1, \dots x_n\}\right]^{-d(1+\varkappa)} \quad \text{for some } \varkappa \geq N-1. \tag{4.31}$$

In a similar way one can consider systems with an infinite group of many-body potentials (i.e., $N = \infty$), which analytically gives rise to the property of ϕ-superstability (4.21) with $\phi(\gamma) := \exp\{\nu|\gamma|\}$, $\nu > 0$. Stability of the whole system can be achieved by employing an infinite sequence of stabilizing potentials $\{V_{n_m}\}_{m\geq 1}$ that behave like (4.31) with $\varkappa_m > n_m - 1$ or just by one of them having an exponential singularity at the diagonal like $V_{n_0}(x_1, \dots x_n) \geq C \exp\left(-\nu \mathrm{diam}\{x_1, \dots x_n\}\right)$.

4.4. **General intensity measures.** Here we place us again in the setting of Section 2, but a *principal* difference is that the spatial regularity condition (2.16) will be dropped. Instead we suppose that the Radon intensity measure σ is *tempered* in the following sense:

$$\int_{\mathbb{R}^d} e^{-\alpha_0|x|}\mathrm{d}\sigma(x) < \infty \quad \text{for some } \alpha_0 > 0. \tag{4.32}$$

Keeping the former Assumptions (**FR**), (**LB**) and substituting (**RC**) by the stronger one

$$A > Mm\left(1 + e^{\alpha_0\vartheta}\right) \tag{4.33}$$

with $\vartheta > 0$ being the same as in (3.8), we claim the existence of Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ supported respectively by $\Gamma^{\mathrm{t}} := \bigcap_{\alpha>\alpha_0}\Gamma_\alpha$. Again, we define $\Phi(\gamma) := \kappa H(\gamma) + \lambda|\gamma|^2$ under the constraint

$$\beta Mme^{\alpha_0\vartheta} < \kappa A + 2\lambda \leq \beta(A - Mm). \tag{4.34}$$

Fixing an arbitrary $\xi \in \Gamma^{\mathrm{t}}$, we start with the exponential bound similar to that in Lemma 3.1

$$\int_{\Gamma_k} \exp\{\Phi(\gamma_k)\}\,\mu_k(\mathrm{d}\gamma_k|\xi) \leq \exp\left\{\Upsilon_k + \frac{1}{2}\beta M\sum_{j\in\partial_g^- k}|\xi_j|^2\right\}, \tag{4.35}$$

but this time with the constant

$$\Upsilon_k := z\exp\{\beta A/2\}\,\sigma(Q_{gk})$$

essentially depending on $k \in \mathbb{Z}^d$. Nevertheless, for $\alpha > \alpha_0$ the following quantity

$$\Upsilon_\alpha := \sup_{k\in\mathbb{Z}^d}\{\Upsilon_k \exp(-\alpha|k|)\}$$

is finite by assumption (4.32). Let $\alpha > \alpha_0$ and $\varepsilon > 0$ be related by

$$\beta Mm\left(1 + e^{\alpha\vartheta}\right) < 2\lambda + \kappa(1-\varepsilon)A. \tag{4.36}$$

Defining the moments $n_k(\mathcal{K}|\xi)$ by (3.7) and mimicking the proof of Lemma 3.3, we get for each $\xi \in \Gamma^{\mathrm{t}}$ and $k \in \mathcal{K}$

$$\begin{aligned} 0 \quad \leq \quad & n_k(\mathcal{K}|\xi) \leq \Upsilon_k + \Upsilon_\varepsilon + \frac{\beta M}{2}\sum_{j\in\mathcal{K}^c\cap\partial_g^- k}|\xi_j|^2 \\ + \quad & \frac{\beta M}{2\lambda + \kappa(1-\varepsilon)A}\sum_{j\in\mathcal{K}\cap\partial_g^- k} n_j(\mathcal{K}|\xi) \end{aligned}$$

with an extra constant

$$\Upsilon_\varepsilon := \frac{\beta\kappa AMm}{8\varepsilon\,[2\lambda+\kappa(1-\varepsilon)A]}. \tag{4.37}$$

Herefrom

$$\begin{aligned}&\sup_{k\in\mathcal{K}}\{n_k(\mathcal{K}|\xi)\exp(-\alpha|k|)\}\\ &\le \left[1-\frac{\beta Mm}{2\lambda+\kappa(1-\varepsilon)A}e^{\alpha\vartheta}\right]^{-1}\left[\Upsilon_\alpha+\Upsilon_\varepsilon+\frac{\beta M}{2}me^{\alpha\vartheta}||\xi_{\mathcal{K}^c}||_\alpha^2\right].\end{aligned} \tag{4.38}$$

Since for $\xi\in\Gamma^{\mathrm{t}}$ the seminorm $||\xi_{\mathcal{K}^c}||_\alpha$ tends to zero as $\mathcal{K}\nearrow\mathbb{Z}^d$, we obtain for each $k\in\mathbb{Z}^d$

$$\begin{aligned}&\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d}\left\{\exp(-\alpha|k|)\int_\Gamma\Phi(\gamma_k)\pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi)\right\}\\ &\le \left[1-\frac{\beta Mm}{2\lambda+\kappa(1-\varepsilon)A}e^{\alpha\vartheta}\right]^{-1}[\Upsilon_\alpha+\Upsilon_\varepsilon].\end{aligned} \tag{4.39}$$

As in the proof of Theorem 2.8, this yields the existence of Gibbs limit points $\mu:=\lim_{N\to\infty}\pi_{\Lambda_N}(\mathrm{d}\gamma|\xi)$ obeying for each $\alpha>\alpha_0$

$$\sup_{k\in\mathbb{Z}^d}\left\{\exp(-\alpha|k|)\int_\Gamma\left[\kappa H(\gamma_k)+\lambda|\gamma_k|^2\right]\mu(\mathrm{d}\gamma)\right\}<\infty \tag{4.40}$$

and hence belonging to $\mathcal{G}^{\mathrm{t}}$. Furthermore, the proof of Theorem 2.9 shows that all $\mu\in\mathcal{G}^{\mathrm{t}}$ must fulfil the same bound (4.40) allowing at most exponential growth of their moments. Because of the absence of spatial regularity for $\sigma(\mathrm{d}x)$, the resulting estimates are *not* expected to be *invariant* under translations in the phase space $\mathbb{R}^d$. Furthermore, the uniform integrability condition

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|e^{-\beta V(x,y)}-1\right|\sigma(\mathrm{d}y)<\infty$$

fails. These are the principal reasons why the situation discussed above does not fit into the framework of Ruelle's superstability method and was not yet covered in the literature. Nevertheless, such spatial irregularity is natural if one considers *disordered* particle systems with random intensities $\sigma(\mathrm{d}x,\omega)$ depending on some external random field $\omega\in\Omega$, what we plan to do elsewhere.

4.5. **Support properties of $\mu\in\mathcal{G}^{\mathrm{t}}$.** There are a few important consequences from the *à-priori* bounds (2.68) and (4.26). Recall that the set of tempered Gibbs measures was introduced by means of the rather moderate restrictions (2.65), (2.66). We now show that all $\mu\in\mathcal{G}^{\mathrm{t}}$ indeed are carried by a much smaller *universal subset* Γ^{s}, which is known (for $P=2$ and $\kappa=0$) as the *Lanford–Lebowitz–Presutti support* (see Definition 3.2 of [29] in the lattice case and respectively Definition 5.2.1 of [27] for configuration spaces).

Let us fix some partition of the phase space $\mathbb{R}^d$ by the elementary cubes Q_{gk}, $k\in\mathbb{Z}^d$. For $b>0$ we define

$$\Gamma(b):=\left\{\gamma\in\Gamma|\ \exists K_\gamma\in\mathbb{Z}_+:\ \kappa|H(\gamma_k)|+\lambda|\gamma_k|^P\le b\log(1+|k|)\text{ if }|k|\ge K_\gamma\right\}, \tag{4.41}$$

which is a Borel subset in Γ.

Proposition 4.4. *For given $P \geq 2$ and $\kappa, \lambda \geq 0$, let us consider those $\mu \in \mathcal{P}(\Gamma)$ which fulfill*

$$\sup_{k\in\mathbb{Z}^d} \int_\Gamma \exp\left\{\kappa|H(\gamma_k)| + \lambda|\gamma_k|^P\right\} \mu(\mathrm{d}\gamma) =: \Psi(\mu) < \infty. \tag{4.42}$$

Then, simultaneously for all such measures (and hence by Theorems 2.9 and 4.2, for all $\mu \in \mathcal{G}^{\mathrm{t}}$), one has $\mu[\Gamma(b)] = 1$ as soon as $b > d$.

Proof. We proceed similarly to the proof of Lemma 3.1 in [29]. Without loss of generality we may assume that at least one of parameters κ, λ is positive, otherwise the result is trivial. The complement of $\Gamma(b)$ in (4.41) can be written as

$$[\Gamma(b)]^c = \bigcap\nolimits_{K\in\mathbb{N}} \bigcup\nolimits_{|k|\geq K} [\Gamma_k(b)]^c, \tag{4.43}$$

where

$$\Gamma_k(b) := \left\{\gamma \in \Gamma \;\middle|\; \kappa|H(\gamma_k)| + \lambda|\gamma_k|^P \leq b\log(1+|k|)\right\}.$$

By Chebyshev's inequality and the estimate (4.42)

$$\mu\left([\Gamma_k(b)]^c\right) \leq \Psi(\mu)\cdot(1+|k|)^{-b}, \tag{4.44}$$

and therefore by (4.43) and (4.44)

$$\mu\left([\Gamma(b)]^c\right) \leq \Psi(\mu) \lim_{K\to\infty} \sum\nolimits_{|k|\geq K} (1+|k|)^{-b}. \tag{4.45}$$

Taking $b > d$ makes the series in (4.45) convergent, which yields the result $\mu\left([\Gamma(b)]^c\right) = 0$. □

Corollary 4.5. *Under the conditions of Theorems 2.9 and 4.2, all $\mu \in \mathcal{G}^{\mathrm{t}}$ are carried by the subset*

$$\Gamma^{\mathrm{s}} := \left\{\gamma \in \Gamma \;\middle|\; \sup_{k\in\mathbb{Z}^d} \left[\left(|H(\gamma_k)| + |\gamma_k|^P\right)\cdot\left(\log(1+|k|)\right)^{-1}\right] < \infty\right\}. \tag{4.46}$$

Next, we claim that for any Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$ its finite volume projections

$$\mu_\Lambda := \mu\circ\mathbb{P}_\Lambda^{-1}, \quad \mu_k := \mu\circ\mathbb{P}_{Q_k}^{-1}, \quad k\in\Lambda\in\mathcal{B}_c(\mathbb{R}^d),$$

satisfy a certain *Ruelle-type bound* (cf. Proposition 5.2 in [43]).

Proposition 4.6. *Under Assumptions* (**FR**) *and* (**SSS**)*, each $\mu \in \mathcal{G}^{\mathrm{t}}$ is locally absolutely continuous with respect to the σ-Poisson measure $\lambda_{z\sigma}$. The corresponding Radon–Nikodym derivatives obey the following estimate for $\lambda_{z\sigma}$-almost all $\gamma_\Lambda \in \Gamma_\Lambda$*

$$\begin{aligned}\rho_{\mu,\Lambda}(\gamma_\Lambda) \;&:= \frac{\mathrm{d}\mu_\Lambda(\gamma_\Lambda)}{\mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda)}\\ &\leq \exp\left\{-\kappa H(\gamma_\Lambda) - \lambda\sum_{k\in\mathcal{K}_\Lambda}|\gamma_k|^P + G_\Lambda|\gamma_\Lambda|\right\} \leq (C_\Lambda)^{|\gamma_\Lambda|}\end{aligned} \tag{4.47}$$

with any $\kappa \in (0,\beta)$, $\lambda \in (0,(\beta-\kappa)\,D)$ and proper $G_\Lambda := G_\Lambda(\kappa,\lambda)$, $C_\Lambda > 0$ being the same for all such μ. In particular, for all $\gamma_k \in \Gamma_k$, $k \in \mathbb{Z}^d$, and some $G > 0$

$$\rho_{\mu,k}(\gamma_k) := \frac{\mathrm{d}\mu_k(\gamma_k)}{\mathrm{d}\lambda_{z\sigma}(\gamma_k)} \leq \exp\left\{-\kappa H(\gamma_k) - \lambda|\gamma_k|^P + G|\gamma_k|\right\}. \tag{4.48}$$

Proof. From the *DLR* equations (2.36) it is easy to see that the Radon–Nikodym derivatives (of course, if such exist) should have the following representation for all $\gamma \in \Gamma_\Lambda \pmod{\lambda_{z\sigma}}$

$$\begin{aligned} \rho_{\mu,\Lambda}(\gamma_\Lambda) &= \exp\{-\beta H(\gamma_\Lambda)\} \int_\Gamma [1/Z_\Lambda(\xi)] \exp\{-\beta W_\Lambda(\gamma_\Lambda|\xi_{\Lambda^c})\}\, \mu(\mathrm{d}\xi) \\ &\le \exp\{-\beta H(\gamma_\Lambda)\} \int_\Gamma \exp\left\{-\beta \sum_{x\in\gamma_\Lambda,\ y\in\xi_{\Lambda^c}} V(x,y)\right\} \mu(\mathrm{d}\xi). \end{aligned} \tag{4.49}$$

So, the only thing needed to check is the validity of the upper bound (4.47), which in turn implies $\rho_{\mu,\Lambda} \in L^1(\lambda_{z\sigma})$ and hence $\mu_\Lambda(\mathrm{d}\gamma_\Lambda) \ll \lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda)$. It is obvious that $\rho_{\mu,\Lambda}(\varnothing_\Lambda) = \mu_\Lambda(\varnothing_\Lambda) \le 1$, so below we may assume that $|\gamma_\Lambda| \ge 1$. The integral in the last line in (4.49) can be estimated by means of the Hölder inequality (3.14) and the exponential bound (4.17)

$$\begin{aligned} &\int_\Gamma \exp\left\{-\beta \sum_{x\in\gamma_\Lambda,\ y\in\xi_{\Lambda^c}} V(x,y)\right\} \mu(\mathrm{d}\xi) \\ \le\ &\int_\Gamma \exp\left\{\beta M \sum_{\substack{k\in\mathcal{K}_\Lambda \\ j\in\partial_g k\cap\mathcal{K}_{\Lambda^c}}} |\gamma_k|\cdot|\xi_j|\right\} \mu(\mathrm{d}\xi) \\ \le\ &\exp\left\{\frac{1}{4\lambda_0}(\beta M m)^2 |\mathcal{K}_\Lambda| \sum_{k\in\mathcal{K}_\Lambda} |\gamma_k|^2\right\} \int_\Gamma \exp\left\{\frac{\lambda_0}{m|\mathcal{K}_\Lambda|} \sum_{\substack{k\in\mathcal{K}_\Lambda, \\ j\in\partial_g k\cap\mathcal{K}_{\Lambda^c}}} |\xi_j|^2\right\} \mu(\mathrm{d}\xi) \\ \le\ &\Psi \exp\left\{\frac{1}{4\lambda_0}(\beta M m)^2 |\mathcal{K}_\Lambda| \sum_{k\in\mathcal{K}_\Lambda} |\gamma_k|^2\right\}, \end{aligned} \tag{4.50}$$

where we put $\kappa = 0$, fix some $\lambda_0 \in (0, \beta D)$ and took the corresponding $\Psi := \Psi(\lambda_0)$ from (4.17). Thus, for $|\gamma_\Lambda| \ge 1$

$$\rho_{\mu,\Lambda}(\gamma_\Lambda) \le \exp\left\{-\beta H(\gamma_\Lambda) + \frac{1}{4\lambda_0}(\beta M m)^2 |\mathcal{K}_\Lambda| \sum_{k\in\mathcal{K}_\Lambda} |\gamma_k|^2 + \log \Psi(\lambda_0)\right\}, \tag{4.51}$$

which together with the strong superstability (4.5) and Young's inequality (4.14) yields the required bound on $\rho_{\mu,\Lambda}$. □

Remark 4.7. In the situation of Section 2, a bound similar to (4.48) can be proved for $P = 2$. In general, the constants G_Λ, C_Λ in (4.47) may *depend* on the geometry of Λ. If $\Lambda := \Lambda_N := [-N, N]^d$, the best control we could get here is that the G_{Λ_N} behave like $\mathcal{O}\left(N^{dp/(p-2)}\right)$ as $N \to \infty$.

The next assertion summarizes different types of regularity for the Gibbs measures $\mu \in \mathcal{G}$ as solutions of the *DLR* equation (2.36). In this respect let us recall the famous result of D. Ruelle, see Corollary 5.3 in [43], where several equivalent descriptions of the superstable Gibbs states $\mu \in \mathcal{G}^{\mathrm{st}}$ (via their support, correlation functions, and local densities) are given. Of course, any pair of the properties (i)–(vi) listed below need not to be equivalent for a general probability measure $\mu \in \mathcal{P}(\Gamma)$.

Theorem 4.8. *Under the hypotheses of Theorem 4.2, the following are equivalent for any $\mu \in \mathcal{G}$:*
(i) μ is supported by the set Γ^{t}, cf. (2.65);
(ii) μ is supported by the smaller subset $\Gamma^{\mathrm{s}} \subset \Gamma^{\mathrm{t}}$, cf. (4.46);
(iii) μ satisfies the a-priori bound (4.17) with some $\kappa = 0$ and some $\lambda \in (0, \beta D)$;
(iv) μ satisfies the a-priori bound (4.17) for all $\kappa \in (0, \beta)$ and $\lambda \in (0, (\beta - \kappa) D)$;
(v) $\mu_{Q_{gk}} \ll \lambda_{z\sigma}$ for each $k \in \mathbb{Z}^d$, with the densities $\rho_{\mu,k}$ obeying the estimate (4.48) with some $\kappa = 0$ and some $\lambda \in (0, \beta D)$;
(vi) $\mu_\Lambda \ll \lambda_{z\sigma}$ for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, with the densities $\rho_{\mu,\Lambda}$ obeying the estimate (4.47) for all $\kappa \in (0, \beta)$ and $\lambda \in (0, (\beta - \kappa) D)$.

Proof. Recall that (i)⇒(iv) follows by Theorem 2.9, (iv)⇒(ii) by Proposition 4.4, and (i)⇒(vi) by Proposition 4.6. The remaining implications (ii)⇒(i), (iv)⇒(iii), and (vi)⇒(v)⇒(iii)⇒(i) are obvious. □

4.6. **Bounds on correlation functions.** Assuming (**FR**) and (**SSS**), let us pick some $\mu \in \mathcal{G}^{\mathrm{t}}$ and consider its finite volume projections μ_Λ, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. Corresponding to the n-particle decomposition $\Gamma_\Lambda = \bigsqcup_{n \in \mathbb{Z}_+} \Gamma_\Lambda^{(n)}$, we have the induced representation $\mu_\Lambda = \sum_{n \in \mathbb{Z}_+} \mu_\Lambda^{(n)}$ with

$$\mathrm{d}\mu_\Lambda^{(n)}(\{x_1, ..., x_n\}) = \frac{z^n}{n!} \rho_{\mu,\Lambda}^{(n)}(\{x_1, ..., x_n\}) \mathrm{d}\sigma_{\mathrm{sym}}^{\otimes n}(x_1, ..., x_n). \tag{4.52}$$

According to (4.47), the *system of densities* $\rho_{\mu,\Lambda}^{(n)} : \Gamma_\Lambda^{(n)} \to \mathbb{R}_+$, $n \in \mathbb{Z}_+$, fulfills the *local* Ruelle bound

$$\rho_{\mu,\Lambda}^{(n)}(\{x_1, ..., x_n\}) \le (C_\Lambda)^n, \quad \{x_1, ..., x_n\} \in \Gamma_\Lambda^{(n)} \pmod{\lambda_{z\sigma}}. \tag{4.53}$$

In much the same way as in Subsection 4.5, one can derive estimates on the *correlation functional* $k_\mu : \Gamma_0 \to \mathbb{R}_+$ of $\mu \in \mathcal{G}^{\mathrm{t}}$ (for its definition see [16, 43]). For $\lambda_{z\sigma}$-almost all $\gamma \in \Gamma_0$ it can be written in the form

$$k_\mu(\gamma) = \int_\Gamma \exp\{-\beta H(\gamma) - \beta W(\gamma|\xi)\} \, \mu(\mathrm{d}\xi), \tag{4.54}$$

where, cf. (2.25),

$$W(\gamma|\xi) := \sum_{x \in \gamma, \ y \in \xi} V(x, y), \quad \gamma \in \Gamma_0, \ \xi \in \Gamma, \tag{4.55}$$

stands for the interaction energy between a pair of configurations, γ and ξ, in the whole $\mathbb{R}^d$. Obviously, $k_\mu(\varnothing) = 1$. By analogy with (4.49)–(4.51), we get for each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and for all nonempty $\gamma_\Lambda \in \Gamma_\Lambda \pmod{\lambda_{z\sigma}}$

$$\begin{aligned} k_\mu(\gamma_\Lambda) &\le \exp\left\{-\beta H(\gamma_\Lambda) + \frac{1}{4\lambda_0} (\beta M m)^2 |\mathcal{K}_\Lambda| \sum_{k \in \mathcal{K}_\Lambda} |\gamma_k|^2 + \log \Psi(\lambda_0)\right\} \\ &\le \exp\left\{-\kappa H(\gamma_\Lambda) - \lambda \sum_{k \in \mathbb{Z}^d} |\gamma_k|^P + G_\Lambda |\gamma_\Lambda|\right\} \le (C_\Lambda)^{|\gamma_\Lambda|}, \end{aligned} \tag{4.56}$$

with any $\kappa \in (0, \beta)$, $\lambda \in (0, (\beta - \kappa) D)$ and some $G_\Lambda := G_\Lambda(\kappa, \lambda)$, $C_\Lambda > 0$. Thus,

$$k_\mu^{(n)}(\{x_1, ..., x_n\}) \le (C_\Lambda)^n \quad \text{for} \quad \{x_1, ..., x_n\} \subset \Lambda, \tag{4.57}$$

where the family $k_\mu^{(n)} := k_\mu \restriction_{\Gamma_0^{(n)}}$, $n \in \mathbb{Z}_+$, are the well-known *correlation functions* of statistical physics (see Section 4.1 of [42]). An estimate with $P = 2$ similar to (4.56) was obtained in Equation 4.29 of [2]. Suppose we could show that $\sup_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} C_\Lambda < \infty$, then this would be the *global* Ruelle bound, cf. Proposition 2.6 in [43]. Note that the correlation functional k_μ and the local densities $\rho_{\mu,\Lambda}$ are related for $\lambda_{z\sigma} - a.e.\ \gamma \in \Gamma_\Lambda$ by the duality

$$\begin{aligned} k_\mu(\gamma_\Lambda) &= \int_{\Gamma_\Lambda} \rho_{\mu,\Lambda}(\gamma_\Lambda \cup \xi_\Lambda) \lambda_{z\sigma}(\mathrm{d}\xi_\Lambda), \\ \rho_{\mu,\Lambda}(\gamma_\Lambda) &= \int_{\Gamma_\Lambda} (-1)^{|\xi_\Lambda|} k_\mu(\gamma_\Lambda \cup \xi_\Lambda) \lambda_{z\sigma}(\mathrm{d}\xi_\Lambda), \end{aligned} \tag{4.58}$$

cf. e.g. Propositions 4.2 and 4.3 in [20]. This, in particular, means that the estimates (4.47) and (4.56) are *equivalent*. Any correlation functional $k : \Gamma_0 \to \mathbb{R}_+$ satisfying the local Ruelle bound (4.56) is admissible in the sense that there exists a *unique* measure μ satisfying (4.53) and $k_\mu = k$. For a general discussion of different *à-priori* bounds for measures on configuration spaces see [21].

Remark 4.9. (i) Using the properties of the so-called K-transform (see [20]), one can show that Lemma 3.3 implies the following bound on the second correlation functions $k_{\Lambda,\xi}^{(2)}(\{x, y\})$ of the local Gibbs measures $\mu_\Lambda(\mathrm{d}\gamma_\Lambda|\xi)$ in the domains $\Lambda \in \mathcal{Q}_c(\mathbb{R}^d)$

$$\limsup_{\Lambda \nearrow \mathbb{R}^d} \iint_{Q_{gk} \times Q_{gk}} |V(x, y)| k_{\Lambda,\xi}^{(2)}(\{x, y\}) \mathrm{d}x\mathrm{d}y \le \Upsilon < \infty,$$

which holds uniformly in $\xi \in \Gamma^{\mathrm{t}}$ and $k \in \mathbb{Z}^d$.

(ii) It is still an *open question* whether $\mathcal{G}^{\mathrm{t}} = \mathcal{G}^{\mathrm{st}}$. Nevertheless, it is clear that any *translation invariant* measure $\mu \in \mathcal{P}(\Gamma)$ obeying the exponential bound (4.42), satisfies for μ-almost all $\gamma \in \Gamma$

$$\sup_{K \in \mathbb{N}} \left\{ K^{-d} \sum_{|k| \le K} \exp\left[\kappa \sum_{\{x,y\} \subset \gamma_k} V(x, y) + \lambda |\gamma_k|^2) \right] \right\} \le \Psi(\gamma), \tag{4.59}$$

which is much stronger than the original Ruelle support condition (2.67). The claim immediately follows from the multidimensional ergodic theorem (cf. e.g. Theorem 14.A8 in [12]) applied to the stationary family of random variables $\Phi_k(\gamma) := H(\gamma_k) + \lambda|\gamma_k|^2$, $k \in \mathbb{Z}^d$, defined on the probability space $(\Gamma, \mathcal{B}(\Gamma), \mu)$.

5. Interactions of Infinite Range

Here we demonstrate how to handle the interactions of infinite range, when the technical Assumption (**FR**) from the previous sections is dropped. It is commonly recognized that a principal difficulty, when compared to the finite range case, is to identify the limit points $\mu := \lim_{\Lambda \nearrow \mathbb{R}^d} \pi_\Lambda(\mathrm{d}\gamma|\xi)$ with solutions of the *DLR* equation. Recall that in Ruelle's approach this problem is solved by establishing global bounds on the finite volume correlation functionals (with empty boundary condition). Here we would like to suggest an alternative method that will be based on the (almost) *continuity* of Gibbs specification in certain spaces of tempered configurations. To be more specific, below we focus on the important class of the so-called *DFR* potentials.

5.1. ***DFR* potentials.** Let us consider a symmetric pair potential (without hard core) $V : \mathbb{R}^{2d} \to \bar{\mathbb{R}}$, which is *finite* and *continuous* everywhere outside the diagonal $\mathcal{D} := \{(x, y) \in \mathbb{R}^{2d} | \, x = y\}$. We suppose that V is of *DFR* type, i.e., there exist constants $0 < r < R$, $0 < \theta \le \varkappa$, and $C_1, C_2, C_3 > 0$ such that

$$V(x, y) \ge C_1 |x - y|^{-(d+\varkappa)}, \qquad \text{if } \; |x - y| < r, \tag{5.1}$$

and

$$|V(x, y)| \le \begin{cases} C_2 |x - y|^{-(d+\theta)}, & \text{if } \; |x - y| > R, \\ C_3, & \text{if } \; |x - y| \in [r, R]. \end{cases} \tag{5.2}$$

A typical example (with $d = \theta = 3$, $\varkappa = 9$), which is especially important in atomic and molecular physics, is given by the *Lennard-Jones* potential

$$V(x, y) := \frac{a}{|x - y|^{12}} - \frac{b}{|x - y|^{6}}, \quad x, y \in \mathbb{R}^3, \;\; a, b > 0. \tag{5.3}$$

Furthermore, we assume that the intensity σ is *locally absolutely continuous* with respect to the *Lebesgue* measure $\mathrm{d}x$ on $\mathbb{R}^d$, i.e.,

$$\sigma(\mathrm{d}x) = \rho(x)\mathrm{d}x \;\; \text{with a density } 0 \le \rho \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x). \tag{5.4}$$

As usual, $L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$ stands for the space of locally integrable functions on $\mathbb{R}^d$. Then the corresponding Poisson measure $\pi_{z\sigma}$ obeys the following property, which will be relevant for constructing the Gibbs states. Let $A \in \mathcal{B}(\mathbb{R}^d)$ be negligible for the Lebesgue measure $\mathrm{d}x$. As is seen from (2.17), the set of configurations *not touching* A has full probability, i.e.,

$$\pi_{z\sigma}\left(\{\gamma \in \Gamma \mid \gamma \subset A^c\}\right) = 1. \tag{5.5}$$

As before, let us pick any partition $\mathbb{R}^d = \bigsqcup_{k \in \mathbb{Z}^d} Q_{gk}$ by the cubes (2.44) of edge length $g \in (0, 1]$. Without loss of generality we may assume that $g < r/\sqrt{d}$, which guarantees that $V(x, y) \ge 0$ whenever $x, y \in Q_{gk}$. By Theorem 2.3 in [41] the corresponding interaction $H(\gamma)$ obeys the *strong superstability* (**SSS**)**,** cf. (4.5). More precisely, there exists a constant $D > 0$ (independent of g) such that

$$H(\gamma) \;\ge\; \sum_{k \in \mathbb{Z}^d} D_g |\gamma_k|^P - E_g |\gamma|, \quad \gamma \in \Gamma_0, \tag{5.6}$$

with $P := 2 + \varkappa/d > 2$, $D_g := D/g^P$ and a proper $E_g > 0$. On the other hand, from (5.1) and (5.2) one may find some $K, L > 0$ (also independent of g) such that for all $k, j \in \mathbb{Z}^d$ and $x \in Q_{gk}$, $y \in Q_{gj}$ the following bounds hold

$$\begin{aligned} V(x, y) &\ge -K_g |k - j|^{-(d+\theta)}, \quad \text{if } \; |k - j| \ge 1, \\ |V(x, y)| &\le L_g |k - j|^{-(d+\theta)}, \quad \text{if } \; |k - j| \ge r/g + \sqrt{d}, \end{aligned} \tag{5.7}$$

with $K_g := g^{-(d+\theta)} K$ and $L_g := g^{-(d+\theta)} L$.

5.2. **Spaces of tempered configurations.** Next we have to introduce the appropriate notion of *temperedness*, which will essentially depend on the decay rate of the pair interaction. By analogy with (2.65) and (2.66), we define the subset of tempered configurations

$$\begin{aligned} \Gamma^{\mathrm{t}} \quad &: \quad = \bigcup\nolimits_{0<\alpha<\theta} \Gamma_\alpha, \\ \Gamma_\alpha \quad &: \quad = \left\{ \gamma \in \Gamma \;\middle|\; |\gamma|_\alpha := \sup_{k\in\mathbb{Z}^d} \left[|\gamma_k|^P \, (1+|k|)^{-\alpha)} \right]^{1/P} < \infty \right\} \end{aligned} \tag{5.8}$$

and respectively the subset of tempered Gibbs measures

$$\begin{aligned} \mathcal{G}^{\mathrm{t}} \quad &: \quad = \bigcup\nolimits_{0<\alpha<\theta} \mathcal{G}_\alpha, \\ \mathcal{G}_\alpha \quad &: \quad = \{ \mu \in \mathcal{G} \mid \mu(\Gamma_\alpha) = 1 \}. \end{aligned} \tag{5.9}$$

Furthermore, for $\alpha > 0$ we set

$$m_\alpha := \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{|k|^{d+\alpha}} < \infty. \tag{5.10}$$

The only difference with the previous scheme is that the exponential weights $\exp\{-\alpha|k|\}$ should be replaced everywhere by the polynomials $(1+|k|)^{-\alpha}$. Clearly, this involves stronger restrictions imposed on the support of $\mu \in \mathcal{G}^{\mathrm{t}}$. In what follows, instead of $|\gamma|_\alpha$ we actually shall consider an equivalent seminorm $|\gamma|_{\alpha,\delta}$ constructed by means of the weights $(1+\delta|k|)^{-\alpha}$ with small enough $\delta \in (0,1)$.

By straightforward arguments one can check that, for each volume $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and boundary condition $\xi \in \Gamma^{\mathrm{t}}$, the specification kernels $\pi_\Lambda(\mathrm{d}\gamma_\Lambda|\xi)$ are well defined by (2.25)–(2.31) as probability measures on Γ^{t}. To this end we observe that the corresponding local Hamiltonians are bounded below by

$$\begin{aligned} H_\Lambda(\gamma_\Lambda|\xi) \quad &\geq \quad \sum_{k\in\mathbb{Z}^d} D_g |\gamma_{\Lambda,k}|^P - C_{g,\Lambda}|\gamma_\Lambda| \left[1 + \sum_{j\in\mathbb{Z}^d} |\xi_{\Lambda^c,j}| \, (1+|j|)^{-(d+\theta)} \right] \\ &\geq \quad -C_{g,\Lambda}|\gamma_\Lambda| \cdot \left[1 + m_{\theta-\alpha} |\xi_{\Lambda^c}|_\alpha^P \right], \end{aligned} \tag{5.11}$$

where the constant $C_{g,\Lambda} > 0$ can be chosen to be the same for all $\xi \in \Gamma_\alpha$ and $0 < \alpha < \theta$ (whereby for $\Lambda \in \mathcal{Q}_c(\mathbb{R}^d)$ there is even the better estimate (5.32)).

Another candidate for the space of tempered configurations might be

$$\tilde{\Gamma}^{\mathrm{t}} := \bigcup\nolimits_{0<\alpha<\theta} \tilde{\Gamma}_\alpha, \tag{5.12}$$

$$\tilde{\Gamma}_\alpha := \left\{ \gamma \in \Gamma \;\middle|\; ||\gamma||_\alpha := \left[\sum_{k\in\mathbb{Z}^d} |\gamma_k|^P \, (1+|k|)^{-(d+\alpha)} \right]^{1/P} < \infty \right\}, \quad \alpha > 0.$$

Note that we have a strict inclusion $\Gamma^{\mathrm{t}} \subset \tilde{\Gamma}^{\mathrm{t}}$. Furthermore, $\Gamma_\alpha \subset \tilde{\Gamma}_{\alpha'}$ as soon as $\alpha' > \alpha$, whereby

$$||\gamma||_{\alpha'}^P \leq m_{\alpha'-\alpha} |\gamma|_\alpha^P \quad \text{for all } \gamma \in \Gamma_\alpha. \tag{5.13}$$

The inverse inclusion $\tilde{\Gamma}_\alpha \subset \Gamma_{\alpha'}$ only holds if $\alpha' > \alpha + d$. However, the set $\tilde{\Gamma}^{\mathrm{t}} \supset \Gamma^{\mathrm{t}}$ turns out to be too large for a good control of all $\mu \in \mathcal{G}$ supported by it (insofar as the estimates (5.23)–(5.25) fail for $\xi \in \tilde{\Gamma}^{\mathrm{t}}$). In particular, for the same proofs to be performed in $\tilde{\Gamma}^{\mathrm{t}}$ we should require that $\theta > d$, which does not include the basic example of the Lennard-Jones potential (5.3).

Our aim is to study the integrability properties of the Lyapunov functional (identical to that in (4.12))

$$\Phi(\gamma_k) := \kappa H(\gamma_k) + \lambda |\gamma_k|^P \geq 0, \tag{5.14}$$

depending on the parameters $\kappa \in [0, \beta)$ and $\lambda \in [0, (\beta - \kappa) D_g)$. Note that (5.6) implies the superstability estimate in each partition cube Q_{gk}

$$\Phi(\gamma_k) \geq (\kappa D_g + \lambda - \delta) |\gamma_k|^P - \delta^{\frac{1}{1-P}} (\kappa E_g)^{\frac{P}{P-1}}, \tag{5.15}$$

valid with any $\delta \in (0, \lambda + \kappa D_g)$. For $\xi \in \Gamma^{\mathrm{t}}$, $k \in \mathbb{Z}^d$, and $\Lambda := \Lambda_{\mathcal{K}} \in \mathcal{Q}_c(\mathbb{R}^d)$ with $\mathcal{K} \Subset \mathbb{Z}^d$, let us define the quantities (the same as in (3.7))

$$n_k(\mathcal{K}|\xi) := \log \left\{ \int_{\Gamma} \exp\left\{\Phi(\gamma_k)\right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \right\}, \tag{5.16}$$

which are nonnegative and finite by (5.11) and (5.32). By the arguments similar to (4.13)–(4.14) we get the following modification of Lemma 3.1

$$\int_{\Gamma_k} \exp\left\{\Phi(\gamma_k)\right\} \mu_k(\mathrm{d}\gamma_k|\xi) \leq \exp\left\{ \Upsilon_\epsilon + \epsilon \sum\nolimits_{j \neq k} |k - j|^{-(d+\theta)} |\xi_j|^P \right\}, \tag{5.17}$$

with any

$$0 < \epsilon < \frac{1}{m_\theta} \min\left\{\kappa D_g + \lambda;\ \ (\beta - \kappa) D_g - \lambda\right\}$$

and the corresponding

$$\Upsilon_\varepsilon := m_\theta \varepsilon^{\frac{2}{2-P}} (\beta K_g)^{\frac{P}{P-2}} + z \exp\left\{\beta E_g\right\} \sup_k \sigma(Q_{gk}). \tag{5.18}$$

Choose $\delta, \varepsilon > 0$ small enough such that

$$\delta + \varepsilon m_\theta < \kappa D_g + \lambda. \tag{5.19}$$

Plugging (5.15) into the right-hand side of (5.17) and going through the proof of Lemma 3.3, we further obtain for each $k \in \mathcal{K}$

$$\begin{aligned} n_k(\mathcal{K}|\xi) \quad &\leq \quad \Upsilon_{\varepsilon,\delta} + \varepsilon \sum\nolimits_{j \in \mathcal{K}^c} |k - j|^{-(d+\theta)} |\xi_j|^P \\ &\quad + \frac{\varepsilon}{\kappa D_g + \lambda - \delta} \sum\nolimits_{j \in \mathcal{K} \setminus \{k\}} |k - j|^{-(d+\theta)} n_j(\mathcal{K}|\xi) \end{aligned} \tag{5.20}$$

with

$$\Upsilon_{\varepsilon,\delta} := \Upsilon_\varepsilon + \varepsilon m_\theta \frac{\delta^{\frac{1}{1-P}} (\kappa E_g)^{\frac{P}{P-1}}}{\kappa D_g + \lambda - \delta}. \tag{5.21}$$

Suppose for a moment that $\sup_{k \in \mathbb{Z}^d} \Phi(\xi_k) < \infty$, which by (2.64) implies $\sup_{k \in \mathbb{Z}^d} n_k(\mathcal{K}|\xi) < \infty$. For simplicity one may take here $\xi = \varnothing$. For any such ξ we immediately see from (5.20) that

$$\begin{aligned} &\sup_{|\mathcal{K}| < \infty} \sup_{k \in \mathcal{K}} \int_{\Gamma} \exp\left\{\Phi(\gamma_k)\right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \\ &\leq \exp\left\{ \left[1 - \frac{\varepsilon m_\theta}{\kappa D_g + \lambda - \delta}\right]^{-1} \left(\Upsilon_{\varepsilon,\delta} + \varepsilon m_\theta \sup_{k \in \mathbb{Z}^d} |\xi_k|^P \right) \right\} =: \Psi(\xi) < \infty. \end{aligned} \tag{5.22}$$

As will be shown in Subsection 5.3, the uniform bound (5.22) suffices to establish the existence of $\mu \in \mathcal{G}^{\mathrm{t}}$.

A more delicate situation occurs in proving the *à-priori* estimates for all $\mu \in \mathcal{G}^{\mathrm{t}}$. To this end we should consider any $\xi \in \Gamma^{\mathrm{t}}$ with possibly $\sup_{k\in\mathbb{Z}^d} \Phi(\xi_k) = +\infty$. For each $k_0 \in \mathcal{K}$ and $\xi \in \Gamma_{\alpha_0}$ with some $\alpha_0 \in (0,\theta)$, we have the following bound implied by (5.20)

$$
\begin{aligned}
n_{k_0}(\mathcal{K}|\xi) &\le \sup_{k\in\mathcal{K}} \left[n_k(\mathcal{K}|\xi)(1+\delta|k_0-k|)^{-\alpha} \right] \\
&\le \left[1 - \frac{\varepsilon m_{\theta-\alpha}(1+\delta)^{\alpha}}{\kappa D_g + \lambda - \delta} \right]^{-1} \left[\Upsilon_{\varepsilon,\delta} + \varepsilon m_{\theta-\alpha} \delta^{-\alpha} (1+|k_0|)^{\alpha} |\xi_{\mathcal{K}^c}|_{\alpha}^{P} \right]. \qquad (5.23)
\end{aligned}
$$

Here we pick first any $\alpha \in (\alpha_0,\theta)$, $\delta \in (0, \lambda + \kappa D_g)$ and then the corresponding

$$0 < \varepsilon < \varepsilon_\alpha := \frac{\kappa D_g + \lambda - \delta}{m_{\theta-\alpha}(1+\delta)^{\alpha}}.$$

An important observation from the definition (5.8) is that $\lim_{\mathcal{K}\nearrow\mathbb{Z}} |\xi_{\mathcal{K}^c}|_\alpha = 0$ whenever $\xi \in \Gamma_{\alpha_0}$ with $\alpha_0 < \alpha$. This yields a modification of Lemma 3.3 saying that for all $\nu \le \lambda\,(1+m_\alpha)^{-1}$ and $k \in \mathbb{Z}^d$

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_\Gamma \exp\left\{ \kappa H(\gamma_k) + \lambda|\gamma_k|^P \right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \quad \le \quad \Psi_\alpha, \qquad (5.24)$$

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_\Gamma \exp\left\{ \nu||\gamma||_\alpha^2 \right\} \pi_{\mathcal{K}}(\mathrm{d}\gamma|\xi) \quad \le \quad \Psi_\alpha, \qquad (5.25)$$

where

$$\Psi_\alpha := \exp\left\{ \left[1 - \frac{\varepsilon m_{\theta-\alpha}(1+\delta)^{\alpha}}{\kappa D_g + \lambda - \delta} \right]^{-1} \Upsilon_{\varepsilon,\delta} \right\} < \infty. \qquad (5.26)$$

Applying the *DLR* equation to (5.24), see the proof of Theorem 2.9, we get the *à-priori* bound

$$\sup_{k\in\mathbb{Z}^d} \int_\Gamma \exp\left\{ \kappa H(\gamma_k) + \lambda|\gamma_k|^P \right\} \mu(\mathrm{d}x) \le \Psi_\alpha, \qquad (5.27)$$

valid for all $\mu \in \mathcal{G}_{\alpha_0}$ with $\alpha_0 \in (0,\alpha)$. By Corollary 4.5 any probability measure on Γ obeying (5.27) is supported by the universal subset $\Gamma^{\mathrm{s}} \subset \bigcap_{0<\alpha<\theta} \Gamma_\alpha \subset \Gamma^{\mathrm{t}}$, which means that indeed

$$\mathcal{G}^{\mathrm{t}} := \bigcup\nolimits_{0<\alpha<\theta} \mathcal{G}_\alpha = \bigcap\nolimits_{0<\alpha<\theta} \mathcal{G}_\alpha,$$

and hence (5.27) holds *uniformly* for all $\mu \in \mathcal{G}^{\mathrm{t}}$. It remains to check that the set $\mathcal{G}^{\mathrm{t}}$ is nonvoid, which will be done in the next subsection.

5.3. **Proof of existence.** The proof follows the same pattern as that of Theorem 2.8, whereby we should more carefully take into account the topological properties of the configuration spaces. A key idea is to use, along with the topology $\mathcal{T}_{\mathrm{loc}}$ of *local setwise* convergence on $\mathcal{P}(\Gamma)$, also the topology of *weak* convergence on $\mathcal{P}(\tilde{\Gamma}_\alpha)$, $\alpha \in (0,\theta)$. This clarifies why for the interaction potential $V(x,y)$ having infinite range we have to assume its continuity at $x \neq y$. Below we point out only those issues which have to be modified.

For a given $\xi \in \Gamma_{\alpha_0} \subset \tilde{\Gamma}_\alpha$ and $0 < \alpha_0 < \alpha < \theta$, we start with the uniform bound, cf. (5.25),

$$\limsup_{\substack{\Lambda \nearrow \mathbb{R}^d \\ \Lambda \in \mathcal{Q}_c(\mathbb{R}^d)}} \int_\Gamma \exp\left\{\nu ||\gamma||_\alpha^2\right\} \pi_\Lambda(\mathrm{d}\gamma|\xi) \le \Psi_\alpha(\xi), \quad \alpha \in (\alpha_0, \theta). \tag{5.28}$$

In particular, this is surely the case if $\xi = \varnothing$. Let us first show that (5.28) implies the local equicontinuity of the family $\{\pi_\Lambda(\mathrm{d}\gamma|\xi) |\ \Lambda \in \mathcal{Q}_c(\mathbb{R}^d)\} \subset \mathcal{P}(\Gamma)$. Let $\Delta \in \mathcal{Q}_c(\mathbb{R}^d)$, $\{B_n\}_{N\in\mathbb{N}} \subset \mathcal{B}_\Delta(\Gamma)$, and $B_n \downarrow \varnothing$ as $n \to \infty$. Set

$$\Gamma_T := \{\gamma \in \Gamma_{\alpha_0} \mid ||\gamma||_\alpha \le T\} \in \mathcal{B}(\Gamma), \quad T > 0. \tag{5.29}$$

Then by Chebyshev's inequality applied to (5.28)

$$\limsup_{\substack{\Lambda \nearrow \mathbb{R}^d \\ \Lambda \in \mathcal{Q}_c(\mathbb{R}^d)}} \pi_\Lambda([\Gamma_T]^c|\xi) \to 0 \quad \text{as } T \to \infty. \tag{5.30}$$

Similarly to (3.23), for each $\Lambda \in \mathcal{Q}_c(\mathbb{R}^d)$ we have

$$\begin{aligned} \pi_\Lambda(B_n|\xi) &\le \pi_\Lambda(B_n \cap [\Gamma_T]^c|\xi) \\ &+ \int_\Gamma \int_{\Gamma_\Delta} \chi_{B_n \cap \Gamma_T}(\eta_\Delta \cup \gamma_{\Delta^c}) \exp\{-\beta H_\Delta(\eta_\Delta|\gamma)\} \lambda_{z\sigma}(\mathrm{d}\eta_\Delta) \pi_\Lambda(\mathrm{d}\gamma|\xi). \end{aligned} \tag{5.31}$$

Note that by (5.6) and (5.7)

$$\begin{aligned} H_\Delta(\eta_\Delta|\gamma) &\ge \sum_{k\in\mathbb{Z}^d} D_g|\eta_{\Delta,k}|^P - \left[E_g|\eta_\Delta| + K_g \sum_{\substack{k,j\in\mathbb{Z}^d \\ k\ne j}} |k-j|^{-(d+\theta)} |\eta_{\Delta,k}| \cdot |\gamma_{\Delta^c,j}| \right] \\ &\ge -|\eta_\Delta| \cdot \left[E_g + K_g 2^\alpha \sup_{k\in\mathcal{K}_\Delta} (1+|k|)^\alpha ||\gamma_{\Delta^c}||_\alpha^P \right], \end{aligned} \tag{5.32}$$

and hence

$$H_\Delta(\eta_\Delta|\gamma) \ge -\mathcal{C}_{\Delta,T} > -\infty \quad \text{for all } \eta_\Delta \cup \gamma_{\Delta^c} \in \Gamma_T. \tag{5.33}$$

Putting (5.30)–(5.33) together, we get

$$\pi_\Lambda(B_n|\xi) \le \pi_\Lambda([\Gamma_{\alpha,T}]^c|\xi) + \exp\{\beta \mathcal{C}_{\Delta,T}\} \lambda_{z\sigma}(B_n),$$

where, uniformly for all $\Lambda \in \mathcal{Q}_c(\mathbb{R}^d)$, the right hand-side can be made arbitrarily small by choosing large enough $T > 0$ and $n \ge n(T)$.

The equicontinuity just proved implies the existence of a limit point

$$\mu := \lim_{N\to\infty} \pi_{\Lambda_N}(\mathrm{d}\gamma|\xi) \in \mathcal{P}(\Gamma) \tag{5.34}$$

in the topology of local setwise convergence. As the interaction has infinite range, this convergence alone is *not enough* to insure that μ will be a Gibbs measure. On the other hand, in (5.28) one could try to employ *Prokhorov's* criterion on weak convergence of measures on Polish spaces, see e.g. Theorem 6.1 in [5]. However, this argument does not apply directly since the level sets (5.29) are *not relatively compact* in the topology $\mathcal{O}_v(\Gamma)$. One more principal difficulty when dealing with the vague topology is that the local Hamiltonians $H_\Lambda(\gamma_\Lambda|\xi)$ are *not continuous* functions of $\gamma, \xi \in \Gamma$.

To overcome these problems, some additional analysis is needed. Let us consider

$$\mathring{\Gamma} := \left\{ \gamma \in \Gamma \;\middle|\; \gamma \cap \partial Q_{gk} = \varnothing, \quad \forall k \in \mathbb{Z}^d \right\} \in \mathcal{B}(\Gamma),$$

$$\mathring{\Gamma}_\Lambda := \left\{ \gamma \in \Gamma \;\middle|\; \gamma_\Lambda \cap \partial Q_{gk} = \varnothing, \quad \forall k \in \mathbb{Z}^d \right\} \in \mathcal{B}(\Gamma_\Lambda), \quad \Lambda \in B_c(\mathbb{R}^d), \tag{5.35}$$

which are the subsets of configurations *not touching* the sites of the partition cubes Q_{gk}. As is seen from (2.27) and (2.31), for each $\xi \in \Gamma^{\mathrm{t}}$ the probability kernel $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ is actually carried by a smaller subset of all $\gamma = \gamma_\Lambda \cup \xi_{\Lambda^c}$ with $\gamma_\Lambda \in \mathring{\Gamma}_\Lambda$. Here we crucially used the same property of the Lebesgue-Poisson measure $\lambda_{z\sigma}$, cf. (5.5). In particular, $\pi_\Lambda(\mathring{\Gamma}_\Delta|\xi) = 1$ for each $\Delta \subseteq \Lambda$. The setwise convergence (5.34) then implies

$$\mu(\mathring{\Gamma}_\Delta) = \lim_{N\to\infty} \pi_{\Lambda_N}(\mathring{\Gamma}_\Delta|\xi) = 1 \quad \text{for all } \Delta \in \mathcal{B}_c(\mathbb{R}^d), \tag{5.36}$$

and hence

$$\mu(\mathring{\Gamma}) = \mu\left(\bigcap_{K\in\mathbb{N}} \mathring{\Gamma}_{\Delta_K}\right) = \lim_{K\to\infty} \mu(\mathring{\Gamma}_{\Delta_K}) = 1 \quad \text{as } \Delta_K \nearrow \mathbb{R}^d. \tag{5.37}$$

Next, we *densely* include each $\tilde{\Gamma}_\alpha$ into a larger space of tempered *multiple* configurations

$$\ddot{\Gamma}_\alpha := \left\{ \gamma \in \ddot{\Gamma} \;\middle|\; ||\gamma||_\alpha := \sum_{k\in\mathbb{Z}^d} \left[|\gamma_k|^P \, (1+|k|)^{-(d+\alpha))} \right] < \infty \right\}. \tag{5.38}$$

Let ρ_v be any metric which is consistent with the vague topology on $\ddot{\Gamma}$, see e.g. (2.4). Then $\ddot{\Gamma}_\alpha$ becomes a *Polish* space with respect to the metric

$$\rho_{v,\alpha}(\gamma,\eta) := \rho_v(\gamma,\eta) + \rho_\alpha(\gamma,\eta), \quad \gamma,\eta \in \ddot{\Gamma}_\alpha, \tag{5.39}$$

where

$$\rho_\alpha(\gamma,\eta) := \left[\sum\nolimits_{k\in\mathbb{Z}^d} |\langle\gamma,\psi_k\rangle - \langle\eta,\psi_k\rangle|^P \, (1+|k|)^{-(d+\alpha)} \right]^{1/P}. \tag{5.40}$$

The additional pseudometric ρ_α is defined by means of a collection of functions $\{\psi_k\}_{k\in\mathbb{Z}^d} \subset C_0(\mathbb{R}^d)$ such that $\psi_k : \mathbb{R}^d \to [0,1]$, $\psi_0(x) = 1$ if $|x| \le g\sqrt{d}$, $\psi_0(x) = 0$ if $|x| \ge 2g\sqrt{d}$, and $\psi_k(x) = \psi_0(x+gk)$ for all $x \in \mathbb{R}^d$. Obviously,

$$||\gamma||_\alpha \le \rho_\alpha(\gamma,\varnothing) \le C_{g,\alpha}||\gamma||_\alpha, \quad \gamma \in \ddot{\Gamma}_\alpha, \tag{5.41}$$

with some constant $C_{g,\alpha} > 0$. The completeness of the metric $\rho_{v,\alpha}$ is checked directly. As a countable dense set in $\tilde{\Gamma}_\alpha$ one can take the set of finite configurations $\sum_{x\in\gamma} n(x)\delta_x$ supported by all possible $\gamma \in \Gamma_0$ with atoms $x \in \mathbb{Q}$ and multiplicities $n(x) \in \mathbb{N}$. Respectively, the space $\tilde{\Gamma}_\alpha$ will be equipped by the metric $\rho_{v,\alpha}$ induced by (5.39).

An important issue (based on Proposition 3.2.6 of [17]) is that the embeddings $(\tilde{\Gamma}_\alpha, \rho_{\nu,\alpha}) \hookrightarrow (\ddot{\Gamma}_{\alpha'}, \rho_{\nu,\alpha'})$ are *compact* for $\alpha < \alpha'$; therefore the level sets (5.24) are relatively compact in $\ddot{\Gamma}_{\alpha'}$. This enables us to apply Prokhorov's criterion (see e.g. 15.4.4 in [16]) to the family $\{\pi_{\Lambda_N}(\mathrm{d}\gamma|\xi)\}_{N\in\mathbb{N}}$ obeying the moment bound (5.28). Thus, for a given $\xi \in \Gamma_{\alpha_0}$ and $\alpha \in (\alpha_0,\theta)$, there exists a limit measure on the Polish space $(\ddot{\Gamma}_\alpha, \rho_{v,\alpha})$

$$\tilde{\mu} := \lim_{M\to\infty} \pi_{\Lambda_{N_M}}(\mathrm{d}\gamma|\xi) \in \mathcal{P}(\ddot{\Gamma}_\alpha), \tag{5.42}$$

such that for all bounded continuous functions $F \in C_{\mathrm{b}}(\ddot{\Gamma}_\alpha)$

$$\lim_{M\to\infty} \int_{\Gamma_\alpha} F(\gamma)\pi_{\Lambda_{N_M}}(\mathrm{d}\gamma|\xi) = \int_{\ddot{\Gamma}_\alpha} F(\gamma)\tilde{\mu}(\mathrm{d}\gamma). \tag{5.43}$$

Now we show that the above measures (5.34) and (5.42) *coincide* and hence μ is supported by *simple* configurations $\gamma \in \Gamma^{\mathrm{t}}$. As we already know, for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and for all bounded *local* functions $G \in L_\infty(\Gamma_\Lambda)$,

$$\lim_{N\to\infty} \int_{\Gamma} G(\gamma)\pi_{\Lambda_N}(\mathrm{d}\gamma|\xi) = \int_{\Gamma} G(\gamma)\mu(\mathrm{d}\gamma). \tag{5.44}$$

Note that $C_{\mathrm{b}}(\ddot{\Gamma}_\alpha)$ and $L_\infty(\Gamma_\Lambda)$ both include the measure determining class $\mathcal{F}C_{\mathrm{b}}(\Gamma)$ of all cylinder functions having the form (2.14). Thus, by comparing (5.43) and (5.44) we immediately obtain that

$$\mu = \tilde{\mu} \in \bigcap\nolimits_{\alpha\in(\alpha_0,\theta)} \mathcal{P}(\tilde{\Gamma}_\alpha).$$

Since the set $\tilde{\Gamma}_\alpha$ is dense in $(\ddot{\Gamma}_\alpha, \rho_{\nu,\alpha})$ and hence any bounded, uniformly continuous function $F : (\tilde{\Gamma}_\alpha, \rho_{\nu,\alpha}) \to \mathbb{R}$ *uniquely* extends to an element from $C_{\mathrm{b}}(\ddot{\Gamma}_\alpha)$, we get that $\mu = \lim_{M\to\infty} \pi_{\Lambda_{N_M}}(\mathrm{d}\gamma|\xi)$ in the *weak* topology on each $\tilde{\Gamma}_\alpha$, $\alpha \in (\alpha_0, \theta)$. Furthermore, applying Fatou's lemma to the continuous functions $\tilde{\Gamma}_\alpha \ni \gamma \to \exp|\langle\gamma, \psi_k\rangle|^P$, we conclude from (5.24) and (5.41) that for all $\lambda_0 \in (0, \lambda m_\psi^{-1}]$ and $k \in \mathbb{Z}^d$

$$\begin{aligned}
\int_{\Gamma} \exp\left\{\lambda_0|\gamma_k|^P\right\}\mu(\mathrm{d}\gamma) &\leq \int_{\Gamma} \exp\left\{\lambda_0|\langle\gamma, \psi_k\rangle|^P|\right\}\mu(\mathrm{d}\gamma) \\
&= \lim_{N\to\infty} \int_{\Gamma} \exp\left\{\lambda_0|\langle\gamma, \psi_k\rangle|^P|\right\}\pi_{\Lambda_N}(\mathrm{d}\gamma|\xi) \\
&\leq \sup_{j\in\mathbb{Z}^d} \lim_{N\to\infty} \int_{\Gamma} \exp\left\{\lambda|\psi_j|^P|\right\}\pi_{\Lambda_N}(\mathrm{d}\gamma|\xi) \leq \Psi_\alpha.
\end{aligned} \tag{5.45}$$

By Corollary 4.5 any $\mu \in \mathcal{P}(\Gamma)$ obeying (5.45) must be supported by the set $\bigcap_{0<\alpha<\theta} \Gamma_\alpha \subset \Gamma^{\mathrm{t}}$.

To guarantee that the limit point μ is Gibbs, we need to establish a proper continuity of the specification π_Λ. For every $\alpha \in (0, \theta)$, $\Lambda := \Lambda_{\mathcal{K}} \in \mathcal{Q}_c(\mathbb{R}^d)$ and $F \in C_{\mathrm{b}}(\tilde{\Gamma}_\alpha)$, let us consider the following map, cf. (2.33),

$$\tilde{\Gamma}_\alpha \ni \eta \to \pi_\Lambda F(\eta) := \int_{\Gamma} F(\gamma)\pi_\Lambda(\mathrm{d}\gamma|\eta). \tag{5.46}$$

We claim that $\pi_\Lambda F$ is continuous in the metric $\rho_{v,\alpha}$ at every point $\eta_0 \in \mathring{\Gamma}_\alpha := \tilde{\Gamma}_\alpha \cap \mathring{\Gamma}$; for the proof of this fact see Proposition 5.2 below. In particular, the set of discontinuities of the function $\pi_\Lambda F$ is of *zero* measure μ, which by the *portmanteau* theorem (see Theorem 2.1 in [5] or 1.4.2 in [16]) allows us to substitute $\pi_\Lambda F$ for F in (5.43). In account of (2.32), we thus can take the limit

$$\begin{aligned}
\int_{\tilde{\Gamma}_\alpha} (\pi_\Lambda F)(\gamma)\mu(\mathrm{d}\gamma) &= \lim_{M\to\infty} \int_{\tilde{\Gamma}_\alpha} (\pi_\Lambda F)(\gamma)\pi_{\Lambda_{N_M}}(\mathrm{d}\gamma|\xi) \\
&= \lim_{M\to\infty} \int_{\tilde{\Gamma}_\alpha} F(\gamma)\pi_{\Lambda_{N_M}}(\mathrm{d}\gamma|\xi) = \int_{\tilde{\Gamma}_\alpha} F(\gamma)\mu(\mathrm{d}\gamma),
\end{aligned} \tag{5.47}$$

for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $F \in C_{\mathrm{b}}(\tilde{\Gamma}_\alpha)$. This shows that μ satisfies the *DLR* equation (2.37) and hence completes the proof. □

Remark 5.1. (i) Note that in the definition (5.38)–(5.40) of $(\ddot{\Gamma}_\alpha, \rho_{v,\alpha})$ we cannot put $\psi_k := \mathbf{1}_{\bar{Q}_{gk}}$ and respectively

$$\rho_\alpha(\gamma, \eta) := \left[\sum\nolimits_{k \in \mathbb{Z}^d} \left(|\bar{\gamma}_k| - |\bar{\eta}_k| \right)^P (1 + |k|)^{-(d+\alpha)} \right]^{1/P}.$$

The reason is that the counting maps $\Gamma \ni \gamma \to |\bar{\gamma}_k| := |\gamma \cap \bar{Q}_{gk}|$ are not vaguely continuous (as pointed out in Subsection 2.1) and thus the corresponding metric $\rho_{v,\alpha}$ would not be complete.

(ii) The optimal choice of the weights $w_\alpha(k,j)$ (typically, $\exp\{-\alpha|k-j|\}$ or $(1+|k-j|)^{-(d+\alpha)}$) determining the set of tempered configurations $\gamma \in \Gamma^{\mathrm{t}}$ strongly depends on the decay of $V(x,y)$ as $|x-y| \to \infty$. More restrictive conditions on V (like the exponential decay or finite range) allow us to consider the larger set Γ^{t}, cf. (2.65). Another property of the weights $w_\alpha(k,j)$, which was essential in the above proofs (more precisely, in (3.15)), is that $\log\left[w_\alpha^{-1}(k,j)\right]$ should be a metric on $\mathbb{Z}^d$. For the latter reason, the weights decaying too quickly (e.g., such as $\exp\{-\alpha|k-j|^2\}$) are not permitted.

5.4. **Almost continuity of the specification.** In the above proof we have crucially used the so-called *almost Feller continuity* of the specification $\Pi = \{\pi_\Lambda\}$, which is the contents of Proposition 5.2. Somewhat surprisingly, it turns out that such regularity of the stochastic kernels $\pi_\Lambda(\mathrm{d}\gamma|\xi)$ holds true, even though the potential $V(x,y)$ itself may be *singular* at the diagonal.

Proposition 5.2. *For each $F \in C_{\mathrm{b}}(\tilde{\Gamma}_\alpha)$, $\alpha \in (0,\theta)$, and $\Lambda \in \mathcal{Q}_c(\mathbb{R}^d)$, the map $(\tilde{\Gamma}_\alpha, \rho_{v,\alpha}) \ni \xi \to \pi_\Lambda F(\xi)$ defined by (5.46) is continuous at every point $\xi \in \mathring{\Gamma}_\alpha$.*

Proof. Recall that

$$\pi_\Lambda F(\xi) = [Z_\Lambda(\xi)]^{-1} \int_{\Gamma_\Lambda} F(\gamma_\Lambda \cup \xi_{\Lambda^c}) \exp\{-\beta H_\Lambda(\gamma_\Lambda|\xi)\} \, \mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda),$$

$$Z_\Lambda(\xi) := \int_{\Gamma_\Lambda} \exp\{-\beta H_\Lambda(\gamma_\Lambda|\xi)\} \, \mathrm{d}\lambda_{z\sigma}(\gamma_\Lambda). \tag{5.48}$$

With the help of Lebesgue's dominated theorem, the claim will follow from the almost continuity property of the functionals

$$\begin{aligned} (\tilde{\Gamma}_\alpha, \rho_{v,\alpha}) &\ni \xi \to F(\gamma_\Lambda \cup \xi_{\Lambda^c}), \\ (\tilde{\Gamma}_\alpha, \rho_{v,\alpha}) &\ni \xi \to W_\Lambda(\gamma_\Lambda|\xi) := \sum_{x \in \gamma_\Lambda} \sum_{y \in \xi_{\Lambda^c}} V(x,y), \end{aligned} \tag{5.49}$$

where we fixed an arbitrary $\gamma_\Lambda \in \Gamma_\Lambda$. Indeed, let a sequence $\{\xi^{(N)}\}_{N \in N} \subset \tilde{\Gamma}_\alpha$ converge to some $\xi \in \mathring{\Gamma}_\alpha$, then by (5.41) $\sup_{N \in \mathbb{N}} ||\xi^{(N)}||_\alpha < \infty$. From (5.7) we see that for $\Lambda := \Lambda_{\mathcal{K}}, \Delta := \Delta_{\mathcal{L}} \in \mathcal{Q}_c(\mathbb{R}^d)$ with $\mathcal{K}, \mathcal{L} \Subset \mathbb{Z}^d$ and large enough $\mathcal{L} \supset \mathcal{K}$ (such that $\mathrm{dist}(\mathcal{K}, \mathcal{L}^c) \geq r/g + \sqrt{d}$)

$$\begin{aligned} |W_\Lambda(\gamma_\Lambda|\xi) - W_\Lambda(\gamma_\Lambda|\xi^{(N)})| \quad &\leq \quad |W_\Lambda(\gamma_\Lambda|\xi_\Delta) - W_\Lambda(\gamma_\Lambda|\xi_\Delta^{(N)})| \\ &+ C_g |\gamma_\Lambda| \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{L}^c} |k-j|^{-(d+\theta)} \left[|\xi_j| + |\xi_j^{(N)}| \right]. \end{aligned} \tag{5.50}$$

The second term in the right hand-side in (5.50) becomes arbitrary small as $\mathcal{L} \nearrow \mathbb{Z}^d$ due to the obvious estimate

(5.51)

$$\sum_{k\in\mathcal{K}}\sum_{j\in\mathcal{L}^c}|k-j|^{-(d+\theta)}\left[|\xi_j|+|\xi_j^{(N)}|\right] \le \frac{C(\mathcal{K})}{[\mathrm{dist}(\mathcal{K},\mathcal{L}^c)]^{d+\theta-a}}\left[||\xi||_\alpha^P+||\xi^{(N)}||_\alpha^P\right],$$

with

$$C(\mathcal{K}) := 2^\alpha \sum_{k\in\mathcal{K}}(1+|k|)^\alpha ,$$

which is uniform in $N\in\mathbb{N}$. Let us examine the first term

$$|W_\Lambda(\gamma_\Lambda|\xi_\Delta) - W_\Lambda(\gamma_\Lambda|\xi_\Delta^{(N)})| \le \sum_{x\in\gamma_\Lambda}\left|\sum_{y\in\xi_{\Delta\setminus\Lambda}}V(x,y) - \sum_{y^{(N)}\in\xi^{(N)}_{\Delta\setminus\Lambda}}V(x,y)\right|.$$

Recall that by construction $\xi\cap(\partial\Lambda\cup\partial\Delta)=\varnothing$. Since $\xi^{(N)}\to\xi$ vaguely, starting from some $N_0\in\mathbb{N}$ the numbers of particles restricted to the domain $\Delta\setminus\Lambda$ should coincide, i.e., $|\xi_{\Delta\setminus\Lambda}| = |\xi^{(N)}_{\Delta\setminus\Lambda}| = n\in\mathbb{N}_0$. Furthermore, by (2.6) one can choose proper enumerations $\{y_j\}_{j=1}^n = \xi_{\Delta\setminus\Lambda}$ and $\{y_j^{(N)}\}_{j=1}^n = \xi^{(N)}_{\Delta\setminus\Lambda}$ such that

$$\left|y_j^{(N)} - y_j\right| \to 0, \quad \text{for all } 1\le j\le n. \tag{5.52}$$

In particular, we observe that for any $\gamma_\Lambda\in\mathring{\Gamma}$

$$\lim_{N\to\infty}\mathrm{dist}(\gamma_\Lambda,\xi^{(N)}_{\Delta\setminus\Lambda}) = \mathrm{dist}(\gamma_\Lambda,\xi_{\Delta\setminus\Lambda})>0.$$

Due to the continuity of the potential $V(x,y)$ for $x\neq y$, this yields the required convergence in (5.50)

$$\lim_{N\to\infty}W(\gamma_\Lambda|\xi_\Delta^{(N)}) = W(\gamma_\Lambda|\xi_\Delta).$$

Similarly one proves that $\lim_{N\to\infty}F(\gamma_\Lambda\cup\xi^{(N)}_{\Lambda^c}) = F(\gamma_\Lambda\cup\xi_{\Lambda^c})$ for each $F\in C_{\mathrm{b}}(\tilde{\Gamma}_\alpha)$ and $\gamma_\Lambda\in\mathring{\Gamma}_\Lambda$. Since the function F is continuous, it would suffice to check that $\rho_{v,\alpha}(\xi^{(N)},\xi)\to 0$ implies $\rho_{v,\alpha}(\gamma_\Lambda\cup\xi^{(N)}_{\Lambda^c},\gamma_\Lambda\cup\xi_{\Lambda^c})\to 0$. Consider any $f\in C_0(\mathbb{R}^d)$ with $\mathrm{supp}f\subseteq\Delta\in\mathcal{Q}_c(\mathbb{R}^d)$. As discussed above (cf. (5.52)), the cut-off operator $\xi\mapsto\xi_{\Delta\setminus\Lambda}$ is continuous in the vague topology $\mathcal{O}_v(\Gamma)$ at each point $\xi\in\mathring{\Gamma}$. Thus

$$\begin{aligned}\langle f,\gamma_\Lambda\cup\xi^{(N)}_{\Lambda^c}\rangle &= \langle f,\gamma_\Lambda\rangle+\langle f,\xi^{(N)}_{\Lambda^c}\rangle\\ &\to \langle f,\gamma_\Lambda\rangle+\langle f,\xi_{\Lambda^c}\rangle = \langle f,\gamma_\Lambda\cup\xi_{\Lambda^c}\rangle,\end{aligned}$$

which means that $\rho_v(\gamma_\Lambda\cup\xi^{(N)}_{\Lambda^c},\gamma_\Lambda\cup\xi_{\Lambda^c})\to 0$. It remains to show that also $\rho_\alpha(\gamma_\Lambda\cup\xi^{(N)}_{\Lambda^c},\gamma_\Lambda\cup\xi_{\Lambda^c})\to 0$. Indeed, by definition (5.40)

$$\begin{aligned}&\rho_\alpha^P(\gamma_\Lambda\cup\xi^{(N)}_{\Lambda^c},\,\gamma_\Lambda\cup\xi_{\Lambda^c}) = \rho_\alpha^P(\xi^{(N)},\,\xi)\\ &+\sum\nolimits_{k:\ \mathrm{supp}\psi_k\cap\Lambda\neq\varnothing}|\langle\xi^{(N)}_{\Lambda^c},\psi_k\rangle-\langle\xi_{\Lambda^c},\psi_k\rangle|^P\,(1+|k|)^{-(d+\alpha)}\\ &-\sum\nolimits_{k:\ \mathrm{supp}\psi_k\cap\Lambda\neq\varnothing}|\langle\xi^{(N)},\psi_k\rangle-\langle\xi,\psi_k\rangle|^P\,(1+|k|)^{-(d+\alpha)},\end{aligned} \tag{5.53}$$

where both series in the right-hand side are finite and tend to zero due to the vague convergence $\xi^{(N)}\to\xi$.

On the other hand, we have the following upper bound

$$\begin{aligned} & |F(\gamma_\Lambda \cup \xi^{(N)}_{\Lambda^c})| \exp\left\{-\beta H_\Lambda(\gamma_\Lambda|\xi^{(N)})\right\} \\ \leq\ & ||F||_{C_{\mathrm{b}}(\tilde{\Gamma}_\alpha)} \exp\left\{\beta|\gamma_\Lambda|\left(E_g + K_g \sum_{k\in\mathcal{K}}\sum_{j\in\mathcal{K}^c}|k-j|^{-(d+\theta)}|\xi_j|\right)\right\} \\ \leq\ & ||F||_{C_{\mathrm{b}}(\tilde{\Gamma}_\alpha)} \exp\left\{\beta|\gamma_\Lambda| \cdot \left[E_g + C(\mathcal{K})K_g||\xi||^P_\alpha\right]\right\}, \end{aligned} \tag{5.54}$$

which is integrable with respect to $\lambda_{z\sigma}(\mathrm{d}\gamma_\Lambda)$. Thus, we can apply Lebesgue's dominated convergence theorem, which yields the required continuity of the integral in (5.48). Convergence of the partition functions $Z_\Lambda(\xi^{(N)}) \geq 1$ is established in the same way. □

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