MR1919377 (2003m:37037) 37D30 (37A25 37C20 37C40)
Dolgopyat, Dmitry (1-PAS)
On mixing properties of compact group extensions of hyperbolic systems. (English summary)
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This is a significant contribution to the field of partially hyperbolic dynamical systems. To the extent that it refers to invariant measures, much of the work to date has concentrated on qualitative properties such as ergodicity, as well as their persistence under perturbation. The present paper obtains a quantitative property, namely a correlation decay rate.

In a partially hyperbolic dynamical system each tangent space splits (equivariantly) into an (exponentially) expanding direction, a contracting direction and a neutral or "slow" center direction. The systems considered here are compact-group extensions of hyperbolic systems, i.e., they are of the form $T(y,x)=(F(y),\tau(y)x)$ on $M:=Y\times X$, where F is a topologically mixing Axiom-A diffeomorphism on the compact manifold Y, X is a transitive G-space for a compact connected simply connected Lie group G, and $\tau\colon Y\to G$ is smooth. (This defines an open set of G-equivariant dynamical systems.) Thus the expanding and contracting directions come from those for F, and X provides the center direction. Without being overly restrictive, this context provides good control over the center direction. As the author points out, Brin showed that these systems are generically ergodic and weakly mixing (and hence Bernoulli). Field, Parry, and Pollicott were able to drop the assumption of volume preservation, and Burns and Wilkinson recently established genericity of stable ergodicity of these (with respect to perturbations that need not be equivariant).

If $f\colon Y\to\mathbb{R}$ is Hölder continuous and μ_f is the corresponding Gibbs state for F then $\mu:=\mu_f\times$ Haar is T-invariant and we can define the correlation function

$$c_{\varphi,\psi}(n):=\int\varphi(y,x)\psi(T^n(y,x))\,d\mu(y,x)\\ -\int\varphi(y,x)\,d\mu(y,x)\int\psi(y,x)\,d\mu(y,x).$$

In hyperbolic dynamics one is often after exponential decay of these correlations, but Dolgopyat defines T to be rapidly mixing if $c_{\varphi,\psi}(n)\to 0$ faster than any power of n. Though faster decay rates are not uncommon, he can establish this rate for large classes of systems, and he shows that it is fast enough to yield the most desirable stochastic properties. (He notes that this property turns out to be independent of f.)

The first result uses an accessibility property, expressed in terms of the Brin transitivity group and a Diophantine condition (discussed at length in the appendix), to obtain (indeed, characterize) rapid mixing. As corollaries one obtains that if F is an Anosov diffeomorphism of an infranilmanifold or G is semisimple then stable ergodicity is equivalent to stable rapid mixing (and, for semisimple G, to ergodicity). And an ergodic such T is rapidly mixing if and only if its factor on $Y\times(X/[G,G])$

is.

These results follow from results presented in Section 4 for subshifts of finite type: a sufficient condition for the desired mixing property, a characterization of it, and a prevalence statement (a generic skew-product over a one-sided subshift of finite type is stably rapidly mixing). An analogous triple of results (mixing-characterization-prevalence) is proved in Section 3 for extensions of expanding maps and their absolutely continuous invariant measure (infinitesimal complete nonintegrability, exponential mixing and stable ergodicity are equivalent and generic among compact extensions in the sense that the complement is a submanifold of positive codimension).

Section 6 discusses applications, principally involving the central limit theorem, and Section 7 poses a number of interesting questions.

Reviewed by *Boris Hasselblatt*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.