Posted Prices vs. Negotiations: An Asymptotic Analysis

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Abstract

We consider two alternatives to optimal auctions: *posted-price* mechanisms and *dynamic auctions.* In posted-price mechanisms, the seller posts a single price and sells the item at this price to a bidder that accepts it. In a dynamic auction, bidders arrive sequentially and each bidder leaves the market before the next bidder arrives. We establish an exact asymptotic characterization of the optimal revenue in each of these mechanisms for general distributions, under a mild condition taken from extreme-value theory. We also devise posted-price and dynamic mechanisms that achieve this optimal revenue. Intuitively, one would expect auctions to perform better compared to posted prices as the values of the bidders are more dispersed; We show that this intuition holds up to a point where more value dispersion causes an opposite phenomenon. Our results also imply that such mechanisms may lose a non-trivial share of the optimal-auction revenue, even in large markets.

keywords: Auctions. Posted prices. Secretary problems. Stopping rules. Dynamic auctions. Extreme Value Theory.

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1 Introduction

The basic model for selling a single item to maximize the seller's revenue is well understood by now. The Myerson auction ([20]) maximizes the seller's expected revenue and, when the bidders values are drawn i.i.d. from a distribution, the Myerson auction simply translates into a Vickrey (second-price) auction with an optimally chosen reserve price. However, in many environments running the optimal auction may be impractical due to environment-specific restrictions. This paper considers two alternative, sub-optimal formats of sale: *posted prices* and *dynamic mechanisms*.

Posting a price is a cheap, natural and popular method to sell goods. The seller posts a price, bidders may accept or reject this price, and the item is given to one of the bidders who accepted the price, and he pays the offered price. The price is determined before any negotiations with the bidders have taken place, just according to the prior beliefs of the seller. While auctions incur longer sale time and higher maintaining costs for the software, posted-price mechanisms are usually much quicker and significantly simpler for the customers (both strategically and cognitively); With posted prices, however, the allocation rule is coarse and cannot always distinguish with sufficient granularity bidders with high valuations. For example, in the online-auctions giant *eBay.com*, sellers can choose between selling their items via ascending-price auctions and posted-price (*"Buy it now"*) mechanisms. Another relevant example is *electricity markets* (see, e.g., [28]); In such markets, when a sudden drop in the electricity supply takes place, the allocation of the remaining electricity should be determined immediately, without long negotiations between the seller and the bidders, and without waiting for the bidders to shape their exact preferences. One solution that have been used in practice is to post take-it-or-leave-it prices for the bidders, and ask for their immediate response.

The second sale method that we discuss in this paper concerns environments with dynamic arrival of bidders. In such environment, the seller handles one customer at a time, without knowing the types of the customers that are yet to arrive. If a customer decides not to buy from the seller, she will leave the market for good. Clearly, the revenue from such dynamic auctions is inferior to optimal auctions where synchronous negotiations with all bidders is possible. It turns out that dynamic auctions can be viewed as posted-price mechanisms, but with *discriminatory prices*. In posted-price mechanisms with discriminatory prices, the seller may post an individual price for each one of the bidders, and the bidder that accepted the highest price wins. Consider, for example, a user who enters a web site of an online retailer like *Expedia.com* aiming to buy an airline ticket; In such cases, price discrimination is reasonable even in the static model, as customers are not aware of offers presented to others. It is not hard to see that optimally designed dynamic auctions and optimal discriminatory posted-price mechanisms are essentially equivalent. On one hand, it was observed in the literature (e.g., [2, 3]) that optimal dynamic auctions offer bidders a series of decreasing take-it-or-leave-it offers, and these offers can be presented to bidders in a static setting too if discrimination is allowed. On the other hand, the optimal static discriminatory-price mechanism can be implemented in a dynamic setting (when bidders are ex-ante identical, see, e.g., [3]), where higher prices are offered to earlier bidders. We therefore refer to dynamic auctions in most parts of this paper as *posted-price mechanisms with discriminatory prices*, as opposed to "symmetric" posted-price mechanisms where all bidders observe the same posted price.

Comparisons between posted-price auctions and optimal auctions were made in several previous works. It is a common belief in the literature that optimally designed auctions are the better solution in environments where the seller's uncertainty about the demand is high, while with lower uncertainty posted prices might perform better (taking into account the additional cost of running optimal auctions). In [19], Paul R. Milgrom asserted that *"When goods are not standardized or when the market-clearing prices are highly unstable, posted prices work poorly, and auctions are usually preferred"*. Ruqu Wang claimed in [27] that *"Intuition suggests that perhaps the more dispersed the value of the object, the more auctions are preferred."* Wang supported this intuition by showing that the optimal sale method (given explicit costs associated with each method) depends on the steepness of the marginal-revenue curve of the buyers' distribution, which roughly corresponds to the dispersion of the value around the mean. Our paper also supports this intuition, comparing the performance of posted-price mechanisms and optimal auctions as a function of the dispersion of the distribution (or, in our case, the shape of the distribution tail). We establish that this intuition is no longer true, however, when the tail of the distribution becomes sufficiently heavy; in this case, heavier tails (or more dispersed distributions) implies that posted-price mechanisms actually achieve a *higher* percentage of the optimal-auction revenue. This U-shaped behavior (to be formally presented in Section 3) can be explained by the following effect; As the tail of the distribution becomes heavier, a better separation between the expected order statistics of the distribution is created, and therefore a single posted price can obtain better results.

Prima facie, the design of optimal posted-price mechanisms (or dynamic mechanisms) with numerous bidders appears to be an easy task. The seller can just post a sufficiently high price, and with high probability some bidder will be willing to pay this price. However, it is not clear what exactly this price should be, how much revenue loss is incurred by running a posted-price mechanism instead of optimal auctions and how the answers to those questions depend on the market size. These are indeed the main questions this paper aims to address.

Our paper compares the expected revenue in optimal auctions, optimal posted-price mechanisms and optimal dynamic auctions (or discriminatory posted-price mechanisms). We focus on single-item auctions in a Bayesian model where the values that the bidders are willing to pay for the item are independently and identically distributed according to a distribution function F that is known to the seller (when a density function exists, we denote it by f). The valuation of bidder i is denoted by $v_i \geq 0$, and a dominant strategy for a bidder is to accept the offered price p when $p \leq v_i$.

We now define the three market mechanisms that we consider in this paper. We denote the optimal revenue from n bidders in these mechanisms, respectively, by r_n^{sym}, r_n^{disc} and r_n^{opt}.

Symmetric posted prices: Let $r_n^{sym}(p)$ be the expected revenue achieved by posting the price $p \in \mathbb{R}$ to all bidders, and allocating the item to one of the bidders who accepted this price and charge him p; In case no bidder accepts, the revenue is 0. We define the *optimal revenue achieved by symmetric posted-price mechanisms* to be

$$r_n^{sym} = \sup_{p \in \mathbb{R}} r_n^{sym}(p). \tag{1}$$

Discriminatory posted prices: Let $r_n^{disc}(p_1, ..., p_n)$ be the expected revenue achieved by offering the price $p_i \in \mathbb{R}$ to each bidder i and selling the item to a bidder that accepted the highest offer. That is, let S be the set of bidders such that each bidder $i \in S$ accepted his personalized price p_i (i.e., $v_i \geq p_i$), then the winning bidder is in $argmax_{i \in S} p_i$ and she pays the price offered to her. Again, if no bidder accepted his offered price, the revenue is 0. We define *the optimal expected revenue achieved by a discriminatory posted-price mechanisms* as

$$r_n^{disc} = \sup_{(p_1,..,p_n) \in \mathbb{R}^n} r_n^{disc}(p_1, ..., p_n). \tag{2}$$

Optimal Auctions: We denote the optimal revenue in optimal auctions among n bidders (i.e., Myerson auctions) by r_n^{opt}. Whenever we use this notation we will also assume that the distribution function is Myerson-regular (that is, a density function f exists and the virtual valuation $\tilde{v}(v) = v - \frac{1-F(v)}{f(v)}$ is non-decreasing).[1]

We note that our results can be applied for the goal of efficiency maximization

[1] In our paper the differences between the Myerson and the Vickrey auction (that always allocate the item) are negligible since with numerous bidders the probability that all the bidders will bid below the reserve price (which is known to be independent of n) is negligible.

via the classic work of Myerson [20], see more details in Section 6.[2] Nonetheless, we mainly discuss revenue maximization in this paper.

1.1 Our Results

This paper provides an exact asymptotic characterization of the optimal expected revenue achieved by each of the above sale methods as a function of the size of the market. For posted-price mechanisms, we also present prices that achieve the optimal results for any number of bidders. Our results are given for general distribution functions, adapting general classes of distributions from *extreme-value theory*.

Extreme value theory is a branch of statistics that studies the behaviour of the largest of independent events. One of the notable results in extreme-value theory tells us that for many distributions, the distribution of the largest value of n independent draws approaches a certain shape as n grows, and this shape depends on a single real parameter γ, called the *extreme value index*. We provide results for almost all distributions for which this is the case (for later reference: we provide results for any distribution except those with $\gamma = 0$ and bounded support, and those with $\gamma = 1$), and give tight expressions for the expected revenue an auctioneer can expect asymptotically. For this wide family of distributions, we present an explicit formula for the optimal revenue in each auction type, parametrized only by the index γ.

While extreme-value theory does not treat all distributions, it does treat most of the standard distributions. For example, all distributions which are defined on a bounded interval $[a, b] \subset \mathbb{R}$ which have positive and differentiable density function on $[a, b]$ are treated. Other examples are Gamma distributions, power-law, log-normal, Pareto, Burr and exponential distributions. We will formally present the set of distributions in Section 2. A detailed introduction to extreme-value theory and the theory of high-order statistics can be found in the textbooks [7, 8].

[2]Myerson [20] showed that maximizing revenue is actually equivalent to maximizing social welfare after performing a transformation on the valuations (i.e., when considering the *virtual* valuations).

We now briefly and informally mention our main results. Our results will be formally discussed in Section 3, after presenting some necessary definitions in Section 2. We present conceptually different results for distributions with bounded and unbounded supports.

Unbounded supports We show that in general, the revenue in optimal auctions, symmetric posted-price mechanisms and discriminatory posted-price mechanisms does not grow at the same rate when the number of bidders grow. We present an exact computation of the ratio between the revenue in these mechanisms as a function of the parameter γ.

For example, consider a power-law distribution $F(x) = 1 - \frac{1}{x^2}$ on the open interval $[1, \infty)$. The expected revenue on the optimal auction turns out to be about $0.89\sqrt{n}$ (ignoring lower order terms), while the optimal expected revenue in discriminatory posted-price mechanisms is about $0.71\sqrt{n}$ and in symmetric posted-price mechanisms it is $0.64\sqrt{n}$. Note that the ratio between those revenues is not converging to 1. In particular, symmetric posted-price mechanisms obtain about 73% of the optimal-auction revenue in this example, discriminatory prices obtain about 80%.

Bounded Support With bounded supports, it is clear that the expected revenue in each sale method can approach the upper bound of the support as n grows. However, we show that while the expected revenue in optimal auctions and in discriminatory posted-price mechanisms converges at the same rate (up to a constant factor), symmetric posted-price mechanisms converge in a much slower rate. In particular, the symmetric posted-price revenue converges to the upper bound in a rate which is slower by a logarithmic factor (in the number of bidders).

For example, in the case where the values of the bidders are distributed uniformly on $[0, 1]$, the optimal expected revenue in the full revelation auction (i.e., the Myerson auction) is known to be around $1 - \frac{2}{n}$ (again, up to lower order terms). With discriminatory prices, the expected revenue becomes is about $1 - \frac{4}{n}$. The optimal revenue with a symmetric posted price is around $1-\frac{\ln n}{n}$, which implies a slower convergence rate to 1 by a logarithmic factor.

In addition to direct revenue-comparison, our results also allow us to compare auctions with respect to the *size of the market*, à la Bulow and Klemperer [4]. Bulow and Klemperer showed that Vickrey auctions with $n+1$ bidders achieve more revenue than n-bidder optimal auctions. We study to what extent the market should be expanded such that the expected revenue in posted-price mechanisms will reach the optimal-auction revenue. (As mentioned, we ignore here the different operating costs of the mechanisms.) For example, for the above power-law distribution ($F(x) = 1-\frac{1}{x^2}$) optimally chosen symmetric posted prices achieve the same expected revenue as optimal auctions when the market is expanded by 89%, and an expansion of 54% is required for discriminatory prices to achieve the optimal-auction revenue.

1.2 Related work

Several existing papers compared auctions and posted-price mechanisms, some of which are close in spirit to ours. Neeman [21], among his other results, bounded from below the worst case ratio between the expected revenue in posted-price mechanisms and the maximal social welfare (i.e., the expected highest order statistic). This bound was given for the worst-case distribution and was independent of the number of bidders, while our results are for a general family of distribution and are sensitive to the size of the market. Kultti [17] showed that auctions and posted prices are practically equivalent when bidders choose in which market they participate. Jackson and Kremer [15] studied discriminatory prices when the number of items is some non-trivial fraction of the number of bidders. Blumrosen, Nisan and Segal [3] studied auctions where the message space available to the bidders is severely limited. [3] showed an asymptotic comparison of optimal and discriminatory posted-price mechanisms, but only for uniform distributions with bounded support. Another related line of research is by [22, 24] and others, who studied the loss in double auctions as the number of bidders grow. Coarse matching models, where the types of bidders are severely discretzied prior to being matched to each other, were discussed by [18, 14]. In a recent paper, Bulow and Klemperer [5] compared static auctions and sequential auctions. They showed that when entry decisions are made before observing any bids, this results in inefficiency. We do not discuss costly entry to the market in our model.

Asymptotic analysis of the revenue in single-item auctions was studied by Fibich et al. (see [9] and the references therein) and by Caserta and de Vries [6], with the emphasis on the asymptotic strategies of the bidders in first-price auctions, revenue equivalence and risk aversion. These works also applied extreme-value theory to auctions, but they did not study posted-price mechanisms.

The dynamic-auction setting studied in this paper is a variant of the *secretary problem*. Our dynamic-auction model is equivalent to the version of the secretary problem where each secretary has some real skill value drawn from a distribution known to the employer, and the employer tries to maximize the expected skill of the secretary it hires.[3] Some of the classic solutions for the secretary problems (e.g., [11, 1, 16]) presented results in the same spirit as ours, but did not handle asymptotics and general distribution functions. A closely related line of research in statistics studied optimal stopping rules (see, e.g., [23]).

We proceed as follows. Section 2 presents the main definitions and notations needed for presenting our results. We continue in Section 3 with an exposition of our results, given with some examples and interpretations. The proofs are given in the subsequent sections; Section 4 proves our results for discriminatory posted prices, Section 5 discusses symmetric posted prices, and optimal auctions are covered in Section 6. We often use in our proofs technical lemmas from the textbook [8]; all the lemmas we use are quoted in Section 3 of the supplementary materials attached to

[3] In many classic paper on secretary problems the goal is to maximize the probability that the secretary with the highest value will be chosen, see, e.g., [11, 1]. The game induced by such settings is also known as the Googol game (see, for instance, [13]).

this paper where we kept the numbering and labeling of [8].

2 Preliminaries

In this section we define the class of distribution functions for which we give results and introduce the main notations that are required for presenting our results.

2.1 Probability Functions

Using *inverse probability functions* turns out to be very useful in our paper. Given a cdf F, we define the inverse probability function to be

$$F^{\leftarrow}(y) = \inf\{x | F(x) \geq y\} \ . \tag{3}$$

Note that for distributions with positive density, the probability that the random variable is smaller than $F^{\leftarrow}(y)$ is exactly y.

If one picks n independent random variables according to a cdf F, very naïvely, one would expect the largest to be approximately $F^{\leftarrow}(1 - \frac{1}{n})$. This motivates the following definition which we use very frequently in the paper:

Definition 1. Let F be a distribution function with unbounded support ($F^{\leftarrow}(1) = \infty$). The *inverse quantile function* $U_F : \mathbb{R}_{>1} \to \mathbb{R}$ is

$$U_F(n) := F^{\leftarrow}\left(1 - \frac{1}{n}\right) \ . \tag{4}$$

For a distribution function with with bounded support ($F^{\leftarrow}(1) < \infty$) the inverse quantile function is

$$U_F(n) := F^{\leftarrow}(1) - F^{\leftarrow}\left(1 - \frac{1}{n}\right) \ . \tag{5}$$

Using the function U_F instead of $F^{\leftarrow}$ or F often simplifies notation enormously.

$U_F(n)$ can be interpreted as the value where $\frac{1}{n}$ of the probability mass lies to its right (when $F^{\leftarrow}(1) = \infty$) or the distance of this value from the upper bound of the support (when $F^{\leftarrow}(1) < \infty$). We note that $\lim_{n\to\infty} U_F(n) \in \{0, \infty\}$ (depending on whether $F^{\leftarrow}(1)$ is finite or not), and furthermore U_F is monotone and positive. We often shorten notation and refer to the inverse quantile function by $U(n)$.

2.2 Domains of Attraction

We now introduce the class of distributions for which we give results, and survey some related results of extreme-value theory.

Let $X_1, \ldots, X_n$ be i.i.d. random variables distributed according to some cdf F. The maximum $\max(X_1, \ldots, X_n)$ of such experiments is a random variable whose cdf is F^n. It is a fundamental fact of extreme value theory that, for many distributions, F^n converges to a single distribution after appropriate rescaling and shifting. Figure 1 illustrates this behaviour in one case.

A distribution is in the *domain of attraction* of another distribution whenever this happens, i.e., there exists scaling constants a_n and shifting constants b_n which make the distribution F^n converge to a single distribution (in Figure 1 the constants b_n can be chosen to be 0).

Definition 2 (Domains of attraction). A cdf F is in the domain of attraction of the cdf G if there exists normalizing constants a_n, b_n such that for all x: $\lim_{n\to\infty} F^n(a_n x + b_n) = G(x)$.

The set $\mathcal{D}(G)$ contains of all cdf's which are in the domain of attraction of G.

Gnedenko [12] showed that if F is in the domain of attraction of G, then there aren't so many choices for G. In fact, G must be (a scaled and shifted version of) one of a single family of distributions G_γ, where $\gamma \in \mathbb{R}$ is a parameter. These distributions

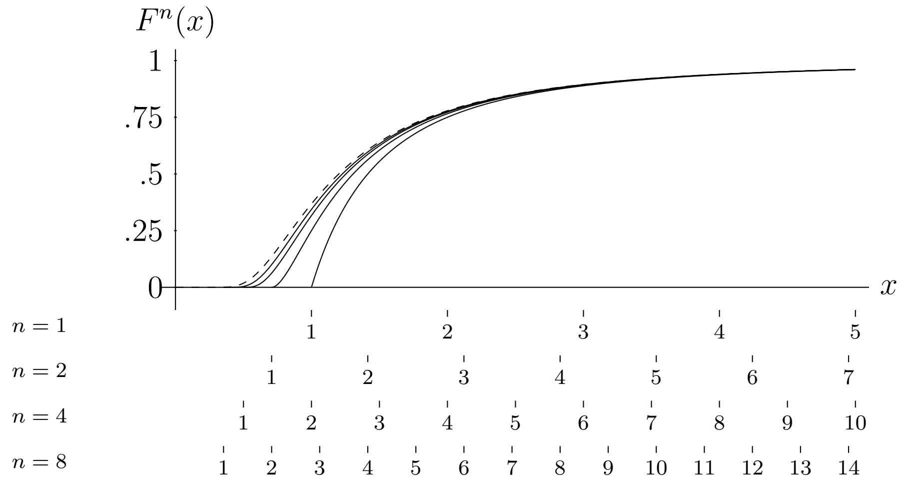

Figure 1: Distribution of the maximum of $n = 1, 2, 4, 8$ i.i.d random variables chosen according to $F(x) = 1 - \frac{1}{x^2}$. The lowest line corresponds to $n = 1$, the second lowest to $n = 2$, and so on. The scale on the x-axis is different for each of the distributions illustrated, and depicted below the graphics. The limiting distribution is drawn dashed, we do not display a scale for it.

are given by

$$G_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma}) \;, \qquad\qquad 1 + \gamma x > 0, \tag{6}$$

where for $\gamma = 0$ this is to be interpreted as $G_0 = \exp(-e^{x})$. In Figure 1, a shifted version of $G_{\frac{1}{2}}$ is indicated with the dashed line.

Almost every standard distribution in the literature is in the domain of attraction of G_γ for some γ. This γ is called the *extreme-value index*, and we give some properties of distributions $F \in \mathcal{D}(G_\gamma)$.

- $\gamma > 0$ implies that the distribution is defined on an unbounded support, i.e., $F(x) < 1$ for all $x \in \mathbb{R}$. Low positive γ implies that F has a lighter tail, and the tail gets heavier when γ increases. We will not be interested in cases where $\gamma > 1$ since for these distributions even $E[X_1]$ does not exist (and, even in the context of posted prices, one can always improve expected revenue by offering a higher posted price). Distributions with $0 < \gamma < 1$ include standard heavy-tail distributions, like power-law, log-normal, Pareto, Burr etc.

- $\gamma < 0$ are distributions on supports which are bounded from above, i.e., $F(b) = 1$ for some $b \in \mathbb{R}$. Examples in the class are all the twice-differentiable functions with positive density, Beta distributions, uniform distributions, etc.

- $\gamma = 0$ is a special case which in some cases requires special treatment. Distributions in $\mathcal{D}(G_0)$ can be bounded or unbounded, and have a light tail (e.g., the exponential distributions and the normal distribution).

Clearly, not every distribution has a density function, and we stress that most of our results are also applicable in such cases. Our results that concern optimal auctions use the concept of virtual valuations (defined by Myerson [20]) that is not well defined unless a density function exists.

2.3 Regular Variation

We now present the family of *regularly varying functions*. It turns out that using this concept we can provide an equivalent, yet more explicit, definition to distributions in the domain of attraction of other distributions (unless $\gamma = 0$). Regularly varying functions are defined as functions that are homogenous of degree γ in the limit.[4]

Definition 3 (Regularly Varying Functions). The set RV_γ, $\gamma \in \mathbb{R}$, consists of the Lebesgue measurable functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ which are eventually positive and satisfy, for all $x \in \mathbb{R}_{>0}$,

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\gamma. \tag{7}$$

We say that f is *regularly varying* with index γ.

The canonical example of a function in RV_γ is x^γ, however, also functions like $x^\gamma \ln(x)$, $x^\gamma / \ln(x)$, and many others are in RV_γ. A fundamental fact of extremal value theory

[4]A complete and excellent introduction to this subject can be found in Appendix B of [8] (which, in turn, has been taken in large parts from [10]).

states that for $\gamma \neq 0$, we have $F \in \mathcal{D}(G_\gamma)$ if and only if $U_F \in \text{RV}_\gamma$ (see Corollary 1.2.10 in [8]), where U_F is as in Definition 1. All our results are given for distributions $F \in \mathcal{D}(G_\gamma)$, and therefore hold for distribution F such that $U_F \in \text{RV}_\gamma$ (when $\gamma \neq 0$).

2.4 Examples

Necessary and sufficient conditions to recognize whether a distributions is in the domain of attraction of some G_γ are known. Here we are satisfied with repeating [8, Theorem 1.1.8][5], which is a simple sufficient condition,[6] due to von Mises [26]. Suppose that the marginal inverse hazard rate converges to a constant γ towards the upper end of the support, i.e., suppose

$$\lim_{x \uparrow F^{\leftarrow}(1)} \Big(\frac{1-F}{f}\Big)'(x) = \gamma. \tag{8}$$

Then, $F \in \mathcal{D}(G_\gamma)$ where γ is the constant appearing in (8).

Following are the leading example distributions that we use in the paper, power-law and uniform distributions. We use them now to demonstrate the above definitions and also in the next section to discuss our general results.

Example 4 (Power-Law Distributions)**.** *Let F be a power-law distribution on $[1, \infty)$ with the form $F(X) = 1 - \frac{1}{x^k}$ where $k > 1$. We get $\frac{1-F}{f} = \frac{x}{k}$, and so (8) converges to $\gamma = \frac{1}{k}$, i.e., F is in the domain of attraction of $G_{\frac{1}{k}}$. Furthermore, $F^{\leftarrow}(y) = (1-y)^{-\frac{1}{k}}$ and the associated function U is $U(n) = n^{\frac{1}{k}}$, therefore U is regularly varying with the index $\frac{1}{k}$.*

Example 5 (Bounded Distributions with Positive Density)**.** *Let F be a twice differentiable distribution over the interval $[0, 1]$ with $f(1) > 0$. One finds that (8) is equivalent to $\lim_{t \to b}\big(-\frac{(1-F(x))f'(x)}{(f(x))^2} - 1\big) = -1$, and so F is in the domain of attraction*

[5]We frequently refer to Theorems in [8], and for the readers convenience repeat them in the supplementary materials.

[6]Condition (8) is not necessary however. For example there are discrete distributions $F \in \mathcal{D}(G_\gamma)$.

of $G_{(-1)}$. In particular, the uniform distribution over $[0,1]$ is in $\mathcal{D}(G_{(-1)})$. For the uniform distribution, one also finds $F^{\leftarrow}(y) = y$ and $U_F(n) = 1/n$, therefore U is regularly varying with the index -1.

3 Exposition of Our Results

This section surveys our formal results and interpret them using examples and figures. Proofs will be given in the sequel of the paper. We consider distributions over unbounded supports ($F \in \mathcal{D}(G_\gamma)$ with $0 < \gamma < 1$) in Section 3.1. Distributions over bounded supports ($\mathcal{D}(G_\gamma)$ with $\gamma < 0$) are discussed in Section 3.2.

3.1 Distributions with unbounded support

We now compare the three families of mechanisms when the bidders valuations are distributed over unbounded supports. The following theorem gives the expected revenue in such mechanisms up to lower order terms in case $0 < \gamma < 1$. We use the standard notation of $o(\cdot)$ to denote low-order terms that we omit.[7] The theorem shows that the expected revenue in each sale method is (again, up to low order terms) $U(c{\cdot}n)$ for some constant c that depends on the parameter γ. The constants are exact in the since that we show mechanisms that achieve such expected revenue and no other mechanism can improve this revenue.

One immediate implication of the theorem is that the revenue in optimal posted-price mechanisms does not converge to the revenue in optimal auctions, even when the number of bidders is very large.

Theorem 6. *Let $F \in \mathcal{D}(G_\gamma)$ for $0 < \gamma < 1$. The optimal expected revenue in the mechanisms that we consider satisfies:*

[7]For example, $o(n)$ denotes functions that are of *strictly* lower asymptotic order than n, e.g., $1000\sqrt{n}$. Similarly, $o(1)$ denotes functions that converge to zero as n grows, e.g., $\frac{1}{\ln n}$.

1. *In symmetric posted-price mechanisms:*

$$r_n^{sym} = U\left(\frac{\phi(\gamma)^{\frac{1}{\gamma}-1}}{(\gamma+\phi(\gamma))^{\frac{1}{\gamma}}}\cdot n\right)(1+o(1)), \tag{9}$$

where $\phi(\gamma)$ *denotes the unique positive solution to the equation* $\exp(x) = 1 + \frac{x}{\gamma}$.[8]

2. *In discriminatory posted-price mechanisms:*

$$r_n^{disc} = U\left((1-\gamma)^{\frac{1-\gamma}{\gamma}}\cdot n\right)(1+o(1)) \ . \tag{10}$$

3. *In optimal auctions, if* F *is additionally Myerson-regular:*

$$r_n^{opt} = U\left((\Gamma(1-\gamma)\cdot(1-\gamma))^{\frac{1}{\gamma}}\cdot n\right)(1+o(1)) \ , \tag{11}$$

where $\Gamma(\cdot)$ *is the gamma function (the extension of the factorial function to real numbers).*[9]

For part 3 of Theorem 6 we additionally need Myerson-regularity (i.e., non-decreasing virtual utility [20]). This is since we prove our results first for social efficiency, and then use Myerson's results regarding virtual valuations to conclude the results for the revenue. We conjecture, however, that our result hold even without Myerson-regularity.[10] Therefore, while incentive issues are in the background in the proofs of the first two parts of the theorem, incentive issues are central in our proof for part 3. In our proof for part 3 of the theorem we prove two lemmas that may be of independent interest; We first prove that a distribution function that is both in $\mathcal{D}(G_\gamma)$ and Myerson-regular must exhibit another property from extreme-value theory called the *von-Mises condition.* Using this lemma, we show the following simple connection

[8] $\phi(\gamma)$ is a decreasing function, Figure 1 in the supplementary materials depicts it.

[9] Formally, $\Gamma(x) = \int_0^\infty t^{z-1}e^{-t}dt$, and for integer x it holds that $\Gamma(x) = (x-1)!$.

[10] We note that our techniques allow us to compare the revenue in posted-price mechanisms to the optimal *efficiency* without requiring monotone virtual values.

between revenue and efficiency in our model, namely, that the optimal revenue is a 1-γ fraction of the optimal efficiency. While this seems to be a fundamental fact, we are not aware of any existing literature showing it. (An analogous result is proven for setting with bounded supports; more details in Section 6.)

Lemma 7. *Consider distribution $F \in \mathcal{D}(G_\gamma), 0 < \gamma < 1$, which is also Myerson-regular. Let $E[X_{n,n}]$ denote the expected highest order statistic of n draws from F, and let $E[X_{n-1,n}]$ denote the expected second-highest order statistic. Then,*

$$E[X_{n-1,n}] = (1-\gamma)E[X_{n,n}](1+o(1)) \ . \tag{12}$$

We will now apply Theorem 6 for a specific distribution.

Example 8 (Power-law distributions)**.** *Consider the setting in Example 4 and recall that for power-law distributions of the form $F(x) = 1 - \frac{1}{x^k}$ we have $\gamma = \frac{1}{k}$ and $U(n) = n^{\frac{1}{k}}$.*

- **Symmetric posted prices**. *Substituting in Eq. (1) shows that $r_n^{sym} \cong \frac{\phi(\frac{1}{k})^{\frac{k}{k-1}}}{\frac{1}{k}+\phi(\frac{1}{k})} n^{\frac{1}{k}}$. When $k = 2$, it is easy to verify that $\phi(\frac{1}{2}) \cong 1.26$, and therefore $r_n^{sym} \cong 0.64\sqrt{n}$.*
- **Discriminatory posted prices.** *By Eq. (2) we get $r_n^{disc} \cong \left(1 - \frac{1}{k}\right)^{\frac{k-1}{k}} \cdot n^{\frac{1}{k}}$. When $k = 2$, we thus have that $r_n^{disc} \cong 0.71\sqrt{n}$.*
- **Optimal auctions.** *By Eq. (3), $r_n^{opt} \cong \Gamma(1-\frac{1}{k}) \cdot (1-\frac{1}{k}) \cdot n^{\frac{1}{k}}$. When $k = 2$, we have that $\Gamma(0.5) \cong 1.772$, and thus $r_n^{opt} \cong 0.89\sqrt{n}$.*

In our proofs, we also present posted prices that achieve the optimal asymptotic revenue. Like the revenue, these prices are also presented in terms of the index γ. Prices that achieve the revenue in Theorem 6 are $p_n^{sym} = U(\frac{n}{\phi(\gamma)})$ in the symmetric posted-price case. In the discriminatory setting, one can use prices $p_i^{disc} = U(\frac{(1-\gamma)^{-1/\gamma}}{1-F(r_{i-1}^{disc})})$, where r_{i-1}^{disc} is the expected revenue achieved with $i-1$ bidders. Inserting our asymptotic formula for r_{i-1}^{disc} reveals that $p_i^{disc} = U(\frac{i}{1-\gamma})(1+o(1))$ as

$i \to \infty$. In the power-law distribution example, these prices will be $p_n^{sym} \cong 0.89\sqrt{n}$ and $p_i^{disc} \cong \sqrt{2}\sqrt{i}$ $(i = 1, ..., n)$, respectively. Note that these prices are not *the* optimal prices, but they are good enough to generate the revenue in Theorem 6 up to a factor of $1 + o(1)$.

3.1.1 Revenue Comparison

Theorem 6 allows us to compare the revenue in each of the sale methods we consider. For instance, for the power-law distribution from Example 8, when $k = 2$, symmetric posted price achieve at most 72% of the revenue in the optimal auction, and discriminatory prices achieve no more than 80% of the optimum. This comparison can be immediately generalized using Theorem 6. We use the fact that $U(cn) = c^\gamma U(n)(1 + o(1))$ (since $\lim_{n\to\infty} U(cn)/U(n) = c^\gamma$) to calculate the fraction of the optimal-auction revenue that can be obtained with posted prices. The left diagram in Figure 2 draws r_n^{sym}/r_n^{opt} and r_n^{disc}/r_n^{opt} (at the limit when $n \to \infty$) as a function of the index γ. It turns out that both diagrams are U-shaped, which can be interpreted as follows. As mentioned in the introduction, intuitively, optimal auctions should perform better in comparison to posted price mechanisms as the uncertainty on the values intensifies, i.e., as $\gamma \to 1$. Allocation mistakes seem to be more severe when the values are more scattered on the support, and posted prices will unavoidably make such errors. This intuition is indeed supported in Figure 2, but only as long as the distribution is sufficiently light-tailed (until $\gamma \cong 0.68$ or $\gamma \cong 0.60$ for discriminatory and symmetric prices, respectively). For distributions with heavier tails, we observe an opposite trend and the posted-price revenue becomes closer to the optimal revenue as γ grows. This can be explained by a different effect of the tail becoming thicker - better separation between the order statistics. With larger γ, the expected highest-order statistic becomes much larger than the expected second order statistic. Actually, we mentioned that the expected 2nd-highest order statistic is about a $(1-\gamma)-$fraction of the expected highest order statistic, see Section 6. This

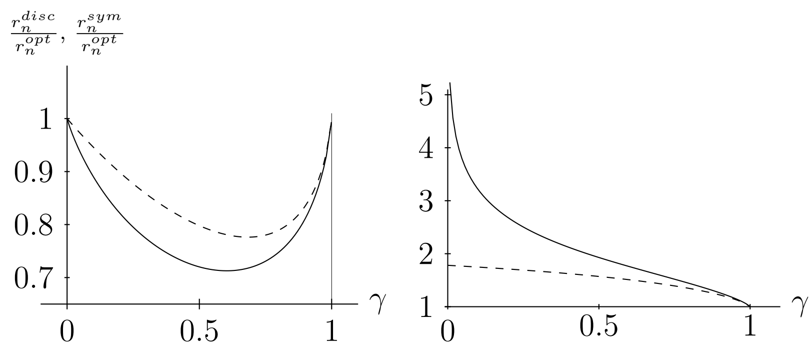

Figure 2: Interpretation of Theorem 6. On the left, we draw the fraction of the revenue achieved by posted prices, symmetric or discriminatory, associated with distributions in the domain of attraction of G_γ, $0 < \gamma < 1$, as a function of γ. More formally, the dashed line is $\lim_{n\to\infty} r_n^{disc}/r_n^{opt}$, and the solid line is $\lim_{n\to\infty} r_n^{sym}/r_n^{opt}$. On the right, the figure shows what number of bidders (as a multiple of n) is required in each posted-price mechanism to achieve the optimal-auction revenue r_n^{opt} (dashed line represents discriminatory prices, and the solid line is for symmetric ones).

enables the seller to post a price that is much larger than the expected 2nd-highest order statistic, and with sufficiently high probability this price will be accepted by at least one of the bidders. With light tail, however, this pricing strategy is impossible as the 2nd-order statistic is close to the highest order statistic and a price that is much higher than these values would be rejected with a probability that is too high.

An immediate corollary from Theorem 6, as seen in Figure 2, is that optimal symmetric (resp., discriminatory) posted-price mechanisms always achieve at least 71% (resp., 78%) of the optimal auction revenue.

Corollary 9. *For every $F \in \mathcal{D}(G_\gamma)$, $0 < \gamma < 1$ we have $\lim_{n\to\infty} \frac{r_n^{sym}}{r_n^{opt}} \geq 0.71$ and $\lim_{n\to\infty} \frac{r_n^{disc}}{r_n^{opt}} \geq 0.78$ and the bounds are tight for distributions with $\gamma = 0.60$ and $\gamma = 0.68$, respectively (up to two digit precision).*

3.1.2 Comparison by Market Expansion

Another form of comparison that our results allow, in the spirit of the work of Bulow and Klemperer [4], is by describing the *market expansion* required for posted-price mechanisms to achieve the optimal-auction revenue. In example 8, we saw that the optimal revenue for the distribution $F(x) = 1 - 1/x^2$ was about $\sqrt{0.41n}, \sqrt{0.5n}, \sqrt{0.77n}$ for symmetric-, discriminatory posted prices and the optimal auction, respectively; that is, one would need $1.89 \cdot n$ bidders in a symmetric posted-price mechanisms to achieve the optimal-auction revenue with n bidders. Similarly, discriminatory posted prices with $1.54 \cdot n$ bidders can achieve the optimal-auction revenue with n bidders. Again, this can be generalized using Theorem 6. The n-bidder optimal-auction revenue can be achieved in symmetric and discriminatory posted-price mechanisms with $\frac{(\gamma+\phi(\gamma))^{\frac{1}{\gamma}}(\Gamma(1-\gamma)\cdot(1-\gamma))^{\frac{1}{\gamma}}}{\phi(\gamma)^{\frac{1}{\gamma}-1}} n$ bidders and $\Gamma(1-\gamma)^{\frac{1}{\gamma}}(1-\gamma)n$ bidders, respectively. Figure 2 (right diagram) depicts these terms as a function of γ. Using this measure to compare the effectiveness of posted-price mechanisms, we can see that a larger market expansion is required when the tail of the distribution becomes lighter. This phenomena is monotonic in γ, both for symmetric and discriminatory posted prices, and it appears to be in contrast to the intuition that posted prices should perform better compared to auctions as the distribution is more dispersed. Note that for light-tail distributions, although the overall posted-price revenue is very close to the optimal-auction revenue (see Figure 2, left side), a large expansion of the market is needed for posted prices to achieve the optimal-auction revenue. In fact, this quotient can be arbitrarily big for symmetric prices.[11] With discriminatory prices, a market expansion of at most a factor of 1.79 is sufficient[12] for every γ.

Remark 10. ***The special case*** $\gamma = 0$***: Distribution with light tails.*** *The case* $\gamma = 0$ *can occur in distributions with both bounded and unbounded support.*[13] *We only provide a simple result for unbounded support, which follows almost immediately from known results. The proof is given in the supplementary materials. The result*

[11]This is because $\Gamma(1) = 1$ and

$$\lim_{\gamma \downarrow 0} \frac{\phi(\gamma)^{\frac{1}{\gamma}-1}}{(\gamma+\phi(\gamma))^{\frac{1}{\gamma}}} = \lim_{\gamma \downarrow 0} \frac{\phi(\gamma)^{\frac{1}{\gamma}-1}}{(\phi(\gamma)(1+\frac{\gamma}{\phi(\gamma)}))^{\frac{1}{\gamma}}} = \lim_{\gamma \downarrow 0} \frac{\phi(\gamma)^{\frac{1}{\gamma}-1}}{(\phi(\gamma))^{\frac{1}{\gamma}} \exp(\frac{1}{\phi(\gamma)})} = \lim_{\gamma \downarrow 0} \frac{1}{\phi(\gamma)\exp(\frac{1}{\phi(\gamma)})} = 0.$$

[12]The exact constant is $\exp(\gamma_{EM})$, where $\gamma_{EM} \approx 0.577$ is the Euler-Mascheroni constant.

[13]In case $\gamma = 0$ and $F^{\leftarrow}(1) < \infty$ we do not provide any results. One such distribution is $F(x) = 1 - (1-x)^{1/(1-x)}$; We are not familiar with "natural" distributions in this family.

says that for such distributions the revenue in all three sale methods that we explore is asymptotically the same. That is, when $F \in \mathcal{D}(G_0)$ and $F^{\leftarrow}(1) = \infty$, the optimal expected revenue satisfies $U(n)(1 + o(1)) = r_n^{opt} \geq r_n^{disc} \geq r_n^{sym} = U(n)(1 + o(1))$.

3.2 Distributions with bounded support

We will now characterize the expected revenue when the bidders' values are distributed on a bounded interval and $\gamma < 0$. With bounded supports, the expected revenue clearly converges to the upper bound of the support in all three mechanisms when the number of bidders grows; however, we will show that the convergence rate in symmetric posted-price mechanisms is asymptotically slower than in optimal auctions by a factor which is logarithmic in the number of bidders. With discriminatory prices, on the other hand, the convergence rate is similar to the rate in an optimal auction (up to a constant factor).

Theorem 11. *Let $F \in \mathcal{D}(G_\gamma)$ for $\gamma < 0$, $0 < F^{\leftarrow}(1) < \infty$. The optimal expected revenue in the mechanisms that we consider satisfies:*

1. *In symmetric posted price mechanisms:*

$$r_n^{sym} = F^{\leftarrow}(1) - U\Big(-\frac{1}{\gamma} \cdot \frac{n}{\ln(n)}\Big)(1 + o(1)) \ . \tag{13}$$

2. *In discriminatory posted-price mechanisms:*

$$r_n^{disc} = F^{\leftarrow}(1) - U\left((1-\gamma)^{\frac{1-\gamma}{\gamma}} \cdot n\right)(1 + o(1)) \ . \tag{14}$$

3. *In optimal full-revelation auction, if F is additionally Myerson-regular:*

$$r_n^{opt} = F^{\leftarrow}(1) - U\big((\Gamma(1-\gamma) \cdot (1-\gamma))^{\frac{1}{\gamma}} \cdot n\big)(1 + o(1)) \ , \tag{15}$$

where, again, $\Gamma(\cdot)$ denotes the Gamma function.

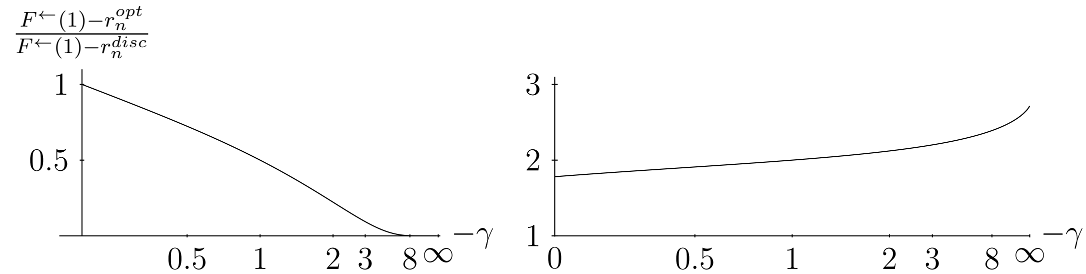

Figure 3: Interpretation of Theorem 11. On the left, we draw the fraction of the gap between the revenue and the upper bound, achieved by a discriminatory posted prices compared to the one in optimal auctions, dependent on γ. On the right, the figure shows the number of bidders (as a multiple of n) which is required to achieve the optimal-auction revenue r_n^{opt} using a discriminatory posted price mechanism. We cannot show symmetric prices, as the factors lost are superconstant. For mapping the full range of γ to the graphics, we used a scale of $\arctan(\cdot)$ on the horizontal axis.

Example 12 (Uniform distributions). *Recall that for the uniform distributions on the interval* $[0, 1]$ *(*$F(x) = x$*) we have* $\gamma = -1$ *and* $U(n) = \frac{1}{n}$.

- **Symmetric posted prices.** *By (13),* $r_n^{sym} \cong 1 - \frac{\ln(n)}{n}$.
- **Discriminatory posted prices.** *From (14) and since* $(1-\gamma)^{\frac{1-\gamma}{\gamma}} = \frac{1}{4}$ *for* $\gamma = -1$, $r_n^{disc} \cong 1 - \frac{4}{n}$.
- **Optimal auctions.** *By (15) (note that* $\Gamma(2) = 1$*) we have,*
$r_n^{opt} \cong 1 - U\left((\Gamma(2) \cdot 2)^{-1} \cdot n\right) \cong 1 - \frac{2}{n}$.

We also present pricing methods for obtaining the asymptotically optimal revenue shown in Theorem 11. The symmetric posted price[14] of $p_n^{sym} = b - U\left(\frac{1}{1-(U(n)/b)^{1/n}}\right)$ and the discriminatory prices $p_i^{disc} = F^{\leftarrow}(1) - U(\frac{(1-\gamma)^{-1/\gamma}}{1-F(r_{i-1}^{disc})})$ will do the job. Similarly to before, the discriminatory prices can be seen to be about $p_i^{disc} \cong F^{\leftarrow}(1) - U(\frac{i}{1-\gamma})$ In the uniform-distribution example, these prices will be $p_n^{sym} = (\frac{1}{n})^{1/n} \approx 1 - \frac{\ln(n)}{n}$ and $p_i^{disc} \cong 1 - \frac{2}{i}$ $(i = 1, ..., n)$, respectively.

We can also compare the expected revenue via the size of the market. Symmetric posted prices achieve only the optimal-auction revenue with a market smaller by

[14] This price p_n^{sym} will do the job when F is continuous, otherwise slightly more complex prices are needed (see Lemma 22 in Section 5.3).

a factor of $-\gamma(\Gamma(1-\gamma)(1-\gamma))^{1/\gamma}\ln(n)$. Note that this factor is bigger than any constant, due to the $\ln(n)$ term. Conversely, the revenue of an optimal auction can be obtained by discriminatory posted prices if the market is expanded by a constant factor (depending only on γ), which is between 1.78 and e for all $\gamma < 0$. The right diagram in Figure 3 shows this revenue comparison as a function of γ.

4 Discriminatory Posted Prices – Proofs

In this section we jointly prove Theorems 6.2 and 11.2. The proof is given in four steps. We first argue that we can study a smooth version of the function U, and then present the solution to an optimization problem that actually models the price determination problem at each stage. We then solve a certain recursive formula, and finally, we show how the theorems can be concluded from the above steps.

4.1 Making U smooth

We now show that for a given function $U \in \mathrm{RV}_\gamma$ one can find a function $\widehat{U} \in \mathrm{RV}_\gamma$ which is differentiable, tail-equivalent, and satisfies some additional "smoothness" condition (as given in (17)). The lemma can be considered to be known (see, for example, [7] Chapter 10), but we were unable to find an explicit proof in the literature. We therefore provide one in the supplementary materials.

Lemma 13. *Let $U \in \mathrm{RV}_\gamma$. There exists a surjective, differentiable function $\widehat{U}$ such that*

$$\lim_{x\to\infty} \frac{\widehat{U}(x)}{U(x)} = 1 \tag{16}$$

and

$$\lim_{x\to\infty} \frac{x\widehat{U}'(x)}{\widehat{U}(x)} = \gamma. \tag{17}$$

The difficult part is to get (16) and (17). Once one has this, it is easy to change $\widehat{U}$ to make it surjective without changing the tail.

We use the above lemma as follows: we would like to say that the revenue with n bidders is roughly $U_F(c_\gamma n)$, for some constant c_γ (whose value is irrelevant for this discussion). Unfortunately, the function U does not have to behave nicely, and for example, it is possible that $U(c_\gamma n) = U(c_\gamma(n+1))$ for some n. As we aim to show the bounds by induction, this would make it hard to get a meaningful step from n to $n+1$. Condition (17) excludes this possibility, and we will in fact measure the revenue as $\widehat{U}(c_\gamma n)$, so that we can do an induction of the revenue with $\widehat{U}$. Furthermore, because $U(c_\gamma n) = \widehat{U}(c_\gamma n)(1 + o(1))$ as $n \to \infty$ this is asymptotically the same.

4.2 An optimization problem

The second lemma gives the solution to an optimization problem which shows up in the proof of Theorems 6.2 and 11.2. One can think of it as follows: suppose $U \in \mathrm{RV}_\gamma$, $0 < \gamma < 1$, and one wants to maximize $\frac{1}{t}\big(\frac{U(tx)}{U(x)} - 1\big)$. Then, pretending that $U(x) = x^\gamma$ will give the right solution as $x \to \infty$ (the solution is easily calculated to be the expression given). In fact, the lemma states somewhat more: the maximization is robust even if there are some minor terms ℓ_1 and ℓ_2 which perturb it.

Lemma 14. *Let $U \in \mathrm{RV}_\gamma$, $\gamma < 1$, $\gamma \neq 0$, and let positive functions $\ell_1(z)$ and $\ell_2(z)$ be given with $\lim_{z\to\infty} \ell_1(z) = \lim_{z\to\infty} \ell_2(z) = 1$.*

If $0 < \gamma < 1$ then

$$\lim_{x\to\infty} \sup_{t>0} \frac{\ell_1(tx)}{t}\Big(\frac{U(tx)\ell_2(x)}{U(x)} - 1\Big)\Big) = \gamma(1-\gamma)^{\frac{1-\gamma}{\gamma}} . \tag{18}$$

If $\gamma < 0$, then Eq. 18 holds when the sup *operator is replaced by* inf*.*

Proof. First note that by setting $t = t_0 := (1-\gamma)^{-1/\gamma}$ we can achieve

$$\lim_{x\to\infty} \frac{\ell_1(t_0 x)}{t_0}\Big(\frac{U(t_0 x)\ell_2(x)}{U(x)} - 1\Big)\Big) = \frac{1}{t_0}\Big(t_0^\gamma - 1\Big) = \gamma(1-\gamma)^{\frac{1-\gamma}{\gamma}} . \tag{19}$$

If we could exchange the lim and the sup (respectively inf) operator in (18), the lemma would follow easily. However, we may not do so immediately, because $\lim_{x\to\infty} \frac{\ell_1(tx)}{t}\left(\frac{U(tx)\ell_2(x)}{U(x)} - 1\right)$ does *not* converge uniformly for all $t > 0$.

However, from the uniform convergence theorem (Theorem B.1.4 in [8]) we see that $\lim_{x\to\infty} \frac{\ell_1(tx)}{t}\left(\frac{U(tx)\ell_2(x)}{U(x)} - 1\right)$ *does* converge uniformly for $t \in [a, b]$ for any $0 < a < b < \infty$. Thus, we first show that it is sufficient to optimize over $t \in [a, b]$ for *some* constants a and b.

For this, we first see that we can use $\frac{1}{2}$ as a lower bound. In case $\gamma > 0$, for $t < \frac{1}{2}$ we get $\frac{U(tx)\ell_2(x)}{U(x)} - 1 \leq \frac{U(\frac{1}{2}x)\ell_2(x)}{U(x)} - 1$ which is negative for large enough x. Since ℓ_1 is non-negative, the same holds for $\frac{\ell_1(tx)}{t}\left(\frac{U(tx)\ell_2(x)}{U(x)} - 1\right)$. Using that (19) shows we can obtain a positive value, we exclude $t < \frac{1}{2}$. Analogously, we can exclude $t < \frac{1}{2}$ in case $\gamma < 0$: in this case the expression would be positive.

Now we give the upper bound. Again, assume first $\gamma > 0$. We apply Prop. B.1.9(5) from [8] for $\epsilon = \frac{1-\gamma}{2}$ and get, for any $x > x_0$ (where x_0 only depends on ℓ_1,ℓ_2, and U)

$$\frac{\ell_1(tx)}{t}\left(\frac{U(tx)\ell_2(x)}{U(x)} - 1\right) \leq \frac{2}{t}\frac{2U(tx)}{U(x)} \leq \frac{4}{t}\frac{3-\gamma}{2}t^{\gamma+\frac{1-\gamma}{2}} \leq 12t^{-\frac{1-\gamma}{2}}. \tag{20}$$

This goes to zero as t goes to infinity, and so in particular this expression is smaller than the one from (19) for $t > t_\gamma$. In case $\gamma < 0$ we get, similarly (using $\epsilon = \frac{1}{3}$)

$$\frac{\ell_1(tx)}{t}\left(\frac{U(tx)\ell_2(x)}{U(x)} - 1\right) \geq \frac{1}{2t}\left(\frac{1}{2}t^{\gamma-\frac{1}{3}} - 1\right) \geq -\frac{1}{2t} \tag{21}$$

which is bigger than (19) for $t > t_\gamma$.

With this, we can now conclude, in case $\gamma > 0$:

$$\lim_{x\to\infty}\sup_{t>0}\frac{\ell_1(tx)}{t}\left(\frac{U(tx)\ell_2(x)}{U(x)} - 1\right)\Big) = \lim_{x\to\infty}\sup_{t\in[\frac{1}{2},t_\gamma]}\frac{\ell_1(tx)}{t}\left(\frac{U(tx)\ell_2(x)}{U(x)} - 1\right)\Big) \tag{22}$$

$$= \sup_{t\in[\frac{1}{2},t_\gamma]}\lim_{x\to\infty}\frac{\ell_1(tx)}{t}\left(\frac{U(tx)\ell_2(x)}{U(x)} - 1\right)\Big) \tag{23}$$

$$= \sup_{t\in[\frac{1}{2},t_\gamma]} \frac{1}{t}(t^\gamma - 1) \tag{24}$$

$$= \gamma(1-\gamma)^{\frac{1-\gamma}{\gamma}}. \tag{25}$$

Here, (23) follows by the uniform convergence theorem (note that it is clear that $\ell_1(tx)$ converges uniformly for $t \in [\frac{1}{2}, t_\gamma]$), (24) by the definition of RV_γ, and (25) because the respective derivative $(\gamma-1)t^{\gamma-2} + t^{-2}$ is zero if and only if $t = (1-\gamma)^{-1/\gamma}$. The case $\gamma < 0$ is finished by replacing sup with inf in the above calculation. □

4.3 A recursion

It is quite straightforward to see that the discriminatory-price problem has a recursive structure: a price is offered to the first bidder, where $n-1$ bidders remain if he rejects the offer; The second offer is made to the second bidder, and $n-2$ bidders remain if he rejects, etc. Optimizing over this recursive structure will lead to the next lemma.

Lemma 15. *Let f be a positive differentiable function with $\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = \gamma \neq 0$, and let $\{s_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ be sequences with $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} c_n = c \in \mathbb{R}$, and $f(s_{n+1}) = f(s_n)(1 + \frac{c_n}{s_n})$. Then, $\lim_{n\to\infty} \frac{s_n}{n} = \frac{c}{\gamma}$.*

Proof. Using the given recursion we see that for any $\gamma' > \gamma$ and big enough n:

$$\ln\Big(1 + \frac{c_n}{s_n}\Big) = \ln\Big(\frac{f(s_{n+1})}{f(s_n)}\Big) = \int_{s_n}^{s_{n+1}} \big(\ln(f(x))\big)' dx \tag{26}$$

$$= \int_{s_n}^{s_{n+1}} \frac{f'(x)}{f(x)} dx \leq \int_{s_n}^{s_{n+1}} \frac{\gamma'}{x} dx = \gamma' \ln\Big(\frac{s_{n+1}}{s_n}\Big), \tag{27}$$

and an analogous bound with $\leq$ replaced by $\geq$ holds in case $\gamma' < \gamma$. Equivalently, we can state

$$s_{n+1} - s_n = s_n \cdot \Big(\frac{s_{n+1}}{s_n} - 1\Big) \geq s_n \cdot \Big((1 + \frac{c_n}{s_n})^{\frac{1}{\gamma'}} - 1\Big), \tag{28}$$

for $\gamma' > \gamma > 0$ or $\gamma' < \gamma < 0$, and also $s_{n+1} - s_n \leq s_n((1 + \frac{c_n}{s_n})^{\frac{1}{\gamma'}} - 1)$ if $\gamma > \gamma' > 0$ or

$\gamma < \gamma' < 0$.

Let $f_{n,\gamma'}(x) = (1 + c_n x)^{1/\gamma'} - 1$. Taylor's theorem with Lagrange remainder gives $f_{n,\gamma'}(\frac{1}{s_n}) = f_{n,\gamma'}(0) + \frac{f'_{n,\gamma'}(0)}{s_n} + \frac{f''_{n,\gamma'}(\xi_n)}{2s_n^2} = \frac{c_n}{\gamma' s_n} + \frac{f''_{n,\gamma'}(\xi_n)}{2s_n^2}$ for some $\xi_n \in [0, \frac{1}{s_n}]$. Thus, for $\gamma' > \gamma > 0$ and $\gamma' < \gamma < 0$ we have $s_{n+1} - s_n \geq s_n \cdot f_{n,\gamma'}(\frac{1}{s_n}) = \frac{c_n}{\gamma'} + \frac{f''_{n,\gamma'}(\xi_n)}{2s_n}$. This converges to $\frac{c}{\gamma'}$ as $n \to \infty$, because $f''_{n,\gamma'}(x) = \frac{c_n^2}{\gamma'}(\frac{1}{\gamma'} - 1)(1 + c_n x)^{\frac{1}{\gamma'}-2}$ (and note that $(1+c_n\xi_n) \in [\frac{1}{2}, 2]$ for n large enough). We get a similar lower bound for $\gamma > \gamma' > 0$ and $\gamma < \gamma' < 0$, which implies $\lim_{n\to\infty} s_{n+1} - s_n = \frac{c}{\gamma}$. From this we can easily obtain that $s_n \leq n\frac{c}{\gamma}(1+\epsilon)$ for any $\epsilon > 0$ and n big enough: we pick n_0 such that $s_{n+1} - s_n \leq \frac{c}{\gamma}(1+\frac{\epsilon}{2})$ for $n > n_0$; then $s_n \leq s_{n_0} + (n - n_0)\frac{c}{\gamma}(1 + \frac{\epsilon}{2}) \leq n\frac{c}{\gamma}(1 + \frac{\epsilon}{2}) + s_{n_0} \leq n\frac{c}{\gamma}(1 + \epsilon)$, as long as n is sufficiently big. A similar calculation gives a lower bound of $s_n \geq n\frac{c}{\gamma}(1 - \epsilon)$, which proves the lemma. □

4.4 Assembling the Parts

Having the machinery, we are now ready to prove Theorems 6.2 and 11.2, claiming that, up to low-order terms, the optimal revenue in discriminatory-price mechanisms is $U\big((1-\gamma)^{\frac{1-\gamma}{\gamma}} n\big)$ if $0 < \gamma < 1$ and $F^{\leftarrow}(1) - U\big((1-\gamma)^{\frac{1-\gamma}{\gamma}} n\big)$ if $\gamma < 0$.

The result follows from the following lemma, whose proof starts with presenting a recursive formula for the optimal revenue with $n + 1$ bidders. We then change some variables and notations to present this recursion using the U functions, which enables us to use the above lemmas.

Lemma 16. *Let $F \in \mathcal{D}(G_\gamma)$ and let $\widehat{U} = \widehat{U}(U_F)$ be as in Lemma 13. Then the expected revenue in an optimal discriminatory posted-price mechanism satisfies*

$$r_n^{disc} = \widehat{U}\big((1-\gamma)^{\frac{1-\gamma}{\gamma}} n\big)(1 + o(1)) \qquad \text{if } 0 < \gamma < 1 \tag{29}$$

$$r_n^{disc} = F^{\leftarrow}(1) - \widehat{U}\big((1-\gamma)^{\frac{1-\gamma}{\gamma}} n\big)(1 + o(1)) \qquad \text{if } \gamma < 0. \tag{30}$$

Proof. Consider first the case $\gamma > 0$, and let r_n be the expected revenue which an auctioneer obtains from n bidders (for example by using an optimal pricing strategy, or any other strategy). If the bidder offers a price of p to the first out of $n+1$ bidders, this bidder will accept with probability $1-F(p)$, generating a revenue of p. Otherwise, the bidder gets an expected revenue of r_n. The auctioneer can simply maximize his revenue in every step. Thus, we have the recursion

$$\begin{aligned} r_{n+1} &= \sup_{p>0}\Big((1-F(p))\cdot p + F(p)\cdot r_n\Big) && (31)\\ &= r_n \cdot \left(1+\sup_{p>0}\left((1-F(p))\Big(\frac{p}{r_n}-1\Big)\right)\right) \ , && (32) \end{aligned}$$

where we picked the representation in (32) with some foresight. We write the revenue as $r_n = \widehat{U}(s_n)$, which is always possible since $\widehat{U}$ is surjective. We can parametrize the price[15] as $p = U(ts_n)$ and get, if $\gamma > 0$,

$$\widehat{U}(s_{n+1}) = \widehat{U}(s_n)\cdot\left(1+\sup_{t>0}\left(\Big(1-F\big(U(ts_n)\big)\Big)\Big(\frac{U(ts_n)}{\widehat{U}(s_n)}-1\Big)\right)\right) \ , \qquad (33)$$

which we can write as

$$\widehat{U}(s_{n+1}) = \widehat{U}(s_n)\cdot\left(1+\sup_{t>0}\left(\frac{\ell_1(ts_n)}{ts_n}\Big(\frac{U(ts_n)\ell_2(s_n)}{U(s_n)}-1\Big)\right)\right) \ , \qquad (34)$$

for $\ell_1(z) = z(1-F(U(z))$ and $\ell_2(z) = \frac{\widehat{U}(z)}{U(z)}$. Both ℓ_1 and ℓ_2 are positive and go to 1 in the limit: for ℓ_1 we use [8, Theorem 1.2.1] and [8, Proposition B.1.9(10)](applied on the function $x \mapsto \frac{1}{1-F(x)}$ which is in $\mathrm{RV}_{1/\gamma}$), for ℓ_2 this is given in Lemma 13. We

[15] We sometimes exploit the fact that only prices of the form $F^{\leftarrow}(y)$ can be optimal. In the mechanisms we study, a buyer with valuation v accepts a price p if $p \leq v$. In case the probability that two different prices $p_1 < p_2$ are accepted is equal for both, we can assume that the seller will always choose the bigger one. The largest value accepted with probability at least $1-y$ equals $\sup_x\{x : \mu([x,\infty)) \geq 1-y\} = \sup_x\{x : \mu((-\infty,x)) \leq y\} = F^{\leftarrow}(y)$, and we see that it is attained if $0<y<1$. Therefore, it is sufficient to consider prices of the form $F^{\leftarrow}(y)$. Furthermore, for $p = F^{\leftarrow}(y)$ we have $F^{\leftarrow}(F(p)) = p$ (this follows from the fact that $F^{\leftarrow}(F(x)) \leq x$ and applying the monotonically increasing $F^{\leftarrow}(\cdot)$ on both sides of $F(F^{\leftarrow}(y)) \geq y$ (both inequalities are well known, see, e.g., [7], page 15)). Thus we can assume that $F^{\leftarrow}(F(p)) = p$ holds for prices offered.

can thus apply Lemma 14 and get the recursion

$$\widehat{U}(s_{n+1}) = \widehat{U}(s_n) \cdot \left(1 + \frac{c_n}{s_n}\right) \tag{35}$$

for some sequence $\{c_n\}_{n\geq 0}$ which converges to $\gamma(1-\gamma)^{\frac{1-\gamma}{\gamma}}$ as $n \to \infty$. Applying Lemma 15 gives that $s_{n+1} = n \cdot (1-\gamma)^{\frac{1-\gamma}{\gamma}}(1+o(1))$, as $n \to \infty$, and thus

$$r_n^{disc} = \widehat{U}((1-\gamma)^{\frac{1-\gamma}{\gamma}} n)(1+o(1)). \tag{36}$$

In case $\gamma < 0$ it is best think of the auctioneer as trying to minimize the gap of his revenue to $F^{\leftarrow}(1)$, i.e., minimizing $F^{\leftarrow}(1) - r_n$. He offers a bidder some new gap, who accepts, with some probability. This gives the following recursion (which is, of course, equivalent to (31)):

$$F^{\leftarrow}(1) - r_{n+1} = \inf_p \Big((1 - F(p)) \cdot (F^{\leftarrow}(1) - p) + F(p) \cdot (F^{\leftarrow}(1) - r_n) \Big) \tag{37}$$

$$= \big(F^{\leftarrow}(1) - r_n\big) \cdot \left(1 + \inf_p \left((1 - F(p)) \Big(\frac{F^{\leftarrow}(1) - p}{F^{\leftarrow}(1) - r_n} - 1 \Big) \right) \right) . \tag{38}$$

We now write $F^{\leftarrow}(1) - r_n = \widehat{U}(s_n)$ and $F^{\leftarrow}(1) - p = U(ts_n)$, which gives

$$\widehat{U}(s_{n+1}) = \widehat{U}(s_n) \cdot \left(1 + \inf_p \left((1 - F(F^{\leftarrow}(1) - U(ts_n))) \Big(\frac{U(ts_n)}{\widehat{U}(s_n)} - 1 \Big) \right) \right) . \tag{39}$$

And as before we apply Lemmas 14 and 15 which gives the result. □

Proof (of Theorems 6.2 and 11.2). Immediately from Lemma 16. □

5 Symmetric Posted Prices – Proofs

Unlike the discriminatory price setting, our result for symmetric posted-price mechanisms are proven in separate proofs for distributions with bounded and unbounded support. We start by presenting some technical tools that we need for the proofs.

5.1 Some Technical Tools

We define the function, $\widetilde{U} : \mathbb{R}_{>0} \to \mathbb{R}$ as

$$\widetilde{U}(n) := U\Big(\frac{1}{1 - \exp(-1/n)}\Big). \tag{40}$$

By Definition 1, in case $F^{\leftarrow}(1) = \infty$ this means $\widetilde{U}(n) = F^{\leftarrow}(\exp(-1/n))$, and in case $F^{\leftarrow}(1) < \infty$, $\widetilde{U}(n) = F^{\leftarrow}(1) - F^{\leftarrow}(\exp(-1/n))$.

If U is regularly varying, $1 - \frac{1}{n} \approx e^{-1/n}$ as $n \to \infty$, implies that $\widetilde{U}$ and U behave roughly the same. We show this in the next lemma.

Lemma 17. *Let $U \in \mathrm{RV}_\gamma$, $\widetilde{U}$ as in (40). Then, $\widetilde{U} \in \mathrm{RV}_\gamma$ and*

$$\lim_{n\to\infty} \frac{U(n)}{\widetilde{U}(n)} = 1 \tag{41}$$

Proof. Using Proposition B.1.9(5) from [8], for any $\epsilon > 0$ (using that $n(1 - e^{-1/n}) \leq n(1 - (1 - \frac{1}{n})) = 1$):

$$\lim_{n\to\infty} \frac{U(n)}{\widetilde{U}(n)} = \lim_{n\to\infty} \frac{U(n)}{U\Big(\frac{1}{1-e^{-1/n}}\Big)} < (1+\epsilon)\Big(n(1 - e^{-1/n})\Big)^{\gamma+\epsilon}, \tag{42}$$

and using the l'Hôpital-rule, we get $\lim_{n\to\infty} n(1-e^{-1/n}) = \lim_{n\to\infty} \frac{1-e^{-1/n}}{1/n} = \lim_{n\to\infty} \frac{e^{-1/n}1/n^2}{-1/n^2} = \lim_{n\to\infty} -e^{-1/n} = 1$. A similar calculation gives $\lim \frac{U(n)}{\widetilde{U}(n)} \geq 1$, which implies (41).

To see that $\widetilde{U} \in \mathrm{RV}_\gamma$ we calculate $\lim_{t\to\infty} \frac{\widetilde{U}(tx)}{\widetilde{U}(t)} = \lim_{t\to\infty} \frac{U(tx)}{U(t)} \frac{\widetilde{U}(tx)}{U(tx)} \frac{U(t)}{\widetilde{U}(t)} = x^\gamma$. □

The following definition describes a notation of a function H that will be helpful in our analysis. To get an intuition for the definition of H, consider first the cdf with unbounded support $F(x) = \exp(-1/x^2)$ (which behaves roughly the same as the power-law distribution $1 - 1/x^2$ for large x). Using the H function will enable us to "pull out" x^2.

Definition 18. Given a distribution F, we define the function $H_F(x)$ as

$$H_F(x) := -\frac{1}{\ln(F(x))} \qquad \text{if } F^{\leftarrow}(1) = \infty \tag{43}$$
$$H_F(x) := -\frac{1}{\ln\big(F\big(b-x\big)\big)} \qquad \text{if b} := F^{\leftarrow}(1) < \infty\ . \tag{44}$$

The function H_F is useful as follows: the probability that in the symmetric price mechanism a price p is rejected by all bidders is $F^n(p) = (\exp(-1/H(p)))^n = \exp(-n/H(p))$ in case $F^{\leftarrow}(1) = \infty$ and $F^n(p) = \exp(-n/H(F^{\leftarrow}(1)-p))$ if $F^{\leftarrow}(1) < \infty$, which will be a useful representation.

We have the following facts regarding the function H.

Lemma 19. *Consider a distribution $F \in \mathcal{D}(G_\gamma)$.*

If $\gamma > 0$, then $H_F \in \mathrm{RV}_{1/\gamma}$ and $\inf\{x|H_F(x) \geq y\} = \widetilde{U}(y)$.

If $\gamma < 0$, then $\lim_{t\downarrow 0} \frac{H_F(tx)}{H_F(t)} = x^{1/\gamma}$ for any $x > 0$, and $\sup\{x|H_F(x) \geq y\} = \widetilde{U}(y)$.

Proof. First consider the case $\gamma > 0$ (and thus $F^{\leftarrow}(1) = \infty$). One checks that H is positive and that

$$\lim_{t\to\infty} \frac{H(tx)}{H(t)} = \lim_{t\to\infty} \frac{\ln(F(t))}{\ln(F(tx))} = \lim_{t\to\infty} \frac{1-F(t)}{1-F(tx)} \frac{\ln(F(t))}{1-F(t)} \frac{1-F(tx)}{\ln(F(tx))} = x^{1/\gamma}\ , \tag{45}$$

where we used that $1 - F(x)$ is regularly varying with index $-1/\gamma$, [8, Theorem 1.2.1(1)] and that $\frac{1-F(tx)}{\ln(F(tx))}$ goes to -1 for any $x > 0$ since $F(tx)$ goes to 1. Furthermore, $\inf\{x|H(x) \geq y\} = \inf\{x| - 1/\ln(F(x)) \geq y\} = \inf\{x|F(x) \geq \exp(-1/y)\} = \widetilde{U}(y)$.

Next, consider the case $\gamma < 0$. We get

$$\lim_{t\downarrow 0} \frac{H(tx)}{H(t)} = \lim_{t\downarrow 0} \frac{1-F(b-t)}{1-F(b-tx)} \frac{\ln(F(b-t))}{1-F(b-t)} \frac{1-F(b-tx)}{\ln(F(b-tx))} = x^{1/\gamma}\ , \tag{46}$$

this time using [8, Theorem 1.2.1(2)]. Furthermore, we get $\sup\{x|H(x) \geq y\} = \sup\{x|-1/\ln(F(b-x)) \geq y\} = \sup\{x|F(b-x) \geq \exp(-1/y)\} = b - \inf\{x|F(x) \geq \exp(-1/y)\} = \widetilde{U}(y)$. □

5.2 Distributions with Unbounded Support

We turn to prove Theorem 6.1 that exhibits the optimal expected revenue obtained by symmetric posted-prices when $0 < \gamma < 1$. We require the following notation:

Definition 20. Define the function $\phi : (0,1] \to \mathbb{R}$ as $x = \phi(\gamma)$ if x is the positive solution to the equation $\exp(x) = 1 + \frac{x}{\gamma}$.

Alternatively, $\phi : [0,1) \to \mathbb{R}_{\geq 0}$ is the inverse of the function $x \mapsto \frac{x}{\exp(x)-1}$: if $\gamma = \frac{x}{\exp(x)-1}$ then $\exp(x) = 1 + \frac{x}{\gamma}$. Since $x \mapsto \frac{x}{\exp(x)-1}$ is continuous and strictly decreasing, ϕ is continuous and strictly decreasing. The function ϕ is drawn in the supplementary materials.

Theorem 6.1 claims that the optimal revenue in symmetric posted-price mechanisms is, up to low order terms, $U\left(\phi(\gamma)^{\frac{1}{\gamma}-1}\left(\gamma+\phi(\gamma)\right)^{-\frac{1}{\gamma}} \cdot n\right)$. It follows from the following lemma and Lemma 17.

Lemma 21. *Let $F \in \mathcal{D}(G_\gamma)$ for $0 < \gamma < 1$. The optimal expected revenue in a symmetric posted-price auction satisfies*

$$r_n = \frac{\phi(\gamma)}{\gamma + \phi(\gamma)} \widetilde{U}\left(\frac{n}{\phi(\gamma)}\right)(1 + o(1)) \ . \tag{47}$$

Furthermore, revenue of this form is achieved by the prices $p_n^* = U\left(\frac{n}{\phi(\gamma)}\right)$.

Proof. We first show that prices $p_n^* = \widetilde{U}(\frac{n}{\phi(\gamma)})$ achieve a revenue of the form (47). The seller obtains a payment of p_n^* if at least one bidder accepts the offer, which happens with probability $1 - F^n(p_n^*)$.

Using the notation from Definition 18, we can write $F^n(p) = (\exp(-1/H(p)))^n = \exp(-n/H(p))$. Proposition B.1.9 (10) in [8] gives $H(p_n^*) = H(\widetilde{U}(\frac{n}{\phi(\gamma)})) = \frac{n}{\phi(\gamma)}(1 + o(1))$, and thus we get that the price p_n^* is accepted with probability $1-\exp\big(-\phi(\gamma)(1+o(1))\big)$ as $n \to \infty$. Since $\phi(\cdot)$ and $\exp(\cdot)$ are continuous functions, Definition 20 implies that the probability that the price is accepted by at least one bidder is of the form

$$1-\frac{1}{\exp\big(\phi(\gamma)(1+o(1))\big)}=1-\frac{1}{1+\phi(\gamma)/\gamma}(1+o(1))=\frac{\phi(\gamma)}{\gamma+\phi(\gamma)}(1+o(1))\,. \quad (48)$$

Since the price was $\widetilde{U}(\frac{n}{\phi(\gamma)})$ this gives us revenue as in Eq. (47).

A price of $U(\frac{n}{\phi(\gamma)})$ instead of $\widetilde{U}(\frac{n}{\phi(\gamma)})$ also achieves such a revenue: for this, it is sufficient to note that this price will only be accepted with higher probability. But because U is non-decreasing and $\frac{1}{1-\exp(-1/n)}\geq\frac{1}{1-(1-\frac{1}{n})}=n$ the new price cannot be higher, and so this is obvious.

We next show that the revenue cannot exceed the revenue in Eq. (47). Consider the optimal prices p_n, which we write in the form $p_n=\widetilde{U}(\frac{n}{x_n})$ for some positive sequence $\{x_n\}_{n\geq 0}$. (This is always possible, as explained in footnote 15.)

First assume that $x_n>\frac{\phi(\gamma)}{c(\gamma)}$ for infinitely many n (where we specify $c(\gamma)>0$ in a moment, which depends only on γ). Then, since the expected revenue is at most the price and $\widetilde{U}$ is monotone, the revenue is infinitely often at most $\widetilde{U}\big(\frac{n}{x_n}\big)\leq\widetilde{U}\big(\frac{c(\gamma)n}{\phi(\gamma)}\big)\leq(c(\gamma))^{\gamma}\widetilde{U}\big(\frac{n}{\phi(\gamma)}\big)(1+o(1))$, by Proposition B.1.9(5). If we choose $c(\gamma)$ such that $(c(\gamma))^{\gamma}$ is a constant smaller than $\frac{\phi(\gamma)}{\gamma+\phi(\gamma)}$, then this is smaller than the revenue in Eq. (47).

As a second case, assume that $x_n<\frac{\phi(\gamma)}{c(\gamma)}$ for all but finitely many n. Here, we compare the expected revenue $r_n(p(x))$ achieved by price $p(x):=\widetilde{U}(\frac{n}{x})$ with the expected revenue $r_n(p_n^*)$ achieved by price p_n^*. We get, using Proposition B.1.9(5) from [8] in the final equation (where we need that $0<x<\frac{\phi(\gamma)}{c(\gamma)}$), that for every $n>n_\epsilon$ depending on $\epsilon>0$:

$$\begin{aligned}\frac{r_n(p(x))}{r_n(p_n^*)}&\leq\frac{(1-\exp(-\frac{n}{H(\widetilde{U}(\frac{n}{x}))}))p(x)}{(1-e^{-\phi(\gamma)})p_n^*}(1+\epsilon)\\&=\frac{(1-e^{-x})\widetilde{U}(\frac{n}{x})}{(1-e^{-\phi(\gamma)})\widetilde{U}(\frac{n}{\phi(\gamma)})}(1+2\epsilon)\\&\leq\frac{(1-e^{-x})}{(1-e^{-\phi(\gamma)})}(1+3\epsilon)\big(\tfrac{\phi(\gamma)}{x}\big)^{\gamma}\cdot\max\Big\{\big(\tfrac{\phi(\gamma)}{x}\big)^{\epsilon},\big(\tfrac{\phi(\gamma)}{x}\big)^{-\epsilon}\Big\}\,.\end{aligned}$$

We assume that $\left(\frac{\phi(\gamma)}{x}\right)^{\epsilon} \geq \left(\frac{\phi(\gamma)}{x}\right)^{-\epsilon}$ (the other case is treated analogously). Then,

$$\frac{r_n(p(x))}{r_n(p_n^*)} \leq \frac{(1+3\epsilon)\phi(\gamma)^{\gamma+\epsilon}}{(1-e^{-\phi(\gamma)})}(1-e^{-x})x^{-\gamma-\epsilon} \tag{49}$$

We can give a good upper bound for the right hand side of (49): assuming $\gamma+\epsilon \in (0,1]$ the derivative of $(1-e^{-x})x^{-\gamma-\epsilon}$ with respect to x is zero if and only if $e^x = 1+\frac{x}{\gamma+\epsilon}$, which, by the remark after Definition 20 has the unique positive solution $x = \phi(\gamma+\epsilon)$ (this must be a maximum, since $(1-e^{-x})x^{-\gamma-\epsilon}$ is positive for $x > 0$ and goes to 0 for $x \to 0$ and $x \to \infty$).

We insert $x = \phi(\gamma+\epsilon)$ into (49) which gives

$$\frac{r_n(p(x))}{r_n(p_n^*)} \leq (1+3\epsilon)\frac{1-e^{-\phi(\gamma+\epsilon)}}{1-e^{-\phi(\gamma)}}\frac{\phi(\gamma)^{\gamma+\epsilon}}{\phi(\gamma+\epsilon)^{\gamma+\epsilon}} \;. \tag{50}$$

Since $\phi(\cdot)$ is continuous, the right hand side of (50) is continuous. Since it is also 1 for $\epsilon = 0$, for every given $\epsilon' > 0$ we can choose ϵ small enough such that $\frac{r_n(p(x))}{r_n(p_n^*)} < 1+\epsilon'$ for large enough n. Thus, the optimal prices $p(x_n)$, for large enough n, generate an expected revenue of at most $r_n(p_n^*)(1+\epsilon')$. □

Proof (of Theorem 6.1). Using Lemma 17 and Proposition B.1.9(5) we see that (for $c = \frac{\phi(\gamma)}{\phi(\gamma)+\gamma}$) $\lim_{x\to\infty} \frac{c\widetilde{U}(x)}{U(c^{1/\gamma}x)} = 1$. Thus, the class of functions given in Eq. (47) equals the class of functions given in Theorem 6 (Eq. (9)). □

5.3 Distributions with Bounded Support

We now prove Theorem 11.1. We will be able to reduce it to the special case where $F^{\leftarrow}(1) = 1$ which is discussed in the following lemma.

Lemma 22. *Let $F \in \mathcal{D}(G_\gamma)$, $\gamma < 0$, $F^{\leftarrow}(1) = 1$. Then, the optimal expected revenue r_n in a symmetric posted-price mechanism satisfies*

$$r_n = 1 - U\Big(\frac{n}{\ell_n}\Big)(1+o(1)) \;, \tag{51}$$

where $\ell_n := -\ln(U(n))$.

Proof. We first show that a revenue of the form (51) can be achieved. We give the proof in case F is continuous here. If F is not continuous, the proof is somewhat technical, and we defer it to Appendix A, Lemma 29.

We claim that prices

$$p_n^* = 1 - \widetilde{U}\left(\frac{n}{\ell_n}\right) \tag{52}$$

achieve a revenue of the form (51). As in the case with unbounded support, the probability that at least one bidder accepts the offer is

$$1 - F^n(p_n^*) = 1 - \exp(-n/H(1-p_n^*)) = 1 - \exp\Big(-n/H\big(\widetilde{U}\big(\tfrac{n}{\ell_n}\big)\big)\Big) \tag{53}$$

$$= 1 - \exp(-\ell_n) = 1 - U(n)\ , \tag{54}$$

where $H(\widetilde{U}(x)) = x$ follows[16] from Lemma 19 and from the fact that F is continuous. It follows that the revenue is at least

$$(1 - F^n(p_n^*))p_n^* \geq (1 - U(n))\Big(1 - \widetilde{U}\big(\frac{n}{\ell_n}\big)\Big) \tag{55}$$

$$\geq 1 - \widetilde{U}\big(\frac{n}{\ell_n}\big) - U(n) = 1 - \widetilde{U}\big(\frac{n}{\ell_n}\big)\Big(1 + \frac{U(n)}{\widetilde{U}\big(\frac{n}{\ell_n}\big)}\Big) \tag{56}$$

Now, $\lim_{n\to\infty} \frac{U(n)}{\widetilde{U}(n)} = 1$, and the fact that $\widetilde{U}$ is monotonically decreasing implies that for any $\epsilon' > 0$ and large enough n:

$$\frac{U(n)}{\widetilde{U}\big(\frac{n}{\ell_n}\big)} \leq \frac{U(n)}{\widetilde{U}(n)}\frac{\widetilde{U}(n)}{\widetilde{U}(\epsilon n)} \leq \epsilon^{-\gamma} + \epsilon\ . \tag{57}$$

Furthermore, we have $\lim_{n\to\infty} \frac{\widetilde{U}(\frac{n}{\ell_n})}{U(\frac{n}{\ell_n})} = 1$ since $\frac{n}{\ell_n} \to \infty$ as $n \to \infty$. In total, we get

[16] This equality fails in general if F is not continuous, and forces us to chose the prices slightly smaller in this case, which introduces the aforementioned technicalities.

that the revenue is of the form (51).

To see that no better revenue than one of the form (51) can be achieved, consider now an optimal sequence of prices $\{p_n\}_{n\geq 0}$. We can write this sequence as

$$p_n = 1 - \widetilde{U}\Big(\frac{n}{(1+\delta_n)\ell_n}\Big). \tag{58}$$

for some other sequence $\{\delta_n\}_{n\geq 0}$ with $\delta_i > -1$ (as explained in footnote 15.

We distinguish two cases:

- We have $\liminf_{n\to\infty} \delta_n \geq 0$. We use that $r_n \leq p_n$ (in the given mechanism the seller never achieves a higher revenue than the requested price), and Proposition B.1.9(5) from [8] to get, for any fixed $\epsilon > 0$:

$$r_n \leq 1 - \widetilde{U}\Big(\frac{n}{(1+\delta_n)\ell_n}\Big) \leq 1 - \widetilde{U}\Big(\frac{n}{(1-\epsilon)\ell_n}\Big) \leq 1 - U\Big(\frac{n}{\ell_n}\Big)(1 - O(\gamma\epsilon)), \tag{59}$$

  which gives us an upper bound on the revenue of the form (51) since ϵ can be chosen arbitrarily low.

- There exists $\delta^* < 0$ such that for infinitely many n, $\delta_n < \delta^*$. In this case, we note that the revenue cannot be higher than the probability that at least one bidder accepts, which is at most, for infinitely many n:

$$\begin{aligned} 1 - F^n(p_n) &= 1 - F^n(F^{\leftarrow}(\exp(-\tfrac{(1+\delta_n)\ell_n}{n}))) && (60)\\ &\leq 1 - \exp(-(1+\delta_n)\ell_n) && (61)\\ &\leq 1 - U(n)^{1+\delta^*} && (62)\\ &= 1 - U\Big(\frac{n}{\ell_n}\Big)\frac{U(n)}{U\Big(\frac{n}{\ell_n}\Big)}U(n)^{\delta^*} && (63)\\ &\leq 1 - U\Big(\frac{n}{\ell_n}\Big)\frac{1}{2}\ell_n^{-\gamma+1}U(n)^{\delta^*} && (64)\end{aligned}$$

  according to Proposition B.1.9(5) from [8] with $\epsilon = \frac{1}{2}$. It is now sufficient to remark that the term $\ell_n^{-\gamma+1}U(n)^{\delta^*} = \big(-\ln(U(n))\big)^{-\gamma+1}U(n)^{\delta^*} = (\ln(m))^{-\gamma+1}m^{-\delta^*}$ for $m = 1/U(n)$ is bigger than any constant, since $m \to \infty$ as $n \to \infty$. Therefore, using such prices, the revenue will be smaller than (51) infinitely often, and the sequence $\{\delta_n\}_{n\geq 0}$ cannot have been optimal. □

The next lemma relates $U(n)$ with $\ln(n)$; we only prove it for $\gamma < 0$, but it holds for any $\gamma \neq 0$.

Lemma 23. *Let $U \in \mathrm{RV}_\gamma$, $\gamma < 0$. Then,* $\lim_{n\to\infty} \frac{\ln(U(n))}{\gamma \ln(n)} = 1$.

Proof. For any $\epsilon > 0$, Proposition B.1.9(5) from [8] gives $t_\epsilon > 0$ such that for all $n > t_\epsilon$:

$$\ln(U(n)) \leq \ln\Big((1+\epsilon)(\tfrac{n}{t_\epsilon})^{\gamma+\epsilon}U(t_\epsilon)\Big) \leq (\gamma+\epsilon)\ln(2n/t_\epsilon) + \ln(U(t_\epsilon)) \tag{65}$$

$$\leq (\gamma+\epsilon)\ln(n) + (\gamma+\epsilon)\ln(2/t_\epsilon) + U(t_\epsilon) \leq (\gamma+2\epsilon)\ln(n) \tag{66}$$

for n big enough. Thus, $\frac{\ln(U(n))}{\gamma\ln(n)} \geq 1 + \frac{2\epsilon}{\gamma}$ for n big enough. The upper bound follows in the same way. □

Proof (of Theorem 11.1). We need to extend Lemma 22 to general upper bounds $F^{\leftarrow}(1)$, and apply Lemma 23 to show that we can use $-\gamma\log(n)$ instead of ℓ_n.

For this, we assume that an auction is given in some fixed currency, with upper bound b. Now, consider an auction in a new currency where one unit has worth $b := F^{\leftarrow}(1)$ of the old currency. In the new currency, the distribution function is $\widehat{F}(x) = F(bx)$, and $\widehat{F}^{\leftarrow}(y) = F^{\leftarrow}(y)/b$. We also have $\widehat{\widetilde{U}}(n) = \widetilde{U}(n)/b$ and $\widehat{U}(n) = U(n)/b$, one easily checks $\widehat{U} \in \mathrm{RV}_\gamma$. We set $\ell_n := -\ln(\widehat{U}(n)) = -\ln(U(n)/b)$. Then, we already showed that setting the prices in the new currency as $\hat{p}_n^* = 1 - \widehat{\widetilde{U}}\big(\frac{n}{\ell_n}\big)$ achieve the revenue as claimed. Those prices are, in the actual currency: $p_n^* = b - \widetilde{U}\big(\frac{n}{\ell_n}\big) = b - U\big(\frac{1}{1-\exp(\ln(U(n)/b)/n)}\big) = b - U\big(\frac{1}{1-(U(n)/b)^{1/n}}\big)$. The revenue in the new currency is $1 - \hat{r}_n = U\big(\frac{n}{\ell_n}\big)(1+o(1))$, i.e., in the actual currency $b - r_n = U\big(\frac{n}{\ell_n}\big)(1+o(1))$.

Proposition B.1.9(5) from [8] gives

$$\lim_{n\to\infty} \frac{U(\frac{n}{-\gamma \ln(n)})}{U(\frac{n}{\ell_n})} < (1+\epsilon)\Big(\frac{\gamma \ln(n)}{\ln(U(n)) - \ln(b)}\Big)^{\gamma\pm\epsilon}$$

which is arbitrarily close to 1, using Lemma 23. A lower bound follows analogously, and we get the theorem. □

6 Optimal Auctions – Proofs

In this section we present the optimal expected efficiency, and then use the classic result by Myerson to show how the revenue maximization problem can be reduced to it. It is well known that the efficient allocation can be achieved in a Bayesian-Nash equilibrium (even in dominant strategies) via the Vickrey auction ([25]).

6.1 Optimal expected efficiency

We now present a characterization for the expected efficiency in efficient auctions. The following result can be obtained easily from [8, Theorem 5.3.2]. We give a proof in the supplementary materials attached to this manuscript.

Theorem 24. *Let $X_1, \ldots, X_n$ be i.i.d. random variables distributed according to $F \in \mathcal{D}(G_\gamma)$. Then, the expected highest order statistic $w_n^{opt} = \mathrm{E}[\max(X_1, \ldots, X_n)]$ satisfies:*

$$w_n^{opt} = U\Big((\Gamma(1-\gamma))^{\frac{1}{\gamma}} \cdot n\Big)(1+o(1)), \qquad \textit{If } 0 < \gamma < 1. \tag{67}$$

$$w_n^{opt} = F^{\leftarrow}(1) - U\Big((\Gamma(1-\gamma))^{\frac{1}{\gamma}} \cdot n\Big)(1+o(1)), \qquad \textit{If } \gamma < 0. \tag{68}$$

6.2 Myerson, regular variation, and von-Mises conditions

We will later use the characterization of the optimal expected social welfare in Theorem 24 for presenting a similar result for the expected revenue. For this, the distributions must exhibit Myerson's regularity condition [20]. Therefore, our result for optimal auctions require both that $F \in \mathcal{D}(G_\gamma)$ and non-decreasing virtual valuations. In this section, we show that the combination of these two condition actually implies another condition, which is well known in extreme-value theory, and is due to von Mises. We believe that that this implication may be of independent interest.

Definition 25. **[von-Mises conditions]** Let $F \in \mathcal{D}(G_\gamma)$, $\gamma < 1$, $\gamma \neq 0$ associated with a density function f. We say that F satisfies the *von Mises conditions* if when $0 < \gamma < 1$ then

$$\lim_{x\to\infty} \frac{xf(x)}{1-F(x)} = \frac{1}{\gamma} \tag{69}$$

and when $\gamma < 0$ then, for $b = F^{\leftarrow}(1)$,

$$\lim_{x\uparrow b} \frac{(b-x)f(x)}{1-F(x)} = -\frac{1}{\gamma}\ . \tag{70}$$

The von Mises conditions imply that $F \in D(G_\gamma)$; however, they aren't necessary conditions (in particular they require F to be differentiable).[17]

Lemma 26. *Let $F \in \mathcal{D}(G_\gamma)$, $\gamma < 1$, $\gamma \neq 0$ and assume F is differentiable and the virtual valuation $\tilde{x}(x) = x - \frac{1-F(x)}{f(x)}$ is non-decreasing for all x. Then F satisfies the von Mises conditions.*

A proof can be find in the appendix. To get an intuition for the proof, assume that the conclusion is wrong. As an example, let $\gamma > 0$, and assume that $\frac{x_i f(x_i)}{1-F(x_i)} > 2/\gamma$ for a sequence $\{x_i\}_{i\geq 0}$ which converges to ∞. Because the virtual valuation is non-decreasing, one can show that this implies $\frac{xf(x)}{1-F(x)} \geq 1.5/\gamma$ for all $x_i \leq x \leq x_i(1+\gamma/2)$. It is then possible to show that this contradicts $F \in \mathcal{D}(G_\gamma)$: integrating yields that $1-F$ declines too fast in the interval $[x_i, (1+\gamma/2)x_i]$.

[17]Compared to (8), (69) and (70) only require the first derivative of F to exist; however, they are not applicable in case $\gamma = 0$.

6.3 Expected Revenue

While Theorem 24 concerned efficiency, we would like to prove a similar result for revenue. Due to Myerson's seminal work, it is known that for maximizing revenue, the seller should allocate the item to the bidder with the highest *virtual valuation*. Given a cdf F (over the support $[a, b]$) with a density function f, let the virtual valuation of a bidder be $\tilde{x}(x) = x - \frac{1-F(x)}{f(x)}$ when the bidder receives the item and 0 otherwise. Let $\tilde{F}$ denote the resulting distribution of the virtual valuations over the support $[a - \frac{1}{f(a)}, b]$. Let the *virtual surplus* be the sum of the virtual valuations of the bidders which receive the item, where in our case it is equal to the virtual valuation of the single bidder that received the item (the seller has a fixed virtual valuation of zero). Recall that F is *Myerson-regular* if $\tilde{x}(x)$ is non-decreasing.

A classic result by Myerson ([20]) shows that in (Bayes-Nash) equilibrium, the expected revenue equals exactly the expected virtual surplus. Therefore, we will now apply Theorem 24 for estimating the expected highest order statistic in the realm of virtual valuations, under the distribution $\tilde{F}$.

We now study how the regularly varying properties change when we move to virtual valuation.

Lemma 27. *Consider a Myerson-regular distribution $F \in \mathcal{D}(G_\gamma)$, and let $b = F^{\leftarrow}(1)$. Then, $\tilde{F} \in \mathcal{D}(G_\gamma)$ and*

$$\lim_{x\to\infty} \frac{\tilde{x}(x)}{x} = 1 - \gamma \qquad \text{if } 0 < \gamma < 1 \tag{71}$$

$$\lim_{x\to b} \frac{b - \tilde{x}(x)}{b - x} = 1 - \gamma \qquad \text{if } \gamma < 0. \tag{72}$$

Proof. If F is both in $\mathcal{D}(G_\gamma)$ and Myerson-regular, then it satisfies the von-Mises properties (Definition 25) by Lemma 26. To see (71) we compute

$$\lim_{x\to b} \frac{\tilde{x}(x)}{x} = \lim_{x\to b} \frac{x - \frac{1-F(x)}{f(x)}}{x} = 1 - \lim_{x\to b} \frac{1 - F(x)}{f(x) \cdot x} = 1 - \gamma, \tag{73}$$

and for (72) analogously

$$\lim_{x \to b} \frac{b - \tilde{x}(x)}{b - x} = \lim_{x \to b} \frac{b - (x - \frac{1 - F(x)}{f(x)})}{b - x} = 1 + \lim_{x \to b} \frac{1 - F(x)}{f(x) \cdot (b - x)} = 1 - \gamma \tag{74}$$

To see that $\tilde{F} \in \mathcal{D}(G_\gamma)$, we first observe that if $\tilde{x}$ is non-decreasing, then for every z and y such that $F(z) = \tilde{F}(y)$ we have that $y = \tilde{x}(z)$. It follows from the definition of the function U (Definition 1) that with non-decreasing virtual values,

$$U_{\tilde{F}}(z) = \tilde{x}(U_F(z)) \qquad \text{when } 0 < \gamma < 1 \tag{75}$$

and

$$F^{\leftarrow}(1) - U_{\tilde{F}}(z) = \tilde{x}(F^{\leftarrow}(1) - U_F(z)) \qquad \text{when } \gamma < 0 \tag{76}$$

When $0 < \gamma < 1$, we get with (75) and (71):

$$\lim_{n \to \infty} \frac{U_{\tilde{F}}(nx)}{U_{\tilde{F}}(n)} = \lim_{n \to \infty} \frac{\frac{U_{\tilde{F}}(nx)}{U_F(nx)}}{\frac{U_{\tilde{F}}(n)}{U_F(n)}} \frac{U_F(nx)}{U_F(n)} = \lim_{n \to \infty} \frac{\frac{\tilde{x}(U_F(nx))}{U_F(nx)}}{\frac{\tilde{x}(U_F(n))}{U_F(n)}} \frac{U_F(nx)}{U_F(n)} \tag{77}$$

$$= \lim_{n \to \infty} \frac{(1 - \gamma)U_F(nx)}{(1 - \gamma)U_F(n)} = x^\gamma \tag{78}$$

Analogously, for $\gamma < 0$:

$$\lim_{n \to \infty} \frac{U_{\tilde{F}}(nx)}{U_{\tilde{F}}(n)} = \lim_{n \to \infty} \frac{\frac{U_{\tilde{F}}(nx)}{U_F(nx)}}{\frac{U_{\tilde{F}}(n)}{U_F(n)}} \frac{U_F(nx)}{U_F(n)} = \lim_{n \to \infty} \frac{\frac{b - \tilde{x}(b - U_F(nx))}{b - (b - U_F(nx))}}{\frac{b - \tilde{x}(b - U_F(n))}{b - (b - U_F(n)))}} \frac{U_F(nx)}{U_F(n)} = x^\gamma. \tag{79}$$

□

We can now show the characterization of the expected revenue in the optimal auctions, proving Theorem 6.3 and Theorem 11.3.

Proof (of Theorems 6.3 and 11.3). We show the proof for the case of unbounded supports (Theorem 6 part 3), and the proof for bounded supports is shown analogously.

The virtual values $\tilde{X}_i$ are random variables drawn from the distribution $\tilde{F}$.

$$E[\text{virtual-surplus}] = \mathrm{E}[\max(\tilde{X}_1, \ldots, \tilde{X}_n)] \tag{80}$$
$$= U_{\tilde{F}}\big((\Gamma(1-\gamma))^{\frac{1}{\gamma}} \cdot n\big)(1+o(1)) \tag{81}$$
$$= \tilde{x}\left(U_F\big((\Gamma(1-\gamma))^{\frac{1}{\gamma}} \cdot n\big)\right)(1+o(1)) \tag{82}$$
$$= (1-\gamma)\left(U_F\big((\Gamma(1-\gamma))^{\frac{1}{\gamma}} \cdot n\big)\right)(1+o(1)) \tag{83}$$
$$= U_F\big((\Gamma(1-\gamma)(1-\gamma))^{\frac{1}{\gamma}} \cdot n\big)(1+o(1)) \tag{84}$$

Where Eq. (81) is due to Theorem 24, Eq. (82) is due to Eq. (75), Eq. (83) is by Lemma 27 and Eq. (84) follows from Definition 3 of regularly varying functions.

Finally, due to Myerson [20], the expected virtual surplus is, in equilibrium, equal to the expected revenue, and with non-decreasing virtual valuation it is achieved in a (dominant strategy) equilibrium. □

We would like to point out an immediate corollary of Eq. (83), that shows a very simple asymptotic connection between the expected highest order statistic and the expected second-highest statistic:

Corollary 28. *Consider distribution $F \in \mathcal{D}(G_\gamma)$ which is also Myerson-regular. Let $E[X_{n,n}]$ denote the expected highest order statistic of F, and let $E[X_{n-1,n}]$ denote the expected second-highest order statistic. Then,*

$$E[X_{n-1,n}] = (1-\gamma)E[X_{n,n}](1+o(1)) \qquad \textit{If } 0<\gamma<1 \tag{85}$$
$$F^{\leftarrow}(1) - E[X_{n-1,n}] = (1-\gamma)\left(F^{\leftarrow}(1) - E[X_{n,n}]\right)(1+o(1)) \qquad \textit{If } \gamma<0 \tag{86}$$

7 Discussion

This paper compares three methods for the sale of a single item: optimal auctions, posted-price mechanisms and dynamic auctions (or discriminatory posted prices).

The main contribution of this paper is an exact asymptotic characterization of the optimal revenue in the three models and a precise comparison of the revenue they obtain. We also present mechanisms that achieve these optimal results. The results are shown for a wide family of distributions denoted by $\mathcal{D}(G_\gamma)$, and parameterized by γ. The index γ classifies distributions according to the shape of their tail. We showed that as γ increases (for $0 < \gamma < 1$), posted price performs worse compares to optimal auctions; however, an opposite trend starts when the tail of the distribution is sufficiently heavy (when $\gamma \cong 0.60$ for symmetric posted prices). Although our results are asymptotic, simulations show that for standard distributions the true expected revenue is indistinguishable from our results after just a few dozens of bidders.

Posted-price mechanisms are often used for the sale of multiple items, and future research should study such environments. We believe that for a fixed number of objects for sale, one would have results in the same spirit to ours. When the number of items grows with the market (for example, selling $c \cdot n$ units where $0 < c < 1$) the results would probably have a different nature and might require different techniques.

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A Missing proofs

Proof (of Lemma 26). Since the virtual valuation is non-decreasing we have, if $x_0 \leq x_1 < F^{\leftarrow}(1)$:

$$x_0 - \frac{1 - F(x_0)}{f(x_0)} \leq x_1 - \frac{1 - F(x_1)}{f(x_1)}, \tag{87}$$

and thus (using that $\frac{1-F(x)}{f(x)} > 0$, since both numerator and denominator are positive)

$$\frac{f(x_1)}{1-F(x_1)} \geq \frac{1}{x_1 - x_0 + \frac{1-F(x_0)}{f(x_0)}} \tag{88}$$

$$\frac{f(x_0)}{1-F(x_0)} \leq \frac{1}{x_0 - x_1 + \frac{1-F(x_1)}{f(x_1)}} \ . \tag{89}$$

The lemma is implied immediately by the following 4 statements:

$$\limsup_{x\to\infty} \frac{xf(x)}{1-F(x)} \leq \frac{1}{\gamma} \qquad \text{(if } 0<\gamma<1\text{)} \tag{90}$$

$$\liminf_{x\to\infty} \frac{xf(x)}{1-F(x)} \geq \frac{1}{\gamma} \qquad \text{(if } 0<\gamma<1\text{)} \tag{91}$$

$$\limsup_{x\to b} \frac{(b-x)f(x)}{1-F(x)} \leq -\frac{1}{\gamma} \qquad \text{(if } \gamma<0\text{)} \tag{92}$$

$$\liminf_{x\to b} \frac{(b-x)f(x)}{1-F(x)} \geq -\frac{1}{\gamma} \qquad \text{(if } \gamma<0\text{)} \ . \tag{93}$$

All these four are proven with very similar calculations, we give the proofs of (90) and (93) explicitly.

Proof of (90): Fix $\epsilon > 0$. The function $1 - F(x)$ is, because of [8, Theorem 1.2.1], regularly varying with index $-1/\gamma$, i.e., $\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-1/\gamma}$. Thus, $\frac{1-F(t(1+\epsilon))}{1-F(t)} \geq \frac{1}{(1+\epsilon)^{1/\gamma}+\epsilon^2}$, for all $t > t_0(\epsilon)$ for some $t_0(\epsilon)$ depending on F and ϵ. We get:

$$\ln\big((1+\epsilon)^{1/\gamma} + \epsilon^2\big) \geq \ln\Big(\frac{1-F(t)}{1-F(t(1+\epsilon))}\Big) \tag{94}$$

$$= -\int_t^{t(1+\epsilon)} \Big(\ln(1-F(s))\Big)' ds \tag{95}$$

$$= \int_t^{t(1+\epsilon)} \frac{f(s)}{1-F(s)} ds \tag{96}$$

$$\geq \int_t^{t(1+\epsilon)} \frac{1}{s - t + \frac{1-F(t)}{f(t)}} ds \tag{97}$$

$$= \ln\Big(s - t + \frac{1-F(t)}{f(t)}\Big)\Big|_{s=t}^{s=t(1+\epsilon)} \tag{98}$$

$$= \ln\Big(\frac{\epsilon t + \frac{1-F(t)}{f(t)}}{\frac{1-F(t)}{f(t)}}\Big) = \ln\Big(1 + \frac{\epsilon t f(t)}{1-F(t)}\Big), \tag{99}$$

where we used (88) to get (97). The above is equivalent to

$$\frac{tf(t)}{1-F(t)} \leq \frac{(1+\epsilon)^{\frac{1}{\gamma}}+\epsilon^2-1}{\epsilon} \ . \tag{100}$$

One checks that $\lim_{\epsilon\to 0} \frac{(1+\epsilon)^{\frac{1}{\gamma}}+\epsilon^2-1}{\epsilon} = \frac{1}{\gamma}$. Since (100) holds for any $\epsilon > 0$ for big enough t, for any $\delta > 0$, if t is big enough, $\frac{tf(t)}{1-F(t)} \leq \frac{1}{\gamma} + \delta$, which is (90).

Proof of (93): The difference to the proof of (90) is that $t \to b$ instead of $t \to \infty$, which has some minor implications on the appearance of the formulas. We again use [8, Theorem 1.2.1] which gives $\lim_{t\downarrow 0} \frac{1-F(b-t(1-\epsilon))}{1-F(b-t)} = (1-\epsilon)^{-1/\gamma}$ in this case, and so $\frac{1-F(b-t(1-\epsilon))}{1-F(b-t)} \geq \frac{1}{(1-\epsilon)^{1/\gamma}+\epsilon^2}$, now for all $0 < t < t_0(\epsilon)$ for some $t_0(\epsilon)$ depending on F and ϵ. Thus:

$$\ln\Big((1-\epsilon)^{1/\gamma}+\epsilon^2\Big) \geq \ln\Big(\frac{1-F(b-t)}{1-F(b-t(1-\epsilon))}\Big) \tag{101}$$

$$= \int_{b-t}^{b-t(1-\epsilon)} -\Big(\ln(1-F(s))\Big)' ds \tag{102}$$

$$= \int_{b-t}^{b-t(1-\epsilon)} \frac{f(s)}{1-F(s)} ds \tag{103}$$

$$\geq \int_{b-t}^{b-t(1-\epsilon)} \frac{1}{s-(b-t)+\frac{1-F(b-t)}{f(b-t)}} ds \tag{104}$$

$$= \ln\Big(s-(b-t)+\frac{1-F(b-t)}{f(b-t)}\Big)\Big|_{s=b-t}^{s=b-t(1-\epsilon)} \tag{105}$$

$$= \ln\Big(\frac{\epsilon t + \frac{1-F(b-t)}{f(b-t)}}{\frac{1-F(b-t)}{f(b-t)}}\Big) = \ln\Big(1+\frac{\epsilon t f(b-t)}{1-F(b-t)}\Big). \tag{106}$$

This is now equivalent to $\frac{tf(b-t)}{1-F(b-t)} \leq \frac{(1-\epsilon)^{\frac{1}{\gamma}}+\epsilon^2-1}{\epsilon}$, and we get $\limsup_{t\to b} \frac{(b-t)f(t)}{1-F(t)} \leq \lim_{\epsilon\to 0} \frac{(1-\epsilon)^{\frac{1}{\gamma}}+\epsilon^2-1}{\epsilon} = -\frac{1}{\gamma}$. □

We next fill in the missing part in Lemma 22.

Lemma 29. *Let $F \in \mathcal{D}(G_\gamma)$, $\gamma < 0$, $F^{\leftarrow}(1) = 1$. Then, there are prices p_n which achieve, in a symmetric posted-price mechanism, a revenue of (let $\ell_n := -\ln(U(n))$)*

$$r_n = 1 - U\Big(\frac{n}{\ell_n}\Big)(1+o(1)) \ . \tag{107}$$

Proof. Let $\delta(x) \geq 0$ be a monotonically decreasing function such that $H(\widetilde{U}(x)) \leq x(1+\delta(x))$ and $\lim_{x\to\infty} \delta(x) = 0$. Such a function exists according to a slight variation on [8, Proposition B.1.9(10)], see Lemma 7 in the supplementary materials. We claim that prices $p_n^* = 1 - \widetilde{U}\left(\frac{n}{\ell_n}(1-\delta(\sqrt{n}))\right)$, achieve a revenue of the form (51). As in the case with unbounded support, the probability that at least one bidder accepts the offer is

$$\begin{aligned}
1 - F^n(p_n^*) &= 1 - \exp(-n/H(1-p_n^*)) && (108)\\
&= 1 - \exp\left(-\frac{n}{H\left(\widetilde{U}\left(\frac{n}{\ell_n}(1-\delta(\sqrt{n}))\right)\right)}\right) && (109)\\
&\geq 1 - \exp\left(-\frac{\ell_n}{(1+\delta(n'))(1-\delta(\sqrt{n}))}\right) && (110)\\
&= 1 - \exp\left(-\frac{\ell_n}{1+\delta(n')-\delta(\sqrt{n})-\delta(n')\delta(\sqrt{n})}\right), && (111)
\end{aligned}$$

where $n' = \frac{n(1-\delta(\sqrt{n}))}{\ell_n}$. Since $n' \geq \sqrt{n}$ for n big enough and δ is monotone, we also get $0 \leq \delta(n') \leq \delta(\sqrt{n})$, and so $1 - F^n(p_n^*) \geq 1 - \exp(-\ell_n) = 1 - U(n)$.

As in the previous proof, it follows that the revenue is at least

$$(1 - F^n(p_n^*))p_n^* \geq 1 - \frac{\widetilde{U}\left(\frac{n}{\ell_n}(1-\delta(\sqrt{n}))\right)}{\widetilde{U}\left(\frac{n}{\ell_n}\right)}\widetilde{U}\left(\frac{n}{\ell_n}\right)\left(1 + \frac{U(n)}{\widetilde{U}\left(\frac{n}{\ell_n}(1-\delta(\sqrt{n}))\right)}\right). \quad (112)$$

The quotient which newly appeared compared to (56) goes to 1 as well when n goes to infinity (because $1-\epsilon < 1-\delta(\sqrt{n}) \leq 1$ for any $\epsilon > 0$ and any big enough n). The last factor can be shown to go to 0 in the same way as in the previous proof. In total, the revenue is of the form (107). □