THE SYMMETRIC GENUS OF THE MATHIEU GROUPS

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ABSTRACT

The (symmetric) genus of a finite group may be defined as the smallest genus of those closed orientable surfaces on which G acts faithfully as a group of automorphisms. In this paper the genus of each of the five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24} is determined, with the help of some computer calculations and a little-known theorem of Ree on permutations.

1. *Introduction*

The *symmetric genus* $\sigma(G)$ of a finite group G was defined by Tom Tucker in [**10**] as the smallest genus of all those closed orientable surfaces on which G acts faithfully as a group of automorphisms, that is, homeomorphisms of the surface onto itself, preserving the local structure but allowing reversal of the surface's orientation. Also, he defined the *strong symmetric genus* $\sigma^0(G)$ of G to be the smallest genus of those surfaces on which G acts faithfully as a group of orientation-preserving automorphisms, so that $\sigma(G) \leqslant \sigma^0(G)$ in general—although in many cases these two numbers are equal (for instance, when G has no subgroup of index two).

By extending a theorem of Hurwitz, Tucker showed that if $\sigma(G) > 1$ (that is, if G does not act faithfully on either the sphere or the torus), then both $|G|/(\sigma(G)-1)$ and $|G|/(\sigma^0(G)-1)$ are bounded above, by 168 and 84 respectively; furthermore, he showed that when values close to these bounds are attained, the group G must have at least one generating-set of a given number of types, determined by the orders of the generators and their products. This information was subsequently used to calculate the symmetric genus and the strong symmetric genus of each of the finite alternating and symmetric groups (see [**3**] and [**2**] respectively).

In a more recent paper [**11**], Andrew Woldar described his attempts to do the same sort of thing for each of the five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24}. His definition of the 'genus' of a finite group coincides with Tucker's definition of the 'strong symmetric genus'—and for a finite simple group this is clearly the same as the 'symmetric genus'—but in general neither of these should be confused with the use of the term 'genus' to describe the smallest genus of all such surfaces into which can be embedded some Cayley graph for the group (see [**10**] also).

Whichever term is chosen, the Riemann–Hurwitz equation is used initially to enumerate the types of generating-sets which need to be considered for a given group. In particular, for any group G whose strong symmetric genus $\sigma^0(G)$ satisfies $6(\sigma^0(G)-1) < |G| < 84(\sigma^0(G)-1)$, the exact value of $\sigma^0(G)$ may be determined by finding from among the following types of sets one which gives the smallest possible value of the quantity M:

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(i) a generating-triple (x, y, z) of elements of orders p, q and r respectively, such that xyz is the identity element, and $M = 1-(1/p+1/q+1/r)$;

(ii) a generating-quadruple (x, y, z, w) of elements of orders p, q, r and s respectively, such that $xyzw$ is the identity element, and

$$M = 2-(1/p+1/q+1/r+1/s).$$

In the optimal case, $\sigma^0(G)$ is then given by $\sigma^0(G) = 1+\frac{1}{2}M|G|$ (see **[10]** or **[11]**).

For the Mathieu groups, Woldar uses character theory and local analysis to establish the existence or non-existence of such sets in each case. Unfortunately, he was unable to settle the question for M_{23} and M_{24}, and also made an error in his calculations for one of the possibilities claimed (namely (2, 6, 7)-generation) for M_{23}; in fact, none of the three types listed for M_{23} in **[11**, Table 1] can be achieved, as will be shown below.

In this paper we determine the symmetric genus (and strong symmetric genus) of all five of the Mathieu groups, using alternative methods for considering the various types of generating-set.

THEOREM. (*a*) *The symmetric genus of the Mathieu group* M_{11} *is* 631, *with a minimal genus action arising from* (2, 4, 11)-*generation of* M_{11}.

(*b*) *The symmetric genus of the Mathieu group* M_{12} *is* 3169, *with a minimal genus action arising from* (2, 3, 10)-*generation of* M_{12}.

(*c*) *The symmetric genus of the Mathieu group* M_{22} *is* 34849, *with a minimal genus action arising from* (2, 5, 7)-*generation of* M_{22}.

(*d*) *The symmetric genus of the Mathieu group* M_{23} *is* 1053361, *with a minimal genus action arising from* (2, 4, 23)-*generation of* M_{23}.

(*e*) *The symmetric genus of the Mathieu group* M_{24} *is* 10200961, *with a minimal genus action arising from* (3, 3, 4)-*generation of* M_{24}.

Here '(p, q, r)-generation' means the same as in **[11]**, namely that the group has a generating-triple of the type (i) given above; equivalently, there is a generating-pair (x, y) of elements of orders p and q respectively whose product xy has order r.

The approach we take is the same as that used in **[3]**, **[10]** and **[11]** for the listing of types, but from then on is based on a little-known and seldom-used theorem of Ree on permutations, plus a small amount of computer calculation (using the CAYLEY system; see **[1]**) where appropriate. Similar ideas were used also for the alternating and symmetric groups in **[2]**.

2. *Further background*

The theorem of Ree is as follows: if $x_1, x_2, \ldots, x_s$ are permutations generating a transitive group on a set of size n, such that $x_1 x_2 \ldots x_s$ is the identity permutation, and if c_i denotes the number of orbits of $\langle x_i \rangle$ for $1 \leqslant i \leqslant s$, then $c_1+c_2+\ldots+c_s \leqslant (s-2)n+2$. In other words, the generating permutations cannot have more than $(s-2)n+2$ cycles between them.

Several proofs of this result are available: the original one (in **[9]**) uses the Riemann–Hurwitz equation; others are given in **[4]** and the references listed there. For an immediate application, consider the possibility that the group M_{23} may be generated by a triple of elements of orders 2, 6 and 7, whose product is the identity element. In the natural action of M_{23} on 23 points, such elements induce permutations with cycle structures $1^7 2^8$, $1^1 2^2 3^2 6^2$ and $1^2 7^3$ respectively, but as $15+7+5 > 23+2$, Ree's transitivity condition is violated. Similarly, M_{23} cannot be generated by a quadruple of elements of orders 2, 2, 2 and 3 whose product is the identity, because $15+15+15+11 > 46+2$; nor can there be a generating-triple of elements of orders 3, 4 and 4 whose product is the identity, as $11+9+9 > 23+2$. Thus none of the three possibilities suggested for M_{23} in **[11]** can occur. Also, the determination of the genus of M_{22} is much more easily achieved using this theorem than by the sort of analysis in **[11]**.

On the other hand, Ree's theorem is not always as effective as this. Although it places an obvious restriction on the cycle structures of possible generators, the inequality can still be satisfied by permutations which generate a proper, imprimitive, or even intransitive subgroup of the group in question, and other means are required to account for types such as these.

In the case of generating-triples, the standard character-theoretic formula for the calculation of class multiplication constants is particularly useful for determining the number of triples of a given type. Specifically, if K_1, K_2 and K_3 are conjugacy classes of elements in the group G, then the number of pairs (x, y) with $x \in K_1$, $y \in K_2$ and $xy \in K_3$ is given by

$$\frac{|K_1|\,|K_2|\,|K_3|}{|G|} \sum_{t=1}^{m} \frac{\chi_t(K_1)\,\chi_t(K_2)\,\overline{\chi_t(K_3)}}{\chi_t(1)},$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of G, $\chi_r(K_s)$ denotes the common value of the character χ_r on the class K_s, and $\overline{\chi_t(K_3)}$ is the complex conjugate of $\chi_t(K_3)$.

This formula is easy to apply, given the character table for G. The problem is to determine which triples (if any) generate the whole group, rather than a proper subgroup, and that may involve some extensive local analysis (as in **[11]**). One way around this problem, which is especially suitable in the case of 'small' groups such as the Mathieu groups, is to enlist the help of a computer. In particular, the CAYLEY system can be used to create a list of representatives of conjugacy classes of triples. For instance, if the elements of the class K_1 can be enumerated, then for any fixed $z \in K_3$, each element x in K_1 can be checked to see whether $x^{-1}z \in K_2$, and the set of such x can be partitioned into equivalence classes under conjugation by $C_G(z)$; alternatively, random conjugates of any chosen element of K_1 can be checked in the same way, until all the triples (with a fixed element $z \in K_3$) are accounted for. Either way, the subgroup generated by each triple can then be analysed, to find its order and any other properties of interest.

An entirely different approach—which was described by this author in **[2]** and used to obtain the results given in **[3]**—involves the use of the 'low-index subgroups' algorithm, due to Dietze, Schaps and Sims, and documented in **[6]**. If n is the least integer for which the group G can be faithfully represented as a transitive permutation group of degree n, then G has a generating-set of the type (i) given earlier if and only if the corresponding abstract triangle group $\Delta = \Delta(p, q, r)$ with presentation

$$\langle x, y, z \mid x^p = y^q = z^r = xyz = 1 \rangle$$

has a subgroup of index n whose core in Δ is a normal subgroup N with Δ/N isomorphic to G. To check whether this happens or not, one can use the low-index subgroups algorithm to enumerate representatives of conjugacy classes of subgroups of index n in Δ, and the Schreier–Sims algorithm (as described in **[7]**) to analyse the permutation group induced by Δ on the cosets of any such subgroup. Both of these algorithms are available in CAYLEY. Similarly, G has a generating-set of type (ii) if and only if G is isomorphic to the permutation group induced on the cosets of some subgroup of index n in the group

$$\langle x, y, z, w \mid x^p = y^q = z^r = w^s = xyzw = 1 \rangle,$$

and again that possibility may be checked by computation. As will be seen in the case of M_{24}, this approach can be much more effective than those using calculations within the group G itself.

3. *Generating-triples for the Mathieu groups*

Before proving the theorem, we verify that each of the five Mathieu groups has a generating-triple (which will turn out to be optimal in the sense of providing the symmetric genus).

First, it is already known that the groups M_{11} and M_{12} can be $(2,4,11)$- and $(2,3,10)$-generated respectively. Explicit generators were given in **[8]**, and also such generation was proved by Woldar in **[11]**. Alternatively, one can show this directly as follows.

Consider the permutations A, B, C and D of the projective line over GF(11) given as generators for M_{12} in the *ATLAS* **[5]**, namely

$$\begin{aligned} A &= (0,1,2,3,4,5,6,7,8,9,10), \\ B &= (1,3,9,5,4)(2,6,7,10,8), \\ C &= (0,\infty)(1,10)(2,5)(3,7)(4,8)(6,9), \\ D &= (2,10)(3,4)(5,9)(6,7). \end{aligned}$$

If, say, u, w and v are defined by

$$\begin{aligned} u &= C^{-1}B^2A^{-3}DA^3B^{-2}C = (1,7)(2,3)(4,5)(9,10), \\ w &= A \quad \text{and} \quad v = u^{-1}A^{-1} = (0,10,8,7)(1,6,5,3), \end{aligned}$$

then (u, v, w) is a triple of elements of orders 2, 4 and 11, fixing the point ∞, and generating a subgroup isomorphic to M_{11}: the conjugates of v by uv^2, uv^2u, $(uv)^2$ and uv^2uv are such that the first two generate a quaternionic subgroup of order 8 fixing the points 1, 6, 7 and ∞; adjoining the third gives the stabilizer of the three points 1, 7 and ∞; and the subgroup generated by all four fixes 7 and ∞ and is transitive on the remaining 10 points. Similarly, if z, x and y are defined by

$$\begin{aligned} z &= CD = (0,\infty)(1,2,9,7,4,8,3,6,5,10), \\ x &= A^{-4}C^{-1}A^2DA^{-2}CA^4 = (0,7)(1,6)(3,9)(8,\infty) \end{aligned}$$

and

$$y = x^{-1}z^{-1} = (0,9,8)(1,3,2)(4,7,\infty)(5,6,10),$$

then (x, y, z) is a triple of elements of orders 2, 3 and 10, generating M_{12}; note that

$$B = z^6, \quad A = (xz^6xz^6xz^5xz^5xz^2x)^2, \quad D = (xz^6)^3xz^{-1}xzx(z^4x)^3 \quad \text{and} \quad C = zD^{-1}.$$

Next, the group M_{22} can be $(2,5,7)$-generated—this was also proved in **[11]**, and explicit generators will be given below—and to complete the picture, we exhibit $(2,4,23)$- and $(3,3,4)$-generation of M_{23} and M_{24} respectively.

Consider the permutations A, B, C and D of the projective line over GF(23) given as generators for M_{24} in [5], namely

$$A = (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22),$$
$$B = (1,2,4,8,16,9,18,13,3,6,12)(5,10,20,17,11,22,21,19,15,7,14),$$
$$C = (0,\infty)(1,22)(2,11)(3,15)(4,17)(5,9)(6,19)(7,13)(8,20)(10,16)(12,21)(14,18),$$
$$D = (1,18,4,2,6)(5,21,20,10,7)(8,16,13,9,12)(11,19,22,14,17).$$

Defining $f = B^{-3}A^{-1}(A^5D^2)^2AB^3$, $g = D$ and $h = (fg)^{-1}$, that is,

$$f = (1,9)(3,22)(7,14)(8,16)(12,17)(13,21)(15,20)(18,19),$$
$$g = (1,18,4,2,6)(5,21,20,10,7)(8,16,13,9,12)(11,19,22,14,17),$$
$$h = (1,6,2,4,19,11,12)(3,22,18,9,21,5,14)(7,10,15,20,13,8,17),$$

we have a $(2,5,7)$-generating-triple (f,g,h) for M_{22}, fixing the points 0 and ∞. Similarly, taking $u = D^{-1}A(A^5D^2)^2A^{-1}D$, $w = z^{12}$ and $v = u^{-1}w^{-1}$, that is,

$$u = (0,14)(1,12)(3,19)(4,5)(6,18)(8,10)(11,22)(15,17),$$
$$w = (0,12,1,13,2,14,3,15,4,16,5,17,6,18,7,19,8,20,9,21,10,22,11),$$
$$v = (0,2,13,1)(3,7,18,17)(4,16)(5,15)(8,21,9,20)(10,19,14,11),$$

gives a $(2,4,23)$-generating-triple (u,v,w) for M_{23}, fixing ∞. Finally, if $x = (A^5D^2)^{-1}ACA^5D^2$, $z = A^5C$ and $y = x^{-1}z^{-1}$, that is,

$$x = (0,22,2)(1,19,16)(3,18,9)(4,12,5)(6,\infty,20)(7,15,21)(8,11,10)(13,17,14),$$
$$z = (0,9,18,\infty)(1,19,22,17)(2,13,14,6)(3,20,11,10)(4,5,16,12)(7,21,15,8),$$
$$y = (0,6,3)(1,5,16)(2,19,17)(8,11,15)(10,20,18)(14,22,\infty),$$

then (x,y,z) is a $(3,3,4)$-generating-triple for M_{24}.

Verification of these claims is left to the reader, but in any case is a simple matter using the CAYLEY system (replacing 0 by 23 and ∞ by 24 to make the permutations acceptable).

4. *Proof of the Theorem*

Once again we note that for a finite simple group G (such as one of the Mathieu groups) the symmetric genus coincides with the strong symmetric genus, as G has no subgroup of index two. In particular, this restricts the types of generating-set which need to be considered in determining the exact value of $\sigma(G)$, and, in fact, if $\sigma(G) < 1+\frac{1}{6}|G|$ then we need consider only those sets of the types (i) and (ii) given in the Introduction. Note also that the smallest positive value of the quantity $M = 2-(1/p+1/q+1/r+1/s)$ obtainable from generation of G by a quadruple of type (ii) is $\frac{1}{6}$, corresponding to the case $(p,q,r,s) = (2,2,2,3)$, with genus $1+\frac{1}{12}|G|$.

We now deal with each of the Mathieu groups in turn.

(*a*) The group M_{11}, of order 7920.

This group has elements of orders 1, 2, 3, 4, 5, 6, 8 and 11 only, with cycle structures 1^{11}, $1^3 2^4$, $1^2 3^3$, $1^3 4^2$, $1^1 5^2$, $2^1 3^1 6^1$, $1^1 2^1 8^1$ and 11^1 respectively in its usual permutation representation on 11 points. Also, (2,4,11)-generation has been established, corresponding to the action of M_{11} on a surface of genus $1+\frac{1}{2}(\frac{7}{44})|M_{11}|$, that is, 631.

As noted by Woldar in [11], the only conceivable generating-triples which could give a value of $M = 1-(1/p+1/q+1/r)$ smaller than 7/44 would be those with $(p,q,r) = (2,3,8)$, $(2,4,5)$, $(2,3,11)$, $(2,4,6)$, $(3,3,4)$, $(2,5,5)$, $(2,4,8)$, $(2,5,6)$ or $(3,3,5)$. Of these, the cases $(2,3,8)$, $(2,4,5)$, $(2,4,6)$, $(3,3,4)$ and $(2,4,8)$ can be eliminated immediately by Ree's theorem, leaving $(2,3,11)$, $(2,5,5)$, $(2,5,6)$ and $(3,3,5)$ to be handled by other means.

Using CAYLEY, it is a straightforward exercise to account for all such triples within M_{11}. Given any element z of order 11, there is just one conjugacy class of pairs (x,y) of elements of orders 2 and 3 whose product xy is the inverse of z, and every such pair generates a subgroup isomorphic to $PSL_2(11)$. If z is any element of order 5, there are forty-five pairs (x,y) of elements of orders 2 and 5 such that $xy = z^{-1}$, and fifteen of these generate subgroups isomorphic to A_5 and ten of them generate subgroups isomorphic to A_6, while the remaining twenty generate subgroups isomorphic to $PSL_2(11)$. The same thing happens for pairs of elements both of order 3 whose product is a fixed element of order 5. And, finally, given any element z of order 6, there are thirty pairs of elements of orders 2 and 5 whose product is z^{-1}, and six of these generate subgroups isomorphic to S_5, while the other twenty-four generate subgroups isomorphic to $PSL_2(11)$.

Alternatively, the low-index subgroups algorithm can be used to eliminate each of these four cases. The triangle group $\Delta(2,3,11)$ has ten conjugacy classes of subgroups of index 11, but for eight of these classes the corresponding permutations (induced on the cosets of any subgroup in the class) generate A_{11}, and for the other two they generate a group isomorphic to $PSL_2(11)$. Similarly, $\Delta(2,5,5)$ has nine conjugacy classes of subgroups of index 11, and the corresponding permutations generate A_{11} in five cases and $PSL_2(11)$ in the other four. Also, $\Delta(3,3,5)$ has ten such classes, with six giving A_{11} and four giving $PSL_2(11)$. And, finally, the group $\Delta(2,5,6)$ has thirty-three such classes, with fourteen giving A_{11}, fifteen giving S_{11}, and four giving $PSL_2(11)$, as the permutation group induced on cosets.

In particular, M_{11} is a quotient of none of these triangle groups, and it follows that the symmetric genus of M_{11} is actually 631.

(*b*) The group M_{12}, of order 95040.

This group has elements of orders 1, 2, 3, 4, 5, 6, 8, 10 and 11, and the cycle structures of elements are 1^{12}, $1^4 2^4$, 2^6, $1^3 3^3$, 3^4, $1^4 4^2$, $2^2 4^2$, $1^2 5^2$, $1^1 2^1 3^1 6^1$, 6^2, $1^1 2^1 8^1$, $4^1 8^1$, $2^1 10^1$ and 11^1. Also, (2,3,10)-generation has been established, corresponding to the action of M_{12} on a surface of genus $1+\frac{1}{2}(\frac{1}{15})|M_{12}|$, that is, 3169. The only possible generating-triples which could give a value of $M = 1-(1/p+1/q+1/r)$ smaller than 1/15 would be those with $(p,q,r) = (2,3,8)$ or $(2,4,5)$, but these can be eliminated, either as in [11] or as follows.

Straightforward calculations within M_{12} using CAYLEY show that for a given element z with cycle structure $4^1 8^1$, there are forty pairs of elements of orders 2 and 3 whose product is z^{-1}, and of these, eight generate intransitive subgroups of order 48, another eight generate transitive but imprimitive subgroups of order 96, sixteen generate imprimitive subgroups of order 432, and the remaining eight generate imprimitive subgroups of order 720. Similarly, there are forty such pairs when z has cycle structure $1^1 2^1 8^1$, and in this case while eight of them generate intransitive subgroups of order 48 and eight generate imprimitive subgroups of order 96, sixteen generate intransitive subgroups of order 432, and the remaining eight generate intransitive subgroups of order 720. Also, there are fifty pairs of elements of orders 2 and 4 whose product is a fixed element of order 5, and of these, five generate transitive subgroups isomorphic to S_5, twenty-five generate intransitive subgroups isomorphic to S_5, and the remaining twenty generate intransitive subgroups isomorphic to A_6.

Alternatively, the low-index subgroups algorithm reveals that the triangle group $\Delta(2,3,8)$ has nine conjugacy classes of subgroups of index 12, but the permutations induced on the cosets of a subgroup in any such class generate an imprimitive group of order 24 (in two cases), 96 (in four cases), 432 (in two cases) or 720 (in the remaining case). Similarly, $\Delta(2,4,5)$ has nine such classes, with one giving S_5 as the permutation group induced on cosets, two giving a direct product $C_2 \times A_6$, two giving $PGL_2(11)$, and the remaining four giving an imprimitive group of order 23040 (being a wreath product of C_2 by A_6).

Thus 3169 is the symmetric genus of M_{12}.

(*c*) The group M_{22}, of order 443520.

This group has elements of orders 1, 2, 3, 4, 5, 6, 7, 8 and 11 only, with cycle structures 1^{22}, $1^6 2^8$, $1^4 3^6$, $1^2 2^2 4^4$, $1^2 5^4$, $2^2 3^2 6^2$, $1^1 7^3$, $2^1 4^1 8^2$ and 11^2. Also, (2, 5, 7)-generation has been established, corresponding to the action of M_{22} on a surface of genus $1+\frac{1}{2}(\frac{11}{70})|M_{22}|$, that is, 34849. The only triples which could give a value of $M = 1-(1/p+1/q+1/r)$ smaller than 11/70 would be those with $(p,q,r) = (2,3,7)$, $(2,3,8)$, $(2,4,5)$, $(2,3,11)$, $(2,4,6)$, $(3,3,4)$, $(2,5,5)$, $(2,4,7)$, $(2,4,8)$, $(2,5,6)$ or $(3,3,5)$. But all of these are eliminated by Ree's theorem—in each case there are just too many cycles for the triple to generate a transitive subgroup. Thus 34849 is the symmetric genus of M_{22}.

(*d*) The group M_{23}, of order 10200960.

This group has elements of orders 1, 2, 3, 4, 5, 6, 7, 8, 11, 14, 15 and 23 only, with cycle structures 1^{23}, $1^7 2^8$, $1^5 3^6$, $1^3 2^2 4^4$, $1^3 5^4$, $1^1 2^2 3^2 6^2$, $1^2 7^3$, $1^1 2^1 4^1 8^2$, $1^1 11^2$, $2^1 7^1 14^1$, $3^1 5^1 15^1$ and 23^1. Also (2, 4, 23)-generation has been established, corresponding to the action of M_{23} on a surface of genus $1+\frac{1}{2}(\frac{19}{92})|M_{23}|$, that is, 1053361.

To prove that this is the symmetric genus of M_{23}, we have to eliminate the possibility of (p,q,r)-generation of M_{23} for each triple (p,q,r) of orders of elements of M_{23} such that $1/p+1/q+1/r$ lies strictly between 73/92 and 1, as well as the possibility of generation by a quadruple of elements of orders 2, 2, 2 and 3 whose product is the identity. In the case of triples, those which need to be considered are as follows: $(2,3,r)$ for $7 \leqslant r \leqslant 23$, $(2,4,r)$ for $6 \leqslant r \leqslant 15$, $(3,3,r)$ for $4 \leqslant r \leqslant 7$, $(2,5,r)$ for $5 \leqslant r \leqslant 8$, $(2,6,6)$, $(2,6,7)$ and $(3,4,4)$; but all of them can be eliminated by Ree's theorem. Also, the quadruple $(2,2,2,3)$ is easily ruled out by Ree's theorem, as was shown in Section 2. Thus $\sigma(M_{23}) = \sigma^0(M_{23}) = 1053361$.

(*e*) The group M_{24}, of order 244823040.

This group has elements of orders 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 21 and 23 only, with the cycle structures of elements being 1^{24}, $1^8 2^8$, 2^{12}, $1^6 3^6$, 3^8, $1^4 2^2 4^4$, $2^4 4^4$, 4^6, $1^4 5^4$, $1^2 2^2 3^2 6^2$, 6^4, $1^3 7^3$, $1^2 2^1 4^1 8^2$, $2^2 10^2$, $1^2 11^2$, $2^1 4^1 6^1 12^1$, 12^2, $1^1 2^1 7^1 14^1$, $1^1 3^1 5^1 15^1$, $3^1 21^1$ and 23^1. Also, (3, 3, 4)-generation has been established, corresponding to the action of M_{24} on a surface of genus $1+\frac{1}{2}(\frac{1}{12})|M_{24}|$, that is, 10200961.

The only triples which could give a value of $M = 1-(1/p+1/q+1/r)$ smaller than $1/12$ would be those with $(p,q,r) = (2,3,7)$, $(2,3,8)$, $(2,4,5)$, $(2,3,10)$ or $(2,3,11)$. Sadly, none of these can be eliminated by Ree's theorem; they can, however, be dealt with very easily using the low-index subgroups algorithm.

The triangle group $\Delta(2,3,7)$ has only one conjugacy class of subgroups of index 24, and the corresponding permutations (induced on the cosets of any subgroup in this class) generate an imprimitive group isomorphic to $PSL_2(7)$. Also, the group $\Delta(2,4,5)$ has only five conjugacy classes of subgroups of index 24, with the corresponding permutations generating an imprimitive group in every case. In contrast to this, $\Delta(2,3,8)$ has 122 conjugacy classes of subgroups of index 24, and the corresponding permutations generate S_{24} in forty-two cases, $PGL_2(23)$ in two cases, and imprimitive subgroups of S_{24} in all the remaining ones. Similarly, $\Delta(2,3,10)$ has ninety such classes, with twenty-seven giving S_{24}, fourteen giving A_{24}, and all others giving imprimitive groups. And, finally, $\Delta(2,3,11)$ has 123 of these classes, with eighty giving A_{24}, five giving $PSL_2(23)$, and all the others giving imprimitive subgroups of S_{24}. In particular, the Mathieu group M_{24} is a quotient of none of these triangle groups.

On the other hand, computations (using CAYLEY) account for all subgroups of M_{24} which can be $(2,3,7)$-, $(2,3,8)$-, $(2,4,5)$-, $(2,3,10)$- or $(2,3,11)$-generated.

Given any element z of order 7 in M_{24}, there are ninety-one pairs (x,y) of elements of orders 2 and 3 respectively such that $xy = z^{-1}$, and while forty-nine of these generate intransitive subgroups isomorphic to $PSL_2(7)$, the remaining forty-two generate transitive but imprimitive subgroups, again isomorphic to $PSL_2(7)$. Similarly, if z is any given element of order 8, there are 112 pairs of elements of orders 2 and 3 whose product is z^{-1}, and of these pairs, sixteen generate intransitive subgroups of order 48, thirty-two generate intransitive subgroups of order 96, sixteen generate transitive but imprimitive subgroups of order 192, thirty-two generate intransitive subgroups of order 432, and the remaining sixteen generate intransitive subgroups of order 720.

There are 960 pairs of elements of orders 2 and 4 whose product is a fixed element of order 5, with 150 of these generating intransitive subgroups isomorphic to S_5, ninety generating transitive but imprimitive subgroups isomorphic to S_5, 120 generating intransitive subgroups of order 160, 180 generating intransitive subgroups isomorphic to A_6, 120 generating intransitive subgroups isomorphic to $PGL_2(11)$, 120 generating transitive but imprimitive subgroups isomorphic to $PGL_2(11)$, and the remaining 180 generating intransitive subgroups of order 1920.

When it comes to $(2,3,10)$-generation, there are 160 pairs to account for, and these pairs generate intransitive subgroups isomorphic to $C_2 \times A_5$ in ten cases, transitive but imprimitive subgroups isomorphic to $C_2 \times A_5$ in another ten cases, intransitive subgroups isomorphic to S_6 in twenty cases, intransitive subgroups isomorphic to $PGL_2(11)$ in forty cases, transitive but imprimitive subgroups isomorphic to $PGL_2(11)$ in forty cases, intransitive subgroups of order 1920 in twenty cases, and intransitive subgroups isomorphic to M_{12} in the remaining twenty cases. Finally, there are 154 pairs of elements of orders 2 and 3 whose product is a fixed element of order 11 in M_{24}, and of these, twenty-two generate intransitive subgroups isomorphic to $PSL_2(11)$, twenty-two generate intransitive subgroups isomorphic to M_{12}, and the remaining 110 generate (primitive) subgroups isomorphic to $PSL_2(23)$.

Thus, on two counts, no value of $M = 1-(1/p+1/q+1/r)$ smaller than $1/12$ can be achieved by (p,q,r)-generation of M_{24}, and so the symmetric genus of M_{24} is 10200961, as claimed.

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References

1. J. J. CANNON, 'An introduction to the group theory language CAYLEY', *Computational group theory* (ed. M. Atkinson, Academic Press, San Diego, 1984) pp. 145–183.
2. M. D. E. CONDER, 'Some results on quotients of triangle groups', *Bull. Austral. Math. Soc.* 30 (1984) 73–90.
3. M. D. E. CONDER, 'The symmetric genus of alternating and symmetric groups', *J. Combinatorial Theory*, Ser. B, 39 (1985) 179–186.
4. M. D. E. CONDER and J. MCKAY, 'A necessary condition for transitivity of a finite permutation group', *Bull. London Math. Soc.* 20 (1988) 235–238.
5. J. H. CONWAY, R. T. CURTIS, S. P. NORTON, R. A. PARKER and R. A. WILSON, *An ATLAS of finite groups* (Oxford University Press, London, 1985).
6. A. DIETZE and M. SCHAPS, 'Determining subgroups of a given finite index in a finitely-presented group', *Canad. J. Math.* 26 (1974) 769–782.
7. J. S. LEON, 'On an algorithm for finding a base and strong generating set for a group given by generating permutations', *Math. Comp.* 35 (1980) 941–974.
8. J. MCKAY and K.-C. YOUNG, 'The non-Abelian simple groups G, $|G| < 10^6$—minimal generating pairs', *Math. Comp.* 33 (1979) 812–814 (plus microfiche supplement).
9. R. REE, 'A theorem on permutations', *J. Combinatorial Theory*, Ser. A, 10 (1971) 174–175.
10. T. W. TUCKER, 'Finite groups acting on surfaces and the genus of a group', *J. Combinatorial Theory*, Ser. B, 34 (1983) 82–98.
11. A. J. WOLDAR, 'Representing M_{11}, M_{12}, M_{22} and M_{23} on surfaces of least genus', *Commun. Algebra* 18 (1990) 15–86.

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