

LOCATING THE RANGE OF AN OPERATOR ON A HILBERT SPACE

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1. Introduction

In classical operator theory it is taken for granted that we can project onto the closure of the range of an operator T on a Hilbert space H . In a constructive development of operator theory, to which this note is a contribution, this projection exists if and only if $\text{ran}(T)$, the range of T , is *located*, in the sense that the distance

$$\rho(x, \text{ran}(T)) \equiv \inf \{ \|x - Ty\| : y \in H \}$$

exists (is computable) for each x in H ; if P is that projection, I is the identity operator on H , and the adjoint T^* of T exists, then $I - P$ is the projection of H on $\ker(T^*)$, the kernel of T^* . (For an example to show that the existence of the adjoint of a bounded operator on a Hilbert space is not automatic in constructive mathematics, see Brouwerian Example 3 in [7]; see also Example 2 below.) This much is neither surprising nor particularly interesting. Of more interest is the observation that, constructively, the existence of the projection on $\ker(T^*)$ is no guarantee of the existence of the projection on the closure of $\text{ran}(T)$, unless we are prepared to accept within our constructive mathematics *Markov's principle*:

If (a_n) is a binary sequence such that $\neg \forall n (a_n = 0)$, then there exists a value N such that $a_N = 1$.

To see this, consider the following Brouwerian counterexample. (For a discussion of the nature and role of Brouwerian counterexamples in mathematics, see page 3 of [6].)

EXAMPLE 1. Let a be a real number such that $\neg(a = 0)$, and define a selfadjoint operator T on the Hilbert space \mathbb{R} by $Tx \equiv ax$. (We must distinguish carefully between the expressions ' $\neg(a = 0)$ ' and ' $a \neq 0$ '. Using ρ to denote the metric on a metric space X , we say that points x and y of X are *distinct* if $\rho(x, y) > 0$, in which case we write $x \neq y$.) Then $\ker(T) = \{0\}$ is located, but $\text{ran}(T)$ is located if and only if $a \neq 0$ —that is, $|a| > 0$. Thus if every selfadjoint operator on \mathbb{R} has located range, then we can prove constructively the statement

$$\forall x \in \mathbb{R} (\neg(x = 0) \Rightarrow x \neq 0),$$

which is easily seen to be equivalent to Markov's principle.

Now, Markov's principle represents a form of unbounded search, and is regarded with at least suspicion by most constructive mathematicians. We would go further,

and view Markov’s principle, and therefore any proposition to which it is constructively equivalent, as essentially nonconstructive. Thus we have a significant constructive question:

If T is an operator, with adjoint T^ , on a Hilbert space, under what circumstances does the locatedness of $\ker(T^*)$ entail that of $\text{ran}(T)$?*

We shall show that, within Bishop’s constructive mathematics (BISH), the answer to this question is intimately tied up with a property of good behaviour of an operator, and with constructive versions of the open mapping theorem. We shall also discuss how far our main theorem on locating ranges is the best possible within BISH.

We shall assume that the reader has access to [1], Chapters 4 and 7 of which provide the basic material on metric, normed and Hilbert spaces upon which our work is founded. We denote by $\langle x, y \rangle$ the scalar product of two vectors x and y in a Hilbert space H , by $\mathcal{B}(H)$ the set of all (bounded linear) operators on H , by $\mathcal{A}(H)$ the set of all elements T of $\mathcal{B}(H)$ such that the adjoint T^* , defined in the usual way, exists, and by $\mathcal{A}_1(H)$ the unit ball of $\mathcal{A}(H)$ —that is, the set

$$\{T \in \mathcal{A}(H) : \|Tx\| \leq \|x\| \text{ for all } x \in H\}.$$

2. Located ranges and good behaviour

In the rest of this paper, except for Example 2, H will be a separable Hilbert space over \mathbb{R} or \mathbb{C} .

An element U of $\mathcal{A}(H)$ is said to be a *partial isometry* if there exists a projection P , called the *initial projection* of U , such that $\|UPx\| = \|Px\|$ and $U(I - P)x = 0$ for each x in X ; the range of P is then called the *initial space* of U . U is a partial isometry if and only if U^*U is a projection. In that case, U^*U is the initial projection of U ; U^* is a partial isometry with initial projection UU^* (the *final projection* of U); and $\text{ran}(UU^*) = \text{ran}(U)$ is called the *final space* of U .

If $T \in \mathcal{A}(H)$, then the positive square root of T^*T is written $|T|$. If $\text{ran}(T^*T)$ is located, then, as classically, T has an exact polar decomposition—that is, $T = U|T|$ for a (unique) partial isometry U whose initial projection is onto the closure of $\text{ran}(|T|)$, and which maps that space onto the closure of $\text{ran}(T)$. To show this, adapt the proof of [2, Theorem 1.1], noting from [3, (4.6)] that if $\text{ran}(|T|)$ is located, then the characteristic function of $\{0\}$ is integrable with respect to the functional calculus measure for $|T|$ (compare pages 315–316 of [12]). In general, the following lemma, which appears as Theorem 1.1 of [2], provides an approximate polar decomposition which is a satisfactory constructive substitute for the exact one.

LEMMA 1. *Let T be an element of $\mathcal{A}(H)$, and $\varepsilon > 0$. Then there exists a partial isometry U on H such that ε is a bound for both $T - U|T|$ and $|T| - U^*T$.*

LEMMA 2. *If $T \in \mathcal{A}(H)$, then $\text{ran}(TT^*)$ is dense in $\text{ran}(T)$.*

Proof. We may assume that $T \in \mathcal{A}_1(H)$. Given $\varepsilon > 0$ and $\xi \in H$, and applying the functional calculus for TT^* , we have

$$|T^*|\xi = (I - (I - TT^*))^{1/2} \xi = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (I - TT^*)^n \xi.$$

For each $N \geq 1$, let

$$p_N(TT^*) \equiv \sum_{n=0}^N \binom{1}{n} (I - TT^*)^n - \sum_{n=0}^N \binom{1}{n} I.$$

Then $p_N(TT^*) = TT^*q_N(TT^*)$ for some polynomial q_N . Moreover, since $\sum_{n=0}^{\infty} \binom{1}{n} = 0$, there exists ν such that

$$\| |T^*| \xi - TT^*q_\nu(TT^*)\xi \| \leq \left\| \sum_{\nu+1}^{\infty} \binom{1}{n} (I - TT^*)^n \xi \right\| + \left| \sum_{n=0}^{\nu} \binom{1}{n} \right| < \varepsilon/2.$$

On the other hand, by Lemma 1, there exists a partial isometry U on H such that $T^* - U^*|T^*|$, and therefore $T - |T^*|U$, has bound $\varepsilon/2$. Now let $x \in H$ and take $\xi \equiv Ux$ in the foregoing; we have

$$\|Tx - TT^*q_\nu(TT^*)Ux\| < \varepsilon,$$

where $TT^*q_\nu(TT^*)Ux \in \text{ran}(TT^*)$.

LEMMA 3. *Let $T \in \mathcal{A}(H)$; then $\text{ran}(T)$ is located if and only if $\text{ran}(T^*)$ is located.*

Proof. If $\text{ran}(T)$ is located, then, by Lemma 2, so is $\text{ran}(TT^*)$. Hence T^* has an exact polar decomposition $T^* = U|T^*|$, where U is a partial isometry whose initial space is the closure of $\text{ran}(|T^*|)$ and whose final space is the closure of $\text{ran}(T^*)$. Since the closure of $\text{ran}(T^*)$ is the range of the projection UU^* , it is located; hence $\text{ran}(T^*)$ itself is located. Interchanging the roles of T and T^* completes the proof.

Vectors e_1, \dots, e_n in a normed space X are said to be *linearly independent* if, for all scalars $\lambda_1, \dots, \lambda_n$ such that $\sum_{i=1}^n |\lambda_i| \neq 0$, we have $\sum_{i=1}^n \lambda_i e_i \neq 0$. A linear subset F of X is *finite-dimensional* if and only if it is spanned by a set of linearly independent vectors.

LEMMA 4. *If e_1, \dots, e_n are linearly independent vectors in H , then*

$$\mathbb{C}^n = \{(\langle x, e_1 \rangle, \dots, \langle x, e_n \rangle) : x \in H\}.$$

Proof. The result holds trivially when $n = 1$. Suppose that it holds for some value k of n , and consider the case where $n = k + 1$. Let $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, and, applying the induction hypothesis, construct $x \in H$ such that $\langle x, e_i \rangle = \xi_i$ for $i = 1, \dots, k$. By the Hahn-Banach theorem and the characterisation of normable linear functionals on H , there exists $y \in H$ such that $\langle y, e_i \rangle = 0$ ($1 \leq i \leq k$) and $\langle y, e_{k+1} \rangle = 1$; see Chapter 2, (6.2), of [6], and Chapter 8, (2.3), of [1]. Let

$$z \equiv x + (\xi_{k+1} - \langle x, e_{k+1} \rangle)y.$$

Then $\langle z, e_i \rangle = \xi_i$ for $1 \leq i \leq k + 1$.

A linear mapping T between normed linear spaces is said to be *well-behaved* if $Tx \neq 0$ whenever x is distinct from each element of $\text{ker}(T)$. The example above shows that if every bounded linear mapping of \mathbb{R} into \mathbb{R} is well-behaved, then Markov's principle holds.

THEOREM 1. *The following are equivalent conditions on an element T of $\mathcal{A}(H)$:*

- (i) $\text{ran}(T)$ is located;

- (ii) T is well-behaved and $\ker(T)$ is located;
- (iii) T^* is well-behaved and $\ker(T^*)$ is located;
- (iv) $\text{ran}(T^*)$ is located.

Proof. In view of Lemma 3, it will suffice to prove the equivalence of (ii) and (iv). To this end, assume (ii) and replace H by $H/\ker(T)$, so that T is a one-one, well-behaved element of $\mathcal{A}(H)$. Let $(V_n)_{n=1}^\infty$ be a sequence of finite-dimensional spaces whose union is dense in H . Given a positive integer n , let $\{e_1, \dots, e_m\}$ be an orthonormal set of linearly independent vectors that spans V_n . Since T is well-behaved, the vectors Te_1, \dots, Te_m are linearly independent. By Lemma 4,

$$\begin{aligned} V_n &= \left\{ \sum_{i=1}^m \langle x, Te_i \rangle e_i : x \in H \right\} \\ &= \left\{ \sum_{i=1}^m \langle T^*x, e_i \rangle e_i : x \in H \right\}. \end{aligned}$$

It follows that $\text{ran}(T^*)$ is dense, and therefore located, in H . Hence (ii) \Rightarrow (iv).

Conversely, assume (iv) and note that as $\ker(T) = \text{ran}(T^*)^\perp$, it is trivial that $\ker(T)$ is located. Given a point x of H distinct from each element of $\ker(T)$, construct y in the closure of $\text{ran}(T^*)$, and z in $\ker(T)$, such that $x = y + z$; then $y = x - z \neq 0$. Choose w in H such that $\|y - T^*w\| < \|y\|/2$. By the Cauchy-Schwarz inequality,

$$|\langle y, T^*w \rangle - \|y\|^2| = |\langle y, T^*w - y \rangle| \leq \|y\|^2/2,$$

and therefore

$$\|Tx\| \cdot \|w\| \geq |\langle Tx, w \rangle| = |\langle Ty, w \rangle| = |\langle y, T^*w \rangle| \geq \|y\|^2/2 > 0.$$

Hence $Tx \neq 0$, T is well-behaved, and (iv) \Rightarrow (ii).

COROLLARY. *If T is an element of $\mathcal{A}(H)$ with complete range and located kernel, then $\text{ran}(T)$ is located.*

Proof. This is an immediate consequence of Theorem 1 and the fact, established in [4], that any linear mapping from a normed space onto a Banach space is well-behaved.

Example 1 shows that we cannot remove from Theorem 1 the condition that T is well-behaved; and that, in the above corollary, we cannot omit the hypothesis that T has complete range. (Note that, for any real number a , the subset $\mathbb{R}a \equiv \{ax : x \in \mathbb{R}\}$ of \mathbb{R} is complete if and only if either $a = 0$ or $a \neq 0$; see [10].)

The following example shows that in Theorem 1 and its corollary we need T to have an adjoint defined on the whole space H .

EXAMPLE 2. Let a be a real number such that $\neg(a = 0)$, and let H be the Hilbert space $(\mathbb{R}a)^- \times \mathbb{R}$, with the scalar product induced by that on \mathbb{R}^2 . Define an element T of $\mathcal{B}(H)$ by

$$T(x, y) \equiv (0, x) \quad (x \in (\mathbb{R}a)^-, y \in \mathbb{R}).$$

Then $\ker(T) = \{0\} \times \mathbb{R}$ is located in H , but $\text{ran}(T) = \{0\} \times (\mathbb{R}a)^-$ is located in H if and only if $a \neq 0$.

In this example, T^* is defined on $\{0\} \times (\mathbb{R}a)^-$, and is defined throughout H if and only if $a \neq 0$.

3. Sequential openness, locating ranges and good behaviour

Is the locatedness of $\ker(T)$ necessary in Theorem 1 and its corollary? In this section we show that the answer to this question is closely connected with the constructive status of the open mapping theorem. We begin with two elementary lemmas, the first of which is proved in [3].

LEMMA 5. *If M and N are orthogonal linear subsets of H such that $M + N$ is dense in H , then M and N are both located.*

A linear mapping T of a normed space X into a normed space is said to be open if there exists $r > 0$ such that $B(0, r) \subset T(B(0, 1))$. It is easy to see that T is open if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in X$ and $\|Tx\| < \delta$, then $\|x - x'\| < \varepsilon$ for some x' in $\ker(T)$. It follows classically that T is open if and only if it is sequentially open, in the following sense: for each sequence (x_n) in X such that (Tx_n) converges to 0, there exists a sequence (y_n) in $\ker(T)$ such that $x_n + y_n \rightarrow 0$ as $n \rightarrow \infty$.

We are now in a position to prove a very general result about located ranges.

THEOREM 2. *If $T \in \mathcal{A}(H)$ is a sequentially open mapping, then $\text{ran}(T)$ is located.*

Proof. By Lemma 2, for each $x \in H$ there exists a sequence (x_n) in H such that $T(T^*x_n - x) \rightarrow 0$ as $n \rightarrow \infty$. Since T is sequentially open, there exists a sequence (y_n) in $\ker(T)$ such that $T^*x_n + y_n \rightarrow x$ as $n \rightarrow \infty$. As x is arbitrary, we conclude that $\text{ran}(T^*) + \ker(T)$ is dense in H . Since $\ker(T)$ is the orthogonal complement of $\text{ran}(T^*)$, it follows from Lemma 5 that $\text{ran}(T)$ is located.

Theorem 2 should be compared with Corollary 1 of [9]: *If T is a sequentially open linear mapping of a normed space onto a Banach space, such that $T(B(0, 1))$ is located, then $\ker(T)$ is located.*

It follows from Theorems 1 and 2 that a sequentially open element of $\mathcal{A}(H)$ is well-behaved. The corollary of the following proposition can be regarded as a significant generalisation of this observation. (Sequential openness appears to be a weaker constructive property than openness; it is not known if this appearance is, in fact, reality.)

PROPOSITION 1. *Let X, Y be normed spaces, and $T: X \rightarrow Y$ an open linear mapping with complete kernel. For each $x \in X$ there exists $x' \in \ker(T)$ such that if $x \neq x'$, then $T(x) \neq 0$.*

Proof. Construct a strictly decreasing sequence (δ_n) of positive numbers converging to 0, such that for each n , if $x \in X$ and $\|T(x)\| < \delta_n$, then $\|x - x'\| < 1/n$ for some $x' \in \ker(T)$. Given $x \in X$, construct an increasing binary sequence (λ_n) such that for each n ,

$$\begin{aligned} \lambda_n = 0 &\Rightarrow \|T(x)\| < \delta_n, \\ \lambda_n = 1 &\Rightarrow T(x) \neq 0. \end{aligned}$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, choose $x_n \in \ker(T)$ such that $\|x - x_n\| < 1/n$; if $\lambda_n = 1 - \lambda_{n-1}$, set $x_k \equiv x_{n-1}$ for all $k \geq n$. Then $\|x_m - x_n\| \leq 2/n$ whenever $m \geq n$, so (x_n) is a Cauchy sequence in the complete space $\ker(T)$; let $x' \in \ker(T)$ be its limit. If $x \neq x'$, then there exists ν such that $\|x - x_n\| > 2/\nu$; whence $\lambda_n = 1$ and therefore $T(x) \neq 0$.

COROLLARY. *If an open linear map between normed spaces has complete kernel, then that map is well-behaved.*

If this corollary held without the openness hypothesis, then we could prove Markov's principle: to see this, consider the linear mapping of Example 1.

Pursuing more directly our discussion of the question heading this section, we now state a remarkable lemma, whose proof is found in [8].

LEMMA 6. *Let T be a linear mapping of a Banach space X into a normed space Y , and let (x_n) be a sequence in X converging to 0. Then for all positive numbers α, β with $\alpha < \beta$, either $\|Tx_n\| > \alpha$ for infinitely many n , or $\|Tx_n\| < \beta$ for all sufficiently large n .*

Next we recall the constructive *Uniform Boundedness Theorem*.

THEOREM 3. *Let (T_n) be a sequence of bounded linear mappings from a Banach space X into a normed space Y , and let (x_n) be a sequence of unit vectors in X such that $\|T_n x_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Then there exists $x \in X$ such that the sequence $(\|T_n x\|)$ is unbounded.*

Proof. The proof is similar to a standard classical one [11].

THEOREM 4 (Hellinger–Toeplitz Theorem). *A selfadjoint linear operator on a Hilbert space is sequentially continuous.*

Proof. Let T be a selfadjoint operator on a Hilbert space H , (x_n) a sequence in H converging to 0, and ε a positive number. By Lemma 6, either $\|Tx_n\| < \varepsilon$ for all sufficiently large n , or there exists a subsequence (x_{n_k}) of (x_n) such that $\|Tx_{n_k}\| > \varepsilon/2$ for all k . In the latter case, there exists a sequence (ξ_n) in X such that $\xi_n \rightarrow 0$ but $\|T\xi_n\| \rightarrow \infty$. Applying the uniform boundedness principle to the bounded linear functionals $x \mapsto \langle x, T\xi_n \rangle$ on H , we construct a vector $z \in H$ such that the sequence $(|\langle z, T\xi_n \rangle|)$ is unbounded. But $\langle z, T\xi_n \rangle = \langle Tz, \xi_n \rangle \rightarrow 0$ as $n \rightarrow \infty$. This contradiction ensures that $\|Tx_n\| < \varepsilon$ for all sufficiently large n .

Using Theorems 2 and 4, we can now prove a weak open mapping theorem for operators on a Hilbert space.

PROPOSITION 2. *Let T be a selfadjoint bounded operator on H with complete range and located kernel. Then T is sequentially open.*

Proof. Replacing H by $H/\ker(T)$, we may assume that T is one-one. By Theorem 4, T^{-1} is sequentially continuous; so T is sequentially open. Reference to Theorem 2 completes the proof.

COROLLARY. *Let T be a selfadjoint bounded operator on H with complete range. Then T is sequentially open if and only if $\text{ran}(T)$ is located.*

Proof. If $\text{ran}(T)$ is located, then so is $\ker(T)$. The result follows from Theorem 2 and Proposition 2.

In Brouwer's intuitionistic mathematics (INT) and Markov's recursive constructive mathematics (RUSS), every sequentially continuous linear mapping from a Banach space into a normed space is uniformly continuous; so in the proof of Proposition 2, T^{-1} is uniformly continuous. Thus within each of these two varieties of constructive mathematics, we have the following version of the open mapping theorem: *Let T be a selfadjoint bounded operator on a Hilbert space such that $\text{ran}(T)$ is complete; then T is open if and only if $\text{ran}(T)$ is located.*

Returning to the question posed at the start of this section, suppose we have a Brouwerian example of a selfadjoint bounded operator T on a Hilbert space such that $\text{ran}(T)$ is complete but not located. In other words, suppose we can prove, within BISH, that the proposition

If T is a selfadjoint bounded operator on a Hilbert space such that $\text{ran}(T)$ is complete, then $\text{ran}(T)$ is located

entails some constructively unacceptable principle, such as the limited principle of omniscience [1, 6]. Then, in view of the last corollary, we have a Brouwerian counterexample to the classical open mapping theorem. Despite extensive analysis [5], no such counterexample has been found. So the question with which we began this section remains unanswered and intimately associated with the constructive status of the open mapping theorem in its full, classical form.

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