# FORMAL LIMITATIONS IN ECONOMIC THEORY AND ALTERNATIVE SET THEORIES

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Abstract. In this paper we discuss some formal limitations in contemporary economic theory and their solution in alternative set theories. The computability of choice functions as well as the existence of economic equilibria and of states of the world may not be ensured in general if the assumed set theory is **ZFC**. We claim that a switch to an alternative set theory may help to get rid of some of these negative results. While this procedure seems a facile form to solve analytical problems, we think that it may have a legitimate importance for economic theory. In fact, alternative set-theoretical frameworks convey different intuitions about how agents behave when solving problems. We discuss the advantages of $\mathbf{AFA}^- + \mathbf{AD} + \mathbf{DC}$ as an alternative set-theoretical universe for economic theory.

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## 1. Introduction

Economic theory is constituted by an ever-growing body of mathematical formulations developed in order to explain both the behavior of economies and of individual agents in them. In most of those formalisms the key question is to find *optimal* results. This is a direct consequence of assuming that the behavior of agents is guided by the principle of *rationality*. Moreover, even in the modeling of uncertain situations, the solutions that provide the explanations sought for by economic theorists are those in which the beliefs and expectations of the agents are consistent with rational behavior [Kreps 1990].

Even if most economic theorists agree with this point of view, some economists have shown concerns about the meaning of the theoretical constructions and particularly about the portion of reality that they represent.[1] The question they pose can be rephrased as: *Does the formal apparatus of economic theory pose extra requirements on the cognitive abilities of the agents, other than their rationality?* If so, can we be sure that these extra requirements do not amount abilities beyond those that can be deemed reasonable?

In this paper we will examine what we see as some of those extra abilities required to reach the conclusions advanced by the theory. Our focus of attention will be on the computational and set-theoretical requirements for the solutions to both individual and collective decision-making problems. It will be shown that some usual innocent looking claims follow only if we assume alternative set-theoretic frameworks (instead of the usual **ZFC** set theory).

Beyond the question of the cognitive requirements on agents we are also interested in the intrinsic mathematical problems that arise in the modeling of economic behavior. In that sense, to choose a set-theoretical framework just to solve an analytical problem may seem an instance of what Bertrand Russell called the "advantage of theft over honest toil". Our answer is that although there is some true to this, in the case of the modeling of agents, the choice of set theory implies to assume a given problem-solving ability that may be absent in another framework. Since one of the main goals of economic theory is to exhibit the form in which agents solve their individual and collective decision-making problems, we may resort to set-theoretic postulates just to represent some cognitive abilities.

On the other hand, the idea of changing the underlying set-theory just in order to make the mathematical constructions isomorphic to their intended real-world counterparts is not new. The entire program of research that Jon Barwise initiated on *situation calculus* and related formalisms, based on his idea that logic is a branch of applied mathematics, is clearly an inspiration for the approach we follow here [Barwise-Etchemendy 1987], [Barwise-Moss 1996],[Barwise-Seligman 1997].

Our goal is to present an approach to economic theory based on choosing as its foundation a set theory that preserves all the results widely accepted while eliminating the problematic ones. The idea is that all the economic entities with their intended properties should be represented in the underlying set theory. This has consequences beyond the mere rephrasing of previous results. New entities will have a right to exist in this framework, which lack legitimate counterparts in **ZFC**.

[1] A particularly influential alternative point of view was forcefully advanced by Herbert Simon, who claimed that human beings are only *bounded rational* and that instead of optimizing when deciding they just use *procedures* to find solutions that satisfy their needs [Simon 1982]. Even this position seems rather "mechanicist" for more radical critics [Mirowski 2002].

As said, an important aspect of this is that the alternative set-theoretical foundation should, on one hand, not give way to undesirable properties or entities, while at the other it should keep all the desirable results. Even if not exactly in these terms, the introduction of *O-minimal* structures as the right models for certain results ([Zame-Blume 1992],[Richter-Wong 2000]) points out toward the need for new foundational formal alternatives to the usual interpretation of theoretical results.

On the other hand, the approach followed here runs in the opposite direction of a wide ranging project pushed forward, in a series of papers of the 1980s and 1990s, by Alain Lewis [Lewis 1985],[Lewis 1990],[Lewis 1991],[Lewis 1992]. The main idea advanced there was that economic theories should be formulated in an *effective* framework. That is, that every entity or property defined in them should be computable. To make his case Lewis tried to show, in an impressive exhibition of scholarship, that the key notions in economic theory are not effective and therefore that they should be discarded. We take issue with this claim and with Lewis' program in general, and a good deal of this paper will be devoted to show how to overcome the limitations denounced by Lewis.

Even if our goal here seems utterly conservative, in the sense of rescuing economic theory of its formal drawbacks, the real objective of this research is pragmatic. We do not claim, as Gödel and other logicians before and after him that there exist a *true* set theory. We just look for the appropriate axioms for the representation of certain intuitions.[2] Our choice of axiomatization of the notion of set responds to the question of which are the foundations of mathematics that are more compatible with the abilities of agents, as represented in economic theory. In this sense, we are more than willing to admit the possibility of an alternative economic theory, with a completely different mathematical foundation [Mirowski 2002].

The set-theoretical framework we think is better fit for the current economic theory is $\mathbf{AFA}^{-} + \mathbf{AD} + \mathbf{DC}$. That is, the theory of sets that can be derived from the axioms of Zermelo-Frenkel set theory (except the axiom of Regularity, which is replaced by the Anti-Foundation axiom) as well as from the axiom of Dependent Choices (weaker than the Axiom Choice) and the Axiom of Determinacy. We claim that a lot of insight is gained in this switch while no important results, currently accepted, are lost.

In section 2 we will introduce a brief description of the problems and goals of economic theory as well as the main difficulties of logical nature we can find there. One of the most discussed problems is that of the computability of choice procedures, which will be analyzed in section 3. In section 4 we consider the problem of existence of competitive equilibria. In section 5 we analyze the problem of finding a common prior in situations of asymmetric information. Finally, in section 6 we will return to our initial discussion of the legitimacy of solving analytic problems in economics by means of a change of the underlying set theory.

## 2. The Main Problems in Economics

Although it is difficult to summarize the core of a discipline in a few words, there exists a consensus that the main questions in economics may be the following:

- How do agents make choices?
- How does so much order arise from individual choices?

[2]See a discussion on the justification of the axioms of set theory in [Maddy 1988b].

These two simple questions have been analyzed under the assumption that agents are *rational*. That is, that they have *preferences*, face *constraints* and choose options in such a form that their elections verify the constraints and are consistent with their preferences. In most models this means that the agents *maximize* their preferences over their constrained sets of options.

This last line of reasoning serves as the basis for the answer of the first question. We conclude that the entire schema of choices of the agents may be derived from the maximization of preferences varying the constraints (which represent the environment in which they have to choose). On the other hand, the second question actually asks for a precise characterization of the environment in which all the agents interact, in order to make their choices mutually consistent.

As an example of how these general problems have been attacked, let us consider an economy $\epsilon$ in which a finite number of agents interact. Each agent $i$ ($i = 1 \dots n$) can be identified with the triple $\langle X_i, \preceq_i, w_i \rangle$, where $X_i$ is the set of possible consumptions of $i$,[3] $\preceq_i$ is a transitive and complete ordering of $X_i$ (representing $i$'s preferences) and $w_i \in X_i$ are the initial endowments of the agent [Debreu 1959],[Arrow-Hahn 1971],[MWG 1995].

An additional information about this economy, in which the only allowed transaction is the exchange of goods, is that the variable representing the environment is the system of prices, $p$.[4] That is, the only information available to the agents about the entire economy is encoded in the prices. Moreover, no agent can modify them.

The decision of an agent $i$ is to choose an amount $x_i^* \in X_i$ that maximizes $\preceq_i$ over $B(p, w_i) = \{x \in X_i : p \cdot x \leq p \cdot w_i\}$. That is, $i$ will choose the amount of goods $x_i^*$ that is among the most preferred of the amounts that have the same or less value than the initial endowments.

If we vary the environmental variable, $p$, we change $x_i^*$. We obtain then a **demand** function, $x_i^*(p)$.[5] On the other hand, a market consistency condition is that the actual prices $p^*$ should verify that $\sum_{i=1}^n x_i^*(p^*) = \sum_{i=1}^n w_i$, that is, that the economy is in **equilibrium**. In other words, at the given equilibrium price, the amounts demanded by all the agents should be equal to the amounts that are available (the endowments). Notice that in order to find $p^*$ we have to know each $x_i^*(\cdot)$, i.e. the demand function, and not only the amount demanded at a given price.

This simple model abstracts the idea that agents are powerless, in the sense of not being able to modify the prices. But, since we do not assume the presence of an authority that enforces the equilibrium, when the number of agents is small this creates incentives for strategic behavior. That is, agents may declare their demands dishonestly, in order to achieve a certain system of prices that allows them to get to a more preferred amount. This raises the question of what conditions ensure that the equilibrium outcome cannot be improved by individual or group deviations.

Finally, one of the more basic considerations in the analysis of economic interactions is whether the agents are fully informed about the characteristics of the other agents. If not, the context is said of *incomplete information*. The only

[3] In general $X_i$ is assumed to be a subset of an Euclidean subspace $\Re^l_+$. We will follow this convention here.

[4] If $X_i \subseteq \Re^l_+$ then $p \in \Re^l_+$, i.e. there are as many prices as goods in the economy.

[5] Notice that for a given price $p$ there might exist several maximal elements for $\preceq_i$ over $B(p, w_i)$. We will not consider this possibility here and will assume that each $x_i^*(p)$ is a singleton.

possibility for the agents to coordinate an equilibrium arises (otherwise than by sheer luck) if they share a *common prior* [Binmore 1990], [Fudenberg-Tirole 1991], [Osborne-Rubinstein 1994]. That is, if all of them initially evaluate the possible situations with the same probability distribution over situations. After that, they may update their evaluations in different ways, according to the information they obtain during their interaction.[6] To achieve this, they have to converge to a shared assessment of the environment, the behavior of the other agents and their beliefs. The objective data plus the common assessment made by the agents is called a *state of the world*.

We can summarize the previous discussion saying that the main questions of economics may be reduced to solving the following formal problems:

1. To compute choice functions (like $x_i^*(p)$).
2. To determine the conditions in which an equilibrium–like $p^*$– exists.
3. To find out the conditions for the convergence of the beliefs of the agents to a common state of the world.

Each of the next three sections is devoted to one of these problems.

## 3. The Existence of Choice Functions

The hypothesis of rationality of the agents allows us to assume that each choice made by an agent is optimal, according to her preferences. Moreover, we assume that this is true for any given choice situation. Otherwise we would not be able to solve a problem like the determination of a market equilibrium. That is, we ask the choice function to be **realizable**, i.e. *computable* [Campbell 1978].

Given the well-known Church's Thesis, we have that the realizability of a choice function amounts to the existence of a Turing machine that yields the choice as an output when the environmental conditions are given as inputs. To make this a bit more precise, let us introduce some definitions.

Given a set of options $\mathbf{X}$, and $\mathcal{F}$ a subset of $2^{\mathbf{X}}$, a choice function $\mathcal{C}: \mathcal{F} \to \mathbf{X}$ is such that for each $\mathbf{B} \in \mathcal{F}$, $\mathcal{C}(\mathbf{B}) \in \mathbf{B}$. In words: for each of its feasible subsets $\mathbf{B}$, the choice function yields only one element in $\mathbf{B}$.

We assume that there exists a transitive and complete order over $\mathbf{X}$, denoted $\preceq$. Then, we say that $\mathcal{C}$ is realizable if there exists a recursive function[7] $f: \mathbf{X} \to N$, such that

- if $x, y \in \mathbf{X}$, $x \preceq y$ if and only if $f(x) \leq f(y)$
- for all $\mathbf{B} \in \mathcal{F}$, $\mathcal{C}(\mathbf{B}) = \{x \in \mathbf{B} : f(y) \leq f(x), \text{for all } y \in \mathbf{B}\}$.

$\mathcal{C}$ is said **recursively realizable** if given its *graph*, $\mathcal{G} = \{\langle \mathbf{B}, \mathcal{C}(\mathbf{B})\rangle\}_{\mathbf{B}\in\mathcal{F}}$ it exists a recursive function $\phi$ such that

$$\phi(\langle \mathbf{B}, \mathcal{C}(\mathbf{B})\rangle) = \begin{cases} 1 & if\ \langle \mathbf{B}, \mathcal{C}(\mathbf{B})\rangle \in \mathcal{G} \\ 0 & otherwise \end{cases}$$

The difference between the problem of the existence of $f$ and that of the existence of $\phi$ is crucial here. The existence of $f$ ensures that $\mathcal{C}$ is *recursively enumerable*

[6] This *Bayesian* conception can be relaxed, just assuming that agents conceive the same set of possible situations and may update it afterwards.

[7] That, is, there exists a Turing machine, that given two natural numbers, $x$ and $y$, answers **yes** to the question *"is $f(x)$ equal to $y$?"* if $y = f(x)$, and **no** otherwise.

(R.E.)[8] while the existence of $\phi$ yields the recursivity of $\mathcal{C}$. That is, if $\mathcal{C}$ is R.E. we know that given a set $\mathbf{B}$ the chosen option may be found, while if it is recursive we know **which** option is chosen in each $\mathbf{B}$.

In the context of our market example, if a demand function $x_i^*(\cdot)$ is R.E. we may determine, given a price $p$, the chosen consumption. If we know more, namely that $x_i^*(\cdot)$ is recursive, we are able to find the price system $p^*$ that yields the market equilibrium ($\sum_{i=1}^n x_i^*(p^*) = \sum_{i=1}^n w_i$).

The problem of whether $\mathcal{C}$ is recursive or not, leads to another, more general problem, which is to find to which class in the *Arithmetic Hierarchy* it corresponds. This hierarchy, which constitutes a form of classifying degrees of *uncomputability*, is defined as follows: given a set of natural numbers $\mathbf{A}$, consider a first-order formula in the language of the theory of numbers $\psi(x)$ such that $\mathbf{A} = \{x : \psi(x)\}$. Then [Putnam 1973], [Ash-Knight 2000]:

- $\mathbf{A}$ is in $\Sigma_0^0$ and in $\Pi_0^0$ if $\psi(\cdot)$ is a recursive predicate.
- $\mathbf{A}$ is in $\Sigma_n^0$ if $\psi(x) \equiv \exists y_1, \forall y_2 \exists y_3 \cdots \Psi(y_1, \ldots, y_n; x)$ where $\Psi(\cdots)$ is a recursive predicate.[9]
- $\mathbf{A}$ is in $\Pi_n^0$ if $\psi(x) \equiv \forall y_1, \exists y_2 \forall y_3 \cdots \Psi(y_1, \ldots, y_n; x)$ where $\Psi(\cdots)$ is a recursive predicate.

A set $\mathbf{A}$ is said recursive if its decision problem ("does $x$ belong to $\mathbf{A}$?") has a **yes** answer if $x \in \mathbf{A}$ and a **no** answer if $x \notin \mathbf{A}$. This answer is provided by a Turing machine, which computes the recursive characteristic function of $\mathbf{A}$. By definition, each recursive set is in $\Sigma_0^0$ and in $\Pi_0^0$.

$\mathbf{A}$ is R.E. if its characteristic function is R.E., i.e., its elements can be generated (enumerated) by a Turing machine. It follows that if $\mathbf{A}$ and its complement $\mathbf{A}^c$ are both R.E., $\mathbf{A}$ is recursive. A well-known result in Recursion Theory shows that any R.E. set $\mathbf{A}$ is in $\Sigma_1^0$, while $\mathbf{A}^c$ is in $\Pi_1^0$.

This correspondence between degrees of recursion and classes in the Arithmetic Hierarchy can be extended beyond $\Sigma_1^0$ and $\Pi_1^0$. A set $\mathbf{A}$ is said $n$-enumerable if it is in $\Sigma_n^0$. That is, it is such that its elements can be enumerated provided that the other elements in the corresponding $n$-ary relation that defines $\mathbf{A}$ have been enumerated. In other words, suppose $\mathbf{A}$ is a $\Sigma_n^0$ set, i.e. $x \in \mathbf{A}$ if and only if $\exists y_1, \forall y_2 \exists y_3 \cdots \Psi(y_1, \ldots, y_n; x)$ is true. But this means that all the admissible elements $y_1, \ldots, y_n$ have been already enumerated, in particular those of $y_1$ (the leading quantified variable). Since it must be shown that at least one value of $y_1$ verifies the expression, its entire course of values should be enumerable. A similar argument shows that $\mathbf{A}$ is in $\Pi_n^0$ (the complement of $\Sigma_n^0$) if the elements of its complement can be enumerated once the corresponding $y_1, \ldots, y_n$ are enumerated as well [Kleene 1943].

A straightforward property of this hierarchy is that if $\mathbf{A}$ is in $\Sigma_n^0$ ($\Pi_n^0$), it is also in $\Sigma_{n+1}^0$ ($\Pi_{n+1}^0$), for all $n$. This inductive characterization is valid even beyond natural ordinals. Given the *hyperarithmetical* sets, which are both in $\Sigma_1^1$ and $\Pi_1^1$, where the superscript 1 indicates that quantification is now defined over infinite sequences of natural numbers, we have that if $\mathbf{A}$ is in $\Sigma_n^0$ ($\Pi_n^0$) it is also in $\Sigma_1^1$ ($\Pi_1^1$).

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[8] This means, that there exists a Turing machine that can generate each and all the elements of the image of $\mathcal{C}$.

[9] The intuition here is that $\Psi(y_1, \ldots, y_n; x)$ defines a function $f^\Psi$ such that for some values of the variables $\bar{y}_1, \ldots, \bar{y}_n, \bar{x}$ yields $\bar{x} = f^\Psi(\bar{y}_1, \ldots, \bar{y}_n)$. Of course, $\bar{y}_1, \ldots, \bar{y}_n, \bar{x}$ are values that verify $\psi(\bar{x})$, i.e. $\exists y_1, \forall y_2 \exists y_3 \cdots \Psi(y_1, \ldots, y_n; \bar{x})$.

Given two sets **A** and **B** we say that **A** is *Turing reducible* to **B** if there exists a Turing machine that translates the problem of enumerating **A** into a decision problem for **B**. That is, if **B** were recursive, then **A** would be recursively enumerable. If **B** is in either $\Sigma^0_n$ or $\Pi^0_n$, **A** will be in $\Pi^0_{n+1}$ or in $\Sigma^0_{n+1}$, respectively. Then, if **A** is reducible to **B**, it is at least as complex as **B**.

With all these notions at hand, we can present the following negative result [Lewis 1985]:[10]

**Theorem 1.** *In the case that $X$ is the recursive representation of a compact and convex subset of $\Re^l_+$ (see* [Moschovakis 1964]*), $\phi$ may not be computable for a generic $\mathcal{C}$, i.e. the graph of $\mathcal{C}$ is not a recursive set.*

**Proof** (sketch): *If we assume that the graph of $\mathcal{C}$ is recursive, its image, $Im(\mathcal{C})$ should also be recursive. On the other hand, $Im(\mathcal{C}) \subseteq [\alpha^-, \alpha^+]$, where $\alpha^-, \alpha^+$ are Gödel numbers corresponding to recursively defined real numbers. But $[\alpha^-, \alpha^+]$ can be Turing reduced to the decision problem of $\mathcal{ALG}([\alpha^-, \alpha^+])$, the set of Gödel codes of algebraic numbers corresponding to elements in $[\alpha^-, \alpha^+]$. If so, the complexity of $Im(\mathcal{C})$ must be at least the same as that of $\mathcal{ALG}([\alpha^-, \alpha^+])$. According to a result in* [Shapiro 1956]*, $\mathcal{ALG}([\alpha^-, \alpha^+])$ is in $\Sigma^0_2$. Since $Im(\mathcal{C})$ is the projection of the graph $\mathcal{G}$ and is $\Sigma^0_2$, $\mathcal{G}$ must also be at least in $\Sigma^0_2$, and therefore it cannot be a recursive set.* $\square$

The key tool in the proof is Shapiro's Theorem $II.15$, which states that the characteristic function of intervals of algebraic real numbers is not computable [Shapiro 1956]. On the other hand, it is a bit at odds with intuition to claim that $[\alpha^-, \alpha^+]$ is Turing reducible to the decision problem of $\mathcal{ALG}([\alpha^-, \alpha^+])$. If we accept this claim, Theorem 1 shows a serious limitation to the ideal of the realizability of economic theory. In fact, a similar result is true in game theory, where not every winning strategy is computable [Rabin 1957]: consider a two-person, zero-sum, perfect information game $\Gamma$, defined in terms of a total recursive function $f : N \rightarrow N$:

- Player $I$ chooses $i \in N$.
- Player $II$, knowing $i$ chooses $j \in N$.
- Player $I$, knowing $i$ and $j$, chooses $k \in N$.

$\Gamma$ ends there. If $f(k) = i + j$, $I$ wins, otherwise, $II$ wins.

Now assume that $f$ enumerates a *simple* set $\mathbf{S} \subset N$, i.e.:

- $\mathbf{S}$ is infinite and R.E.,
- $N - \mathbf{S}$ is infinite and there does not exist a R.E. set $\mathbf{W} \subseteq N$ such that $\mathbf{W} \subseteq N - \mathbf{S}$.

Then we have:

**Theorem 2.** $\Gamma$ *has no computable winning strategy.*

[10] The tools of *Constructive Analysis* have been also applied to the analysis of the computability of choice functions and equilibria. This variant of Analysis has been characterized as the result of applying Intuitionistic Logic to classical mathematics [Bridges 1994]. In this sense, every entity to be defined must be *constructed* (not just shown to be contradiction-free). Using this approach, a negative result for economic theory has been found [Richter-Wong 1999]. On the other hand, rather simple alternative definitions of convexity help allow again to obtain positive results [Bridges 1992].

**Proof**: *Assume that $II$ has a computable winning strategy. Given $i$, $II$ chooses $j$ such that $i + j \notin \mathbf{S}$ ($\mathbf{S}$ is infinite in $N$). That is, $II$'s strategy can be described by a function $\tau : N \to N$ such that $i + \tau(i) \notin \mathbf{S}$, for all $i \in N$. It follows that for every choice $k$ of $I$, since $f(k) \in \mathbf{S}$ and $i + j = i + \tau(i) \notin \mathbf{S}$, $II$ wins. But then, since $w(i) = i + \tau(i)$ is R.E., we have that its image, $\mathbf{W}$, is a R.E. set such that $\mathbf{W} \subseteq N - \mathbf{S}$. Absurd, since $\mathbf{S}$ is simple.* □

This result indicates that even if it is possible to describe the game for each possible sequence of plays and determine its winner (by means of $f$), there is no recursive function that yields 1 if the choice of $II$ leads her to win and 0 otherwise.

A possibility we want to explore is whether the change of the underlying set theory may lead to a positive result for the computability of choice functions. As a first step, let us consider, like in Theorem 2, a *Gale-Stewart* game, i.e. a zero-sum, perfect information game in which two players choose natural numbers and one wins if she can lead the sequence to be in a certain set and looses otherwise. To define this game, $\Gamma_{\mathcal{C}}$, we will consider the same prerequisites as those for Theorem 1: we have the recursive presentation of $\mathcal{C}$ over $X$, $f$ and a recursive presentation of the domain of $\mathcal{C}$, $\mathcal{F}$, i.e. a recursive function $F$ such that $F(\mathbf{B}) = 1$ if $\mathbf{B} \in \mathcal{F}$.

In $\Gamma_{\mathcal{C}}$, Agent $I$ (the *spoiler*) chooses a subset $\mathbf{B} \in \mathcal{F}$. $II$ replies with an element $x \in \mathbf{B}$, and player $I$ chooses $y \in \mathbf{B}$. $I$ wins if $f(y) \geq f(x)$ and $II$ wins otherwise. If $II$ has a winning strategy it must consist in choosing $x^* = \mathcal{C}(\mathbf{B})$, for any $\mathbf{B} \in \mathcal{F}$.

In $\Gamma_{\mathcal{C}}$, $II$ has always a winning strategy as indicated by *Zermelo's Theorem*, which claims that every game, in which the number of moves is finite the players have complete information of possible actions and payoffs, is **determined**, i.e. one of the players can always win [Steinhaus 1965].

Theorem 1 may be reinterpreted as stating–very much like Theorem 2–that winning strategies for the representation of choice functions are not necessarily computable. But, since $f$ is recursive, we can define a recursive relation $P$ over $\mathcal{F} \times X \times X$ such that $P(\mathbf{B}, x, y)$ is equivalent to, $x, y \in \mathbf{B}$ and $f(x) \geq f(y)$. Then, a pair $\langle \mathbf{B}, x \rangle \in \mathcal{G}$ must verify $\forall y P(\mathbf{B}, x, y)$. Although this expression does not correspond to a set in $\Sigma^0_2$ (it is in $\Pi^0_1$, if $\mathbf{B}$ is replaced by its corresponding index under $F(\cdot)$) it characterizes $\mathcal{G}$. Even if indeed $\mathcal{G}$ were not recursive, the following result gives a clue of how to make it recursive in a winning strategy of $\Gamma_{\mathcal{C}}$ [Blass 1972]:

**Proposition 1.** *If $\mathbf{A}$ is an hyperarithmetical set–it is both $\Pi^1_1$ and $\Sigma^1_1$–it is recursive in every winning strategy of an associated game.*

This means that, if there exists a winning strategy $\sigma$ for one of the two players in a Gale-Stewart game based on $\mathbf{A}$, then for a given $\mathbf{a}$, $\mathbf{a}$ is in $\mathbf{A}$ if and only if $\sigma(\mathbf{a})$ leads to a win.[11] Notice that this result can be also applied to the non-computable winning strategy in Theorem 2: we define a game (which is not the same $\Gamma$!) to characterize $\tau(\cdot)$. If a winning strategy exists in this new game, then $\tau$ is recursive in it.

Proposition 1 can be trivially applied to $\Gamma_{\mathcal{C}}$, since its class, be it either $\Sigma^0_2$ as claimed by Lewis, or $\Pi^0_1$ as we argued above, is trivially hyperarithmetical due to the properties of the Arithmetic Hierarchy. This means that, if there is a winning

[11] By $\sigma(\mathbf{a})$ we denote the series of choices prescribed by $\sigma$ given a potential candidate for being an element of $\mathbf{A}$.

strategy for $II$ in $\Gamma_{\mathcal{C}}$, this winning strategy defines a recursive set. That is, $\mathcal{C}$ will be computable.[12]

Therefore, the only condition that needs to be met to ensure the computability of $\mathcal{C}(\cdot)$ is the existence of a winning strategy for $II$. As we have shown, Zermelo's Theorem ensures its existence. But on the other hand, notice that our characterization of $\Gamma_{\mathcal{C}}$, based on the hypotheses of Lewis, assumes that $x^*$ has in fact a corresponding Gödel number. In the more general case, we have to think that agents may have to come up with successive approximations to the actual chosen $x^*$. In this case we can no longer maintain the assumption of a finite (bounded) number of moves required to the application of Zermelo's Theorem. Therefore, a more general condition is needed, ensuring the existence of winning strategies. One form to achieve this is by including in our set theory the **Axiom of Determinacy**(**AD**). **AD** states that each Gale-Stewart game is **determined**, i.e. there exists a winning strategy for it.[13] On the other hand, the addition of **AD** to our underlying set theory forces us to drop the Axiom of Choice, although it is widely assumed that it is consistent with the weaker **DC** (Axiom of Dependent Choices).[14] That is, we can assume that for any binary relation $R \subseteq \mathbf{X} \times \mathbf{X}$ (for any $\mathbf{X} \neq \emptyset$), if for any $x \in \mathbf{X}$ there exists $y \in \mathbf{X}$ such that $\langle x, y \rangle \in R$, then there exists a countable sequence $(x_n)_{n \in N} \subseteq X$ such that $\langle x_n, x_{n+1} \rangle \in R$ for every $n$.[15]

Therefore, if we say that a set is **win-recursive** if there exists a winning strategy for an associated game that makes the set recursive, we have that:

**Lemma 1.** *In* $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$, $\mathcal{C}$ *is win-recursive.*

To switch to this set theory means, in the context of choice functions in economics, that we assume that the internal deliberation process that leads to a choice must reach a result, and moreover, that under the same conditions the result must also be the same. A trivial form in which this could happen is if agents compute their choices with a certain bound in precision. Once an outcome has that degree of precision (say in solving a maximization problem using an approximation method), it is considered the chosen option. This may be a formal form to present Norbert Wiener's quip that arithmetic in economics does not require more than two decimal digits [Wiener 1964]. In this case the range of values is discrete, requiring therefore only a rather simple set theory.

Another form to ensure the computability of $\mathcal{C}(\cdot)$ is by means of an *oracle*. An *oracle* for a function $\alpha : N \to N$ is a device that, given $x \in N$, responds with the value $\alpha(x)$ [Enderton 1977]. So, a Turing machine that requires as an intermediate step of its computation the value of an arbitrary function over $N$ can be empowered

[12] Related arguments, in rather different contexts, lead to similar conclusions [Canning 1992], [Mihara 1997].

[13] The exact characterization of **AD** is a bit more complicated than this, but for our purposes this is enough [Mycielski-Steinhaus 1962], [Fenstad 1971], [Jech 1973],[Mycielski 1992].

[14] The incorporation of **AD** and the loss of **AC** has important consequences, one of them being that every set of real numbers becomes Lebesgue-measurable [Jech 1973].

[15] **DC** is, in **ZF**, equivalent to the *Principle of Recursive Constructions*, i.e., given $\mathbf{X} \neq \emptyset$ and the set of all finite sequences in $\mathbf{X}$, denoted ${}^{<\omega}\mathbf{X}$, if we have a function $G : {}^{<\omega}\mathbf{X} \to \mathcal{P}(\mathbf{X}) \setminus \{\emptyset\}$, where $\mathcal{P}(\mathbf{X})$ is the class of subsets of $\mathbf{X}$, then there exists $f_G : N \to \mathbf{X}$ such that $f_G(0) \in G(\emptyset)$ and $f_G(n) \in G(f_G(0), \ldots, f_G(n-1))$, for all $n > 0$ [Just-Weese 1996]. The importance of this equivalence for us is that the proof of Proposition 1 is based on the recursive construction of an ordering of the elements of a hyperaritmethical set $\mathbf{A}$ and another for those in its complement $\mathbf{A^c}$. Our argument would be meaningless if the very tool we use is not available in the new framework.

allowing it to consult an oracle for that function. Oracles are required, in fact, to perform Turing-reductions, i.e. to reduce the complexity of computations.

If we assume that computability is extended by means of oracles, we can make the following claim:

**Lemma 2.** *Given a choice function $\mathcal{C}$, with graph $\mathcal{G}$, the function $\phi$ such that*

$$\phi(\langle \mathbf{B}, \mathcal{C}(\mathbf{B})\rangle) = \begin{cases} 1 & if\ \langle \mathbf{B}, \mathcal{C}(\mathbf{B})\rangle \in \mathcal{G} \\ 0 & otherwise \end{cases}$$

*can be computed by a Turing machine with an oracle for $x^*$ the function that yields the winning strategy for $\Gamma_{\mathcal{C}}$, given the index of $\mathbf{B}$.*

The proof is straightforward. It follows from Proposition 1 and the characterization of a game (either with a finite number of moves or not) $\Gamma_{\mathcal{C}}$, associated to $\mathcal{C}$. It is rather immediate to conjecture that Lemmas 1 and 2 indicate a certain equivalence between "win-recursivity with oracles" and "win-recursivity in $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$". The intuition here is that agents able to solve a (economic) choice problem act as if they were computing with oracles, and that these oracles are able to yield the winning strategies for Gale-Stewart games.

In any case, these results indicate that a crucial step in the determination of a choice is the internal deliberation process (represented by a game $\Gamma_{\mathcal{C}}$). If there exists a plan to reach a conclusion in that process (a winning strategy), and this plan can be specified (be it trough **AD** or by means an oracle), it becomes trivial to compute a choice for each possible set of options.

We cannot leave the topic of the (un)computability of choice functions without a brief analysis of another kind of complexity. On one hand, by the arguments we gave using **AD**, the complexity of $\mathcal{C}$ equals that of a Turing machine. Another kind of complexity is defined in terms of the number of steps needed to get to a result. Given a choice set $\mathbf{B}$, if the number of bits in its description is $m$, we say that it is *polynomial* if the number of steps needed to calculate its Gödel number is $\mathcal{O}(t^m)$, i.e. a polynomial of the type $\alpha_m t^m + \alpha_{m-1} t^{m-1} + \cdots + \alpha_1 t + \alpha_0$. It would be highly desirable that if $\mathcal{C}(\mathbf{B})$ is describable with, say, $r$ bits, the number of steps to compute its Gödel number were also polynomial, i.e. its complexity were $\mathcal{O}(t^r)$. It seems that it may not be so [Friedman 1984]:

**Theorem 3.** *The outcome of an optimization problem over polynomial inputs is polynomial if and only if $\mathbf{P} = \mathbf{NP}$.*

That is, the encoding of the outcome of an optimization over a polynomial input is polynomial only in case that all the problems that have solutions that can be checked out in a polynomial number of steps can also be solved in a polynomial number of steps.[16] As it is well known, whether $\mathbf{P}$ is equal or not to $\mathbf{NP}$ is an open problem, but there is a consensus among experts that the answer is for the negative. If so, according to Friedman's result, we should expect that the solution to an optimization problem over polynomial inputs will not be polynomial. In other words, the optimization may be *intractable*, being practically feasible only for small inputs. If we recall that $\mathcal{C}(\mathbf{B})$ is the solution to a maximization problem we can see the relevance of this problem for economic theory.

[16]The intuition is that, while the actual solution may be easily describable, its computation–as the result of the process of maximization–may require a high number of steps.

While the assumed $\Sigma_2^0$ complexity of choice functions may be "reduced" by means of a change of set theory (from **ZFC** to **ZF** + **AD** + **DC**),[17] there is no equivalent change, to our knowledge, that may reduce the complexity of encoding $\mathcal{C}(\mathbf{B})$ if $\mathbf{P} \neq \mathbf{NP}$. This branch of complexity theory involves only finite numbers and therefore it must be common to any set theory we might choose.[18]

## 4. Existence of Equilibria

Market equilibria are fundamental entities in economic theory. If an economy is not in equilibrium, there are few characteristics of that economy that can be ascertained. On the other hand, the mere existence of an equilibrium does not ensure (if not accompanied by proofs of stability and/or uniqueness) that the economy will end up there. But, in any case, to prove their existence is the first step towards the development of a more general model of economic interactions.

Kenneth Arrow and Gerard Debreu, applying the notion of *Nash equilibrium* found the first general proof of existence of equilibria. Further work by those authors led to a refinement of the proof. They got rid of the need of specifying a fictitious game (in order to find a Nash equilibrium) by using the main formal tool behind Nash theorem. That is, they used Kakutani's fixed-point theorem to determine the conditions that ensure the existence of a fixed point. The key of the proof is to define an *excess demand* function. In our leading example, $z(p) = \sum_{i=1}^n x_i^*(p) - \sum_{i=1}^n w_i$. Since $z(p) \in \Re_+^l$, if the price of good $j$ is such that its corresponding entry in the excess demand vector is $z_j(p) > 0$, it is increased from $p_j$ to $p_j^{'} > p_j$. If instead $z_j(p) < 0$ then $p_j^{'} < p_j$, while if $z_j(p) = 0$, $p_j$ remains unchanged. Therefore, we have a correspondence $F$ that takes $p$ as argument and yields $p^{'}$ (given $p$ it determines $z(p)$ and then changes the prices according to the signs of its coordinates). An equilibrium is just a $p^*$ such that $p^* \in F(p^*)$. Standard properties of preferences, which translate into the functional form of $F(\cdot)$, ensure the existence of such $p^*$. Variants of this argument, involving all the available fixed-point theorems, have been applied to prove the existence of equilibria in different market structures.[19]

If we consider $\mathbf{E}$, the class of economies with a finite number of agents and endowments in a finite-dimensional Euclidean space, we can distinguish the corresponding space of prices $\Delta_{\mathbf{E}}$. Then, the existence of an equilibrium amounts to claim that there exists a well defined correspondence $\mathcal{EQ} : \mathbf{E} \rightarrow \Delta_{\mathbf{E}}$, such that for $\epsilon = \{\langle X_i, \preceq_i, \omega_i, \rangle\}_{i=1}^n$, if $p^* \in \mathcal{EQ}(\epsilon)$, then $\sum_{i=1}^n x_i^*(p^*) - \sum_{i=1}^n w_i \ = \ 0$. Nice as this sounds, the following result casts doubts on its meaningfulness [Lewis 1992]:

**Proposition 2.** *The graph of $\mathcal{EQ}$ is not recursive.*

**Proof**: *Since each equilibrium is found considering the individual demand functions, $x_i^*(\cdot)$, $\mathcal{EQ}$ is Turing-reducible to their sum. But each demand function is a choice function. Then, the complexity of $\mathcal{EQ}$ is at least that of a $\mathcal{C}$ (as discussed in*

[17] In the new set-theoretic framework a winning strategy makes an associated game $\mathcal{C}(\cdot)$ recursive, according to Lemma 1. Therefore, some variables can be replaced by functions of the parameters of the problem.

[18] Unless $\mathbf{P} = \mathbf{NP}$ is shown undecidable [Tsuji-Da Costa-Doria 1998]. If so, there might be models of our set-theory in which it is true and others in which not. Extra axioms may make the claim decidable. But this is not an alternative that most theoretical computer scientists consider seriously [Pudlák 1996].

[19] Even the topology-free Fixed-Point Theorem of Tarski has been shown useful [Tohmé 2003].

*section 3). According to Theorem 1, a $\mathcal{C}$ is in the class $\Sigma_2^0$. Therefore, $\mathcal{EQ}$ cannot be either in $\Sigma_0^0$ or $\Pi_0^0$, i.e. be recursive.* □

This result, obviously true if we accept Theorem 1, can be easily disposed again by means of **AD**. On one hand, we get rid of the uncomputability of $\mathcal{C}$, as we argued in section 3. On the other, just consider a game between "the economy" and a fictitious player, usually known in the literature as the "Walrasian auctioneer". For each economy $\epsilon$, say with commodity space $\Re^l_+$, the auctioneer begins by proposing a $p \in \Re^l_+$. The economy responds by declaring the vector of demands $(x_1^*(p), \ldots, x_n^*(p))$. If $z(p) = 0$ the auctioneer wins, otherwise he proposes another system of prices $p^{'} \in \Re^l_+$ and the economy responds with the corresponding demands, etc. The game ends when the auctioneer wins, i.e. at a $p^*$ such that $z(p^*) = 0$. Notice that in this game, the conditions of Zermelo's Theorem do not apply, but the existence of a winning strategy for the auctioneer is ensured by **AD**. Therefore, $z(\cdot)$ is recursive in the winning strategy of the auctioneer. As a consequence:

**Lemma 3.** *In* $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$*,* $\mathcal{EQ}$ *is win-recursive.*

Another line of attack to the problem of existence of equilibria has also a long tradition in economics. Francis Y. Edgeworth initiated this approach to the analysis of exchange economies by introducing the notion of the **core** of an economy, that is, the set of resource allocations that cannot be improved upon for at least one agent without impoverishing the others. Edgeworth advanced the idea, later known as the *Core Equivalence Conjecture*, that the core of an competitive economy with a large number of traders is identical to the set of equilibrium allocations. The various proofs of this conjecture have been frequently cast as limit results. The key idea in those proofs is to make the relative importance of each agent decrease when the number of agents increases. One of the approaches consisted in assuming that in the limit there exists a continuum of agents, for example indexing them with the real numbers in the closed interval $[0, 1]$, a setting in which measure theory can be applied [Aumann 1966].

In the case in which the number of agents is uncountable, the economy is seen as the limit of sequences of economies in which the measure of non-competitiveness converges to zero. In terms of the cores of those economies it means that they become "smaller" until, in the limit economy, the core has measure zero and coincides with the set of equilibria in the economy [Hildenbrandt 1974]. A closely related approach consisted in assuming that in the limit economy each agent is indexed by an infinitesimal number. In that setting the tools of non-standard analysis were applied to find an analogous result [Brown-Robinson 1975].

Besides measure theory and non-standard analysis other mathematical tools have been applied to proving versions of the Core Equivalence Theorem. This includes proofs using only elementary arguments. They begin by defining an economy like the one we presented in section 2. That is, a $n$-agent economy $\epsilon_n$ is:

$$\epsilon_n \ = \ \{\langle X_i, \preceq_i, w_i\rangle\}_{i=1}^n$$

An *allocation* is defined as $x \ \in \ X$, where $X \ = \ X_1 \times \ldots \times X_n$. An allocation $x$ is said *feasible* if $\sum_{i=1}^n x_i \ = \ \sum_{i=1}^n w_i$. Then, the core of $\epsilon$ is defined as:

$$\mathcal{COR}(\epsilon) \ = \ \{\bar{x} \in X : \text{ there is no } S \subseteq \{1, \ldots, n\} \text{ and } x \in X \text{ such that}$$

$$\sum_{i\in S} x_i = \sum_{i \in S} w_i \text{ and } \bar{x}_i \preceq_i x_i \text{ for all } i \in S\},$$

that is, $\bar{x}$ is in the core only if it is no other allocation, feasible for at least one group of agents, that makes them better off.

As the number of agents in $\epsilon_n$ is finite, we do not expect the core to coincide with an equilibrium allocation. But, if we consider a sequence of economies that converges in the sense that their cores shrink, then the result is that in the limit, the core has a single allocation $\bar{x}$, which coincides with the equilibrium allocation, $x^*(p^*)$.

The interest in elementary proofs resides in that they can be developed, in principle at least, without the aid of advanced mathematical tools. Even so, they have been exposed to a certain amount of criticism. As briefly noted in the Introduction, Alain Lewis pushed forward for the exclusive use of *effective* methods in economic theory. A first step in that direction was for him to restrict the mathematical arguments to be part of only *ordinary mathematics*, meaning the fragment of **absolute** first-order logic formulas in the language of set theory. That is, formulas of the form $\psi(x_1, \ldots, x_n)$, with free variables $x_1, \ldots, x_m$ such that for any model of **ZF**, $\mathcal{M}$, for all constants $a_1, \ldots, a_m \in \mathcal{M}$, $\mathcal{M} \models \psi(a_1, \ldots, a_m)$ if and only $\mathcal{M} \models \psi_{|\mathcal{M}}(a_1, \ldots, a_m)$, where $\psi_{|\mathcal{M}}$ is $\psi$ with its bounded (i.e. in the scope of a quantifier) variables also restricted to be in $\mathcal{M}$ [Just-Weese 1996].[20]

Lewis concern was that the proofs and arguments should be effective, that is, that they must only involve recursively defined steps. Since ordinary mathematics does not use the non-effective **AC** he found that **ZF** was the right setting for his program. In the cases in which a direct proof of non-recursiveness was not available he tried to show that there exist a model of **ZF** in which the involved notions are not true (indicating that they are not part of ordinary mathematics) and therefore that they could not be effective.

Using this kind of arguments, Lewis made the claim that even the elementary core equivalence results were not effective [Lewis 1991]. Let us analyze this claim. If we let $\mathbf{M}_{\epsilon_n} = \max_{S \subseteq \{1,\ldots,n\}} \max_{j=1,\ldots,l}\{\sum_{i\in S} w_i^j\}$ we have that [Anderson 1978]:

**Lemma 4.** *If $\bar{x} \in \mathcal{COR}(\epsilon_n)$ there exists a system of prices $p^*$ such that:*

- $\sum_{i=1}^n |p^* \cdot (\bar{x}_i - w_i)| \leq 2\mathbf{M}_{\epsilon_n}$.
- $\sum_{i=1}^n |\inf\{p^* \cdot (x_i - w_i) : \bar{x}_i \preceq_i x_i\}| \leq 2\mathbf{M}_{\epsilon_n}$.

Of course, if $\bar{x} \in \mathcal{COR}(\epsilon)$ already coincides with an equilibrium allocation $\langle x_1^*(p^*), \ldots, x_n^*(p^*)\rangle$ corresponding to the system of prices $p^*$ this lemma follows immediately. More generally, if we have a sequence of economies $\{\epsilon_n\}_{n>1}$ such that:

- $\frac{\mathbf{M}_{\epsilon_n}}{n} \to 0$.
- $\sup_n \frac{\max_{k=1\ldots l}\{\sum_{i=1}^n w_i^k\}}{n} < \infty$.

with a few additional technical conditions, including Lemma 4, we have the following result [Anderson 1981]:

**Theorem 4.** *There exists a sequence of prices $\{p_n^*\}_{n>1}$ such that for all $S^n \subseteq \{1, \ldots, n\}$ we have that:*

[20] A whole research program, known as *Reverse Mathematics* consists in searching for a set-theoretical foundation, less general than **ZF**, for ordinary mathematics [Friedman 1981].

$$\frac{\max_{k=1,\ldots,l}\{|\sum_{i\in S^n}\bar{x}_i^j - x_i^j(p_n^*)|\}}{n} \to 0.$$

That is, when $n$ grows, the number of goods for which the core allocation differs from the equilibrium allocation decreases. In the limit the core coincides with the equilibrium allocation. But we have that [Lewis 1992]:

**Proposition 3.** *There exists a model of* **ZF**, $\mathcal{M}$, *in which, given the conditions of Theorem 4, every sequence* $\{\epsilon_n\}_{n>1}$ *is such that for each* $n$, $\mathcal{COR}(\epsilon_n)$ *is either* $\emptyset$ *or includes at most one allocation.*

The proof given by Lewis is that if at least for one $n$, $|\mathcal{COR}(\epsilon_n)| > 1$, then there must exist a choice function $g : \omega \to \cup_n \mathcal{COR}(\epsilon_n)$, such that $g(n) \in \mathcal{COR}(\epsilon_n)$. Since this requires the application of **AC**, and this axiom is independent of **ZF**, there exists a model of **ZF** in which $g$ cannot be defined.

This argument can be criticized on several grounds, but let us concentrate on Lewis' ultimate goal of showing that the Core Equivalence cannot be proven effectively. In this sense, if a non-effective axiom is required then we cannot expect the outcome to be effective. But, in fact, the existence of $g$ does not require the full **AC** but only a much weaker form, called Axiom of Countable Choice (**CC**) that allows to select an element from a each set in a countable class.

**CC** can be derived from **DC** [Just-Weese 1996]. Therefore, if we assume **DC** it is enough to ensure the existence of $g$. Of course **DC** is independent of **ZF** and Proposition 3 will still stand. But as remarked above, **DC** is equivalent in **ZF** to the Principle of Recursive Constructions. That is, it is effective. And therefore, non-trivial sequences fulfilling the conditions of Theorem 4 can be effectively defined in **ZF** + **DC**.

Interestingly enough, Lewis overlooks a more interesting source of possible problems for the validity of Theorem 4. The proof of Lemma 3 uses a variant of Hahn-Banach theorem, Minkowski's theorem, that ensures that two disjoint convex sets can be separated by an hyperplane. Only one variant of the Hahn-Banach theorem can be proved in **ZF**, the so-called Finite Extension Lemma (FEL) [Schechter 1997].[21] But FEL is too weak to imply Minkowski's Theorem ($\Re^l$ is not a convex hull of the union of any of its hyperspaces and a finite number of points). Therefore Lemma 3 requires more than **ZF** to be true. But then, it is known that **DC** implies the Hahn-Banach theorem in separable spaces [Bridges 1994]. Since Euclidean spaces like $\Re^l$ are separable, in **ZF** + **DC** Lemma 3 can be effectively established.

Another form of the Core Equivalence result can be obtained considering **market games**. This approach permits us to translate the structure of an economy into a coalitional game form. Let us consider an economy

$$\epsilon_n = \{\langle X_i, u_i, w_i\rangle\}_{i=1}^n$$

where each $u_i : X_i \to \Re$ is a representation of the preferences $\preceq_i$, i.e., for $x, y \in X_i$, $u_i(x) \leq u_i(y)$ if and only if $x \preceq_i y$.

[21] The version of FEL that is of interest for us is as follows: *Suppose that over a linear subspace* $\mathbf{H} \subseteq \Re^l$, *a linear map* $f^0 : \mathbf{H} \to \Re^l$ *is such that* $f^0 \leq p$, *where* $p$ *is a convex map* $p : \Re^l \to \Re^l$. *Suppose furthermore that there exists* $\mathbf{S} = \{s_1, \ldots, s_m\} \subset \Re^l$ *such that* $\Re^l$ *is the convex hull of* $\mathbf{H} \cup \mathbf{S}$. *Then,* $f^0$ *can be extended to* $f : \Re^l \to \Re^l$, *such that* $f \leq p$ *over* $\Re^l$.

Then, a market game is a cooperative (or coalitional) game $\Gamma = \langle \mathbf{I}, A(\mathbf{I}), \mu \rangle$, where $\mathbf{I} = \{1, \ldots, n\}$ is the set of agents, $A(\mathbf{I}) \subseteq 2^{\mathbf{I}}$ the class of coalitions of agents and $\mu : A(\mathbf{I}) \rightarrow \Re$ is the payoff function. Given $\epsilon_n$, a game $\Gamma$ can be defined such that for each coalition $\mathbf{S} \subseteq \mathbf{I}$, $\mu(\mathbf{S}) = \max_{x^{\mathbf{S}}} \sum_{i \in \mathbf{S}} u_i(x_i^{\mathbf{S}})$, where $x^{\mathbf{S}} = \sum_{i \in \mathbf{S}} x_i^{\mathbf{S}}$ such that $\sum_{i \in \mathbf{S}} x_i^{\mathbf{S}} = \sum_{i \in \mathbf{S}} w_i$. In words, the payoff to a coalition $\mathbf{S}$ is the best sum of utilities that the members of the coalition can achieve in a redistribution of their endowments.

The core of the market game $\Gamma$ is

$$\mathcal{COR}(\Gamma) = \{(\bar{\mu}_1, \ldots, \bar{\mu}_n) : \text{for all } \mathbf{S} \in A(\mathbf{I}), \sum_{i \in \mathbf{S}} \bar{\mu}_i \geq \mu(\mathbf{S})\}$$

That is, the core consists of the vector of individual payoffs that cannot be improved upon by any coalition. Given the endowments associated to $\Gamma$, $\{w_i\}_{i \in \mathbf{I}}$, the core corresponds to their redistribution among the agents in $\mathbf{I}$ such that no individual nor coalition could get a higher payoff in another distribution without reducing the payoffs corresponding to other agents. In **ZFC** we have that [Shapley-Shubik 1969]:

**Theorem 5.** *Every market game has a non-empty core.*

A proof of this result is based on the simple fact that if $x^*$ is an equilibrium allocation in the economy $\epsilon_n$, then if for each $i$, $\bar{\mu}_i = u_i(x_i^*)$, then $(\bar{\mu}_1, \ldots, \bar{\mu}_n) \in \mathcal{COR}(\Gamma_n)$. And this can be proven entirely in **ZF**. Of course, this last proof assumes that a market game is associated with an economy. When this connection is not assumed a proof can be given using again a form of Minkowski's Theorem. We already discussed that this obtains in $\mathbf{ZF} + \mathbf{DC}$.

A Core Equivalence result requires to consider a countable sequence of market games, $\{\Gamma_n\}_{n<\omega}$, in which each $\Gamma_n$ is $\langle \mathbf{I}_n, A(\mathbf{I}_n), \mu^n \rangle$ and the sizes of the cores shrink. The limit of this sequence must be a game $\Gamma^*$ such that $\mathcal{COR}(\Gamma^*)$ is a singleton (corresponding to the equilibrium allocation). In **ZF** this sequence may not exist [Lewis 1990]:

**Proposition 4.** *There exists a model $\mathcal{M}$ of* **ZF** *in which there exists an infinite class of market games $\{\Gamma_n\}_{n<\omega}$ all of which verify that $\mathcal{COR}(\Gamma_n) = \emptyset$.*

**Proof** *(Sketch): Consider a model of* **ZF**, *$\mathcal{M}$ such that a measure $\Upsilon$ is defined verifying that $\Upsilon(\mathbf{S})$ is either 1 or 0, for each $\mathbf{S} \subseteq \omega$. Furthermore, $\Upsilon(\mathbf{S}) = 0$ for every finite set $\mathbf{S}$. Then, for each $\mathbf{I}_n$ define a game $\langle \mathbf{I}_n, A(\mathbf{I}_n), \mu^n \rangle$, where $\mu^n$ is the restriction of $\Upsilon$ on $2^{\mathbf{I}_n}$ (i.e. $\mu^n(\mathbf{S}) = 0$, for every $\mathbf{S} \subseteq \mathbf{I}_n$, $\mathbf{S} \neq \mathbf{I}_n$, while $\mu^n(\mathbf{I}_n) = 1$). In each of these games no allocation yields a better outcome than the grand coalition of all the agents. That is, the core is empty.* □

A model of **ZF** with this property was originally obtained by Paul Cohen, using his method of **forcing**. Cohen produced a model in which every Boolean Algebra $2^\omega$ admits a real measure $\Upsilon : 2^\omega \rightarrow \Re$ which is either 0 or 1, that vanishes over all the finite subsets of $\omega$ [Jech 1973].[22]

In turn, the existence of a bi-valued measure $\Upsilon$ over the Boolean Algebra $2^\omega$ that vanishes over all the finite sets implies in **ZF** that there exists a non-measurable

[22]The *forcing conditions* that yield $\mathcal{M}$ are the finite functions $f : \omega \times \omega \rightarrow \{0, 1\}$. Then, real numbers are defined as

$$\alpha_n = \{m \in \omega : \exists f, f(n, m) = 1\}$$

$\mathcal{M}$ is the submodel of Cohen's that contains all the sets $\alpha_n$, but not the collection of all of them $\bar{\mathbf{A}} = \{\alpha_n : n \in \omega\}$ [Pincus 1973],[Jech 1973].

subset of $\omega$ [Sikorski 1969]. On the other hand, in $\mathbf{ZF}+\mathbf{AD}+\mathbf{DC}$ every set of real numbers is measurable. [23] In fact, we have that

**Proposition 5.** *In $\mathbf{ZF}+\mathbf{AD}+\mathbf{DC}$, given a sequence of economies $\{\epsilon_n\}_{n\in\omega}$, with each $\epsilon_n = \{\langle X_i, u_i, w_i\rangle\}_{i=1}^n$ such that:*

- *$\sum_{i=1}^n w_i = \mathbf{w}$ for all $n$ (i.e. the available endowments are the same for every economy in the sequence).*
- *For each pair of economies $\epsilon_n$ and $\epsilon_{n+1}$ and for each $i \leq n$, the corresponding endowments $w_i^n$ and $w_i^{n+1}$ verify that $w_i^{n+1} \leq w_i^n$.*
- *$\Upsilon(\mathcal{COR}(\epsilon_n)) \to_n 0$ in any suitable notion of measure $\Upsilon$ over Euclidean spaces.*

*Then, to each $\epsilon_n$ there corresponds a market game $\Gamma_n$, with a non-empty core, such that the sequence of market games $\langle\Gamma_n\rangle_{n<\omega}$, converges to a limit game $\Gamma^* = \langle\omega, A(\omega), \mu\rangle$ in which $\mathcal{COR}(\Gamma^*)$ is a singleton $\{(\bar{\mu}_i)_{i\in\omega}\}$ and there exists an allocation $x^*$ that verifies for each $i$, $u_i(x_i^*) = \bar{\mu}_i$, which is an equilibrium allocation in a competitive economy $\epsilon^*$, the limit of $\{\epsilon_n\}_{n\in\omega}$.*

**Proof**: *Let us consider a sequence of market games $\langle\Gamma_n\rangle_{n<\omega}$, where each $\Gamma_n$ obtains from the economy $\epsilon_n$. Notice that each $\Gamma_n$ has an associated payoff function $\mu^n$ that verifies that for every pair of finite sets $\mathbf{S}, \mathbf{T} \in 2^\omega$ such that $\mathbf{S} \subseteq \mathbf{T}$, $\mu^n(\mathbf{S}) \leq \mu^n(\mathbf{T})$. This follows from the definition:*

$$\mu^n(\mathbf{T}) \;=\; \max_{x^{\mathbf{T}}} \sum_{i\in\mathbf{T}} u_i(x_i^{\mathbf{T}})$$

*such that $\sum_{i\in\mathbf{T}} x_i^{\mathbf{T}} = \sum_{i\in\mathbf{T}} w_i$. But notice that $\sum_{i\in\mathbf{T}} u_i(x_i^{\mathbf{T}}) = \sum_{i\in\mathbf{S}} u_i(x_i^{\mathbf{S}}) + \sum_{i\in\mathbf{T}\setminus\mathbf{S}} u_i(x_i^{\mathbf{T}\setminus\mathbf{S}})$ while $\sum_{i\in\mathbf{S}} x_i^{\mathbf{S}} + \sum_{i\in\mathbf{T}\setminus\mathbf{S}} x_i^{\mathbf{T}\setminus\mathbf{S}} = \sum_{i\in\mathbf{S}} w_i + \sum_{i\in\mathbf{T}\setminus\mathbf{S}} w_i$. It follows that $\mu^n(\mathbf{S}) + \mu^n(\mathbf{T}\setminus\mathbf{S}) \leq \mu^n(\mathbf{T})$. On the other hand, consider any finite coalition $\mathbf{S}$ which is in both $\Gamma_n$ and $\Gamma_{n+1}$. Then $\mu^{n+1}(\mathbf{S}) \leq \mu^n(\mathbf{S})$. We know that*

$$\mu^{n+1}(\mathbf{S}) \;=\; \max_{{}^{(n+1)}x^{\mathbf{S}}} \sum_{i\in\mathbf{S}} u_i({}^{(n+1)}x_i^{\mathbf{S}})$$

*such that $\sum_{i\in\mathbf{S}} {}^{(n+1)}x_i^{\mathbf{S}} = \sum_{i\in\mathbf{S}} w_i^{n+1}$, where each $w_i^{n+1}$ is the initial endowment of agent $i$ in the economy $\epsilon_{n+1}$ while ${}^{(n+1)}x_i$ is a possible allocation to $i$ in that economy. But since total endowments remain constant we have that $\sum_{i\in\mathbf{S}} w_i^{n+1} \leq \sum_{i\in\mathbf{S}} w_i^n$ (otherwise, there would be a redistribution of resources in the economy). It follows then that*

$$\mu^{n+1}(\mathbf{S}) \;=\; \max_{{}^{n+1}x^{\mathbf{S}}} \sum_{i\in\mathbf{S}} u_i({}^{n+1}x_i^{\mathbf{S}}) \;\leq\; \max_{{}^{n}x^{\mathbf{S}}} \sum_{i\in\mathbf{S}} u_i({}^{n}x_i^{\mathbf{S}}) \;=\; \mu^n(\mathbf{S})$$

.

*Consider now the limit game $\Gamma^*$ with payoff $\mu$. Since this payoff function must verify that for each finite $\mathbf{S}$, $\mu^n(\mathbf{S}) \to \mu(\mathbf{S})$. It could be that $\mu(\mathbf{S}) = 0$, but then, this is not possible, because it leads to contradiction in the presence of $\mathbf{AD}$.*[24] *Therefore, it must exist a finite $\mathbf{S}$ such that $\mu(\mathbf{S}) \neq 0$. In particular, it must exist an agent $i$*

[23] Alternatively, it has been shown that $\mathbf{ZF}+$ "existence of large cardinals", also supports the conclusion that every *reasonably definable* set of real numbers is measurable [Shelah-Woodin 1990]. But each finite coalition in our market games– in which each agent is identified by a natural (therefore real) number–constitutes a "reasonably definable" set of real numbers (it can be defined by simple enumeration).

[24] *The existence of a measure that vanishes over all finite subsets of natural numbers implies that some subsets of real numbers do not verify the* Baire property *(which we do not define here). But* **AD** *implies that every set of real numbers verifies this property* [Jech 2003].

*such that $\mu(\{i\}) \neq 0$. That means that $i$'s endowment in $\epsilon^*$ is $w_i^* > 0$. By the usual Archimedean property,*[25] *there must exist only a finite number of agents with this property, say $n^*$. That means that $\mathcal{COR}(\Gamma^*)$ must coincide with $\mathcal{COR}(\Gamma_{n^*})$ (up to a renaming of the agents). To see this suppose that $(\bar{\mu}_i)_{i\in\omega} \in \mathcal{COR}(\Gamma^*)$ is such that $\bar{\mu}_j > 0$, for $j$ with $w_j^* = 0$. This would mean that there is at least one agent $k$ with $w_k^* > 0$ receiving $\mu_k$ such that $\mu(\{k\}) > \bar{\mu}_k$. Contradiction.*

*So we have that the sequence of market games converges to a finite market game, which we know has a non-empty core. Now suppose that $|\mathcal{COR}(\Gamma^*)| > 1$. Say that $\bar{\mu} = (\bar{\mu}_i)_{i\in\omega}$ and $\bar{\mu}^{'} = (\bar{\mu}_i^{'})_{i\in\omega}$ are both in $\mathcal{COR}(\Gamma^*)$. Since for each $\mathbf{S}$, $\sum_{i\in\mathbf{S}} \bar{\mu}_i \geq \mu(\mathbf{S})$ as well as $\sum_{i\in\mathbf{S}} \bar{\mu}_i^{'} \geq \mu(\mathbf{S})$, we have that for any $0 \leq \alpha \leq 1$, $\alpha\bar{\mu} + (1-\alpha)\bar{\mu}^{'}$. Then, there exists a homeomorphism between $[0,1]$ and $\mathcal{COR}(\Gamma^*)$. But, for every $\bar{\mu} \in \mathcal{COR}(\Gamma^*)$ there exists a $x \in \prod_{i\in\omega} X_i$ such that $\bar{\mu}_i = u_i(x_i)$ if $w_i^* > 0$ and $\bar{\mu}_i = 0$ otherwise. But then $x$ is in $\mathcal{COR}(\epsilon^*)$. But this means that $\Upsilon(\mathcal{COR}(\epsilon_n)) > 0$. Contradiction. Therefore, there is only one allocation $\bar{\mu} \in \mathcal{COR}(\Gamma^*)$. Finally, notice that for every $\epsilon_n$, its equilibrium allocation ${}^n x^*$ verifies that $\bar{\mu}$ such that $\bar{\mu}_i = u_i({}^n x_i^*)$ is in $\mathcal{COR}(\Gamma_n)$. Therefore, $\mathcal{COR}(\Gamma^*)$, being equivalent to $\mathcal{COR}(\Gamma_{n^*})$, includes the allocation corresponding to the equilibrium allocation ${}^{n^*}x^*$.* $\square$

This result shows that in $\mathbf{ZF}+\mathbf{AD}+\mathbf{DC}$ every economy with an infinite number of agents is actually equivalent to a finite one. That is, only a finite number of agents determine the final outcome. It is interesting to note that in another collective-decision making context (the theory of *Social Choice*) we have a similar result:

**Proposition 6.** *In $\mathbf{ZF}+\mathbf{AD}+\mathbf{DC}$, consider a society $\mathcal{S} = \langle \mathbf{S}, \mathbf{X}, \{\preceq_i\}_{i\in\mathbf{S}}\rangle$, where $\mathbf{S}$ is the set of agents, $\mathbf{X}$ the set of options and each $\preceq_i$ is a reflexive, antisymmetric, transitive and complete ordering for $\mathbf{X}$. Assume that $\mathbf{S}$ is countable. Then, let $\preceq_{\mathbf{S}}$ be a social preference that verifies the conditions of Arrow's Theorem (see* [Arrow 1951]*). Then, there exist an $i \in \mathbf{S}$ (the* dictator*) such that $\preceq_i \equiv \preceq_{\mathbf{S}}$.*

This result can be derived from the properties of ultrafilters over $\omega$. A family of sets $\mathcal{F} \subseteq 2^{\mathbf{S}}$ is said an *ultrafilter* over $\mathbf{S}$ if: (*i*) $\mathbf{S} \in \mathcal{F}$; (*ii*) $\mathbf{A} \cap \mathbf{B} \in \mathcal{F}$, if $\mathbf{A}, \mathbf{B} \in \mathcal{F}$; (*iii*) if $\mathbf{A} \in \mathcal{F}$, and $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{B} \in \mathcal{F}$; (*iv*) if each $\mathbf{A} \subseteq \mathbf{S}$ verifies either $\mathbf{A} \in \mathcal{F}$ or $\mathbf{A} \in 2^{\mathbf{S}} \setminus \mathcal{F}$. A ultrafilter $\mathcal{F}$ is said *free* if $\cap_{\mathbf{A}\in\mathcal{F}} \mathbf{A} = \emptyset$. Otherwise it is called *principal* and verifies that there exist a singleton $\{\mathbf{a}\}$ such that $\mathcal{F} = \{\mathbf{A} : \{\mathbf{a}\} \subseteq \mathbf{A}\}$.

A well-known result in social choice theory is that the set of *decisive* agents (the coalitions that can impose their preferences over the society) constitutes an ultrafilter [Kirman-Sondermann 1972]. If $\mathbf{S}$ is finite, it is easy to show that every ultrafilter in $2^{\mathbf{S}}$ is principal. Therefore, since this is true for the class of decisive sets there exists an agent $i$ that is in each decisive set and moreover, is decisive by himself. This $i$ is the dictator.

In $\mathbf{ZF}+\mathbf{AD}$ every ultrafilter over $\omega$ is principal [Just-Weese 1996]. Therefore, there exists also a dictator in the case that $\mathbf{S} = \omega$. That is, in the context of $\mathbf{AD}$, the society $\mathcal{S}$ behaves like a finite society.

[25] *The construction of a non-Archimedean field (the* hyperreal *line) obtains as a non-standard model of the real numbers. But Robinson's construction requires the existence of a non-principal ultrafilter over $\omega$* [Goldblatt 1998]. *As we will repeat below, in $\mathbf{ZF}+\mathbf{AD}$ every ultrafilter over $\omega$ is principal.*

## 5. Existence of States of the World

The dynamics of beliefs has been always part of the explanatory mechanisms of economists. No sound analysis of complex situations, both in macro and microeconomics, can disregard the importance of the beliefs held by the agents, and moreover, how they evolve in time. Despite this fact, these considerations were always used as additions to the actual theoretical constructions, merely as parts of their intended interpretations. Even John von Neumann shied away from beliefs and deemed the games of incomplete information as ill-defined.

It was not until John Harsanyi postulated that agents interact in an implicitly agreed-on environment (a common prior), that beliefs became legitimate part of the modeling toolbox of economists [Harsanyi 1967]. In fact, he advanced the idea that agents in an interaction build their beliefs incorporating the possible beliefs of the *others*.

The entire description of the physical resources and the beliefs of the agents in an interaction is called the **state of the world**, $\alpha$. The description of the physical aspects of the context is summarized by the *state of the nature* of the situation, $s$, while the beliefs that are held in $\alpha$ can be represented as $\mathcal{B}(\alpha)$. Then, we have that a state of the world is a fixed point in a description operator:

$$\alpha \ = \ \langle s, \mathcal{B}(\alpha) \rangle$$

To see whether this $\alpha$ exists, the usual procedure, is to try to find it as a limit in a stepwise construction, which begins with the state of nature and then adds the beliefs of the agents:

$$\mathcal{B}^0(\alpha) = s$$

$$\mathcal{B}^1(\alpha) = \mathcal{B}^0(\alpha) \times bel\left(\mathcal{B}^0(\alpha)\right)$$

$$\mathcal{B}^2(\alpha) = \mathcal{B}^1(\alpha) \times bel\left(\mathcal{B}^1(\alpha)\right)$$

$$\ldots$$

Therefore, the conjecture is that $\mathcal{B}(\alpha)$ is the **limit** of this sequence, i.e.:

$$\mathcal{B}(\alpha) = \mathcal{B}^\omega(\alpha) = \lim_{n<\omega} \mathcal{B}^n(\alpha)$$

This construction has been shown to be sound, usually by means of topological assumptions about the space of states of nature [Mertens-Zamir 1984], [Brandenburger-Dekel 1993], [Dekel-Gul 1997]. But a critical assumption that is *always* included in the proofs of the conjecture, is that agents have *consistent* beliefs. That is, that the belief operator verifies that, if $\beta$ is a limit ordinal:

$$\lim_{\beta} \mathcal{B}^\beta(\alpha) = \mathcal{B}(\lim_{\beta} \alpha^\beta)$$

This consistency requirement arises naturally in any Bayesian model. Such a probabilistic structure generates a potential tree, where the probability of an outcome is identified with the measure of the entire branch that ends in that outcome. This measure, in turn, allows to reconstruct, via *backtracking*, the probabilities at each intermediate node [Chuaqui 1991]. That is, the sequence of probabilities at nodes along a branch is consistent. Therefore, consistency constitutes a necessary condition of Bayesian belief updating.

While Bayesian reasoning, and therefore the consistency of beliefs in a sequence, can be considered a reasonable condition to ask of rational agents, it is easy to imagine situations where consistency leads to absurd results. In fact, a number of examples exhibit that agents may form their beliefs discontinuously, but even so end up converging to a single state of world. The most interesting for the foundations of economic theory is known as *Newcomb's paradox* [Dekel-Gul 1997]. Let us assume a human agent that has to play against a Genie who claims that can predict the human's choices. There are two boxes **A** and **B**, the first translucent, the second opaque. The Genie offers the human to take either both boxes or only box **B**. The agent can see that box **A** has \$1000 inside, but the Genie tells him that if he chooses **B** he will leave \$1000000 instead. Otherwise, if he predicts that the agent will grab both boxes, he will leave **B** empty. The final decision made by the agent depends on what he believes about the powers of the Genie. A possibility is to think that the Genie may be able to predict correctly the agent's choice. According to that, the agent should choose only box $B$ and, if so, the Genie should leave the million dollars inside the box. But then, at the moment of grabbing only box **B** the agent reflects that either his belief was right and therefore there are \$ 1000000 inside the box or he was wrong and therefore the box is empty. Then, he looses nothing grabbing *both* boxes. In other words, the belief that constitutes part of the state of the world (i.e. that both boxes have money inside) is inconsistent with the beliefs he held in the process.

While examples like this seem far from real-world decision-making situations, just consider that for many authors, the well-known Prisoner's Dilemma can be seen as a two-sided version of Newcomb's problem (i.e. each player conceives the other as making decisions based on a prediction about her own decision) [Sobel 1991]. In fact, many situations in which strategic uncertainty plays a role can be seen as variants of this same phenomenon.

Notice that various forms of non-monotonic reasoning processes can be represented as a discontinuous sequences of intermediate beliefs. Therefore, in the case that neither the number of possible intermediate beliefs nor that of steps is bounded, no proofs of termination of the reasoning process can be given, at least in **ZFC**. Even so, since obviously the agents reach some state of the world, we have to explain the convergence to it, without the assumption of consistency of $\mathcal{B}$. One way of handling the problem is by assuming that agents are able to handle non-well founded objects like $\alpha$ when $\mathcal{B}$ does not warrant the convergence of the iterative construction.

This assumption amounts to drop from **ZF** the Regularity Axiom,

$$\exists x F(x) \rightarrow \exists y \Big[ F(y) \wedge \forall z \neg \big( z \in y \wedge F(z) \big) \Big]$$

and to replace it with the so-called *Solution Lemma*, that states that every general system of equations $\varepsilon$ has a unique solution $\bar{sol}$:

- A *general system of equations* is a $\varepsilon = \langle \mathbf{X}, \mathbf{A}, \mathbf{e} \rangle$, where $\mathbf{X}$ is a set of indeterminates, $\mathbf{A}$ a set of "constants", $\mathbf{X} \cap \mathbf{A} = \emptyset$ and $\mathbf{e} : \mathbf{X} \rightarrow \mathcal{V}(\mathbf{X} \cup \mathbf{A})$, provides the equations (where $\mathcal{V}(\mathbf{X} \cap \mathbf{A})$ is the class of sets build up from elements in $\mathbf{A}$). Equations have the following form: $x = \mathbf{e}(x) \in \mathcal{V}(\mathbf{X} \cap \mathbf{A})$.
- A solution to $\varepsilon$ is a function $\bar{sol}$ on $\mathbf{X}$, such that $\bar{sol}(x) = \bar{sol}(\mathbf{e}(x))$. This $\bar{sol}$ is a *substitution* function, which assigns to each indeterminate a set in

$\mathcal{V}(\mathbf{A})$ (i.e. the class of sets without indeterminates and constructed entirely of elements in $\mathbf{A}$).

The new set theory that obtains, **ZFC** $-$ Regularity $+$ Solution Lemma, is called **AFA** (for *antifoundation*) [Aczel 1988],[Devlin 1993][Barwise-Moss 1996].

In the case of states of the world, we take $\alpha \in \mathbf{X}$ and $s \in \mathbf{A}$, i.e. our indeterminates are states of the world, and the constants are states of nature. Then we can prove the following result:

**Theorem 6.** *In* **AFA**, *a state of the world $\alpha$, given its underlying state of nature $s$, obtains as a fixed point of the belief formation operator $\mathcal{B}(\cdot)$.*

The key of the proof is just to show that $\alpha$ is either a fixed point or the process of belief formation is endless. Since every general system of equations has a solution (a bounded sequence), it cannot be endless. Therefore $\alpha$ has to be a fixed point.

This is different from the fixed point theorems we briefly mentioned in section 4, since in all those cases we had either continuity of the operators (like in Kakutani's theorem) or at least the property of being *increasing* in a structure in which each set has a unique maximal element (like in Tarski's fixed point theorem). Here we cannot ensure almost nothing about the state of the world, but at least we want to know whether it exists, which in this case means just that it can be described as a set.

Notice that the change from **ZFC** to **AFA** is not as radical as the one from **ZFC** to $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$. In fact, every object that can be defined in **AFA** ends up being an element in the universe of **ZFC**. The only advantage in this change is that it simplifies the argument. Using *coalgebraic* methods–i.e. characterizations of the "solutions of equations", particularly in the case of circular definitions–we could achieve similar results.[26] In any case, **AFA** seems to be the most intuitive form of ensuring the existence of states of the world.

For an alternative approach to the characterization of states of the world, consider the space of possible belief structures that can be constructed over a fixed space of states of nature $\mathbf{S}$. It is easy to conceive a game in which player $I$ chooses a state of nature $s \in \mathbf{S}$, $II$ replies with a belief $\phi_1 \in \mathcal{B}(s)$ to which $I$ responds with $\phi_2 \in \mathcal{B}(\langle s, \mathcal{B}(s)\rangle)$, to which $II$ answer is $\phi_3 \in \mathcal{B}(\langle s, \mathcal{B}^2(s)\rangle)$ etc. $II$ wins if it can lead the sequence to fixed-point, otherwise $I$ wins. Notice that this game is isomorphic to a Gale-Stewart game, except for the fact that since we do not assume continuity, the number of steps in the game could be transfinite, unless we can again apply the **AD**. But for this to ensue, each belief structure must be codified (at least in principle) by a member of a countable family. This depends, of course, on the language in which these beliefs are expressed.

Consider the class $\mathcal{B}^{\mathbf{ORD}}(\mathbf{S}) = \cup_{\beta \in \mathbf{ORD}} \mathcal{B}^{\beta}(\mathbf{S})$, where $\mathcal{B}^{\beta}(\mathbf{S})$ is the class of belief structures constructed over $\mathbf{S}$ for an ordinal level $\beta$. Let $\mathcal{L}_{\mathcal{B}^{\mathbf{ORD}}(\mathbf{S})}$ be a language such that for each $\phi \in \mathcal{B}^{\mathbf{ORD}}(\mathbf{S})$ there exists a formula $\Phi$ expressed in $\mathcal{L}_{\mathcal{B}^{\mathbf{ORD}}(\mathbf{S})}$ such that $\lceil \Phi \rceil = \phi$, that is, the interpretation of $\Phi$ is the belief structure $\phi$.

It has been shown that not every $\phi \in \mathcal{B}^{\mathbf{ORD}}(\mathbf{S})$ can be defined in a language that does not permit inconsistencies [Brandenburger-Keisler 1999]. This indicates that completeness in a the characterization of states of the world is somehow hard to achieve, if we do not restrict the class of admissible beliefs. For our purposes it

[26]But this requires to use category-theoretical methods instead of set-theoretical ones [Moss 2001].

is enough for each belief structure $\phi$ to be such that the corresponding formula $\Phi$ has a finite Gödel code. That is, the belief structure $\phi$ must be represented by a formula in $\mathcal{L}_{\mathcal{B}^{\mathbf{ORD}}(\mathbf{S})}$ that can be coded by a natural number. Let us call a belief structure $\phi \in \mathcal{B}^{\mathbf{ORD}}(\mathbf{S})$ *finitely representable* if it has this property. Then we have this trivial result:

**Lemma 5.** *(*$\mathbf{ZF}+\mathbf{AD}+\mathbf{DC}$*) If each possible belief structure in an incomplete information context with underlying state of the world $s \in \mathbf{S}$ is finitely representable, there exists a state of the world $\alpha = \langle s, \mathcal{B}(\alpha)\rangle$. Moreover, even if $\mathcal{B}$ is not continuous, it can be unfolded in $\omega$ steps.*

Admittedly, this sounds very restrictive, but on the other hand, there are no natural examples in economics in which beliefs cannot be represented by a finite expression. Since the usual form of representing beliefs is by means of probability distributions, if no finite expression existed for a given distribution or hierarchy of distributions, it would mean that it is not possible to compress its informational content in order to obtain a manageable form. While it is easy to infer that most of all possible probability structures cannot be represented by an expression shorter than its full extent (an almost immediate consequence of Chaitin's theorem [Chaitin 1974]), it is also true that it is quite unlikely that they could represent the beliefs held by any rational agent.

## 6. Discussion

In the previous sections we presented a variety of limitations to the theoretical answers that have been given to the main problems in economics. It is not clear that they will (or should) affect the work of most economists. Although the existence of holes in the edifice of economic theory is somewhat worrisome, it is also true that none of the problems discussed here amounts to its demolition. In fact, most economists use the existence of choice functions, equilibria and states of the world as just metaphors and make models in which the difficulties discussed here are disregarded.[27]

On the other hand, even if the problems discussed here may have small impact in the practice of economics, they ask for more refined tools in the analysis of economic phenomena. In fact, Lewis, in his analysis of economic theory concludes, rather inconclusively, that the set-theoretical principles required by contemporary economic theory are far more demanding than what is needed in physics, both classic and quantum [Lewis 1990].[28] Be it true or not, the search for the right set-theoretical foundations for economic theory should be pursued more actively. At least, it should be advisable for economists to know that the validity of their conclusions depends on very deep properties of the formal tools they use.

In this sense, we have shown that a change in the underlying set theory may lead to the elimination of undesirable results. As admitted at the beginning of the paper, this seems a too easy technique to solve analytical problems. But as said, if economic theory represents something, it must be the problem solving ability of decision-making agents. Our claim is that this ability may correspond to certain properties of the set-theoretic universe in which our models are represented. It is

[27] A clear exception is the complexity of computations (of the **NP** sort).

[28] According to Lewis, all the principles of physics are derivable in **ZF**. This is contrary to the conclusions in [Benioff 1976].

interesting, in this light, to discuss what abilities do our alternative set theories imply.

Let us consider **ZF** + **DC** + **AD**. In this framework we can still use our finite mathematics and much of elementary calculus. What we add here is that agents can always find winning strategies in Gale-Stewart games. This means that we are assuming a very simplified universe of sets, that satisfies our intuition that economic theories describe "tame" environments, and therefore degenerate cases do not have place in them.

This result would be trivial if **ZF** + **AD** + **DC** were not consistent. But consistency proofs have not been found even for **ZF**. The strongest result in that direction for **ZF**+**AD**+**DC** indicates that it is consistent in the context of certain additional axioms of *large cardinals* ([Martin-Steel 1989]) and, moreover, it has been shown by Woodin that it is equiconsistent with **ZF** + "There are infinite Woodin cardinals" [Jech 2003].[29] In fact, set theorists use normally large cardinals just to build *inner models* of **ZF** as well as of some other set theories, as a form to evaluate the consistency of the theory up to a very large cardinal. For our purposes it suffices to know that these results make the set-theoreticians to believe that **ZF** + **AD** + **DC** has a model.

Moreover, in these mild environments we do not admit something like a finite coalition in a market game without a payoff. In fact, in **ZF** + **DC** + **AD** every set of reals is measurable. Therefore, if we increase the number of agents we will always have a sequence of games that ends up converging to a game $\Gamma$ such that the allocations supported in its core are the equilibria of the economy. In fact, every use we made of this particular system of axioms ends up showing some sort of "finitization" of the theory, making the problems more tractable, at least in principle.[30]

In the case in which problems arise with the information held by the agents, we saw that **AFA** allows us to ensure the convergence of beliefs, even if the process is discontinuous. An additional interest we may have in using **AFA** in economics is that it can be interpreted as indicating that agents are able to conceive states of the world as integral constructions instead of just processes that converge to them. This legitimates the methodology of applied economists, that just take states of the world (or equivalently, the **types** of agents in them) as data and not as objects that are to be derived.

In a few words, we think that **AFA**$^-$+**DC**+**AD** (where **AFA**$^-$ is **AFA** without **AC**) is a intuitively good arena for economic theory. It remains to be proven that it is consistent. Our conjecture is that in fact it is consistent since, for example, Regularity is not used in the consistency proof in [Martin-Steel 1989].

This procedure of choosing the right set theory for economic theory has still to be checked out, to see whether these ideas are sound enough. In any case, the study of this possibility as well as its extension to other economic problems

[29]The "large cardinals" axioms postulate the existence of sets that verify any given set-theoretic predicate $P(\cdot)$ in the class $\Sigma^1_2$. If $\kappa$ is a cardinal such that $P(\kappa)$ is true in the universe of sets, it must be an uncountable, strongly *inaccessible* cardinal [Woodin 2001]. For our purposes it is enough to indicate that this means that for every definable property of sets, there must exist infinite very large sets verifying them.

[30]**ZF**+ "large cardinals" is more difficult to interpret in terms of cognitive abilities, although it can be speculated that agents able to conceive (at least) inaccessible cardinals may also be able to handle discontinuous processes, which seems interesting for the problem of the existence of states of the world.

(e.g. the treatment of economies with a large number of goods or with other types of dynamical structures) is matter of further work. It may be that in those environments, in which uncertainty is quintessential, a relaxation of **AD** may be needed. A possibility is to consider, instead of Gale-Stewart games, games in which the players have imperfect information, called *Blackwell games*. The corresponding variant of determinacy, called *Blackwell determinacy*([Martin 1998], [Löwe 2002]) may be useful to retain results that otherwise require the validity of deep measure -theoretic assumptions.

Finally, let us briefly address the cognitive abilities of economic agents, as represented in $\mathbf{AFA}^- + \mathbf{DC} + \mathbf{AD}$. On one hand, the equivalence of the results on computability with those that follow from assuming oracles, shows that agents are extremely powerful in this sense. On the other hand, an oracle is far from an infallible device. It just fills in the gaps of undecibility. Therefore, an agent just yields a decision, not necessarily the "correct" one. In any case, the current debate in cognitive science is far from settled, and the once central computational model of the mind is nowadays under strong criticism [Clark 1996].

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