PROJECTIONS OF BODIES AND HEREDITARY PROPERTIES OF HYPERGRAPHS

BÉLA BOLLOBÁS AND ANDREW THOMASON

ABSTRACT

We prove that for every n-dimensional body K, there is a rectangular parallelepiped B of the same volume as K, such that the projection of B onto any coordinate subspace is at most as large as that of the corresponding projection of K.

We apply this theorem to projections of finite set systems and to hereditary properties. In particular, we show that every hereditary property of uniform hypergraphs has a limiting density.

1. *Projections of bodies*

Let K be a body in $\mathbb{R}^n$, and let $(v_1, \ldots, v_n)$ be the standard basis for $\mathbb{R}^n$. Denote the volume of K by $|K|$. Furthermore, given a subset $A \subseteq [n] = \{1, 2, \ldots, n\}$ with d elements, denote by K_A the orthogonal projection of K onto the subspace spanned by $\{v_i : i \in A\}$, and by $|K_A|$ its (d-dimensional) volume. Thus $K_{[n]} = K$. By the term *box* we shall mean a rectangular parallelepiped whose sides are parallel to the coordinate axes.

For the purposes of this paper, a *body* can be taken to be a compact subset of $\mathbb{R}^n$ which is the closure of its interior. It would be effortless to rewrite our results and their proofs in terms of arbitrary product measures and measurable subsets of $\mathbb{R}^n$, using outer measures on the projections. We choose not to write down these extensions, in order to avoid irrelevant technicalities cluttering up an otherwise simply stated theorem, which is the following.

THEOREM 1. *Let K be a body in $\mathbb{R}^n$. Then there is a box B in $\mathbb{R}^n$* with $|B| = |K|$ *and $|B_A| \leqslant |K_A|$ for every $A \subseteq [n]$.*

It is immediate that if bounds on the volumes of the $(n-1)$-dimensional projections of a box B are given, then a bound on the total volume ensues, because $\prod_{i=1}^n |B_{[n]\setminus\{i\}}| = |B|^{n-1}$. From Theorem 1, it then follows that the inequality $\prod_{i=1}^n |K_{[n]\setminus\{i\}}| \geqslant |K|^{n-1}$ holds for any body K. This is the Loomis–Whitney inequality [14] (see also [7, page 95] and [12, page 162]; the inequality was rediscovered by Allan [3], who gave a more streamlined proof).

A valuable consequence of Theorem 1 is that if the volume of a box can be bounded in terms of the volumes of a certain collection of projections, then the same bound will be valid for all bodies. We shall, in fact, prove Theorem 1 by examining some collections of projections for which there is such a bound on the volume of a box.

By a *cover* of $[n]$ we mean a multiset $\mathscr{C}$ of subsets of $[n]$ such that each element $i \in [n]$ is in at least one of the members of $\mathscr{C}$. A *k-cover* is a cover in which each element

Received 22 June 1993; revised 26 September 1993, 15 January 1994.

1991 *Mathematics Subject Classification* 05D05, 05C30, 05C65, 05C80, 28A75, 51M16, 51M25.

Bull. London Math. Soc. 27 (1995) 417–424

of $[n]$ is in exactly k of the members of $\mathscr{C}$. We emphasise that the sets in a cover *need not be distinct*; for example, $\{[n],[n],[n]\}$ is a 3-cover of $[n]$. A *uniform cover* of $[n]$ is a k-cover for some $k \geqslant 1$. The 1-cover $\{[n]\}$ of $[n]$ is said to be *trivial*, all other covers being non-trivial.

A uniform cover of $[n]$ which is not the disjoint union of two uniform covers of $[n]$ is said to be *irreducible*. It is important to note that there are only a finite number of irreducible uniform covers of $[n]$. One way of seeing this is to consider an infinite sequence $\Gamma_0 = (\mathscr{C}_0^{(j)})_{j=1}^{\infty}$ of distinct uniform covers of $[n]$. Enumerate the subsets $A_1, A_2, \ldots, A_{2^n}$ of $[n]$, and then select infinite sequences $\Gamma_i = (\mathscr{C}_i^{(j)})_{j=1}^{\infty}$, $1 \leqslant i \leqslant 2^n$, so that Γ_i is a subsequence of Γ_{i-1} and so that the number of copies of A_i in $\mathscr{C}_i^{(j)}$ is a non-decreasing function of j. It is easy to see this can be done, and it is clear from the construction that the covers in Γ_{2^n} are nested. Hence if $\mathscr{C}'$ and $\mathscr{C}$ are any two terms of Γ_{2^n}, with $\mathscr{C}' \subseteq \mathscr{C}$, then $\mathscr{C}$ is not irreducible, it being the disjoint union of the two uniform covers $\mathscr{C}'$ and $\mathscr{C}\backslash\mathscr{C}'$. In fact, writing $D(n)$ for the number of irreducible covers of $[n]$, Huckeman, Jurkat and Shapley proved (see Graver [**11**]) that $D(n) \leqslant (n+1)^{(n+1)/2}$ for all n. Related results have been proved by Alon and Berman [**4**] and Füredi [**10**].

Clearly, if B is a box and $\mathscr{C}$ is a k-cover of $[n]$, then $\prod_{A\in\mathscr{C}}|B_A| = |B|^k$. In view of this, the next result, Theorem 2, can be considered as a special case of Theorem 1. However, in order to prove Theorem 1, we must first prove Theorem 2 directly and use it to derive Theorem 1.

THEOREM 2. *Let K be a body in $\mathbb{R}^n$, and let $\mathscr{C}$ be a k-cover of $[n]$. Then*

$$|K|^k \leqslant \prod_{A\in\mathscr{C}} |K_A|.$$

Proof. The proof is by induction on n, the case $n = 1$ being trivial. For the general case, for each $x\in\mathbb{R}$ let $K(x)$ be the section of K consisting of points with nth coordinate equal to x, so that $|K| = \int |K(x)|\,dx$. We define $\mathscr{C}' = \{A\in\mathscr{C} : n\in A\}$ and $\mathscr{C}'' = \mathscr{C}\backslash\mathscr{C}'$, so that $|\mathscr{C}'| = k$. Then $\{A\backslash\{n\} : A\in\mathscr{C}'\} \cup \mathscr{C}''$ forms a k-cover of $[n-1]$, so, by the induction hypothesis,

$$\prod_{A\in\mathscr{C}'} |K(x)_{A\backslash\{n\}}| \prod_{A\in\mathscr{C}''} |K(x)_A| \geqslant |K(x)|^k.$$

Moreover, we have

$$|K_A| = \int |K(x)_{A\backslash\{n\}}|\,dx \quad \text{for } A\in\mathscr{C}',$$
$$|K_A| \geqslant |K(x)_A| \quad \text{for } A\in\mathscr{C}''.$$

So we obtain, by means of Hölder's inequality,

$$\begin{aligned}
|K| = \int |K(x)|\,dx &\leqslant \int \Bigg[\prod_{A\in\mathscr{C}'} |K(x)_{A\backslash\{n\}}| \prod_{A\in\mathscr{C}''} |K(x)_A|\Bigg]^{1/k} dx \\
&\leqslant \Bigg[\prod_{A\in\mathscr{C}''} |K_A|\Bigg]^{1/k} \int \prod_{A\in\mathscr{C}'} |K(x)_{A\backslash\{n\}}|^{1/k}\,dx \\
&\leqslant \Bigg[\prod_{A\in\mathscr{C}''} |K_A|\Bigg]^{1/k} \prod_{A\in\mathscr{C}'} \Bigg[\int |K(x)_{A\backslash\{n\}}|\,dx\Bigg]^{1/k} \\
&= \Bigg[\prod_{A\in\mathscr{C}} |K_A|\Bigg]^{1/k},
\end{aligned}$$

as promised.

Proof of Theorem 1. For every integer $k \geqslant 1$, and for every non-trivial irreducible k-cover $\mathscr{C}$ of $[n]$, Theorem 2 tells us that $\prod_{A\in\mathscr{C}}|K_A| \geqslant |K|^k$; it also tells us (by applying it to K_S) that $\prod_{i\in S}|K_{\{i\}}| \geqslant |K_S|$ for every $S \subseteq [n]$. In this way we have written down a finite set of inequalities involving the numbers $\{|K_A|: A \subseteq [n]\}$. Let $\{x_A: A \subseteq [n]\}$ be a collection of positive numbers with $x_A \leqslant |K_A|$ and $x_{[n]} = |K|$, which are minimal subject to satisfying all the above inequalites with x_A in place of $|K_A|$ for all A. Note that we are applying only a finite number of constraints to the x_A, but that nevertheless $\prod_{A\in\mathscr{C}} x_A \geqslant |K|^k$ for *every* k-cover $\mathscr{C}$, any such cover being a disjoint union of irreducible uniform covers.

The fact that the numbers $x_{\{i\}}$, $i\in[n]$, are minimal means that for each i there is an inequality involving $x_{\{i\}}$ in which equality holds. If the inequality is of the first kind, then we have a k_i-cover $\mathscr{C}_i$ of $[n]$ with $\{i\}\in\mathscr{C}_i$ and $\prod_{A\in\mathscr{C}_i} x_A = |K|^{k_i}$. The same is, in fact, true if the inequality is of the second kind, namely $\prod_{i\in S} x_{\{i\}} = x_S$ for some $S \subseteq [n]$, because in this case the minimality of x_S implies $\prod_{A\in\mathscr{C}} x_A = |K|^{k_i}$ for some k_i-cover $\mathscr{C}$ of $[n]$ containing S, and we can take $\mathscr{C}_i = (\mathscr{C}\backslash\{S\}) \cup \{\{i\}: i\in S\}$. Now we let $\mathscr{C} = \bigcup_{i=1}^n \mathscr{C}_i$ and $k = \sum_{i=1}^n k_i$. Then $\mathscr{C}$ is a k-cover of $[n]$ containing $\{i\}$ for all $i\in[n]$, and $\prod_{A\in\mathscr{C}} x_A = |K|^k$. But $\mathscr{C}' = \mathscr{C}\backslash\{\{i\}: i\in[n]\}$ is a $(k-1)$-cover of $[n]$, so $\prod_{A\in\mathscr{C}'} x_A \geqslant |K|^{k-1}$, from which we conclude $\prod_{i=1}^n x_{\{i\}} \leqslant |K|$. Since $\{\{i\}: i\in[n]\}$ is a 1-cover of $[n]$, the reverse inequality also holds here, and hence, in fact, equality holds.

Finally, we observe that for any $A \subseteq [n]$, the set $\{A\} \cup \{\{i\}: i\notin A\}$ is a 1-cover of $[n]$, so

$$|K| \leqslant x_A \prod_{i\notin A} x_{\{i\}} \leqslant \prod_{i\in A} x_{\{i\}} \prod_{i\notin A} x_{\{i\}} = |K|,$$

whence $x_A = \prod_{i\in A} x_{\{i\}}$. It follows that the box B of side length $x_{\{i\}}$ in the direction of v_i satisfies $|B| = |K|$ and $|B_A| = x_A \leqslant |K_A|$ for all $A \subseteq [n]$.

2. *Further results on projections of bodies*

Theorem 1 has many consequences. Some of these can also be derived directly from Theorem 2. For example, let $\mathscr{C}$ be a collection of subsets of $[n]$ (but not necessarily a uniform cover). Suppose that to each $A\in\mathscr{C}$ we can associate a positive real weight $w(A)$ in such a way that, for each $i\in[n]$, $\sum\{w(A): i\in A\in\mathscr{C}\} = 1$. Then

$$|K| \leqslant \prod_{A\in\mathscr{C}} |K_A|^{w(A)}.$$

Another consequence of Theorem 1, which is not a direct consequence of Theorem 2, can be stated as follows. Consider real-valued functions of 2^n variables, these variables being indexed by the 2^n subsets of $[n]$; for example, $f((x_A)_{A\subset[n]}) = -1+\sum x_A^2$. Given such a function $f\colon \mathbb{R}^{\mathscr{P}(n)} \to \mathbb{R}$ and a body $K \subseteq \mathbb{R}^n$, we call K *f-dominated* if $f((|K_A|)_{A\subset[n]}) \leqslant 0$. So, in our example, K would be f-dominated if $\sum_A |K_A|^2 \leqslant 1$. Theorem 1 shows us how to find the largest volume of a body which is f-dominated, provided f is monotone.

THEOREM 3. *Let $f_i\colon \mathbb{R}^{\mathscr{P}(n)} \to \mathbb{R}$, $i = 1,\dots,s$, be monotone increasing functions in each of the 2^n variables. Then*

$$\sup\{|K|: K \text{ is } f_i\text{-dominated}, 1 \leqslant i \leqslant s\}$$
$$= \sup\{|B|: B \text{ is a box and } B \text{ is } f_i\text{-dominated}, 1 \leqslant i \leqslant s\}.$$

This theorem follows immediately from Theorem 1. An equivalent formulation of it is the following. Let $\pi\colon \mathbb{R}_+^n \to \mathbb{R}^{\mathscr{P}(n)}$ be given by $(x_1, \ldots, x_n) \mapsto (\prod_{i\in A} x_i)_{A\subset[n]}$, and set $g_i = f_i\pi\colon \mathbb{R}_+^n \to \mathbb{R}$. Then

$$\sup\{|K| : f_i \text{ is } K\text{-dominated}, 1 \leqslant i \leqslant s\} = \sup\left\{\prod_1^n x_j : g_i(x) \leqslant 0,\ i = 1, \ldots, s\right\}.$$

It should be noted, as suggested earlier, that the proof of Theorems 1 and 2 (and hence their extensions) carry over trivially to the case of general product spaces and product measures. The changes needed are purely cosmetic, and the results do not differ materially from those in stated form.

It is natural to ask whether Theorem 1 can be strengthened so as to preserve containment, in the following sense. The theorem demonstrates the existence of a map $K \mapsto B(K)$ which associates with a body K a box $B = B(K)$ such that $|B| = |K|$ and $|B_A| \leqslant |K_A|$ for all $A \subset [n]$. Can this map be chosen so that $B(K') \subset B(K)$ whenever $K' \subset K$? It is easy to see that it is impossible, even in the case $n = 2$, to find a box $B(K)$ which will work simultaneously for all choices of $K' \subset K$. Nevertheless, in $\mathbb{R}^2$ it is possible, given K' and K, to choose boxes $B(K)$ and $B(K')$ so that $B(K') \subset B(K)$. If this were to remain true in $\mathbb{R}^n$ it would offer the possibility of a direct proof of Theorem 1 by induction. However, even this weakened form of containment fails when $n = 3$, as the following example shows. Take K' to be a $12 \times 1 \times 1$ box, and let K be the union of K' with a $3 \times 3 \times 3$ cube, the two overlapping in a $3 \times 1 \times 1$ box. Suppose there were an $a \times b \times c$ box $B(K)$ containing $B(K')$. Note that $B(K')$ must equal K' since K' is already a box. Then $abc = 36$, and $ab \leqslant 12$. Thus $c \geqslant 2$, and likewise $b \geqslant 2$, so $a \leqslant 9$, contradicting $B(K') \subset B(K)$.

It is also natural to ask whether the following converse to Theorem 1 is true: namely, that for any set of positive numbers $\{x_A : A \subseteq [n]\}$, there is a body K with $|K_A| = x_A$ for each $A \subseteq [n]$, provided that for each $C \subseteq [n]$ there is a box B^C, with $|B^C| = x_C$ and $|B^C_A| \leqslant x_A$ for each $A \subseteq C$. The condition is necessary by Theorem 1, but it turns out to be sufficient only in dimension at most three (provided we extend the definition of a body to include pieces of n-dimensional measure zero; that is, drop the condition that a body be the closure of its interior). In $\mathbb{R}^4$, the following is a counter-example. Let $x_A = 2$ if $|A| = 1$, $x_{\{1,2\}} = x_{\{2,3\}} = x_{\{3,4\}} = x_{\{4,1\}} = 1$, $x_{\{1,3\}} = x_{\{2,4\}} = 4$, $x_A = 2$ if $|A| = 3$ and $x_{[4]} = 1$. It is easy to check the existence of a box B^C for each $C \subseteq [n]$ satisfying the stated constraints. Nevertheless, no body K of the desired kind can exist. For it is easy to see (or by referring to Theorem 4 below) that for each A of order 3, K_A must be a box of dimensions $(1/2) \times 2 \times 2$. Hence the projections of both $K_{\{1,2,3\}}$ and $K_{\{1,2,4\}}$ in the $\{1,2\}$ plane will be $(1/2) \times 2$ rectangles. However, these two rectangles will not overlap, thereby forcing $|K_{\{1,2\}}| > x_{\{1,2\}} = 1$.

The cases of equality in Theorem 2 can be characterised as follows. Each uniform cover $\mathscr{C}$ of $[n]$ partitions $[n]$ into equivalence classes, two elements being in the same class if they lie in exactly the same subsets in $\mathscr{C}$. We call these classes the *equivalence classes of the cover* $\mathscr{C}$.

THEOREM 4. *Let K be a body in $\mathbb{R}^n$, and let $\mathscr{C}$ be a k-cover of $[n]$ such that*

$$\prod_{A\in\mathscr{C}} |K_A| = |K|^k.$$

Then $K = \prod K_E$, the product being over all equivalence classes E of the cover $\mathscr{C}$.

Proof. As Theorem 2 holds for general product measures, we may as well assume that each equivalence class consists of just one element. Furthermore, we may replace the compact set K by its interior without the result being affected.

The proof is again by induction on n. Let us examine the proof of Theorem 2, noting that all the weak inequalities in the final sentence must be equalities. First, we observe that for all $A \in \mathscr{C}''$, and for almost all x, $|K_A| = |K(x)_A|$ holds. But then $K_A = K(x)_A$ for almost all, and hence for all, x. Secondly, for almost all x we have $\prod_{A \in \mathscr{C}'} |K(x)_{A \setminus \{n\}}| \prod_{A \in \mathscr{C}''} |K(x)_A| = |K(x)|^k$, and so by the induction hypothesis $K(x) = \prod_{i=1}^{n-1} K(x)_{\{i\}}$. For any x for which this equation holds, and for any $A \in \mathscr{C}''$, $K_A = K(x)_A = \prod_{i \in A} K(x)_{\{i\}}$. Therefore $K(x)_{\{i\}} = K_{\{i\}}$ for almost all, and hence for all, x. But then $K(x) = \prod_{i=1}^{n-1} K_{\{i\}}$ for all x, so $K = \prod_{i=1}^{n} K_{\{i\}}$.

Note that Theorem 4 would be false if we did not require the condition that a body be the closure of its interior, since the addition of, say, a 4-dimensional subset to K would have no effect if $|A| \geqslant 5$ for every $A \in \mathscr{C}$; however, the other results in this paper would not be affected if this condition were dropped.

Finally, observe that, by Theorem 2, if the sizes of the projections corresponding to a uniform cover are known, then the volume is bounded. Even if the cover is not uniform, the volume is still bounded provided the cover *generates* a uniform cover, in the sense that a uniform cover can be constructed from the original cover by taking non-negative multiples of the subsets in that cover. Thus the cover $\mathscr{C}$ generates the uniform cover $\mathscr{D}$ if $A \in \mathscr{C}$ for every $A \in \mathscr{D}$. Not all collections of projections will yield bounds valid on the total volume; for example, if $n = 3$, no bound on $|K|$ can be derived from $|K_{\{1,2\}}|$ and $|K_{\{1,3\}}|$ alone. For which covers is it true, then, that a bound on the volume of a body can be derived from bounds on its projections in the directions of that cover? The next theorem shows that these covers are precisely the covers that generate uniform covers.

THEOREM 5. *Let $\mathscr{C}$ be a cover of $[n]$ which does not generate a uniform cover. Then for any real constant N, there is a body K in $\mathbb{R}^n$ with $|K_A| \leqslant 1$ for all $A \in \mathscr{C}$ and $|K| > N$.*

Proof. Associate with each subset $A \subseteq [n]$ its characteristic vector $v_A = \sum_{i \in A} v_i$. Let L be the positive cone generated by $\mathscr{C}$, namely $L = \{\sum_{A \in \mathscr{C}} \lambda_A v_A : \lambda_A \geqslant 0\}$. The fact that $\mathscr{C}$ generates no uniform cover imples that the point $v_{[n]}$ is not in L, and this point is therefore separated from L by a hyperplane passing through the origin: there are constants α_i, $1 \leqslant i \leqslant n$, such that $\sum_{i=1}^{n} \alpha_i = \alpha > 0$ and $\sum_{i \in A} \alpha_i \leqslant 0$ for all $A \in \mathscr{C}$.

Let K be the box whose ith side has length r^{α_i}, where $r > 0$. Then $|K_A| \leqslant 1$ but $|K| = r^{\alpha} > N$ if r is large enough.

3. *Projections of set systems*

We offer here a discrete application of Theorem 1. For any set $\mathscr{F}$ of subsets of $[n]$, and for any subset $A \subseteq [n]$, we may define the *projection* $\mathscr{F}_A$ *of* $\mathscr{F}$ *onto* A by $\mathscr{F}_A = \{F \cap A : F \in \mathscr{F}\}$. The Sauer–Shelah lemma [**18**, **20**] states that $|F| \leqslant \sum_{i=0}^{t-1} \binom{n}{i}$ if $|\mathscr{F}_A| < 2^t$ for all $A \subseteq [n]$ with $|A| = t$, and this bound is easily seen to be tight. See [**5**] for further discussion of this result. We give now a bound for $|\mathscr{F}|$ in the more general setting where $|\mathscr{F}_A|$ is bounded for all A in some uniform cover of $[n]$. Our bound is not the best possible, but it is useful and very easily applied.

THEOREM 6. *Let $\mathscr{C}$ be a k-cover of* $[n]$. *Let $\mathscr{F}$ be a set of subsets of* $[n]$, *and let* $c \geqslant 1$ *be a constant such that* $|\mathscr{F}_A| \leqslant c^{|A|}$ *for all* $A \in \mathscr{C}$. *Then* $|\mathscr{F}| \leqslant c^n$.

Proof. To each subset $A \subseteq [n]$ we associate its characteristic vector $v_A \in \mathbb{R}^n$, namely $v_A = \sum_{i \in A} v_i$. Let I be the unit cube, and let $I + v_A$ be the translate of I by v_A. Set $K = \bigcup_{F \in \mathscr{F}} (I + v_A)$, so that $|K| = |\mathscr{F}|$, and $|K_A| = |\mathscr{F}_A|$ for all $A \subseteq [n]$. Then by Theorem 2,

$$|\mathscr{F}|^k = |K|^k \leqslant \prod_{A \in \mathscr{C}} |K_A| \leqslant \prod_{A \in \mathscr{C}} c^{|A|} = c^{\sum_{A \in \mathscr{C}} |A|} = c^{kn},$$

because $\mathscr{C}$ is a k-cover of $[n]$.

We can create a probabilistic version of Theorem 6 as follows. Let $\mathscr{F}$ be a set system on $[n]$, and suppose we choose a random subset $F \subseteq [n]$ by selecting elements independently and at random, each with probability p. It is natural to ask for the probability that F be in $\mathscr{F}$, that is, the probability that the event $\mathscr{F}$ occurs. We denote this probability by $\Pr(\mathscr{F})$. If we now assign measure $1-p$ to the interval $[0,1]$ and measure p to the interval $[1,2]$ in each coordinate direction, extending these via product measures to higher dimensions, then the body K of the previous proof now satisfies $|K| = \Pr(\mathscr{F})$. In this manner we obtain the following theorem.

THEOREM 7. *Let $\mathscr{C}$ be a k-cover of* $[n]$. *Let $\mathscr{F}$ be a set of subsets of* $[n]$, *and let F be a randomly chosen subset of* $[n]$, *elements being selected independently with probability p. Let* $c < 1$ *be a positive constant such that* $\Pr(F \cap A \in \mathscr{F}_A) \leqslant c^{|A|}$ *for all* $A \in \mathscr{C}$. *Then* $\Pr(F \in \mathscr{F}) \leqslant c^n$.

4. *Hereditary properties of hypergraphs*

A *property* $\mathscr{P}$ of r-uniform hypergraphs is an infinite class of r-uniform hypergraphs which is closed under isomorphism. $\mathscr{P}$ is *hereditary* if every induced subgraph of every member of $\mathscr{P}$ is also in $\mathscr{P}$. Let $\mathscr{P}^n$ be the set of hypergraphs in $\mathscr{P}$ with vertex set $[n]$. We are interested in the rate of growth of $\mathscr{P}^n$ with n. It is convenient to define the constant c_n by $|\mathscr{P}^n| = 2^{c_n \binom{n}{r}}$. Note that eventually $0 < c_n < 1$ unless $\mathscr{P}$ is a *trivial* class, consisting of hypergraphs with no edges only, or complete hypergraphs only, or all hypergraphs. The consequence of our earlier work on projections is that the sequence (c_n) is monotone decreasing.

THEOREM 8. *Let $\mathscr{P}$ be a hereditary property of r-uniform hypergraphs, and let* $|\mathscr{P}^n| = 2^{c_n \binom{n}{r}}$. *Then* $c_{n-1} \geqslant c_n$ *for* $n \geqslant 2$. *In particular,* $\lim_{n \to \infty} c_n$ *exists.*

Proof. Denote by $[n]^{(r)}$ the set of r-subsets of $[n]$. We identify a hypergraph with the subset of $[n]^{(r)}$ which is its edge set. Then $\mathscr{P}^n$ becomes a set system on $[n]^{(r)}$. Let $A(i)$ be the set of r-subsets of $[n] \backslash \{i\}$. Then $\mathscr{P}^n_{A(i)}$, the projection of the set system $\mathscr{P}^n$ onto $A(i)$, is the set of hypergraphs induced by the hypergraphs in $\mathscr{P}^n$ on the vertex set $[n] \backslash \{i\}$. Since $\mathscr{P}$ is hereditary, $|\mathscr{P}^n_{A(i)}| \leqslant |\mathscr{P}^{n-1}|$ for every such $A(i)$.

Now the sets $A(i)$, $1 \leqslant i \leqslant n$, form an $(n-1)$-cover of $[n]^{(r)}$. Also, $|A(i)| = \binom{n-1}{r}$ and $|\mathscr{P}^n_{A(i)}| \leqslant |\mathscr{P}^{n-1}| = 2^{c_{n-1}|A(i)|}$. By Theorem 6, $|\mathscr{P}^n| \leqslant 2^{c_{n-1}|[n]^{(r)}|} = 2^{c_{n-1}\binom{n}{r}}$.

The case $r = 2$ of Theorem 8 answers a question of Scheinermann and Zito **[19]**, who asked if $\lim_{n \to \infty} c_n$ always exists. In fact, the monotonicity is not needed to show that the limit exists, which can be seen (for any r) by the following argument. Choose a value of m so that c_m is near to $\liminf_{n\to\infty} c_n$. Cover $[n]$ by m-subsets so that each r-subset of $[n]$ appears in exactly one m-subset (this usually cannot be done exactly, but it can be done approximately, which is good enough). Since the hypergraphs induced on the m-subsets are in $\mathscr{P}$, we deduce that $c_n \leqslant c_m$ (or, if we were using an approximate construction, we deduce that c_n is not much larger than c_m). This observation appears in Alekseev [1].

Earlier, the limit had been evaluated for the property of K_n-free graphs by Erdős, Kleitman and Rödl [9], by using the method of Kleitman and Rothschild [13]. Their result was generalised separately by Erdős, Frankl and Rödl [8], who considered the property of graphs not containing a given graph H as a subgraph, and by Prömel and Steger [17], who worked with the property of graphs not containing H as an induced subgraph. For related results see [15] and [16]. It turns out that for any graph property $\mathscr{P}$, the limit equals $1-1/r$, where r is the maximal integer for which there is an integer $0 \leqslant s \leqslant r$, such that $\mathscr{P}$ contains every graph which has a vertex partition into r classes, s of which span complete subgraphs and the remainder spanning independent sets. This was proved by Alekseev [2]; a transparent proof relating this result to the Erdős–Stone theorem appears in [6].

ACKNOWLEDGEMENT. The authors wish to thank Martin Dyer and Zoltán Füredi for helpful conversations.

References

1. V. E. ALEKSEEV, 'Hereditary classes and coding of graphs', *Probl. Cybernet.* 39 (1982) 151–164 (in Russian).
2. V. E. ALEKSEEV, 'On the entropy values of hereditary classes of graphs', *Discrete Appl. Math.* 3 (1993) 191–199.
3. G. R. ALLAN, 'An inequality involving product measures', *Radical Banach algebras and automatic continuity*, Lecture Notes in Math. 975 (ed. J. M. Bachar *et al.*, Springer, New York, 1981) 277–279.
4. N. ALON and K. A. BERMAN, 'Regular hypergraphs, Gordon's lemma and invariant theory', *J. Combin. Theory Ser. A* 43 (1986) 91–97.
5. B. BOLLOBÁS, *Combinatorics* (Cambridge University Press, 1987).
6. B. BOLLOBÁS and A. THOMASON, 'Hereditary properties of graphs', to appear in the volume in honour of Paul Erdős on his 80th birthday (ed. R. L. Graham and J. Nešetřil).
7. YU. D. BURAGO and V. A. ZALGALLER, *Geometric inequalities* (Springer, Berlin, 1988).
8. P. ERDŐS, P. FRANKL and V. RÖDL, 'The asymptotic enumeration of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent', *Graphs Combin.* 2 (1986) 113–121.
9. P. ERDŐS, D. J. KLEITMAN and V. RÖDL, 'Asymptotic enumeration of K_n-free graphs', *Internat. Colloq. Combin., Atti Convegni Lincei (Rome)* 17 (1976) 3–17.
10. Z. FÜREDI, 'Indecomposable regular graphs and hypergraphs', *Discrete Math.* 101 (1992) 59–64.
11. J. E. GRAVER, 'A survey of the maximum depth problem for indecomposable exact covers', *Infinite and finite sets (Keszthely, 1973)*, Proc. Colloq. Math. Soc. János Bolyai 10 (North-Holland, Amsterdam, 1976) 731–743.
12. H. HADWIGER, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie* (Springer, Berlin, 1957).
13. D. J. KLEITMAN and B. L. ROTHSCHILD, 'Asymptotic enumeration of partial orders on a finite set', *Trans. Amer. Math. Soc.* 205 (1975) 205–220.
14. L. H. LOOMIS and H. WHITNEY, 'An inequality related to the isoperimetric inequality', *Bull. Amer. Math. Soc.* 55 (1949) 961–962.
15. H. J. PRÖMEL and A. STEGER, 'Excluding induced subgraphs: quadrilaterals', *Random Structures Algorithms* 2 (1991) 55–71.
16. H. J. PRÖMEL and A. STEGER, 'Excluding induced subgraphs II: extremal graphs', *Discrete Appl. Math.* 44 (1993) 283–294.
17. H. J. PRÖMEL and A. STEGER, 'Excluding induced subgraphs III: a general asymptotic', *Random Structures Algorithms* 3 (1992) 19–31.

18. N. Sauer, 'On the density of families of sets', *J. Combin. Theory Ser. A* 13 (1972) 145–147.
19. E. R. Scheinermann and J. Zito, 'On the size of hereditary classes of graphs', *J. Combin. Theory Ser. B* 61 (1994) 16–39.
20. S. Shelah, 'A combinatorial problem; stability and order for models and theories in infinitary languages', *Pacific J. Math.* 41 (1972) 247–261.

Department of Pure Mathematics and Mathematical Statistics
16 Mill Lane
Cambridge CB2 1SB