RIBBON KNOTS IN S^4

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ABSTRACT

Our main results are several new obstructions to knotted 2-spheres' in S^4 being ribbon knots and the application of these to characterize fibered ribbon knots in S^4. For knots which bound punctured $S^1 \times S^2$, all of the known ribbon invariants vanish. We produce a new obstruction which detects the first known non-ribbon knots in this class. Finally, we show that ribbon knots are naturally associated with links whose groups are free, but not on their meridians.

0. *Introduction*

One of the outstanding questions in classical knot theory ($S^1 \hookrightarrow S^3$) asks whether every slice knot is a ribbon knot. For knotted 2-spheres in S^4, this question takes on a different flavor because, while every knot is a slice knot **[14]**, it is known that the corresponding notion of *ribbon* is more restrictive **[29, 30]**. One is left, therefore, with the question of exactly which knots in S^4 *are* ribbon knots.

This paper presents several new obstructions to knotted 2-spheres' being ribbon knots. These are applied to *characterize fibered ribbon knots* and consequently it is shown that no non-trivial twist-spun knot is ribbon. We also investigate the class of knots which bound punctured $S^1 \times S^2$ in S^4. These knots are interesting because all of the known ribbon invariants vanish for them **[11, 24]**. We herein exhibit the first known non-ribbon knots in this class, thus answering Problem 3 of **[10]**. These are also the first known examples of '0-slice knots' which are not ribbon. The class of 0-slice knots was introduced by Paul Melvin in his thesis, where he showed that the Gluck construction on a 0-slice knot yielded S^4 **[17]**. Furthermore for these knots (which bound punctured $S^1 \times S^2$) we present evidence for a 'ribbon conjecture' analogous to a corresponding conjecture for genus one knots in S^3. Lastly we note that to each ribbon knot is associated a link whose group is a free group, though not freely generated by any set of meridians of the link.

The necessary notation is reviewed in §1 while in §2 the main theorem (2.2) is introduced and proven. In §3, we accomplish the characterization of fibered ribbon knots. In §4, we investigate the class of knots which have a punctured $S^1 \times S^2$ as a Seifert manifold. In §6, we relate ribbon knots to links with free groups, and in §7 we close with a few questions for the reader.

1. *Notation*

We shall work entirely in the smooth category. Homology coefficients, if not specified, should be understood to be $\mathbb{Z}$. If G is a group, then $H_*(G)$ will denote $H_*(K(G,1))$ where $K(G,1)$ is a connected cell-complex with $\pi_1 \cong G$ and $\pi_n \cong 0$ for $n \geqslant 2$. The *cohomological dimension* of G, denoted cd G, can be taken to mean simply the minimum dimension of a $K(G,1)$. A space Y is said to be *aspherical* if $\pi_n(Y) = 0$ for $n > 1$.

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The unit Euclidean n-ball will be denoted variously by D^n, B^n (or I if $n = 1$) and its boundary by S^{n-1}. The interior of a closed subset A will be denoted by int A and if M is a closed manifold then M° will stand for M minus the interior of a small ball. If ϕ is a diffeomorphism of M then $S^1 \times_\phi M$ will denote the mapping torus of ϕ. If K^n is a knot in S^{n+2}, then by a *Seifert manifold* for K^n we shall mean an embedded, orientable, compact $(n+1)$-manifold whose boundary is K^n. We shall define a *knot* in S^4 as the image of a smooth embedding of a 2-sphere. The symbol $\mathcal{N}(A)$ will denote an open tubular neighborhood of A in some ambient manifold.

If $\mathbb{F}$ is a field, then an *n-dimensional $\mathbb{F}$-duality space* X will be a space together with a class $[X]_{\mathbb{F}} \in H_n(X; \mathbb{F})$ which induces isomorphisms $H^i(X; \mathbb{F}) \to H_{n-i}(X; \mathbb{F})$, for all i, via cap product. An *$(n+1)$-dimensional $\mathbb{F}$-duality pair* (A, X) will be a pair of spaces $(A, \partial A = X)$ together with a class $[A, X]_{\mathbb{F}} \in H_{n+1}(A, X; \mathbb{F})$ which induces isomorphisms $H^i(A, X; \mathbb{F}) \to H_{n+1-i}(A; \mathbb{F})$ as above, with the restriction that X be an n-dimensional $\mathbb{F}$-duality space with class $[X]_{\mathbb{F}} = \partial_*([A, X]_{\mathbb{F}})$.

Connected-sum will be denoted by $\#$ and boundary connected-sum by $\natural$.

2. *Obstructions to a knot's being ribbon*

The concept of a ribbon knot (as applied to 2-spheres in S^4) was first introduced by T. Yajima in [**29**].

DEFINITION. A knot K in S^4 is a *ribbon knot* if it bounds an immersed 3-disc $f: D^3 \looparrowright S^4$ with only (transverse) double points of a certain form. Namely, the singularity set $S(f)$ is $\{A_i, B_i\}$ where the A_i are properly-embedded 2-discs in D^3, the B_i are embedded 2-discs interior to D^3, all these 2-discs are disjoint and $f(A_i) = f(B_i)$ for each i (see below).

The property we shall actually investigate is the following (possibly) weaker one.

DEFINITION. $K \hookrightarrow S^4 = \partial B^5$ is a *homotopy-ribbon knot* if it bounds a properly-embedded 3-disc D_K in B^5 such that the *exterior* of D_K (that is $B^5 - \mathcal{N}(D_K)$) has a handlebody decomposition consisting of 0, 1 and 2-handles. (This compact 5-manifold will be denoted by A in the sequel.)

THEOREM 2.1 (L. Hitt [**10**]). *If $K \hookrightarrow S^4$ is a ribbon knot then K is a homotopy-ribbon knot.*

The proof is identical to the corresponding result for knots in S^3. Notice that $\partial A = X$ where X is the compact 4-manifold obtained by (framed) surgery on the 2-sphere $K \hookrightarrow S^4$. It follows from 2.1 that $\pi_1(X) \cong \pi_1(S^4 - \mathcal{N}(K))$, the *knot group*.

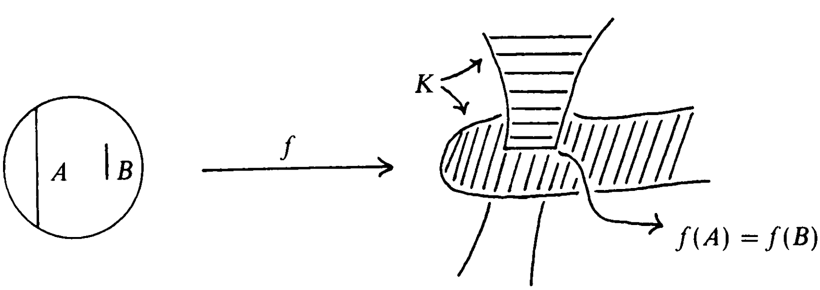

FIG. 1

Before stating our main theorem, we define a class of groups, first introduced by R. Strebel and J. Howie [**26, 13**]. The reader is warned that Strebel's actual definition of an E-group is in terms of projective $\mathbb{Z}G$-resolutions, not complexes; so presumably our notion is more restrictive. Howie calls our sub-class of groups $\mathcal{N}$-groups.

DEFINITION. A group G is an *E-group* if it is the fundamental group of some connected 2-complex Y (not necessarily finite) where $H_1(Y)$ is torsion-free and $H_2(Y) \cong 0$ (or, equivalently, $H_2(Y; C)$ vanishes for all abelian groups C).

THEOREM 2.2. *Let K be a homotopy-ribbon knot in S^4 and let $G = \pi_1(S^4 - \mathcal{N}(K)) \cong \pi_1(X)$ where X is as above. Then the following hold:*

(1) *$[G, G]$ is an E-group; in particular $H_1([G, G])$ is torsion-free and $H_2([G, G]) = 0$;*

(2) *if $\tilde{X}$ is the regular $\mathbb{Z}$-cover of X, then, for any abelian group C, the 'natural map' $i : \tilde{X} \to K([G, G], 1)$ induces the zero map $i_* : H_3(\tilde{X}; C) \to H_3([G, G]; C)$.*

REMARKS. That $H_1([G, G])$ must be torsion-free is a result of Hitt [**11**]. He used this fact to exhibit non-ribbon knots. By 'natural map' above, we mean one resulting from $K([G, G], 1)$ being built from $\tilde{X}$ by adding cells. Actually any continuous map $f : \tilde{X} \to K([G, G], 1)$ inducing an isomorphism on π_1 will do. We remind the reader that if $f, g : X \to K([G, G], 1)$ induce the same map on π_1 then they are homotopic.

Proof. (1) By Theorem 2.1, $X = \partial A$ where A is built with 0, 1 and 2-handles, and so the inclusion $j : X \to A$ induces an isomorphism on fundamental groups. But A has the homotopy-type of a connected 2-complex A_0 and

$$H_*(A_0) \cong H_*(B^5 - \mathcal{N}(D_K)) \cong H_*(S^1)$$

by Alexander duality. Hence $G \cong \pi_1(X) \cong \pi_1(A) \cong \pi_1(A_0)$ is an E-group. Similarly, $\tilde{X} = \partial \tilde{A}$ where $\tilde{A}$ is the regular $\mathbb{Z}$-cover of A, and $\tilde{A}$ has a handle-decomposition with 0, 1 and 2-handles. Thus $[G, G] \cong \pi_1(\tilde{X}) \to \pi_1(\tilde{A})$ is an isomorphism and $\tilde{A}$ has the homotopy-type of $\tilde{A}_0$, a connected 2-complex. The following lemma of J. F. Adams shows that $H_2(\tilde{A}_0; C) = 0$ for any abelian group C and therefore that $[G, G]$ is an E-group.

LEMMA 2.3 (J. F. Adams [**1**]; see also [**26**]). *Let A_0 be a connected 2-complex with $H_2(A_0; C) = 0$ for C a cyclic group. Let $\tilde{A}_0$ be a regular $\mathbb{Z}$-cover of A_0. Then $H_2(\tilde{A}_0; C) = 0$ as well.*

It follows immediately that $H_1([G, G]) \cong H_1(\tilde{A}_0)$ is torsion-free. To see that if $[G, G]$ is an E-group then $H_2([G, G]) = 0$, recall the exact sequence, due to Hopf,

$$\pi_2(Y) \xrightarrow{p} H_2(Y) \xrightarrow{i_*} H_2(\pi_1(Y)) \longrightarrow 0,$$

where Y is a connected space, p is the Hurewicz homomorphism and i_* is induced by the identity map on π_1 [**9**]. Since $[G, G]$ is an E-group, $[G, G] = \pi_1(Y)$ where $H_2(Y) = 0$ and so, *a fortiori*, $H_2([G, G]) = 0$.

Proof of (2). The commutative diagram of groups

$$\begin{array}{ccc} \pi_1(\tilde{A}) & & \\ {\scriptstyle j_*}\uparrow & \searrow^{(j_*)^{-1}} & \\ \pi_1(\tilde{X}) & \xrightarrow[\text{id.}]{} & [G,G] \end{array}$$

induces one of spaces

$$\begin{array}{ccc} \tilde{A} & & \\ {\scriptstyle j}\uparrow & \searrow^{\psi} & \\ \tilde{X} & \xrightarrow[i^{\#}]{} & K([G,G],1) \end{array}$$

and hence the map $i_*: H_3(\tilde{X}; C) \to H_3([G,G]; C)$ factors through $j_*; H_3(\tilde{X}; C) \to H_3(\tilde{A}; C)$. But $H_3(\tilde{A}; C) \cong 0$ since $\tilde{A}$ has the homotopy-type of a 2-complex.

Remark (a). There are known to be much stronger conditions on G than those of Theorem 2.2 for G to be the group of a ribbon knot. For example, G must have deficiency 1 (and in fact be Wirtinger with respect to some meridian [**30**]). These results are, in practice, not useful in detecting 'ribbonness'. Note also that condition (2) of Theorem 2.2 is a restriction on the knot K, not solely on its group G.

Remark (b). The techniques of Adams and Strebel [**1**, **26**] actually show that *any* member of the derived series of an E-group is an E-group; thus, for K to be ribbon, *every* group in the derived series of G must be an E-group. This is to be seen as a first approximation to G having deficiency 1.

Remark (c). In Theorem 2.2, even if we drop the hypothesis that K be homotopy-ribbon and simply assume that $j: X \to A$ induces an isomorphism on π_1, the conclusions of part (2) will still hold for any *field* coefficients $\mathbb{F}$. For, since $H_*(A) \cong H_*(S^1)$ and $H_*(\tilde{X}; \mathbb{F})$ is finitely-generated, a beautiful theorem of Milnor ensures that $(\tilde{A}, \tilde{X})$ is a 4-dimensional $\mathbb{F}$-duality pair for any field $\mathbb{F}$ [**18**]. Thus the ∂ map in the exact sequence

$$0 \longrightarrow H_4(\tilde{A}, \tilde{X}; \mathbb{F}) \xrightarrow{\partial} H_3(\tilde{X}; \mathbb{F}) \xrightarrow{j_*} H_3(\tilde{A}; \mathbb{F})$$

is an isomorphism and so the inclusion-induced map j_* is the zero map. This is enough to prove Theorem 3.0 below.

3. *Fibered ribbon knots*

In this section we shall use Theorem 2.2 to characterize fibered ribbon knots.

DEFINITION. The knot $K \hookrightarrow S^4$ is *fibered* if its exterior fibers over S^1 with fiber a punctured, orientable 3-manifold M°. That is to say, $S^4 - \mathcal{N}(K) = S^1 \times_\phi M^\circ$ for some diffeomorphism ϕ. (It follows that the framed surgery on K, denoted X, is diffeomorphic to $S^1 \times_\phi M$ for some connected, closed, orientable M.)

We shall establish the following result.

THEOREM 3.0. *Let K be a fibered knot in S^4 with fiber M°.*

(A) *If K is homotopy-ribbon then M is diffeomorphic to $\#_r (S^1 \times S^2) \# P$ where P is a homotopy-sphere.*

(B) *If M is diffeomorphic to $\#_r (S^1 \times S^2)$ then K is homotopy-ribbon.*

COROLLARY 3.1. *If $K \hookrightarrow S^4$ is a non-trivial twist-spun knot then K is not a ribbon knot.*

Proof. Zeeman has shown that an m-twist-spun knot is fibered with fiber M° where M is the m-fold cyclic cover of S^3 branched along some knot in S^3 [33]. By Theorem 3.0, to be ribbon, M would have to be of the form $\#_r (S^1 \times S^2) \# P$ with P simply-connected. But recent work of Steve Plotnick shows that no such m-fold cover can contain an $S^1 \times S^2$ factor [22], and hence M is simply-connected. Then, by the Smith conjecture, K is unknotted.

Proof of 3.0. (B) This follows easily from the fact that any orientation-preserving diffeomorphism of $\#_r (S^1 \times S^2)$ extends to one, ψ, of $\natural_r (S^1 \times B^3)$ [15]. Let $N = S^1 \times_\psi \left(\natural_r (S^1 \times B^3) \right)$, a compact, connected 5-manifold whose boundary is X (with X as above). Examining the exact sequence

$$H_k(\natural(S^1 \times B^3)) \xrightarrow{\psi_* - \mathrm{id}} H_k(\natural(S^1 \times B^3)) \longrightarrow H_k(N) \longrightarrow H_{k-1}(\natural(S^1 \times B^3))$$

we see immediately that $H_k(N)$ is trivial for $k > 2$ and for $k = 2$ is isomorphic to the kernel of $\psi_* - \mathrm{id}_*$ on $H_1(\natural(S^1 \times B^3))$. Since the inclusion $j : \#(S^1 \times S^2) \to \natural(S^1 \times B^3)$ is an isomorphism on H_1 and since ψ extends ϕ, the map $\psi_* - \mathrm{id}$ will be an isomorphism just as $\phi_* - \mathrm{id}$ is. It follows that $H_k(N)$ is trivial for all $k > 1$ and that $\pi_1(N) \cong \pi_1(X)$ is normally generated by a single 'meridional' element. Therefore, by adding a single 2-handle to $\partial N = X$, we arrive at a contractible 5-manifold $\mathscr{B}$. Furthermore, if we choose the framed curve for the 2-handle so as to exactly reverse the surgery that produced X from S^4, then the boundary of $\mathscr{B}$ will be S^4. Hence $\mathscr{B} \cong B^5$ [19], and the cocore D of the 2-handle will have the knot K as its boundary. Since $B^5 - \mathcal{N}(D) = N$ can be built with 0, 1 and 2-handles, it follows that K is homotopy-ribbon.

(A) We shall require a lemma about the nature of H_3 of 3-manifold groups.

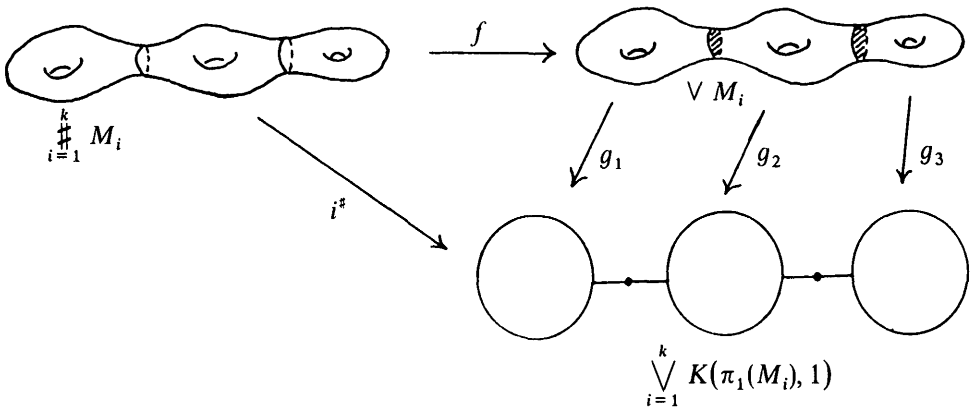

FIG. 2

LEMMA 3.2. *Let M be a closed, connected, orientable 3-manifold. The 'natural map' $i^\#: M \to K(\pi_1(M), 1)$ induces the zero map $i_*: H_3(M) \to H_3(\pi_1(M))$ only if $M = \#_r (S^1 \times S^2) \# P$ where P is a homotopy-sphere.*

Proof. We can assume that $M \cong \#_{i=1}^{k} M_i$ where each M_i is $S^1 \times S^2$ or has vanishing π_2 [6]. The map $i^\#: M \to K(\pi_1(M), 1)$ can be described in the following way (see diagram).

First map M to $\bigvee_{i=1}^{k} M_i$ by adding $(k-1)$ 3-cells to 'the' separating 2-spheres of the prime decomposition; then add whatever cells are necessary to *each* M_i to arrive at a $K(\pi_1(M_i), 1)$. Clearly $\bigvee_{i=1}^{k} K(\pi_1(M_i), 1)$ is a $K(\pi_1(M), 1)$, so that

$$i_*([M]) = (g_{1_*}([M_1]), g_{2_*}([M_2]), \ldots, g_{k_*}([M_k]))$$

where each $g_i: M_i \to K(\pi_1(M_i), 1)$ is the 'natural' map. If $i_*([M]) = 0$ then each $g_{i_*}([M_i]) = 0$. Assume that, for some i, M_i is not $S^1 \times S^2$, so that $\pi_2(M_i) = 0$. Then we have the exact sequence

$$H_3(\tilde{M}_i) \xrightarrow{d_*} H_3(M_i) \xrightarrow{g_{i_*}} H_3(\pi_1(M_i), 1) \longrightarrow 0,$$

where $\tilde{M}_i$ is the universal cover of M_i, d is the covering map and g_i is induced by the 'natural' map [9]. By our hypothesis, d_* is an epimorphism; but this implies that the degree of $d: \tilde{M}_i \to M_i$ is one, and hence that M_i is a homotopy-sphere. This completes the proof of Lemma 3.2.

We return to the proof of Theorem 3.0(A). If K is a ribbon knot then Theorem 2.2 implies that $i_*([\tilde{X}]_{\mathbb{Z}}) = 0$ where $i_*: H_3(\tilde{X}: \mathbb{Z}) \to H_3([G, G]; \mathbb{Z})$. If K is fibered, then $\tilde{X} \cong \mathbb{R} \times M$ and $[G, G] \cong \pi_1(M)$, so that $i_*([M]_{\mathbb{Z}}) = 0$ where $i_*: H_3(M; \mathbb{Z}) \to H_3(\pi_1(M); \mathbb{Z})$. Then, by Lemma 3.2, $M \cong \#_r (S^1 \times S^2) \# P$ where P is a homotopy-sphere.

Proposition 6.1 (a) was first due to J. Hillman [7]. We note that the construction below applies to *homotopy*-ribbon knots in all higher dimensions as well.

PROPOSITION 6.1. *Let K be a ribbon knot with $\pi_1(S^4 - K)$* not *infinite cyclic. Then K is a sublink of a link $L = \{K, L_1, \ldots, L_n\}$ such that*

(a) *$\pi_1(S^4 - \mathcal{N}(L))$ is a free group,*

(b) *$\pi_1(S^4 - \mathcal{N}(L))$ is not free on any set of meridians of L.*

Proof. Recall that a *meridian* of $L = \{K, L_1, \ldots, L_n\}$ is any element of $\pi_1(S^4 - \mathcal{N}(L), *)$ which is obtained by joining the boundary of a fiber of $\mathcal{N}(L)$ to the base point $*$. To prove (a), simply recall the 5-manifold A provided by Theorem 2.1:

$$A = B^5 \cup \text{1-handles} \cup \text{2-handles} = \left(\natural_{n+1} (S^1 \times B^4)\right) \cup (\text{2-handles}).$$

Since $\partial A = X$, the surgery on K, we can also think of A as a handlebody built from X. Splitting A at the $\#_{n+1} (S^1 \times S^3)$ level yields

$$A = \left(\natural_{n+1} (S^1 \times B^4)\right) \cup_{\#(S^1 \times S^3)} (X \times I \cup \text{3-handles}).$$

The attaching spheres of the 3-handles form a link of 2-spheres $\{L_1, \ldots, L_n\}$ in X such that the framed surgery on $\{L_1, \ldots, L_n\}$ yields $\partial_+(X \times I \cup \text{3-handles}) = \#_{n+1} (S^1 \times S^3)$. Thus $\pi_1(X - \mathcal{N}(L_1, \ldots, L_n))$ is a free group. Note that there is a framed circle γ in X such that $X - \mathcal{N}(\gamma) \cong S^4 - \mathcal{N}(K)$. Since we can assume that the $\{L_1, \ldots, L_n\}$ miss γ, we get a link $L = \{K, L_1, \ldots, L_n\}$ in S^4 whose group

$$\begin{aligned}\pi_1(S^4 - \mathcal{N}(K) - \mathcal{N}(L_1, \ldots, L_n)) &\cong \pi_1(X - \mathcal{N}(\gamma) - \mathcal{N}(L_1, \ldots, L_n)) \\ &\cong \pi_1(X - \mathcal{N}(L_1, \ldots, L_n))\end{aligned}$$

is a free group. If this group were free on meridians $\{m_K, m_1, \ldots, m_n\}$, then $\pi_1(S^4 - \mathcal{N}(K))$ would be isomorphic to $\mathbb{Z}$; this contradicts the hypothesis.

7. *Questions*

(1) Is there a knot K in S^4 whose *group* is the group of a ribbon knot but such that K is *not* ribbon?

(2) If G is the group of a ribbon knot, must G have a 2-dimensional $K(G, 1)$? (This is implied by the Whitehead conjecture.)

(3) If G is the group of the knot K and G has a finite 2-dimensional $K(G, 1)$, then is K a ribbon knot? (I feel this should be false.)

(4) Call a Seifert manifold M° for K *incompressible* if the 2-maps $v_+, v_- : M^\circ \to S^4 - \mathcal{N}(M^\circ)$ induce monomorphisms on π_1. Must every