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# A two-stage logarithmic goal programming method for generating weights from interval comparison matrices☆

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**Abstract**

A two-stage logarithmic goal programming (TLGP) method is proposed to generate weights from interval comparison matrices, which can be either consistent or inconsistent. The first stage is devised to minimize the inconsistency of interval comparison matrices and the second stage is developed to generate priorities under the condition of minimal inconsistency. The weights are assumed to be multiplicative rather than additive. In the case of hierarchical structures, a nonlinear programming method is used to aggregate local interval weights into global interval weights. A simple yet practical preference ranking method is investigated to compare the interval weights of criteria or rank alternatives in a multiplicative aggregation process. The proposed TLGP is also applicable to fuzzy comparison matrices when they are transformed into interval comparison matrices using $\alpha$-*level* sets and the extension principle. Six numerical examples including a group decision analysis problem with a group of comparison matrices, a hierarchical decision problem and a fuzzy decision problem using fuzzy comparison matrix are examined to show the applications of the proposed methods. Comparisons with other existing procedures are made whenever possible.


*Keywords:* Analytic hierarchy process (AHP); Interval comparison matrix; Fuzzy comparison matrix; Goal programming; Preference ranking; The extension principle; Group decision making

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## 1. Introduction

Most real-world decision problems involve multiple criteria that are often in conflict in general and it is sometimes necessary to conduct trade-off analysis in multiple criteria decision analysis (MCDA). As such, the estimation of the relative weights of criteria plays an important role in a MCDA process. Among many frameworks developed for weight estimation, pairwise comparison matrices provide a natural framework to elicit preferences from decision makers and have been used in several weight generation methods. However, due to the complexity and uncertainty involved in real-world decision problems and the inherent subjective nature of human judgments, it is sometimes unrealistic and infeasible to acquire exact judgments. It is more natural or easier to provide fuzzy or interval judgments for parts or all of the judgments in a pairwise comparison matrix. A number of techniques have been developed to use such a fuzzy or interval comparison matrix to generate weights.

For instance, Van Laarhoven and Pedryce [37] considered treating elements in a comparison matrix as fuzzy numbers having triangular membership functions and employed the logarithmic least-squares method to generate fuzzy weights. Buckley [10] extended the method to trapezoidal membership functions and hierarchical analysis. Boender et al. [6] found a fallacy in the normalization procedure of Van Laarhoven and Pedryce's method for generating fuzzy weights and subsequently modified the method. Xu and Zhai [43] discussed the problem of extracting fuzzy weights from a fuzzy judgment matrix also using the logarithmic least-squares method based on a distance definition in a fuzzy judgment space. Xu [42] used the same distance definition to develop a fuzzy least-squares priority method. Leung and Cao [24] proposed a fuzzy consistency definition by considering a tolerance deviation and determined fuzzy local and global weights using the extension principle. Buckley et al. [11] directly fuzzified Saaty's original procedure of computing weights in hierarchical analysis to get fuzzy weights in fuzzy hierarchical analysis. Csutora and Buckley [13] presented a Lambda–Max method, which is the direct fuzzification of the $\lambda_{\max}$ method, to find fuzzy weights.

Saaty and Vargas [32] proposed interval judgments for the AHP method as a way to model subjective uncertainty and used a Monte Carlo simulation approach to find out weight intervals from interval comparison matrices. They also pointed out difficulties in using this approach. Arbel [1,2] interpreted interval judgments as linear constraints on local priorities and formulated the prioritization process as a linear programming (LP) model. Kress [22] found that Arbel's method is ineffective for inconsistent interval comparison matrices because no feasible region exists in such circumstances. Salo and Hämäläinen [33,34] extended Arbel's approach to hierarchical structures. Their method found the maximum and minimum feasible values for all interval priorities and incorporated the resulting intervals into further synthesis of global interval priorities. Arbel and Vargas [3,4] formulated the hierarchical problem as a nonlinear programming model in which all local priorities in a hierarchy are included as decision variables and also established a connection between Monte Carlo simulation and Arbel's LP approach. Moreno-Jiménez [29] studied the probability distribution of possible rankings of the alternatives in an interval comparison matrix of size $n = 2$ or 3. Islame et al. [20] used a Lexicographic Goal Programming (LGP) to find out weights from inconsistent pairwise interval comparison matrices and explored its properties and advantages as a weight estimation technique. Haines [18] proposed a statistical approach to extract preferences from interval comparison matrices. Two specific distributions on a feasible region were examined and the mean of the distributions was used as a basis for assessment and ranking. Mikhailov [26–28] developed a fuzzy preference programming (FPP) method to derive crisp priorities from interval or fuzzy comparison matrices and extended the method to the case of group decision making.

Our literature review shows that only Monte Carlo simulation, LGP and FPP methods could be used to generate weights from both consistent and inconsistent interval comparison matrices. All the other existing methods mentioned above are only applicable to consistent interval comparison matrices. As pointed out by Saaty and Vargas [32], Monte Carlo simulation is rather complicated and time consuming in computation. Since the number of simulations is always limited, the accuracy of the resultant priority intervals may not be satisfactory. In general, weight intervals generated by Monte Carlo simulations are narrower than the real priority intervals. Although Islame et al's LGP and Mikhailov's FPP methods can both be used to generate weights from inconsistent interval comparison matrices, the former is defective in theory because using the upper or lower triangular judgments of an interval comparison matrix could always lead to different priority rankings, the latter requires the decision maker (DM) to predetermine the values of all tolerance parameters, and both methods can only generate a crisp set of priorities in the presence of inconsistency. Since judgments in an interval comparison matrix are imprecise, it is more natural and logical that an interval weight vector should be generated than an exact priority vector that is only a point estimate. However, how to generate a valid estimate for weighs in the presence of inconsistent interval comparison matrices and how to develop an effective method that is applicable to both consistent and inconsistent interval comparison matrices still remains unsolved.

This paper is devoted to investigating the above issues. A simple yet pragmatic two-stage logarithmic goal programming method is proposed to generate weights from both consistent and inconsistent interval comparison matrices. The first stage is devised to minimize the inconsistency that may exist in interval comparison matrices and the second stage is developed to generate interval priorities under the condition of minimal inconsistency. In the case of hierarchical structures, a nonlinear programming method is proposed to aggregate local interval weights into global interval weights. A simple yet practical preference ranking method is extended to compare the interval weights of criteria or rank alternatives in a multiplicative aggregation process. Since fuzzy comparison matrices can be transformed into interval comparison matrices using $\alpha$-*level* sets and the extension principle, fuzzy comparison matrices will be handled as interval comparison matrices. Six numerical examples including a group decision analysis problem with a group of comparison matrices, a hierarchical (AHP) decision problem and a fuzzy comparison matrix are provided to show the applications of the proposed methods.

The paper is organized as follows. Section 2 addresses the method of two-stage logarithmic goal programming (TLGP) for generating priorities from interval comparison matrices and explores some of its properties. Section 3 discusses the aggregation problem of interval weights and a nonlinear programming method is proposed. Section 4 focuses on the problem of comparing or ranking interval weights and a simple and practical preference ranking method is investigated. Section 5 presents five numerical studies to show the applications of the proposed methods. Section 6 discusses the extension of TLGP to fuzzy comparison matrices, which are transformed into interval comparison matrices using $\alpha$-*level* sets and the extension principle. The paper is concluded in Section 7.

## 2. A two-stage logarithmic goal programming method for generating interval weights

Suppose the decision maker provides interval judgments instead of precise judgments for a pairwise comparison. For example, it could be judged that criterion $i$ is between $l_{ij}$ and $u_{ij}$ times as important as criterion $j$ with $l_{ij}$ and $u_{ij}$ being nonnegative real numbers and $l_{ij} \leqslant u_{ij}$. Then, an interval comparison matrix can be represented by

$$A = \begin{bmatrix} 1 & [l_{12}, u_{12}] & \cdots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \cdots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \vdots & \vdots \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & \cdots & 1 \end{bmatrix}, \tag{1}$$

where $l_{ij} = 1/u_{ji}$ and $u_{ij} = 1/l_{ji}$. About the above interval comparison matrix, we have the following definitions:

**Definition 1.** Given an interval comparison matrix $A = (a_{ij})_{n\times n}$ with $l_{ij} \leqslant a_{ij} \leqslant u_{ij}$ and $a_{ii} = l_{ii} = u_{ii} = 1$ for $i, j = 1, \ldots, n$, if the following convex feasible region $S_w = \{w = (w_1, \ldots, w_n) \mid l_{ij} \leqslant w_i/w_j \leqslant u_{ij}, \sum_{i=1}^{n} w_i = 1, w_i > 0\}$ is nonempty, then $A$ is said to be a consistent interval comparison matrix; otherwise, $A$ is said to be inconsistent.

Let $W = (w_1, \ldots, w_n)$ be weight vector, on which two different types of constraints may be imposed. One is the additive constraint, namely, $\sum_{i=1}^{n} w_i = 1$. The other is the multiplicative constraint, i.e. $\prod_{i=1}^{n} w_i = 1$, which is equivalent to

$$\sum_{i=1}^{n} \ln w_i = 0. \tag{2}$$

Such a multiplicative constraint is widely used in multiplicative AHP [5,35] and is also used throughout the paper. Since interval judgments may be interpreted as constraints on weights, accordingly, (1) may be expressed as

$$l_{ij} \leqslant w_i/w_j \leqslant u_{ij}, \quad i, j = 1, \ldots, n, \tag{3}$$

which can be equivalently expressed as

$$\ln l_{ij} \leqslant \ln w_i - \ln w_j \leqslant \ln u_{ij}, \quad i, j = 1, \ldots, n. \tag{4}$$

Inequality (4) holds only for consistent judgments and does not hold for inconsistent judgments. To generate a set of unified inequality constraints holding for both consistent and inconsistent judgments, deviation variables $p_{ij}$ and $q_{ij}$ are introduced into (4)

$$\ln l_{ij} - p_{ij} \leqslant \ln w_i - \ln w_j \leqslant \ln u_{ij} + q_{ij}, \quad i, j = 1, \ldots, n, \tag{5}$$

where $p_{ij}$ and $q_{ij}$ are both nonnegative real numbers, but only one of them can be positive, i.e. $p_{ij}q_{ij} = 0$. For consistent judgments, both $p_{ij}$ and $q_{ij}$ are set to be zero. In the presence of inconsistent judgments, only one of $p_{ij}$ and $q_{ij}$ may be unequal to zero. So, (5) holds for both consistent and inconsistent judgments. It is desirable that the deviation variables $p_{ij}$ and $q_{ij}$ are kept to be as small as possible, which means to minimize the inconsistency of interval comparison matrices, thus leading to the following objective function and goal programming (GP) model:

$$\text{Min } J = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (p_{ij} + q_{ij}) \tag{6}$$

$$\text{s.t. } \ln w_i - \ln w_j + p_{ij} \geqslant \ln l_{ij}, \quad i, j = 1, \ldots, n, \tag{7}$$

$$\ln w_i - \ln w_j - q_{ij} \leqslant \ln u_{ij}, \quad i, j = 1, \ldots, n, \tag{8}$$

$$\sum_{i=1}^{n} \ln w_i = 0, \tag{9}$$

$$p_{ij}, q_{ij} \geqslant 0 \quad \text{and} \quad p_{ij} q_{ij} = 0, \quad i, j = 1, \ldots, n. \tag{10}$$

Note that Bryson [8] ever developed a goal programming (GP) method for generating priority vector from crisp comparison matrix. But here we use logarithms instead of original judgments. Since the value of $\ln w_i$ is nonnegative when $w_i \geqslant 1$ and negative when $w_i < 1$, the following nonnegative variables are introduced:

$$x_i = \frac{\ln w_i + |\ln w_i|}{2}, \quad i = 1, \ldots, n, \tag{11}$$

$$y_i = \frac{-\ln w_i + |\ln w_i|}{2}, \quad i = 1, \ldots, n. \tag{12}$$

Based on $x_i$ and $y_i$, $\ln w_i$ can be expressed as

$$\ln w_i = x_i - y_i, \quad i = 1, \ldots, n, \tag{13}$$

where $x_i y_i = 0$. Thus, the above GP model (6)–(10) can be further expressed and simplified as

$$\text{Min } J = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (p_{ij} + q_{ij}) \tag{14}$$

$$\text{s.t. } x_i - y_i - x_j + y_j + p_{ij} \geqslant \ln l_{ij}, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n, \tag{15}$$

$$x_i - y_i - x_j + y_j - q_{ij} \leqslant \ln u_{ij}, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n, \tag{16}$$

$$\sum_{i=1}^{n} (x_i - y_i) = 0, \tag{17}$$

$$x_i, y_i \geqslant 0, \quad x_i y_i = 0, \quad i = 1, \ldots, n, \tag{18}$$

$$p_{ij}, q_{ij} \geqslant 0, \quad p_{ij} q_{ij} = 0, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n \tag{19}$$

or

$$\text{Min } J = \sum_{i=2}^{n} \sum_{j=1}^{i-1} (p_{ij} + q_{ij}) \tag{20}$$

$$\text{s.t. } x_i - y_i - x_j + y_j + p_{ij} \geqslant \ln l_{ij}, \quad i = 2, \ldots, n, \quad j = 1, \ldots, i-1, \tag{21}$$

$$x_i - y_i - x_j + y_j - q_{ij} \leqslant \ln u_{ij}, \quad i = 2, \ldots, n, \quad j = 1, \ldots, i-1, \tag{22}$$

$$\sum_{i=1}^{n} (x_i - y_i) = 0, \tag{23}$$

$$x_i, y_i \geqslant 0, \quad x_i y_i = 0, \quad i = 1, \ldots, n, \tag{24}$$

$$p_{ij}, q_{ij} \geqslant 0, \quad p_{ij} q_{ij} = 0, \quad i = 2, \ldots, n, \quad j = 1, \ldots, i-1. \tag{25}$$

About the above GP models, there exist the following theorems.

**Theorem 1.** *$A = (a_{ij})_{n\times n}$ is a consistent interval comparison matrix if and only if $J^* = 0$, where $J^*$ is the optimal value of objective function* (14) *or* (20).

**Proof.** If $A$ is a consistent interval comparison matrix, then the convex feasible region $S_w$ is nonempty, which means that $l_{ij} \leqslant w_i/w_j \leqslant u_{ij}$ holds for all the judgments, equivalently, $\ln l_{ij} \leqslant \ln w_i - \ln w_j \leqslant \ln u_{ij}$ for $i, j = 1, \ldots, n$. So $p_{ij} = 0$, $q_{ij} = 0$ for all the judgments, which is equivalent to $J^* = 0$.

If $J^* = 0$, then $p_{ij} = 0$ and $q_{ij} = 0$ hold for all $i, j = 1, \ldots, n$. Accordingly, inequality (4) holds for all the judgments. This means that (3) holds for all $i, j = 1, \ldots, n$. In other words, there is no contradiction among all the judgments. So, the convex feasible region $S_w$ cannot be empty when $J^* = 0$. By Definition 1, $A$ is a consistent interval comparison matrix. □

**Theorem 2.** *GP models* (14)–(19) *and* (20)–(25) *are equivalent.*

**Proof.** Consider a reciprocal pair of interval judgments, say, $l_{ij} \leqslant a_{ij} \leqslant u_{ij}$ and $1/u_{ij} \leqslant a_{ji} \leqslant 1/l_{ij}$. With the introduction of deviation variables, the above reciprocal pair of interval judgments can be transformed to the following pair of inequality constraints:

$$\ln l_{ij} - p_{ij} \leqslant \ln w_i - \ln w_j \leqslant \ln u_{ij} + q_{ij}, \tag{26}$$

$$-\ln u_{ij} - p_{ji} \leqslant \ln w_j - \ln w_i \leqslant -\ln l_{ij} + q_{ji}. \tag{27}$$

Inequality (27) may be further written as

$$\ln l_{ij} - q_{ji} \leqslant \ln w_i - \ln w_j \leqslant \ln u_{ij} + p_{ji}. \tag{28}$$

Let $p_{ij} = q_{ji}$ and $q_{ij} = p_{ji}$. Then, inequality constraints (26) and (28) are indeed equivalent. Besides, since $p_{ij} + q_{ij} = p_{ji} + q_{ji}$, the contributions of deviation variables $p_{ij}, q_{ij}$ and $p_{ji}, q_{ji}$ to their respective objective functions are also equivalent. Since the above discussion applies to all the reciprocal pairs of interval judgments, it can be concluded that models (14)–(19) and (20)–(25) are in fact equivalent.

Note that Theorem 1 shows how to check if an interval comparison matrix is consistent or not. Theorem 2 ensures that using the upper or lower triangular judgments of an interval comparison matrix will always lead to the same results, which is the very difference of our method from Islame et al's LGP method.

Since Models (14)–(19) and (20)–(25) are equivalent in nature, we will consider only GP model (14)–(19) in the rest of this paper. Generally speaking, there may be multiple solutions to the GP model, which leads to intervals of weights. In order to find a feasible interval for each weight $w_i (i = 1, \ldots, n)$, we keep the optimal objective function value unchanged and use it as a constraint to construct the following pairs of GP models:

$$\text{Min/Max } \ln w_i = x_i - y_i \tag{29}$$

$$\text{s.t. } x_i - y_i - x_j + y_j + p_{ij} \geqslant \ln l_{ij}, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n, \tag{30}$$

$$x_i - y_i - x_j + y_j - q_{ij} \leqslant \ln u_{ij}, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n, \tag{31}$$

$$\sum_{i=1}^{n} (x_i - y_i) = 0, \tag{32}$$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (p_{ij} + q_{ij}) = J^*, \tag{33}$$

$$x_i, y_i \geqslant 0, \qquad x_i y_i = 0, \qquad i = 1, \ldots, n, \tag{34}$$

$$p_{ij}, q_{ij} \geqslant 0, \quad p_{ij} q_{ij} = 0, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n, \tag{35}$$

where $J^*$ is the optimal value of the objective function of GP model (14)–(19). Note that the complementarity constraints $x_i y_i = 0$ ($p_{ij} q_{ij} = 0$) can always be satisfied without $x_i$ and $y_i$ ($p_{ij}$ and $q_{ij}$) being simultaneously selected as basic variables in a simplex method. The optimal objective values of the above pairs of GP models (29)–(35) consist of the possible intervals of the logarithmic weights $\ln w_i (i = 1, \ldots, n)$, which are denoted by the logarithmic weight intervals $[\ln w_i^{\mathrm{L}}, \ln w_i^{\mathrm{U}}]$ $(i = 1, \ldots, n)$. Accordingly, the weight intervals $[w_i^{\mathrm{L}}, w_i^{\mathrm{U}}]$ can be obtained from logarithmic weight intervals, where $w_i^{\mathrm{L}} = \exp(\ln w_i^{\mathrm{L}})$ and $w_i^{\mathrm{U}} = \exp(\ln w_i^{\mathrm{U}})$. Since the whole solution process for generating weights includes two stages, the method is thus referred to as the two-stage logarithmic goal programming (TLGP) method. □

**Theorem 3.** *If $J^* = 0$, then TLGP degenerates to solving the following pairs of GP models*:

$$\text{Min/Max } \ln w_i = x_i - y_i \tag{36}$$

$$\text{s.t. } x_i - y_i - x_j + y_j \geqslant \ln l_{ij}, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n, \tag{37}$$

$$x_i - y_i - x_j + y_j \leqslant \ln u_{ij}, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n, \tag{38}$$

$$\sum_{i=1}^{n} (x_i - y_i) = 0, \tag{39}$$

$$x_i, y_i \geqslant 0, \quad x_i y_i = 0, \quad i = 1, \ldots, n. \tag{40}$$

The proof of Theorem 3 is straightforward. This theorem shows that if an interval comparison matrix $A$ has already been known to be consistent, then only the GP model of the second stage will need to be solved.

## 3. Synthesis of interval weights

Suppose interval weights for upper-level criteria and lower-level alternatives have all been obtained, as shown in Table 1, where $[w_j^{\mathrm{L}}, w_j^{\mathrm{U}}]$ is the interval weight of criterion $j (j = 1, \ldots, m)$ and $[w_{ij}^{\mathrm{L}}, w_{ij}^{\mathrm{U}}]$ is the interval weight of alternative $A_i$ with respect to the criterion $j (i = 1, \ldots, n; j = 1, \ldots, m)$.

Salo and Hämäläinen [34] showed by an example that interval arithmetic was unsuitable for the synthesis of interval weights and had to be rejected. They thus proposed a hierarchical decomposition method that decomposes a hierarchical composition problem into a series of linear programming problems over feasible regions. Their method does not provide information on the interval weights shown in Table 1 and therefore is not adopted in this paper.

Bryson and Mobolurin [9] suggested a linear programming method to aggregate additively the weights of each alternative with respect to different criteria, in which the weights of the criteria were treated as decision variables and a pair of LP models was constructed to capture, respectively, the lower and upper bounds of the composite weight of each alternative.

Much earlier, Dubois and Prade [14] had investigated at length the problem of additive aggregation of interactive fuzzy weights and developed effective computational formulas for the additions of interactive LL fuzzy weights without the need of solving any linear programming.

Table 1
Synthesis of interval weights

| Alternative | Criterion 1 | Criterion 2 | … | Criterion $m$ | Composite weight |
|---|---|---|---|---|---|
| | $[w_1^{\mathrm{L}}, w_1^{\mathrm{U}}]$ | $[w_2^{\mathrm{L}}, w_2^{\mathrm{U}}]$ | … | $[w_m^{\mathrm{L}}, w_m^{\mathrm{U}}]$ | |
| $A_1$ | $[w_{11}^{\mathrm{L}}, w_{11}^{\mathrm{U}}]$ | $[w_{12}^{\mathrm{L}}, w_{12}^{\mathrm{U}}]$ | … | $[w_{1m}^{\mathrm{L}}, w_{1m}^{\mathrm{U}}]$ | $[w_{A_1}^{\mathrm{L}}, w_{A_1}^{\mathrm{U}}]$ |
| $A_2$ | $[w_{21}^{\mathrm{L}}, w_{21}^{\mathrm{U}}]$ | $[w_{22}^{\mathrm{L}}, w_{22}^{\mathrm{U}}]$ | … | $[w_{2m}^{\mathrm{L}}, w_{2m}^{\mathrm{U}}]$ | $[w_{A_2}^{\mathrm{L}}, w_{A_2}^{\mathrm{U}}]$ |
| $\vdots$ | $\vdots$ | $\vdots$ | … | $\vdots$ | $\vdots$ |
| $A_n$ | $[w_{n1}^{\mathrm{L}}, w_{n1}^{\mathrm{U}}]$ | $[w_{n2}^{\mathrm{L}}, w_{n2}^{\mathrm{U}}]$ | … | $[w_{nm}^{\mathrm{L}}, w_{nm}^{\mathrm{U}}]$ | $[w_{A_n}^{\mathrm{L}}, w_{A_n}^{\mathrm{U}}]$ |

Since the interval weights in Table 1 satisfy the multiplicative constraint (2) rather than additive constraint, the following pairs of nonlinear programming (NLP) models are therefore suggested for the synthesis of interval weights:

$$\text{Min } w_{A_i}^{\mathrm{L}} = \prod_{j=1}^{m} (w_{ij}^{\mathrm{L}})^{w_j} \tag{41}$$

$$\text{s.t. } W \in \Omega_W, \tag{42}$$

$$\text{Max } w_{A_i}^{\mathrm{U}} = \prod_{j=1}^{m} (w_{ij}^{\mathrm{U}})^{w_j} \tag{43}$$

$$\text{s.t. } W \in \Omega_W, \tag{44}$$

where $W = (w_1, \dots, w_m)$, $\Omega_w = \Omega_w = \{W = (w_1, \dots, w_m) | w_j^{\mathrm{L}} \leqslant w_j \leqslant w_j^{\mathrm{U}}, \prod_{i=1}^{m} w_j = 1\}$, and $w_{A_i}^{\mathrm{L}}$ and $w_{A_i}^{\mathrm{U}}$ are, respectively, the lower and upper bounds of the composite weight $w_{A_i}$, which constitute an interval denoted by $w_{A_i} = [w_{A_i}^{\mathrm{L}}, w_{A_i}^{\mathrm{U}}]$ $(i = 1, \dots, n)$. The global interval weight for each alternative can be generated by repeating the above synthesis processes until reaching the top level, which represents the goal of decision analysis.

## 4. Comparisons and ranks of interval weights

In the case of interval comparison matrices, since judgments are partly or completely imprecise, it is more logical and acceptable to use interval weights to represent imprecision than an exact priority vector that is only a point estimate. However, interval weights can lead to greater complexity and difficulty in comparison and ranking. In order to compare or rank global interval weights, Salo and Hämäläinen [34] required the decision maker (DM) to provide information on the revision of interval comparison matrices until one interval weight absolutely dominates all other weights or a pairwise dominance relation is found. However, it is not always feasible to require DM to provide extra information especially when DM fails or is unwilling to do so. Ishibuchi and Tanaka [19] used the comparison rule for interval numbers to define order relations. Their approach is probably the most prominent in the analysis and comparison of interval numbers. But it fails when one interval number is nested in another one. Moreover, due to the existence of uncertainty, preference relationships among interval numbers are not likely to be 100 percent certain. So, it would be more desirable to provide the degrees of preference along with preference relations among interval weights. This section is devoted to dealing with this problem and a simple and practical preference ranking approach [41] is further extended to compare the weights of criteria or rank alternatives in a multiplicative aggregation process.

Let $a = [a_1, a_2]$ and $b = [b_1, b_2]$ be two interval weights, whose relationships are as shown in Fig. 1. We refer to the degree of one interval weight being greater than another one as *the degree of preference*.

**Definition 2.** The degree of preference of $a$ over $b$ (or $a > b$) is defined as

$$P(a > b) = \frac{\max(0, a_2 - b_1) - \max(0, a_1 - b_2)}{(a_2 - a_1) + (b_2 - b_1)}. \tag{45}$$

The degree of preference of $b$ over $a$ (or $b > a$) can be defined in the same way. That is

$$P(b > a) = \frac{\max(0, b_2 - a_1) - \max(0, b_1 - a_2)}{(a_2 - a_1) + (b_2 - b_1)}. \tag{46}$$

Due to the fact that $a_2 - b_1$ and $a_1 - b_2$ are the maximum and the minimum of $a - b$, respectively, the degrees of preference can also be equivalently defined as

$$P(a > b) = \frac{\max(0, \max(a - b)) - \max(0, \min(a - b))}{\max(a - b) - \min(a - b)}, \tag{47}$$

$$P(b > a) = \frac{\max(0, \max(b - a)) - \max(0, \min(b - a))}{\max(a - b) - \min(a - b)}. \tag{48}$$

It is obvious that $P(a > b) + P(b > a) = 1$ and $P(a > b) = P(b > a) \equiv 0.5$ when $a = b$, i.e. $a_1 = b_1$ and $a_2 = b_2$. If $a = [a_1, a_2]$ and $b = [b_1, b_2]$ are two interval weights with certain constraint such as $a > b$ or $a < b$, then (47) and (48) should be used to calculate the degrees of preference. For example, $w_1 = [0.558, 0.750]$ and $w_2 = [0.391, 0.585]$ are two possible interval weights of alternatives E05 and S10 (see [30] for details). Although they have an intersection, the relationship between the alternatives considered in the problem does not allow for the preference S10 $\succ$ E05 to appear, which means that $w_1$ and $w_2$ have to satisfy the constraint of $w_1 \geqslant w_2$. In this situation, i.e. $w_1 \geqslant w_2$, we have $\min(w_1 - w_2) = 0$ and $\max(w_1 - w_2) = 0.750 - 0.391 = 0.359$. Accordingly, from (47) and (48) we get $P(w_1 > w_2) = 1$ and $P(w_2 > w_1) = 0$.

As can be seen from (45)–(48), the degrees of preference are defined to be directly proportional to the maximum nonnegative distance between two interval weights. This in fact involves an implied assumption that the interval weights are uniformly distributed within their intervals. Based on the above definition of degree of preference, we have the following definition and properties about interval weights.

**Definition 3.** If $P(a > b) > P(b > a)$, then $a$ is said to be superior to $b$ to the degree of $P(a > b)$, denoted by $a \overset{P(a>b)}{\succ} b$; if $P(a > b) = P(b > a) = 0.5$, then $a$ is said to be indifferent to $b$, denoted by $a \sim b$; If $P(b > a) > P(a > b)$, then $a$ is said to be inferior to $b$ to the degree of $P(b > a)$, denoted by $a \overset{P(b>a)}{\prec} b$.

**Property 1.** $P(a > b) = 1$ *if and only if* $a_1 \geqslant b_2$.

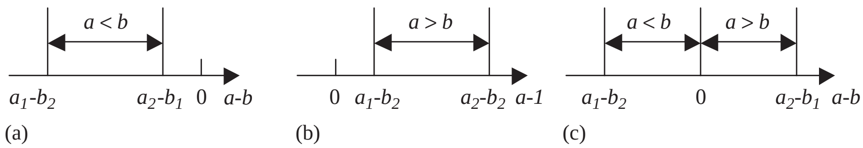


Fig. 1. Relationships between two interval weights $a$ and $b$.

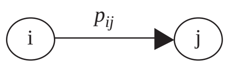


Fig. 2. Preference representation for interval weights $w_i$ and $w_j$.

**Property 2.** *If* $a_1 \geqslant b_1$ *and* $a_2 \geqslant b_2$, *then* $P(a > b) \geqslant 0.5$ *and* $P(b > a) \leqslant 0.5$.

**Property 3.** *If* $b$ *is nested in* $a$, *i.e.* $a_1 \leqslant b_1$ *and* $a_2 \geqslant b_2$, *then* $P(a > b) \geqslant 0.5$ *if and only if* $\frac{a_1+a_2}{2} \geqslant \frac{b_1+b_2}{2}$.

**Property 4.** *If* $P(a > b) \geqslant 0.5$ *and* $P(b > c) \geqslant 0.5$, *then* $P(a > c) \geqslant 0.5$.

Property 1 shows that if two interval weights do not overlap, then the one on the upper end will 100 percent dominate the other one on the lower end. Property 2 is similar to the comparison rule for interval numbers, but our ranking approach provides information on degrees of preference of one interval weight being preferred to another one whilst the latter does not. Property 3 shows how to compare two interval weights when one interval weight is included in the other. Property 4 shows that the preference relations are transitive. With the help of the above properties, a complete ranking order for interval weights could be achieved. The complete implement process is outlined below:

*Step* 1: Calculate the matrix of degrees of preference

$$P_D = \begin{array}{c} \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{array} \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & \cdots & w_n \end{array} \\ \begin{bmatrix} - & p_{12} & \cdots & p_{1n} \\ p_{21} & - & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & - \end{bmatrix} \end{array}, \tag{49}$$

where

$$p_{ij} = P(w_i > w_j) = \frac{\max(0, w_i^{\mathrm{U}} - w_j^{\mathrm{L}}) - \max(0, w_i^{\mathrm{L}} - w_j^{\mathrm{U}})}{(w_i^{\mathrm{U}} - w_i^{\mathrm{L}}) + (w_j^{\mathrm{U}} - w_j^{\mathrm{L}})}, \quad i, j = 1, \ldots n, \quad i \neq j. \tag{50}$$

*Step* 2: Draw a directed diagram

If $p_{ij} \geqslant 0.5$, then draw an arrow from node $i$ to $j$. Such an arrow means that interval weight $w_i$ is preferred to interval weight $w_j$ with a degree of preference of $p_{ij}$, as shown in Fig. 2 for an example.

*Step* 3: Find a complete preference ranking order for all interval weights from the directed diagram using the property of transitivity.

Alternatively, the following simple row–column elimination method can be used to generate a complete preference ranking order. In the matrix of degrees of preference given in (49), first find a row where all elements (except for the diagonal element) are larger than 0.5. If this row corresponds to $w_i$, then $w_i$ is the most preferred interval weight. Eliminate the $i$th row and $i$th column (thus $w_i$) in the matrix from further consideration. In the reduced matrix, if $w_j$ stands out as the most preferred interval weight among the remaining intervals, then $w_j$ should be ranked the second or $w_i \overset{p_{ij}}{\succ} w_j$ if $p_{ij} > 0.5$, and $w_i$ should be indifferent to $w_j$ or $w_i \sim w_j$ if $p_{ij} = p_{ji} = 0.5$. Eliminate the $j$th row and $j$th column and repeat the above process until all intervals are ranked.

## 5. Numerical examples

In this section, we offer five numerical examples that are examined using the proposed TLGP and ranking approaches and show their potential applications. Comparisons with other existing procedures will also be made whenever possible.

**Example 1.** Consider the following interval comparison matrix, which was examined by Arbel and Vargas [3,4] and Haines [18].

$$A = \begin{bmatrix} 1 & [2,5] & [2,4] & [1,3] \\ \left[\frac{1}{5}, \frac{1}{2}\right] & 1 & [1,3] & [1,2] \\ \left[\frac{1}{4}, \frac{1}{2}\right] & \left[\frac{1}{3}, 1\right] & 1 & \left[\frac{1}{2}, 1\right] \\ \left[\frac{1}{3}, 1\right] & \left[\frac{1}{2}, 1\right] & [1,2] & 1 \end{bmatrix}.$$

It has been known that $A$ is a consistent interval comparison matrix, which can be further confirmed using the GP model given in Eqs. (14)–(19) or (20)–(25). So, we can directly solve GP model (36)–(40). Table 2 shows the results and the corresponding weight intervals are recorded in Table 3, from which it is clear that criterion 1 is the most important because its minimum weight is greater than the maximum weights of all the other criteria. Table 4 gives the matrix of degrees of preference for the interval weights $w_1$, $w_2$, $w_3$ and $w_4$. It can be seen from Table 4 that $w_1$ is preferred over $w_2, w_3$ and $w_4$ to a degree of 100%, $w_2$ over $w_3$ and $w_4$ to a degree of 88.21% and 63.62%, respectively, and $w_4$ over $w_3$ to a degree of 76.42%. To provide a complete ranking order for the four interval weights, a directed diagram is depicted in Fig. 3, from which it is quite clear that the ranking order is $w_1 \overset{100\%}{\succ} w_2 \overset{63.62\%}{\succ} w_4 \overset{76.42\%}{\succ} w_3$, which is the same as the rankings given by Arbel and Vargas [3,4] using the average weights of all the vertices and by Haines [18] using the expected weights, but our ranking order provides the information about the degrees of preference, which reflects the uncertain nature of the ranking.

Table 2
The logarithmic weight intervals generated from Example 1

| $\ln W$ | $\ln w_1$ | $\ln w_2$ | $\ln w_3$ | $\ln w_4$ |
|---|---|---|---|---|
| $\ln w_i^{\mathrm{L}} = \min \ln w_i$ | 0.5198604 | −0.2746531 | −0.6931472 | −0.3760194 |
| $\ln w_i^{\mathrm{U}} = \max \ln w_i$ | 0.8958797 | 0.1013663 | −0.1732868 | 0 |

Table 3
The weight intervals in Example 1

| $W$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|
| $w_i^{\mathrm{L}} = \min w_i$ | 1.6818 | 0.7598 | 0.5000 | 0.6866 |
| $w_i^{\mathrm{U}} = \max w_i$ | 2.4495 | 1.1067 | 0.8409 | 1.0000 |

Table 4
The matrix of degrees of preference in Example 1

| $p_{ij}$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|
| $w_1$ | — | 1.0000 | 1.0000 | 1.0000 |
| $w_2$ | 0 | — | 0.8821 | 0.6362 |
| $w_3$ | 0 | 0.1179 | — | 0.2358 |
| $w_4$ | 0 | 0.3638 | 0.7642 | — |

**Example 2.** Consider the following interval comparison matrix, which was investigated by Kress [22] and Islam et al. [20].

$$A = \begin{bmatrix} 1 & [1,2] & [1,2] & [2,3] \\ \left[\frac{1}{2},1\right] & 1 & [3,5] & [4,5] \\ \left[\frac{1}{2},1\right] & \left[\frac{1}{5},\frac{1}{3}\right] & 1 & [6,8] \\ \left[\frac{1}{3},\frac{1}{2}\right] & \left[\frac{1}{5},\frac{1}{4}\right] & \left[\frac{1}{8},\frac{1}{6}\right] & 1 \end{bmatrix}.$$

Kress [22] showed that this interval comparison matrix is inconsistent and hence cannot be solved using Arbel's preference programming method and its variants. This can be confirmed by solving GP model (14)–(19) or (20)–(25), which leads to $J^* = 1.791759$. Theorem 1 ensures that $A$ is an inconsistent interval comparison matrix.

Islam et al. [20] used lexicographic goal programming (LGP) to get a point estimate for the priority vector from the upper triangular judgments of $A$, i.e. $W = (0.3030, 0.4545, 0.1515, 0.0910)$, which shows that $w_2 \succ w_1 \succ w_3 \succ w_4$. However, if the lower triangular judgments of $A$ are used, which provide completely the same information as the upper triangular part, then a different point estimate would be obtained. That is $W = (0.3636, 0.3636, 0.1818, 0.0909)^{\mathrm{T}}$, which implies that $w_2 = w_1 \succ w_3 \succ w_4$. These two different ranking orders show that LGP method is defective in theory.

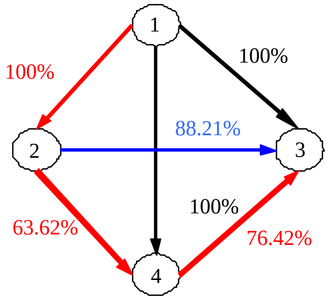


Fig. 3. Preference relations in Example 1.

Table 5
The logarithmic weight intervals generated from Example 2

| $\ln W$ | $\ln w_1$ | $\ln w_2$ | $\ln w_3$ | $\ln w_4$ |
|---|---|---|---|---|
| $\ln w_i^{\mathrm{L}} = \min \ln w_i$ | 0.1469467 | 0.3465736 | −0.3465736 | −1.207078 |
| $\ln w_i^{\mathrm{U}} = \max \ln w_i$ | 0.575646 | 0.8047189 | 0.4023595 | −0.7225929 |

Table 6
The weight intervals in Example 2

| $W$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|
| $w_i^{\mathrm{L}} = \min w_i$ | 1.1583 | 1.4142 | 0.7071 | 0.2991 |
| $w_i^{\mathrm{U}} = \max w_i$ | 1.7783 | 2.2361 | 1.4953 | 0.4855 |

Table 7
The matrix of degrees of preference in Example 2

| $p_{ij}$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|
| $w_1$ | — | 0.2525 | 0.7607 | 1.0000 |
| $w_2$ | 0.7475 | — | 0.9496 | 1.0000 |
| $w_3$ | 0.2393 | 0.0504 | — | 1.0000 |
| $w_4$ | 0 | 0 | 0 | — |

The proposed two-stage logarithmic goal programming method can overcome the above-mentioned drawbacks and estimate a valid set of weights. Table 5 shows the algorithmic weights generated from the inconsistent interval comparison matrix $A$, and the corresponding interval weights are shown in Table 6, from which it can be seen that $w_1$, $w_2$ and $w_3$ all absolutely dominate $w_4$, so $w_4$ should be ranked the last. In order to yield a complete ranking order, the matrix of degrees of preference for the interval weights is generated as shown in Table 7. The corresponding directed diagram is depicted in Fig. 4. It is clear in Table 7 that $w_1$ is preferred over $w_3$ and $w_4$ to a degree of 76.07% and 100%, respectively, $w_2$ over $w_1$, $w_3$ and $w_4$ to a degree of 74.75%, 94.96% and 100%, respectively, and $w_3$ is absolutely

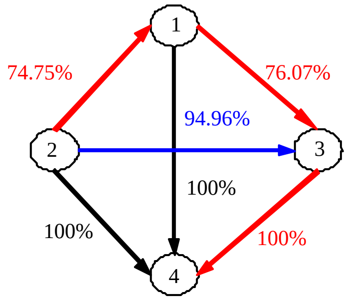


Fig. 4. Preference relations in Example 2.

Table 8
The logarithmic weight intervals generated from Example 3

| $\ln W$ | $\ln w_1$ | $\ln w_2$ | $\ln w_3$ | $\ln w_4$ |
|---|---|---|---|---|
| $\ln w_i^{\mathrm{L}} = \min \ln w_i$ | 0.7225929 | −0.3465736 | −0.5756463 | −0.7033527 |
| $\ln w_i^{\mathrm{U}} = \max \ln w_i$ | 1.151293 | 0.1732868 | −0.1469467 | −0.2746531 |

preferred to $w_4$. The final ranking is therefore $w_2 \overset{74.75\%}{\succ} w_1 \overset{76.07\%}{\succ} w_3 \overset{100\%}{\succ} w_4$, which is the same as the ranking given by Islam et al. [20]. But our ranking provides richer information about the uncertain nature of the priorities.

**Example 3.** Consider the following interval comparison matrix, which was investigated by Saaty and Vargas [32].

$$A = \begin{bmatrix} 1 & [2,4] & [3,5] & [3,5] \\ \left[\frac{1}{4},\frac{1}{2}\right] & 1 & \left[\frac{1}{2},1\right] & [2,5] \\ \left[\frac{1}{5},\frac{1}{3}\right] & [1,2] & 1 & \left[\frac{1}{3},1\right] \\ \left[\frac{1}{5},\frac{1}{3}\right] & \left[\frac{1}{5},\frac{1}{2}\right] & [1,3] & 1 \end{bmatrix}.$$

It can be confirmed using Theorem 1 that $A$ is an inconsistent interval comparison matrix because $J^* = 0.6931472$. Tables 8 and 9 show the logarithmic weight intervals and the corresponding interval weights generated from the above inconsistent interval comparison matrix using the two-stage logarithmic goal programming method. It is clear in Table 8 that interval weight $w_1$ absolutely dominates all the other weights, so criterion 1 is the most important and should be ranked the first. To produce a complete ranking order for all the four interval weights, Table 10 records the matrix of their degrees of preference. The final ranking is thus generated to be $w_1 \overset{100\%}{\succ} w_2 \overset{80.05\%}{\succ} w_3 \overset{65.1\%}{\succ} w_4$.

Saaty and Vargas [32] used the Monte Carlo simulation approach to extract the interval weights from the above inconsistent interval comparison matrix. Their findings are listed in Table 11 and lead to the same ranking: $w_1 \overset{100\%}{\succ} w_2 \overset{83.44\%}{\succ} w_3 \overset{64.78\%}{\succ} w_4$, but with slightly different degrees of preference.

Table 9
The weight intervals in Example 3

| $W$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|
| $w_i^{\mathrm{L}} = \min w_i$ | 2.0598 | 0.7071 | 0.5623 | 0.4949 |
| $w_i^{\mathrm{U}} = \max w_i$ | 3.1623 | 1.1892 | 0.8633 | 0.7598 |

Table 10
The matrix of degrees of preference in Example 3

| $p_{ij}$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|
| $w_1$ | — | 1.0000 | 1.0000 | 1.0000 |
| $w_2$ | 0 | — | 0.8005 | 0.9295 |
| $w_3$ | 0 | 0.1995 | — | 0.6510 |
| $w_4$ | 0 | 0.0705 | 0.3490 | — |

Table 11
The intervals of priorities obtained by Saaty and Vargas [32] in Example 3

| $W$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|
| $w_i^{\min}$ | 0.4374 | 0.1654 | 0.1111 | 0.1011 |
| $w_i^{\max}$ | 0.5696 | 0.2708 | 0.1971 | 0.1633 |
| Average ($\bar{w}_i$) | 0.5093 | 0.2131 | 0.1496 | 0.1280 |

**Example 4.** Consider a group decision analysis situation that was taken from Wang and Xu [40]. Suppose six decision makers give the following comparison matrices with regard to the same decision problem, respectively.

$$A_1 = \begin{bmatrix} 1 & 3 & 5 & 4 & 7 \\ \frac{1}{3} & 1 & 3 & 2 & 5 \\ \frac{1}{5} & \frac{1}{3} & 1 & \frac{1}{2} & 3 \\ \frac{1}{4} & \frac{1}{2} & 2 & 1 & 3 \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 4 & 3 & 5 & 8 \\ \frac{1}{4} & 1 & 4 & 3 & 6 \\ \frac{1}{3} & \frac{1}{4} & 1 & 1 & 5 \\ \frac{1}{5} & \frac{1}{3} & 1 & 1 & 7 \\ \frac{1}{8} & \frac{1}{6} & \frac{1}{5} & \frac{1}{7} & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & \frac{1}{2} & 3 & 2 & 5 \\ 2 & 1 & 5 & 1 & 2 \\ \frac{1}{3} & \frac{1}{5} & 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 1 & 5 \\ \frac{1}{5} & \frac{1}{2} & 2 & \frac{1}{5} & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 3 & 5 & 2 & 6 \\ \frac{1}{3} & 1 & 1 & 3 & 2 \\ \frac{1}{5} & 1 & 1 & 4 & 5 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 1 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{5} & 2 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 2 & 6 & 3 & 3 \\ \frac{1}{2} & 1 & 2 & 5 & 4 \\ \frac{1}{6} & \frac{1}{2} & 1 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{5} & 2 & 1 & 5 \\ \frac{1}{3} & \frac{1}{4} & 1 & \frac{1}{5} & 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 2 & 5 & 4 & 9 \\ \frac{1}{2} & 1 & 3 & 2 & 6 \\ \frac{1}{5} & \frac{1}{3} & 1 & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & 1 & 1 & 3 \\ \frac{1}{9} & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix}.$$

Table 12
The logarithmic weight intervals generated from Example 4

| $\ln W$ | $\ln w_1$ | $\ln w_2$ | $\ln w_3$ | $\ln w_4$ | $\ln w_5$ |
|---|---|---|---|---|---|
| $\ln w_i^{\mathrm{L}} = \min \ln w_i$ | 0.4759092 | −0.1386294 | −0.9574984 | −0.8481054 | −1.318335 |
| $\ln w_i^{\mathrm{U}} = \max \ln w_i$ | 1.396943 | 1.104292 | 0.1021651 | 0.3008155 | −0.1175573 |

Table 13
The weight intervals in Example 4

| $W$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ |
|---|---|---|---|---|---|
| $w_i^{\mathrm{L}} = \min w_i$ | 1.6095 | 0.8706 | 0.3839 | 0.4282 | 0.2676 |
| $w_i^{\mathrm{U}} = \max w_i$ | 4.0428 | 3.0171 | 1.1076 | 1.3510 | 0.8891 |

Using the minimum and maximum values for the same element in the above six comparison matrices, an interval comparison matrix can be constructed as follows:

$$A = \begin{bmatrix} 1 & \left[\frac{1}{2}, 4\right] & [3, 6] & [2, 5] & [3, 9] \\ \left[\frac{1}{4}, 2\right] & 1 & [1, 5] & [1, 5] & [2, 6] \\ \left[\frac{1}{6}, \frac{1}{3}\right] & \left[\frac{1}{5}, 1\right] & 1 & \left[\frac{1}{2}, 4\right] & \left[\frac{1}{2}, 5\right] \\ \left[\frac{1}{5}, \frac{1}{2}\right] & \left[\frac{1}{5}, 1\right] & \left[\frac{1}{4}, 2\right] & 1 & \left[\frac{1}{2}, 7\right] \\ \left[\frac{1}{9}, \frac{1}{3}\right] & \left[\frac{1}{6}, \frac{1}{2}\right] & \left[\frac{1}{5}, 2\right] & \left[\frac{1}{7}, 2\right] & 1 \end{bmatrix}.$$

Based on this interval comparison matrix, the interval weights of the group of decision makers can be generated. It turns out that $A$ is a consistent interval comparison matrix, which can be confirmed by solving GP model (14)–(19) or (20)–(25) in the first stage, leading to $J^* = 0$. The logarithmic weight intervals were derived by solving GP model (36)–(40). The results are presented in Table 12. The corresponding interval weights are recorded in Table 13. Table 14 shows the matrix of degrees of preference for the five interval weights $w_1$ to $w_5$. The final ranking is thus given by $w_1 \overset{69.27\%}{\succ} w_2 \overset{84.35\%}{\succ} w_4 \overset{58.74\%}{\succ} w_3 \overset{62.44\%}{\succ} w_5$.

**Example 5.** Consider a hierarchy of criteria, which is taken from Islam et al. [20] and shown in Fig. 5. A person is interested in investing his money to any one of the four portfolios: bank deposit (BD), debentures (DB), government bonds (GB), and shares (SH). Out of these portfolios he has to choose only one based upon four criteria: return (Re), risk (Ri), tax benefits (Tb), and liquidity (Li).

The interval comparison matrices for the four criteria as well and for the four portfolios are summarized in Tables 15–19. The five interval comparison matrices all turn out to be inconsistent. The proposed two-stage logarithmic goal programming method was used to generate both local and global priorities. The results are reported in Table 20, where the global interval weights are obtained by solving NLP model (41)–(44). To yield a complete ranking to help the decision maker (DM) invest his money in the best

Table 14
The matrix of degrees of preference in Example 4

| $p_{ij}$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ |
|---|---|---|---|---|---|
| $w_1$ | — | 0.6927 | 1.0000 | 1.0000 | 1.0000 |
| $w_2$ | 0.3073 | — | 0.9174 | 0.8435 | 0.9933 |
| $w_3$ | 0 | 0.0826 | — | 0.4126 | 0.6244 |
| $w_4$ | 0 | 0.1565 | 0.5874 | — | 0.7015 |
| $w_5$ | 0 | 0.0067 | 0.3756 | 0.2985 | — |

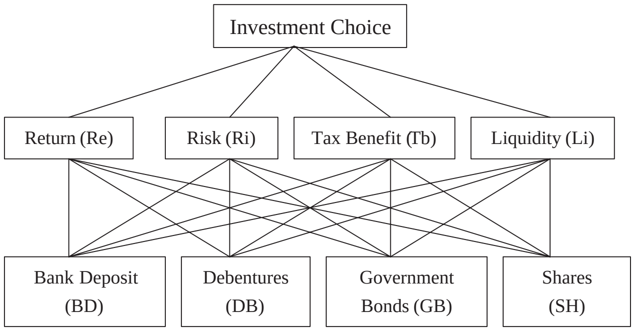


Fig. 5. Hierarchy structure.

Table 15
Interval comparison matrix for the four criteria with respect to “investment choice (Ic)’’

| Ic | Re | Ri | Tb | Li |
|---|---|---|---|---|
| Re | 1 | [3,4] | [5,6] | [6,7] |
| Ri | | 1 | [4,5] | [5,6] |
| Tb | | | 1 | [3,4] |
| Li | | | | 1 |

Table 16
Interval comparison matrix for all the alternatives with respect to “return (Re)”

| Re | BD | DB | GB | SH |
|---|---|---|---|---|
| BD | 1 | $\left[\frac{1}{4}, \frac{1}{3}\right]$ | [3,4] | $\left[\frac{1}{6}, \frac{1}{5}\right]$ |
| DB | | 1 | [6,7] | $\left[\frac{1}{5}, \frac{1}{4}\right]$ |
| GB | | | 1 | $\left[\frac{1}{7}, \frac{1}{6}\right]$ |
| SH | | | | 1 |

Table 17
Interval comparison matrix for the four portfolios with respect to "risk (Ri)"

| Ri | BD | DB | GB | SH |
|---|---|---|---|---|
| BD | 1 | [3,4] | [4,5] | [6,7] |
| DB | | 1 | [3,4] | [5,6] |
| GB | | | 1 | [4,5] |
| SH | | | | 1 |

Table 18
Interval comparison matrix for the four portfolios with respect to "tax benefits (Tb)"

| Tb | BD | DB | GB | SH |
|---|---|---|---|---|
| BD | 1 | 1 | $\left[\frac{1}{6}, \frac{1}{5}\right]$ | $\left[\frac{1}{4}, \frac{1}{3}\right]$ |
| DB | | 1 | $\left[\frac{1}{6}, \frac{1}{5}\right]$ | $\left[\frac{1}{4}, \frac{1}{3}\right]$ |
| GB | | | 1 | [4,5] |
| SH | | | | 1 |

Table 19
Interval comparison matrix for the four portfolios with respect to "liquidity (Li)"

| Li | BD | DB | GB | SH |
|---|---|---|---|---|
| BD | 1 | [3,4] | 6 | [6,7] |
| DB | | 1 | [3,4] | [3,4] |
| GB | | | 1 | [3,4] |
| SH | | | | 1 |

Table 20
Local and global priority intervals in Example 5

| Portfolio | Re | Ri | Tb | Li | Global priority |
|---|---|---|---|---|---|
| | [2.5718, 4.2426] | [1.2910, 2.0933] | [0.4777, 0.7746] | [0.2357, 0.3888] | |
| BD | [0.5774, 0.7401] | [2.4719, 4.0536] | [0.4855, 0.4855] | [3.0274, 3.8337] | [0.2321, 10.3091] |
| DB | [1.4142, 1.8841] | [1.2209, 1.9168] | [0.4855, 0.4855] | [1.1892, 1.5651] | [1.9283, 45.1463] |
| GB | [0.2268, 0.3021] | [0.5774, 0.9306] | [2.9130, 2.9130] | [0.5046, 0.6389] | [0.0008, 0.0817] |
| SH | [3.0214, 4.2426] | [0.2193, 0.3618] | [1.4565, 1.4565] | [0.3195, 0.4495] | [0.5508, 137.2216] |

portfolio, Table 21 records the matrix of degrees of preference for the four portfolios, from which it is clear that investment in shares (SH) is the best choice for the decision maker because the four portfolios are ranked as SH $\overset{75.21\%}{\succ}$ DB $\overset{84.27\%}{\succ}$ BD $\overset{100\%}{\succ}$ GB.

Table 21
The matrix of degrees of preference in Example 5

| $p_{ij}$ | BD | DB | GB | SH |
|---|---|---|---|---|
| BD | — | 0.1573 | 1.0000 | 0.0665 |
| DB | 0.8427 | — | 1.0000 | 0.2479 |
| GB | 0 | 0 | — | 0 |
| SH | 0.9335 | 0.7521 | 1.0000 | — |

## 6. Extension of the TLGP method to fuzzy comparison matrices

Since uncertainties can also be well modeled by using fuzzy numbers, we deal with in this section fuzzy comparison matrices using the proposed TLGP method. The ranking approach for intervals discussed in Section 4 will also be extended to handle fuzzy rankings.

Let $\tilde{A}$ be an upper triangular fuzzy comparison matrix expressed by

$$\tilde{A} = (\tilde{a}_{ij})_{n\times n} = \begin{bmatrix} 1 & (l_{12}, m_{12}, u_{12}) & \cdots & (l_{1n}, m_{1n}, u_{1n}) \\ & 1 & \cdots & (l_{2n}, m_{2n}, u_{2n}) \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}, \tag{51}$$

where $\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})$ $(i = 1, \ldots, n-1;\ j = i+1, \ldots, n)$ are triangular fuzzy numbers with their membership functions defined by

$$\mu_{\tilde{a}_{ij}}(x) = \begin{cases} \dfrac{x - l_{ij}}{m_{ij} - l_{ij}}, & x \in [l_{ij}, m_{ij}], \\ \dfrac{u_{ij} - x}{u_{ij} - m_{ij}}, & x \in [m_{ij}, u_{ij}], \\ 0 & \text{otherwise}. \end{cases} \tag{52}$$

According to Zadeh's extension principle [45], the above fuzzy comparison elements $\tilde{a}_{ij}$ can be represented by using $\alpha$-*level* sets as

$$\tilde{a}_{ij} = \bigcup_{\alpha} \alpha \cdot (a_{ij})_\alpha, \quad 0 < \alpha \leqslant 1, \tag{53}$$

where $\alpha$ is a real number and $(a_{ij})_\alpha$ is an interval defined by

$$(a_{ij})_\alpha = \{x | \mu_{\tilde{a}_{ij}}(x) \geqslant \alpha\} = [l_{ij} + \alpha(m_{ij} - l_{ij}), u_{ij} - \alpha(u_{ij} - m_{ij})]. \tag{54}$$

Since inverse fuzzy numbers $\tilde{a}_{ji} = 1/\tilde{a}_{ij}$ $(i = 2, \ldots, n;\ j = 1, \ldots, i-1)$ are usually no longer precise triangular fuzzy numbers, their $\alpha$-*level* sets cannot be easily determined. So, we use $(a_{ij})_\alpha = [1/(a_{ji})_\alpha^{\mathrm{U}}, 1/(a_{ji})_\alpha^{\mathrm{L}}]$ $(i = 2, \ldots, n;\ j = 1, \ldots, i-1)$ instead of them. Thus, $A_\alpha = ((a_{ij})_\alpha)_{n\times n}$ constitutes an interval comparison matrix. By setting different levels of confidence, namely $1-\alpha$, fuzzy comparison matrix, $\tilde{A}$ is accordingly transformed into interval comparison matrix $A_\alpha = ((a_{ij})_\alpha)_{n\times n}$ with different $\alpha$-*level* set comparison elements $\{(a_{ij})_\alpha \,|\, 0 < \alpha \leqslant 1\}$. The proposed TLGP method can, therefore, be used to derive the priorities from the interval comparison matrix, $A_\alpha$ for different $\alpha$-*levels*.

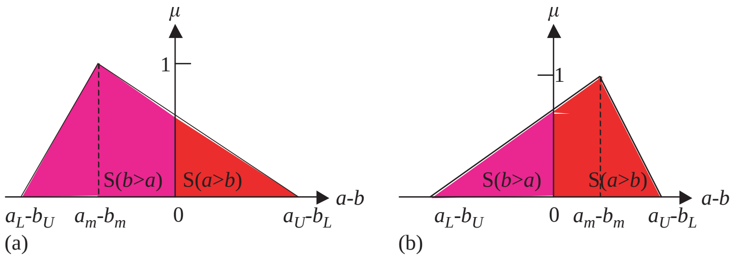


Fig. 6. The typical relationships between two triangular fuzzy numbers: (a) $a_U > b_L$ and $a_m \leqslant b_m$ and (b) $a_m > b_m$ and $a_L < b_U$.

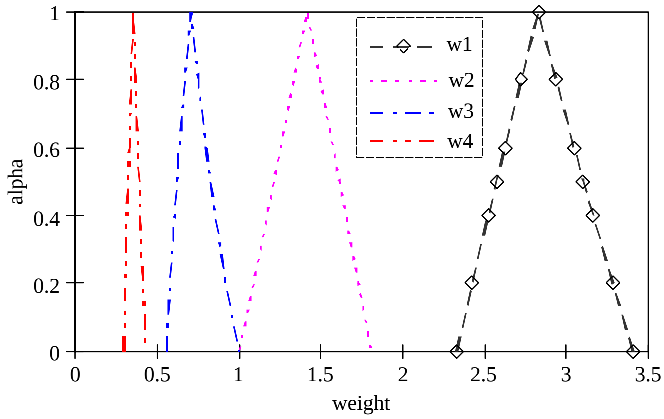


Fig. 7. Weights generated from the fuzzy comparison matrix.

Another possible way of transforming fuzzy comparison matrices into interval comparison matrices is to use interval approximations of fuzzy numbers. The interested reader may refer to [12,17,21] for details.

For different $\alpha$-*level* sets, different interval weights can be generated, which constitute a fuzzy weight by the extension principle (see Fig. 7 for an example). Therefore, a fuzzy ranking approach is required to compare or rank fuzzy weights. A variety of approaches for comparing or ranking fuzzy numbers have been proposed. Useful surveys of fuzzy rankings can be found in [7,15,25,38,39], and so on. However, there has been no approach that can be universally accepted so far. Some approaches even contradict one another. In what follows, the ranking approach for interval numbers will be further extended to deal with fuzzy numbers.

Let $a = (a_L, a_m, a_U)$ and $b = (b_L, b_m, b_U)$ be two triangular fuzzy numbers, whose possible relationships include $a_L \geqslant b_U$, $a_U \leqslant b_L$, $(a_U > b_L) \cap (a_m \leqslant b_m)$ and $(a_m > b_m) \cap (a_L < b_U)$. The latter two relationships are shown in Fig. 6.

Accordingly, the degrees of preference can be defined in the form of area rather than distance as

$$P(a > b) = \frac{S(a > b)}{S(a > b) + S(b > a)}, \tag{55}$$

$$P(b>a)=\frac{S(b>a)}{S(a>b)+S(b>a)}, \tag{56}$$

where $S(a>b)$ and $S(b>a)$ are areas of $a>b$ and $b>a$, respectively (see Fig. 6). Both (55) and (56) can be precisely expressed as follows:

$$P(a>b) = \begin{cases} 1 & \text{if } a_L \geqslant b_U, \\ 0 & \text{if } a_U \leqslant b_L, \\ \dfrac{(a_U-b_L)^2}{(a_U-b_L+b_m-a_m)(a_U-a_L+b_U-b_L)} & \text{if } (a_U>b_L)\cap(a_m\leqslant b_m), \\ 1-\dfrac{(b_U-a_L)^2}{(b_U-a_L+a_m-b_m)(a_U-a_L+b_U-b_L)} & \text{if } (a_m>b_m)\cap(a_L<b_U), \end{cases} \tag{57}$$

$$P(b>a) = \begin{cases} 0 & \text{if } a_L \geqslant b_U, \\ 1 & \text{if } a_U \leqslant b_L, \\ 1-\dfrac{(a_U-b_L)^2}{(a_U-b_L+b_m-a_m)(a_U-a_L+b_U-b_L)} & \text{if } (a_U>b_L)\cap(a_m\leqslant b_m), \\ \dfrac{(b_U-a_L)^2}{(b_U-a_L+a_m-b_m)(a_U-a_L+b_U-b_L)} & \text{if } (a_m>b_m)\cap(a_L<b_U), \end{cases} \tag{58}$$

which are both extremely useful in comparing and ranking fuzzy numbers. The formulas for trapezoidal fuzzy numbers can be derived in a very similar way.

Note that the use of areas for comparing fuzzy numbers is not new and has been investigated by some researchers, the interested reader may refer to [16,23,31,36,44] for details. But it has not been reported yet to define and calculate areas from the angle of fuzzy arithmetic. This offers a more intuitive way of computing areas. Although our approach appears to be the same as the one proposed by Tseng and Klein [36], our approach provides the analytical formulas for the computation of degrees of preference, which makes it particularly useful when a large number of fuzzy numbers needs to be compared or ranked. However, the derived analytical formulas (57) and (58) are not suitable for the comparisons of nonnormal or irregular fuzzy numbers. This is the disadvantage of our ranking approach.

To illustrate the application of the proposed TLGP method to fuzzy comparison matrices, we examine the following illustrative example:

**Example 6.** Consider the following fuzzy comparison matrix

$$\tilde{A} = \begin{bmatrix} 1 & (1,2,3) & (3,4,5) & (7,8,9) \\ & 1 & (1,2,3) & (3,4,5) \\ & & 1 & (1,2,3) \\ & & & 1 \end{bmatrix}.$$

Table 22
Interval weights generated by the TLGP method under different levels of alpha

| $\alpha$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|---|
| 1.0 | 2.8284 | 1.4142 | 0.7071 | 0.3536 |
| 0.8 | [2.7238, 2.9361] | [1.3343, 1.4935] | [0.6696, 0.7495] | [0.3406, 0.3671] |
| 0.6 | [2.6219, 3.0474] | [1.2535, 1.5724] | [0.6360, 0.7978] | [0.3282, 0.3814] |
| 0.5 | [2.5718, 3.1045] | [1.2126, 1.6119] | [0.6204, 0.8247] | [0.3221, 0.3888] |
| 0.4 | [2.5223, 3.1628] | [1.1712, 1.6513] | [0.6056, 0.8538] | [0.3162, 0.3965] |
| 0.2 | [2.4246, 3.2829] | [1.0871, 1.7301] | [0.5780, 0.9199] | [0.3046, 0.4124] |
| 0 | [2.3286, 3.4087] | [1.0000, 1.8092] | [0.5537, 1.0000] | [0.2934, 0.4295] |

Using $\alpha$-*level* sets and the extension principle, the above fuzzy comparison matrix can be transformed into the following interval comparison matrix:

$$A_\alpha = \begin{bmatrix} 1 & [1+\alpha, 3-\alpha] & [3+\alpha, 5-\alpha] & [7+\alpha, 9-\alpha] \\ & 1 & [1+\alpha, 3-\alpha] & [3+\alpha, 5-\alpha] \\ & & 1 & [1+\alpha, 3-\alpha] \\ & & & 1 \end{bmatrix},$$

where $0 < \alpha \leqslant 1$. For different $\alpha$-*levels*, the interval weights generated from the above interval comparison matrices are shown in Table 22 and plotted in Fig. 7, where the seven sets of (interval) weights all produce exactly the same ranking $w_1 \overset{100\%}{\succ} w_2 \overset{100\%}{\succ} w_3 \overset{100\%}{\succ} w_4$, which is also the final ranking. For other fuzzy examples, they can be dealt with in the same way.

## 7. Concluding remarks

The use of pairwise comparisons to generate relative weights of criteria in multiple criteria decision analysis requires human judgments. Because of the complexity of real-world decision problems and the subjective nature of human judgments, interval comparison matrices can provide a more realistic framework to account for such uncertainty. This is especially the case in a group decision situation. However, how to derive weights from interval comparison matrices, especially from inconsistency interval comparison matrices, is still subject to further investigation.

In this paper, a pragmatic two-stage logarithmic goal programming method was proposed to generate weights from both consistent and inconsistent interval comparison matrices. The first stage was devoted to minimizing the inconsistency of an interval comparison matrix and the second stage was developed to generate priority intervals with the inconsistency being kept to be minimal. The proposed approach provides a rational procedure for interval weight generation. In the case of hierarchical structures, a nonlinear programming method was suggested to aggregate local interval weights into global interval weights. A simple and practical preference ranking method was extended to compare the interval weights of criteria or rank alternatives in a multiplicative aggregation process. Fuzzy comparison matrices were transformed into interval comparison matrices so that the proposed methods are also applicable to them. Six numerical examples including a group decision analysis problem with a group of comparison matrices, a hierarchical (AHP) decision problem and a fuzzy decision problem using fuzzy comparison matrix were examined and showed the applications of the proposed methods.

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