Boolean Completeness in Two-valued Set Logic

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Dedicated to Professor Ivo G. Rosenberg on the occasion of his 65th birthday, and based on his remarkable Completeness Theorem

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This paper discusses the functional completeness problems in r-valued set logic, which is the logic of functions mapping r-tuples of subsets into subsets over r values. It is shown that r-valued set logic is isomorphic to 2r-valued logic, meaning that the well known completeness criteria (described by Ivo G. Rosenberg) in multiple valued Post algebras apply to set valued logic. Since Boolean functions are convenient choice as building blocks in the design of set logic functions, we introduce the notion of Boolean completeness of a set. A set is Boolean complete if it becomes complete ones all Boolean functions are added to the set. Finally, this paper gives a full description of Boolean complete sets, Boolean maximal sets, Boolean bases and Boolean Sheffer functions for the case of two-valued set logic.

Keywords: Bio-computing; set logic; functional completeness; Boolean functions; bases

1. INTRODUCTION

The works of Higuchi, Kameyama and Aoki on biological computing, that is on computing based on the interaction between enzymes and substrata, suggest the interest of studying set-valued functions and switching devices. Bio-switching devices introduced and studied in

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[6, 1, 2] use the specificity of the reaction between enzymes and substrata in order to compute multi-valued switching functions. This kind of circuitry allows ultra-high-valued data processing and a high degree of computing parallelism.

We consider the set $r = \{e_0, e_1, \ldots, e_{r-1}\}$ as the set of fundamental values of an $r$-valued set logic. The bio-circuits mentioned above operate on the set of subsets of $r$, denoted as usual by $P(r)$ (thus $X \in P(r) \iff X \subseteq r$), and, therefore, can be described as set valued functions of the form $f : f : P(r)^n \to P(r)$ that map $n$-tuples of subsets of $r$ into a subset of $r$. The set of all such function $f$s is referred to as $r$-valued set logic.

The number of $n$-variable functions $f : P(r)^n \to P(r)$ is quite considerable: there are $2^{2^r}$ such functions; for $n = 1$ and $r = 2$ there are 256 functions while for $n = 1$ and $r = 3$ we find $2^{2^4} = 16,780,032$ such one-variable functions.

A small fraction of these functions are Boolean functions that is, functions that can be constructed from constants and variables, using union, intersection and complementation. The set $P(r)$ is a Boolean algebra ($P(r), \emptyset, r, \cup, \cap, \overline{\cdot}$) when equipped with set-theoretical operations $X \cup Y$, $X \cap Y$ and $\overline{X}$, which denote the union, intersection and complementation, respectively. The first two are binary operations while the complement is a unary operation. The number of $n$-ary $r$-valued Boolean functions of set logic is $2^{2^r}$ (cf. [15]). For $n = 1$ we find 16 Boolean functions for $r = 2$ and 64 such functions for $r = 3$.

Let $k$ be a fixed positive integer and let $E_k = \{0, 1, \ldots, k-1\}$. The set of $n$-ary $k$-valued logical functions (i.e., maps $f : E_k^n \to E_k$) is denoted by $P_k(n)$. The union of $P_k(n)$ for $n = 0, 1, 2, \ldots$ is denoted $P_k$. The number of $n$-ary $k$-valued logical functions is $k^e$.

Every $r$-valued set logic function can be regarded as a $k$-valued logic function for $k = 2^r$, as follows. Without loss of generality we may use characteristic binary vectors to represent the elements of $P(r)$ as binary numbers. A subset $X \in P(r)$ is represented as binary number $x_0 \ x_1 \cdots x_{r-1}$ determined by $x_i = 1$ if and only if $e_i \in X$, for $i = 0, 1, \ldots, r - 1$. Next, $X \in P(r)$ is mapped into the decimal number $x$ which has binary representation $x_{r-1}x_{r-2}\cdots x_1x_0$, $x = 2^{r-1}x_{r-1} + 2^{r-2}x_{r-2} + \cdots + 2x_1 + x_0$. 

Example 1 For \( r = 2 \) the elements of \( \mathcal{P}\{e_0, e_1\} \) are represented in the following way:

<table>
<thead>
<tr>
<th>Set</th>
<th>binary</th>
<th>decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>00</td>
<td>0</td>
</tr>
<tr>
<td>( {e_0} )</td>
<td>01</td>
<td>1</td>
</tr>
<tr>
<td>( {e_1} )</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>( {e_0, e_1} )</td>
<td>11</td>
<td>3</td>
</tr>
</tbody>
</table>

Example 2 For \( r = 2 \), \( k = 2^r = 4 \), the operations \( \cup, \cap, \text{ and } - \) are represented by the following tables.

| \( \cup \) | 0 1 2 3 | \( \cap \) | 0 1 2 3 | \(-\) | 3 |
|-----------|---------|-----------|---------|-----|
| 0         | 0 1 2 3 | 0 0 0 0   | 0       | 3   |
| 1         | 1 1 3 3 | 1 0 1 1   | 1       | 2   |
| 2         | 2 3 2 3 | 2 0 2 2   | 2       | 1   |
| 3         | 3 3 3 3 | 3 1 2 3   | 3       | 0   |

In general, \( \bar{x} = r - 1 - x \), while \( x \cup y = u \) and \( x \cap y = v \) are determined by \( u_i = \max(x_i, y_i) \), \( v_i = \min(x_i, y_i) \) for \( i = 0, 1, \ldots, r - 1 \). We refer to these functions as union, intersection and complement functions in \( P_k \), \( k = 2^r \). For example, \( 1 \cup 2 = 01 \cup 10 = 11 = 3 \), \( 1 \cap 3 = 01 \cap 11 = 01 = 1 \).

The closure of the set of these three functions and constants is denoted by \( BF \) and contains all functions that correspond to Boolean functions in \( r \)-valued set logic. For simplicity, we refer to \( BF \) as the set of Boolean functions in \( P_k \). Also, let \( BF_k(n) \) be the set of all Boolean \( n \)-ary functions in \( P_k \). It is well known that \( |BF_k(n)| = 2^{2^n} \). For example, \( |BF_4(1)| = 16, |BF_4(2)| = 256, |BF_4(3)| = 65536 \).

In the sequel, let \( X(n) \) denotes the set of \( n \)-ary functions of given set \( X \).

### 2. Functional Completeness in Set Logic

A subset \( F \) of \( P_k \) is said to be closed if it contains all compositions (or superpositions) of its members. The composition includes permuting
variables in a function, identifying two variables, and replacing variables by functions from $F$. A more formal definition of superposition is given in [7]. If the functions from $F$ are treated as circuits then the composition is creation of new functions by using the output of some functions as input to other ones, where it is allowed to use multiple copies of same output or to permute input "wires".

A closed set $F$ is $P_k$-maximal if there is no closed set $G$ such that $P_k \supset G \supset F$ (proper inclusion). A subset $X$ is complete in $P_k$ if $P_k$ is the least closed set containing $X$ (in other words, if the functions in $X$ can produce by composition any function in $P_k$). It is well known that a subset $X$ of functions is complete in $P_k$ if and only if it is contained in no $P_k$-maximal set (completeness condition) (cf. [5, 10]).

Investigations of completeness and related topics, usually called functional completeness problem, are mathematically important and have a wide range of applications including their direct relationship to logical circuit design. A complete set $X$ in $P_k$ is called a base of $P_k$ if no proper subset of $X$ is complete in $P_k$. The rank of a base is the number of its elements. A function $f$ is Sheffer for $P_k$ if $\{f\}$ is a base (of rank 1) of $P_k$. Clearly $f$ is Sheffer for $P_k$ iff it belongs to no $P_k$-maximal set.

To describe $P_k$-maximal sets, we need the following essential concept of "functions preserving a relation" (cf. [10]). Let $h \geq 1$. An $h$-ary relation $\rho$ on $E_k$ is a subset of $E_k^h$ (i.e., a set of $h$-tuples over $E_k$) whose elements are written as columns. Given row $n$-vectors $a_i = (a_{i1}, \ldots, a_{in})$ ($i = 1, 2, \ldots, h$) we write $(a_1, \ldots, a_n)^T \in \rho$ to indicate that $(a_{1j}, \ldots, a_{nj})^T \in \rho$ for all $j = 1, 2, \ldots, n$, where $T$ denotes the transpose (this means that the $h \times n$ matrix with rows $a_1, \ldots, a_h$ has all columns in $\rho$). We say that an $n$-ary $f \in P_k$ preserves $\rho$ if $f((a_1), \ldots, f(a_h))^T \in \rho$ whenever $(a_1, \ldots, a_h)^T \in \rho$.

Then the set of functions preserving $\rho$ is denoted by $\text{Pol}\ \rho$:

$$\text{Pol}\ \rho = \{f \mid (a_1, \ldots, a_n)^T \in \rho \rightarrow (f(a_1), \ldots, f(a_n))^T \in \rho\}.$$ 

All $P_k$-maximal sets are of the form $\text{Pol}\ \rho$ for some relation $\rho$. Their full description is given by Rosenberg [10–14] and the following is the list of all relations that correspond to $P_k$ maximal sets. They are grouped into six classes:

$(R_1)$ Every partial order on $E_k$ having a greatest and a least element.

$(R_2)$ Every relation $\{(x, s(x)) \mid x \in E_k\}$ where $s$ is a permutation of $E_k$ with $k/p$ cycles of the same prime length $p$.

$(R_3)$ Every 4-ary relation $\{(a_1, a_2, a_3, a_4) \mid a_i \in E_k, a_1 + a_2 = a_3 + a_4\}$, where $(E_k, +)$ is a $p$-elementary abelian group ($p$ prime).
(Ra) Every non-trivial equivalence relation on $E_k$.
(R5) Every central relation on $E_k$. For $1 \leq h \leq k - 1$, a central relation is formed as follows (note that this definition is new):
- Choose $t$ combinations $(c_{i1}, \ldots, c_{ih})$ of $h$ out of $k$ elements, $i = 1, 2, \ldots, t$, such that $\{c_{11}, \ldots, c_{1h}, \ldots, c_{t1}, \ldots, c_{th}\} \neq E_k$ (i.e., there exist at least one element from $E_k$ which is not part of any chosen combination),
- for each combination form all $h!$ permutations; let $I$ be the set of all $h!$ permutations,
- a central relation is $E_k^h - I$, where $E_k^h$ denotes the set of all $h$-tuples over $E_k$, and "-" is set difference (i.e., eliminate permutations from $I$ from $E_k^h$).
(R6) Every relation determined by a $h$-regular $(2 < h \leq k)$ family of equivalences $T$ on $E_k$, where $T = \{\theta_1, \ldots, \theta_m\}$ is a family of equivalence relations, $h^m \leq k$, is $h$-regular if the following conditions are satisfied:
1. Each equivalence relation $\theta_j$ has exactly $h$ equivalence classes, $1 \leq j \leq m$,
2. If $e_j$ is an arbitrary equivalence class of $\theta_j$ then the intersection $\cap \{e_j | 1 \leq j \leq m\}$ is non-empty.

The relation determined by $T$ is the $h$-ary relation $\lambda_T$ containing all the $h$-tuples $(a_1, \ldots, a_h) \in E_k^h$ such that for each $j$, $1 \leq j \leq m$ at least two elements among $a_1, \ldots, a_h$ are $\theta_j$-equivalent.

Theorem 1 (Jablonski [5] and Rosenberg [12]) A set $F$ of functions from $P_k$ is complete if and only if for every relation $\rho$ described above there exists an $f \in F$ not preserving $\rho$.

We have established mapping from $n$-ary functions of $r$-valued set logic to $n$-ary functions of $2^r$-valued logic. From this mapping, it immediately follows that above functional completeness criteria in $P_k$ applies also to $r$-valued set logic with $k = 2^r$.

3. BOOLEAN COMPLETENESS IN SET LOGIC

In the sequel, we suppose that $k = 2^r$, $r$ positive integer.

In [1, 2] the question of constructing all set logic functions using Boolean functions is studied. Since the set is incomplete, some
functions are added to the set of Boolean functions to form a complete set. For example, in [1] bio-output is added while in [2] literal function is added. In these considerations Boolean functions are considered "cheap" elements in the design of set logic functions. Following these investigations, we are interested in characterizing all sets of functions which become complete when all Boolean functions are added to them. We define the notion of Boolean completeness as follows.

**Definition 1** A subset $F$ of $P_k$ is said to be Boolean complete in $P_k$ if the set $BF \cup F$ is complete in $P_k$.

**Definition 2** A Boolean complete set $F$ in $P_k$ is called a Boolean base of $P_k$ if no proper subset of $F$ is Boolean complete in $P_k$. The rank of a Boolean base is the number of its elements.

**Definition 3** A non-Boolean function $f \in P_k$ is a Boolean Sheffer for $P_k$ if $\{f\}$ is a Boolean base (of rank 1) of $P_k$.

**Definition 4** A maximal set $F$ in $P_k$ is said to be a Boolean maximal set in $P_k$ if $BF \subseteq F$.

**Theorem 2** (Boolean completeness theorem in a general form) A subset $F$ of non-Boolean functions in $P_k$ ($k = 2^r$) is Boolean complete in $P_k$ if and only if $F$ is contained in no Boolean maximal set in $P_k$.

**Proof** According to completeness theorem of Jablonski [5], a subset $F$ of $P_k$ is complete if it is contained in no maximal set of $P_k$. If for a maximal set $M$, $BF - M \neq \emptyset$, then $BF$ contains a function $f \in P_k$, not belonging to $M$. So, for each maximal set $M$ in $P_k$ which is not a Boolean maximal set in $P_k$ there exists a Boolean function not belonging to $M$. Hence follows the statement.

### 4. BOOLEAN COMPLETENESS IN TWO-VALUED SET LOGIC

In this section we give a full description of Boolean complete sets, Boolean bases and Boolean Sheffer functions for the case $r = 2, k = 4$, i.e., in $P_4$. 
There are exactly 82 maximal sets in $P_4$, divided into six classes $R_1 - R_6$. We refer to these classes also as partial order, self-dual, linear, equivalence, central and semi-degenerate classes of maximal sets, respectively. To determine which of 82 maximal sets are Boolean maximal sets, it is sufficient to check for each of them whether it contains all constants, union, intersection and complement functions.

4.1. Partial Order Class $R_1$

There are exactly 18 different partial orders on $E_4$ having a greatest and a least element. Among them, there are 12 linear orders and six partial orders in which only one pair of elements are noncomparable. Let the partial order relation be denoted by $\prec$. Then an unary function $f$ preserves partial order $\prec$ if and only if $x \prec y \rightarrow f(x) \prec f(y)$ for every $x, y \in E_4$.

**Lemma 1** The complement function does not preserve any partial order.

**Proof** Let $a$ be the least element in given partial order. Let $b = a = 3-a$. Clearly $b \neq a$ and $b = a$. Then, $a \prec b \rightarrow a \prec b$ is not satisfied since $b = a$ contradicts the choice of least element $a$.

Hence $\bar{x} \in BF - M$ for every of 18 maximal sets $M$ determined by partial orders, and class $R_1$ contains no Boolean maximal set in $P_4$.

4.2. $R_2$-Selfdual Class

There are three relations on $E_4$ belonging to $R_2$, corresponding to permutations $s_1 = (1\ 0\ 3\ 2)$, $s_2 = (2\ 3\ 0\ 1)$ and $s_3 = (3\ 2\ 1\ 0)$.

**Lemma 2** The constant functions do not belong to any selfdual class.

**Proof** Consider unary constant function $f(x) = a$. Let $(b\ c)^T \in \rho$ where $\rho$ is any of three relations in selfdual class $(c = s_i(b))$. Then $(f(b)\ f(c))^T = (a\ a)^T \notin \rho$. Therefore constant function does not preserve $\rho$.

Hence $a \in BF - M$ for every maximal set in selfdual class, and class $R_2$ contains no Boolean maximal set in $P_4$. 

4.3. $R_3$-Linear Functions

For $r = 2$, the class $R_3$ contains only one maximal set $L$, which is also known as the set of linear functions in $E_4$. It can be also defined as follows. A function is linear if there are $a_0, \ldots, a_n \in E_4$ so that 
\[ f(x_1, \ldots, x_n) = a_0 + a_1x_1 + \cdots + a_nx_n, \]
where $x+y$ and $xy$ denote $x+y(\text{mod } 4)$ and $xy(\text{mod } 4)$.

**Lemma 3** $BF-L \neq \emptyset$.

**Proof** Follows from $|L(n)| = 2^{2(n+1)} < 2^{2^{n+1}} = |BF(n)|$.

Thus $L$ is not a Boolean maximal set in $P_4$.

4.4. $R_4$-Equivalence Class

There are 13 different non-trivial partitions on $E_4$ and consequently there are 13 non-trivial equivalences on $E_4$. These equivalence relations are: 
\[ \alpha_1 = \{(01)\{23\}\}, \alpha_2 = \{(02)\{13\}\}, \alpha_3 = \{(03)\{12\}\}, \alpha_4 = \{(0)\{123\}\}, \alpha_5 = \{(1)\{023\}\}, \alpha_6 = \{(2)\{013\}\}, \alpha_7 = \{(3)\{012\}\}, \]
\[ \alpha_8 = \{(0)\{1\}\{23\}\}, \alpha_9 = \{(0)\{2\}\{13\}\}, \alpha_{10} = \{(0)\{3\}\{12\}\}, \alpha_{11} = \{(2)\{3\}\{01\}\}, \alpha_{12} = \{(1)\{3\}\{02\}\}, \alpha_{13} = \{(1)\{2\}\{03\}\}. \]

Let the corresponding maximal sets be $C_i = \text{Pol} \alpha_i, 1 \leq i \leq 13$. $C_i$ contains functions $f$ satisfying the following property: if $x_j$ and $y_j$ are equivalent under $\alpha_i$ for $1 \leq j \leq n$ then $f(x_1, \ldots, x_n)$ and $f(y_1, \ldots, y_n)$ are also equivalent under $\alpha_i$.

**Lemma 4**

(a) $BF-C_i \neq \emptyset$ for $i = 3, 4, \ldots, 13$.

(b) $BF \subseteq C_i$ for $i = 1, 2$.

**Proof**

(a) Consider the complement function. In order to have it included in a maximal set $C_i$, the following property should be satisfied: if $x$ and $y$ are equivalent under $\alpha_i$ then $\overline{x}$ and $\overline{y}$ should also be equivalent under $\alpha_i$. Thus if 0 and 1 are equivalent then $3 = \overline{0}$ and $2 = \overline{1}$ should also be equivalent. The last is not satisfied for relations $\alpha_6, \alpha_7$ and
α_{11}. By considering other choices of x and y we may conclude that the complement function does not belong to C_i for 4 \leq i \leq 12. From 0 \cap 1 = 0 and 3 \cap 1 = 1 it follows that the intersection function does not belong to C_3 and C_{13} (pairs (0, 3) and (1, 1) are equivalent while (0, 1) is not).

(b) Follows from the fact that the Boolean functions constants, complement, union and intersection which form a complete set on the class of Boolean functions belong to both C_1 and C_2. Therefore C_1 and C_2 are the only Boolean maximal sets from the equivalence class. We give their corresponding relations in full notation.

\[ \alpha_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 1 & 0 & 1 & 2 & 3 & 2 & 3 \end{pmatrix} \]
\[ \alpha_2 = \begin{pmatrix} 0 & 0 & 2 & 2 & 1 & 1 & 3 & 3 \\ 0 & 2 & 0 & 2 & 1 & 3 & 1 & 3 \end{pmatrix} \]

4.5. \( R_5 \)-Central Class

There are 40 different central relations on \( E_4 \). Among them, according to [13], there are 14 unary relations, 22 binary relations and 4 ternary relations.

The unary central relations are all proper non-empty subsets of \( E_4 \), i.e. (0), (1), (2), (3), (01), (02), (03), (12), (13), (23), (012), (013), (023), (123).

**Lemma 5** For each unary central relation there exist a constant function which does not preserve it.

**Proof** For each proper subset of \( E_4 \) there exist an element \( u \) which does not belong to it. Then the constant \( u \) obviously does not preserve the relation. For example, for subset (023) the constant function \( \text{I} \) does not preserve the relation (\( \text{I}(0) = \text{I} \) is a contradiction). If we denote by \( P_{i(1), \ldots, i(m)} \) the set of all permutations of different elements \( i(1), \ldots, i(m) \) from \( E_4 \), \( 2 \leq m \leq 4 \), then the complete list of
binary and ternary central relations on $E_4$ is the following:

$$d_{15} = E_4^2 - P_{12} - P_{13} - P_{23},$$
$$d_{16} = E_4^2 - P_{13} - P_{23}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{pmatrix}$$

$$- \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{pmatrix},$$

$$d_{17} = E_4^2 - P_{12} - P_{23},$$
$$d_{18} = E_4^2 - P_{12} - P_{13},$$
$$d_{19} = E_4^2 - P_{23},$$
$$d_{20} = E_4^2 - P_{12},$$
$$d_{21} = E_4^2 - P_{13},$$
$$d_{22} = E_4^2 - P_{02} - P_{03} - P_{23},$$
$$d_{23} = E_4^2 - P_{03} - P_{23},$$
$$d_{24} = E_4^2 - P_{02} - P_{23},$$
$$d_{25} = E_4^2 - P_{02} - P_{03},$$
$$d_{26} = E_4^2 - P_{02},$$
$$d_{27} = E_4^2 - P_{03},$$
$$d_{28} = E_4^2 - P_{01} - P_{03} - P_{13},$$
$$d_{29} = E_4^2 - P_{03} - P_{13},$$
$$d_{30} = E_4^2 - P_{01} - P_{13},$$
$$d_{31} = E_4^2 - P_{01} - P_{03},$$
$$d_{32} = E_4^2 - P_{01},$$
$$d_{33} = E_4^2 - P_{01} - P_{02} - P_{12},$$
$$d_{34} = E_4^2 - P_{02} - P_{12},$$
$$d_{35} = E_4^2 - P_{01} - P_{12},$$
$$d_{36} = E_4^2 - P_{01} - P_{02},$$
$$d_{37} = E_4^2 - P_{123},$$
$$d_{38} = E_4^2 - P_{012},$$
$$d_{39} = E_4^2 - P_{023},$$
$$d_{40} = E_4^2 - P_{013}.$$

**Lemma 6** All maximal sets determined by relations $d_{15} - d_{40}$ are not Boolean complete sets in $P_4$.

**Proof** For $15 \leq i \leq 40$, $i \neq 20$, $i \neq 27$, it can be easily seen that the complement function does not preserve relation $d_i$. For example, the pair $(0 \ 1)^T$ belongs to $d_{16}$ while $(0 \ 1)^T = (3 \ 2)^T$ does not. On the other hand, the intersection function does not preserve relations $d_{27}$ ($1 \cap 2 = 0, 3 \cap 3 = 3$) and $d_{20}$ ($3 \cap 2 = 2, 1 \cap 2 = 1$).

Hence there is no weak maximal set in $P_4$ in the central class.

### 4.6. $R_6$-Semidegenerate Class

There are seven different relations on $E_4$ belonging to the family $R_6$. The complete list of them is the following:

$$q_1 = E_4^3 - P_{023} - P_{123},$$
$$q_2 = E_4^3 - P_{013} - P_{123},$$
$$q_3 = E_4^3 - P_{012} - P_{123},$$
$$q_4 = E_4^3 - P_{013} - P_{023},$$
\[
q_5 = E_4^3 - P_{012} - P_{023}, \quad q_6 = E_4^3 - P_{012} - P_{013}, \\
q_7 = E_4^3 - P_{0123}.
\]

First six relations correspond to the case \( h = 3, \ m = 1 \) (in the general definition of class \( R_6 \)) and equivalence relations \( \{\{01\} \{2\} \{3\}\}, \{\{02\} \{1\} \{3\}\}, \{\{03\} \{1\} \{2\}\}, \{\{12\} \{0\} \{3\}\}, \{\{13\} \{0\} \{2\}\}, \text{and} \{\{23\} \{0\} \{1\}\} \) while the last relation correspond to the case \( h = 4, m = 1, \) and equivalence relation \( \{\{0\} \{1\} \{2\} \{3\}\} \).

Let the corresponding maximal sets be \( G_i = \text{Pol}(q_i), \ i = 1, 2, \ldots, 7. \)

**Lemma 7** \( BF-G_i = \emptyset, \) for \( i = 1, 2, \ldots, 7. \)

**Proof** It can be easily observed that the complement function does not preserve relations \( q_1 - q_6 \) (for example, it does not preserve \( q_1 \) and \( q_2 \) because \( 0 = 3, \ 1 = 2, \ 2 = 1 \)), while the intersection operation does not preserve relation \( q_7 \) (\( 1 \cap 0 = 0, \ 1 \cap 1 = 1, \ 2 \cap 3 = 2, \ 3 \cap 3 = 3 \)). \( \square \)

From above considerations the following theorem easily follows.

**Theorem 3** \( P_4 \) has exactly two weak maximal sets: \( C_1 \) and \( C_2. \)

**Corollary 1** A subset \( F \) of non-Boolean functions in \( P_4 \) is Boolean complete in \( P_4 \) if and only if \( F \subseteq C_1 \neq \emptyset \) and \( F \subseteq C_2 \neq \emptyset. \)

**Corollary 2** \( |B| \leq 2 \) for any weak base \( B \) in \( P_4. \)

### 5. Enumeration of Weak Bases in \( P_4 \)

**Lemma 8** \( |C_1(n)| = |C_2(n)| = 2^{2^n - 2^n} \) for \( n \geq 0. \)

**Proof** \( C_1(n) \) is the set of functions \( f : E_4^n \to E_4 \) preserving the relation \( \alpha_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 1 & 0 & 1 & 2 & 3 & 2 & 3 \end{pmatrix}. \) Partition the set \( E_4^n \) into the \( n \)-dimensional blocks in such a way that two points \( (x_1, \ldots, x_n) \) and \( (\beta_1, \ldots, \beta_n), \) \( x_i, \beta_i \in E_4, \) belong to the same block iff for every \( i = 1, 2, \ldots, n, \) either both \( x_i \) and \( \beta_i \) belong to \( \{0, 1\} \) or both of them belong to \( \{2, 3\}. \) If \( f \in C_1 \) then the values \( f(x_1, \ldots, x_n) \) in the points of a fixed block are all from the same subset, either \( \{0, 1\} \) or \( \{2, 3\}. \)

The number of blocks is \( 2^n \) and each block contains \( 2^n \) points. For each block the set of values, either \( \{0, 1\} \) or \( \{2, 3\}, \) can be chosen in two
different ways. It follows that the sets of values for all $2^n$ blocks can be chosen in $2^{2^n}$ ways. For every such choice, there are $2^{2^n}$ ways for each of the $2^n$ blocks to choose the values of $f(x_1, \ldots, x_n)$ in the points of that block from a two-element subset. So, there are $2^{2^n}(2^{2^n})^{2^n} = 2^{2^{2^n}}$ functions in $C_1(n)$. The cardinality of $C_2(n)$ is the same by symmetry. ■

**Lemma 9** \[ |C_1(n) \cap C_2(n)| = 2^{2^{2^n}}, \text{ for } n \geq 0. \]

**Proof** $C_1(n) \cap C_2(n)$ is the set of functions preserving both relations $\alpha_1$ and $\alpha_2$. Consider two different partitions of the set $E_q^n$. The first partition $\pi_1$ is the same as in the proof of Lemma 8. The second partition $\pi_2$ of $E_q^n$ is performed in such a way that two points $(\chi_1, \ldots, \chi_n)$ and $(\beta_1, \ldots, \beta_n)$, $\chi_i, \beta_i \in E_q$, belong to the same block iff for every $i = 1, 2, \ldots, n$, either both $\chi_i$ and $\beta_i$ belong to $\{0, 2\}$ or both of them belong to $\{1, 3\}$. It is easy to see that any two blocks from different partitions, $\pi_1$ and $\pi_2$, have exactly one common point.

Note that all intersections $\{0, 1\} \cap \{0, 2\} = \{0\}$, $\{0, 1\} \cap \{1, 3\} = \{1\}$, $\{2, 3\} \cap \{0, 2\} = \{2\}$, $\{2, 3\} \cap \{1, 3\} = \{3\}$ are single sets. So after choosing the set of values for every block of the partition $\pi_1$ either $\{0, 1\}$ or $\{2, 3\}$ and for every block of the partition $\pi_2$ (either $\{0, 1\}$ or $\{2, 3\}$), the function $f$ is uniquely determined. Since the sets of values for all blocks of the partition $\pi_1$ (and also for all blocks of the partition $\pi_2$) can be chosen in $2^{2^n}$ different ways, it follows that the number of functions $f$ preserving both relations $\alpha_1$ and $\alpha_2$ is $(2^{2^n})^2 = 2^{2^{2^n}}$. ■

**Corollary 3** $C_1 \cap C_2 = BF$.

**Proof** According to Lemma 4(b), $BF \subseteq C_1$ and $BF \subseteq C_2$. Hence follows $BF(n) \subseteq C_1(n)$ and $BF(n) \subseteq C_2(n)$, for every $n \geq 0$, and consequently $BF(n) \subseteq C_1(n) \cap C_2(n)$. Since $|BF(n)| = 2^{2^{2^n}}$, according to Lemma 9, $|C_1(n) \cap C_2(n)| = |BF(n)|$, for every $n \geq 0$, we obtain that $C_1(n) \cap C_2(n) = BF(n)$. Taking the sums for $n \geq 0$ we conclude $C_1 \cap C_2 = BF$. ■

Let $b_1(n)$ and $b_2(n)$ be the numbers of weak bases in $P_4$ or ranks 1 and 2, respectively, containing $n$-ary functions.

**Corollary 4**

(i) $b_1(n) = |C_1(n) \cap C_2(n)|$,

(ii) $b_2(n) = |C_1(n) - BF(n)| \cdot |C_2(n) - BF(n)|$. 


Theorem 4

(a) The number of weak Sheffer n-ary functions in $P_4$ is

$$b_1(n) = 2^{22n+1} - 2^{22n+2} + 1 + 2^{2n+1}.$$  

(b) The number of two-element weak bases containing n-ary functions in $P_4$ is

$$b_2(n) = 2^{22n+1} + 2^{2n+1} - 2^{22n+2} + 2^{2n+1} + 2^{2n+2}.$$  

Proof

(a) According to Corollary 4(i), $b_1(n) = |C_1(n) \cap C_2(n)| = |P_4(n)| - |C_1(n)| - |C_2(n)| + |C_1(n) \cap C_2(n)| = 4^{4n} - 2 \cdot 2^{2n+2n} + 2^{2n+1} = 2^{2n+1} - 2^{2n+2n+1} + 2^{2n+1}$. 

(b) The number of non-Boolean functions in $C_1(n)$ is $|C_1(n) - BF(n)| = |C_1(n)| - |BF(n)| = 2^{2n+2n} - 2^{2n+1}$. The same is the number of non-Boolean functions in $C_2(n)$. According to Corollary 4(ii), $b_2(n) = (2^{2n+2n} - 2^{2n+1})^2 = 2^{2n+1} + 2^{2n+1} - 2^{2n+2n+1} + 2^{2n+2}$. 

Obviously,

$$\lim_{n \to \infty} \frac{b_1(n)}{2^{22n+1} - 2^{2n+1}} = 1,$$

so Theorem 4(a) implies the following statement.

Corollary 5  Almost all non-Boolean set valued functions are weak Sheffer functions.

6. CONCLUSION

Lemmas 1, 2, and 3 can be easily generalized for the set valued functions in $P_k$, where $k = 2^r$ and $r$ is an arbitrary integer $> 2$. It appears that there are $2^r - 2$ weak maximal sets in equivalence class. For larger $r$, most of relations belong to central group (cf. [13]). Listing all weak maximal sets, classification and enumeration of weak bases for $2^r$-valued logic (or equivalently $r$-valued set logic) for $r > 2$ remains an open problem for further study.
References


