Clifford Fourier Transform on Vector Fields

Julia Ebling and Gerik Scheuermann, *Member, IEEE*

***Abstract*— Image processing and computer vision have a lot of useful tools for the analysis of scalar fields. There are robust methods for feature extraction and the computation of derivatives. Furthermore, interpolation and the resulting effects for the application of filter can be analyzed in detail. It seems to be sensible to try to apply these methods to vector fields, too, in order to get a solid theoretical base for feature extraction. Recently, we introduced the Clifford convolution which is an extension of the classical convolution on scalar fields. Clifford convolution is a unified notation for the convolution of scalar and vector fields. It has nice geometric properties which were used for pattern matching on vector fields. In image processing, convolution and Fourier transform are closely connected by the convolution theorem. In this paper, we extend the Fourier transform to general elements of the Clifford Algebra, called multivectors, which include scalars and vectors. The resulting convolution theorems for the Clifford convolution and the derivation theorems are extensions of the theorems for convolution and Fourier transform on scalar fields. The Clifford Fourier transform allows a frequency analysis of vector fields and of the behavior of vector valued filters. In frequency space, vectors are transformed into general mulitvectors of the Clifford Algebra. Many basic vector valued pattern such as source, sink, saddle points and potential vortices can be described by a few multivectors in frequency space.**

***Index Terms*— Flow visualization, vector fields, Fourier transform, convolution, Clifford algebra**

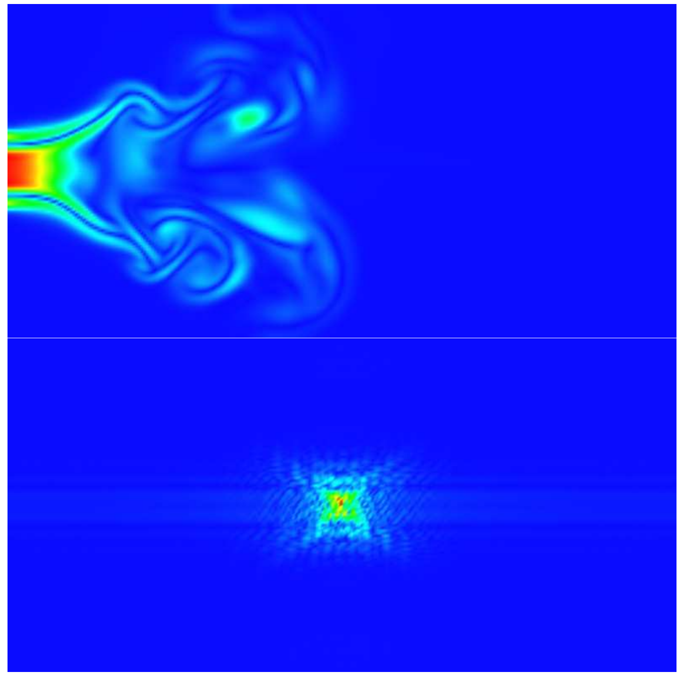

Fig. 1. Turbulent swirling jet entering a fluid at rest. Top: Color coding of the absolute magnitude of the vectors. Deep blue indicates low magnitude and deep red high magnitude. (2): (fast) Clifford Fourier transform of the dataset. Zero frequency is in the middle of the image. Vectors transform to multivectors from Clifford algebra in frequency domain, therefore color coding of the magnitude of the multivectors.

I. Introduction

THERE are several ways of visualizing vector fields. They can be visualized directly using glyphs, streamline ore texture based methods [12]. Then there exist automatic approaches for analysis and subsequent visualization. Classical methods for that are topology [15] and feature detection [11], [13] and verification [10]. While topological methods are good for separation of regions with different flow behavior and an overall visualization of the flow in 2D, a convincing presentation of 3D topology is still missing. Attempts for feature detection usually try to give an analytic model of a feature and create an algorithm from there. Some features like vortices are not well defined, then the limitations of the model play an important role. Furthermore, these approaches often have severe robustness problems in terms of noise. This becomes especially important when dealing with measured data instead of simulations. Then there is the derivation which a lot of the algorithms use. In order to get sensible results, the vector field has often to be smoothed beforehand.

Image processing and computer vision have longer had to deal with this kind of problems. Methods for analysis, feature extraction and derivation computation are well know. Convolution based approaches are pretty robust in terms of white or noise because of the inherent averaging. Furthermore, most image processing methods allow a precise analysis of the accuracy of their results concerning sampling, interpolation, smoothing and noise. Therefore it seems to be sensible to try to apply them to vector fields. First attempts for that concentrated on

The authors are with the FB Informatik, University of Leipzig, 04315 Leipzig, Germany.
E-mail:{ebling,scheuer}@informatik.uni-leipzig.de

Manuscript received May 2004.

transferring convolution to vector fields. Thus, some pattern matching algorithms for vector fields were created.

Heiberg et al. [6] defined a scalar convolution on vector fields based on the scalar product of two vectors. The similarity measure thus obtained was combined with different filter directions and the orientation tensor. We [3] defined the Clifford convolution and used its geometric properties for pattern matching. The Clifford convolution is superior as it is defined for multivector fields and thus it gives a unified notation for the convolution of scalar, vector and multivector fields. Another advantage is that the discretized derivative can be computed by a Clifford convolution with a vector valued filter. The Clifford convolution is an extension of the convolution of scalar fields from image processing and the scalar convolution on vector fields defined by Heiberg et al.

From a theoretical point of view, a central question is if it is possible to generalize the Fourier transform to (multi-)vector fields and to obtain convolution and derivation theorems similar to the original ones. This would open up a new path for analyzing vector fields using frequency domain. Image processing provides a theoretical and well established basis for the analysis of scalar fields. Moreover the tools from image processing are mostly robust. With Fourier transform and convolution extended to multivector fields, many successful tools from image processing could be applied directly to vector fields. Furthermore, the existence of a fast Fourier transform would speed up the convolution computation.

In this paper we define a Clifford Fourier transform and prove the generalizations of convolution, correlation and derivation theorem. These theorems are extensions of the corresponding theorems for convolution and Fourier transform on scalar fields. Then we analyze some basic vector valued patterns such as source, sink, saddle point and swirl in frequency domain.

II. Related Work

An obvious approach to image processing on vector fields is to decompose the vector field into its components. The resulting scalar fields can be processed separately with known tools from image processing such as convolution and Fourier transform in this case. But a vector is more than its separated components and the scalar fields of the components are not independent. Thus, the convolution is not invariant to changes of the coordinate system. Granlund and Knutson [5] have investigated this approach in 2D.

Another approach for the definition of the convolution is to define a multiplication of vectors and thus convey the convolution to vector fields. Heiberg et al. [6] define convolution on vector fields with the scalar product of two vectors:

$$(\mathbf{h} *_s \mathbf{f})(\mathbf{x}) = \int_{E^d} < \mathbf{h}(\mathbf{x'}), \mathbf{f}(\mathbf{x}-\mathbf{x'}) > d\mathbf{x'}$$

$\mathbf{f}$ is the normalized vector field and $\mathbf{h}$ the filter. As the scalar product is used, it gives an approximation of the cosine of the angle between the structure in the vector field and the direction of the filter. Heiberg et al. do not formulate or use a Fourier transform.

In an earlier paper [3], we defined the Clifford convolution on multivector fields by using the Clifford product. Clifford algebra works with multivectors. A multivector in 3D consists of the sum of a scalar, a 3D vector, a 3D bivector and a trivector (figure 2). Scalar and vector are as usual. In 3D the unit bivectors can be identified with a planar direction and a limited oriented area. The unit trivector gives the volume spanned by three orthogonal unit vectors building a right hand system. As scalars and vectors are part of multivectors, a scalar or vector valued field can be regarded as part of a multivector field. Each of them can be transformed into a multivector field by retaining the scalars respectively vectors and initializing the other components of the multivectors with zero. This way these fields can be used with Clifford convolution, too. Thus, Clifford Convolution is a unifying framework for the use of scalar, vector and multivector valued filter. Scalar filter like gradient or smoothing filter from image processing can be applied to vector fields within this framework just like vector valued filter for pattern matching. The scalar convolution on vector fields defined by Heiberg et al. is a special case of the Clifford convolution. Clifford algebra and Clifford convolution are explained in further detail in section III and IV.

Extensions of the Fourier transform arose from a different area of research, namely texture segmentation. While giving extensions of the analytic signal to 2D and 3D, Bülow [2] and Felsberg [4] defined some Fourier transforms, again with the help of some Clifford algebra.

Bülow [2] used a Clifford algebra where $\mathbf{e}_j^2 = -1$ and defined the d dimensional Fourier transform by using all $\mathbf{e}_1, ..., \mathbf{e}_d$ in the Fourier kernel:

$$\mathscr{F}_b\{\mathbf{f}\}(\mathbf{u}) = \int_{E^d} \mathbf{f}(\mathbf{x}) \prod_{k=1}^{d} e^{(-\mathbf{e}_k 2\pi x_k u_k)} |d\mathbf{x}|$$

The product has to be performed in the fixed order of the indices. Furthermore the corresponding convolution theorems are rather complicated. The complex form of the kernel and the non-commutativity of the multiplication also presented a problem when trying to establish a fast version of this Fourier transform.

Felsberg [4] extended the analytic signal to the monogenic signal by using embedded functions to get more phases. Therefore he regarded vectors build by spatial coordinates and the corresponding value of the function. He defined his Fourier transform for arbitrary multivector valued functions $\mathbf{F} \in G_2$ (1D) and $\mathbf{F} \in G_3$ (2D) as

$$\mathscr{F}\{\mathbf{F}\}(\mathbf{u}) = \int_{x_1} e^{(-2\pi \boldsymbol{i}_2 <\mathbf{x},\mathbf{u}>)} \mathbf{F}(\mathbf{x}) dx_1$$

with $\mathbf{x} = x\mathbf{e}_1$ in 1D and

$$\mathscr{F}\{\mathbf{F}\}(\mathbf{u}) = \int_{x_1}\int_{x_2} e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)} \mathbf{F}(\mathbf{x}) dx_1 dx_2$$

with $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ in 2D. $\boldsymbol{i}_2$ and $\boldsymbol{i}_3$ are defined in section III-C. Felsberg defined the convolution only for quaternion (1D) and spinor (2D) valued filter on vector valued functions. Therefore no vector valued filter for pattern matching on vector fields could be applied. As the convolution was restricted, so was the convolution theorem. Concerning other theorems like derivation and Parseval's theorem, only special cases of the multivector valued functions were regarded, too. We build on this approach by using Clifford Convolution and proving theorems for arbitrary multivector valued functions.

III. Clifford Algebra

Clifford algebra extends the classical description of an Euclidean n-space as a real n-dimensional vector space with scalar product to a real algebra. From now on, scalar or vector valued fields are always assumed to be special multivector fields.

A. Clifford Algebra in 3D

For the 3-dimensional Euclidean vector space E^3, we get a 8-dimensional $\mathbb{R}$-algebra G^3 with the basis $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ as a real vector space. The elements of the algebra are called multivectors. The multiplication of multivectors is defined as associative, bilinear and by the equations

$$\begin{aligned} 1\mathbf{e}_j &= \mathbf{e}_j, & j &= 1,2,3 \\ \mathbf{e}_j\mathbf{e}_j &= 1, & j &= 1,2,3 \\ \mathbf{e}_j\mathbf{e}_k &= -\mathbf{e}_k\mathbf{e}_j, & j,k &= 1,2,3,\ j \neq k \end{aligned}$$

This multiplication is not commutative, but a mixture of commutation and anticommutation of parts of the multivectors. A multiplication of vectors is described by this Clifford multiplication, too. The usual vectors $(x,y,z) \in \mathbb{R}^3$ are identified with

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \in E^3 \subset G^3.$$

An arbitrary multivector $\mathbf{A}$ can be written as

$$\mathbf{A} = \alpha + \mathbf{a} + \boldsymbol{i}_3(\mathbf{b} + \beta)$$

with $\alpha, \beta \in \mathbb{R}$, $\mathbf{a}, \mathbf{b} \in E^3$, $\boldsymbol{i}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, $(\boldsymbol{i}_3)^2 = -1$.

The grade projectors $<>_j : G^3 \to G^3$ are the maps

$$\begin{aligned} <\mathbf{A}>_0 &= \alpha, \quad <\mathbf{A}>_1 = \mathbf{a}, \\ <\mathbf{A}>_2 &= \boldsymbol{i}_3\mathbf{b}, \quad <\mathbf{A}>_3 = \boldsymbol{i}_3\beta. \end{aligned}$$

The Clifford multiplication of two vectors $\mathbf{a}, \mathbf{b} \in E^3$ results in

$$\mathbf{ab} = <\mathbf{a},\mathbf{b}> + \mathbf{a} \wedge \mathbf{b},$$

where $<,>$ is the inner product and $\wedge$ the outer product. Furthermore, we have

$$\begin{aligned} <\mathbf{ab}>_0 = <\mathbf{a},\mathbf{b}> &= \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\omega \\ \|<\mathbf{ab}>_2\| = \|\mathbf{a}\wedge\mathbf{b}\| &= \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\omega \end{aligned}$$

where ω is the angle between $\mathbf{a}$ and $\mathbf{b}$. $<\mathbf{ab}>_2$ corresponds to the plane through $\mathbf{a}$ and $\mathbf{b}$ as it is the corresponding bivector.

The 2D Clifford algebra G_2 is defined analog, see Hestenes [7] and Scheuermann [15]. An example for the multiplication of multivectors in 2D can be found in the appendix of an earlier paper of the authors [3].

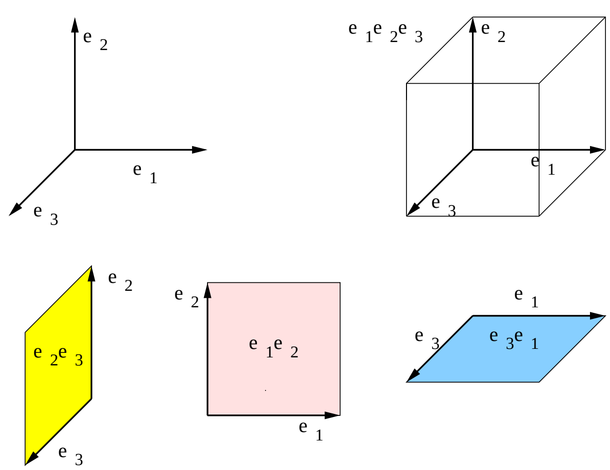

Fig. 2. Unit vectors, bivectors and trivector in G_3

B. Integral and derivative

Let $\mathbf{F}$ be a multivector valued function (field) of a vector variable $\mathbf{x}$ defined on some region of the Euclidean space E^d. If the function is only scalar or vector valued, we will call it scalar or vector field.

The Riemann integral of a multivector valued function $\mathbf{F}$ is defined as

$$\int_{E^d} \mathbf{F}(\mathbf{x})|d\mathbf{x}| = \lim_{\substack{|\Delta x_j| \to 0 \\ n \to \infty}} \sum_{i=1}^{n} \mathbf{F}(x_j\mathbf{e}_j)\Delta x_j$$

Note that $|d\mathbf{x}|$ used. This is done to make the integral grade preserving as $d\mathbf{x}$ is vector valued within Clifford algebra. The integral can be discretized into sums using quadrature formulas.

The directional derivative of $\mathbf{F}$ in direction $\mathbf{r}$ is

$$\mathbf{F_r}(\mathbf{x}) = \lim_{h\to 0} \frac{[\mathbf{F}(\mathbf{x}+h\mathbf{r}) - \mathbf{F}(\mathbf{x})]}{h}$$

with $h \in I\!R$. With the vector derivative

$$\nabla = \sum_{j=1}^{d} \mathbf{e}_j \frac{\partial}{\partial x_j},$$

which is again vector valued, the total derivative of $\mathbf{F}$ can be defined as

$$\nabla\mathbf{F}(\mathbf{x}) = \sum_{j=1}^{d} \mathbf{e}_j \mathbf{F}_{\mathbf{e}_j}(\mathbf{x})$$

using the directional derivatives $\mathbf{F}_{\mathbf{e}_j}(\mathbf{x})$. Analogous is the application of the derivation from the right:

$$\mathbf{F}(\mathbf{x})\nabla = \sum_{j=1}^{d} \mathbf{F}_{\mathbf{e}_j}(\mathbf{x})\mathbf{e}_j$$

This is not quite the same as the application of the derivation from left because of the non-commutativity of the multiplication in Clifford algebra. Curl and divergence of a vector valued function $\mathbf{f}$ can be computed as

$$div\mathbf{f} = <\nabla, \mathbf{f}> = \frac{(\nabla\mathbf{f} + \mathbf{f}\nabla)}{2}$$

$$rot\mathbf{f} = \nabla \wedge \mathbf{f} = \frac{(\nabla\mathbf{f} - \mathbf{f}\nabla)}{2}$$

C. Complex numbers and duals

1) 3D:

$$\{\alpha + \boldsymbol{i}_3\beta | \alpha, \beta \in I\!R\} \subset G_3$$

is isomorph to the complex numbers C as $(\boldsymbol{i}_3)^2 = -1$. From this it follows that for every scalar $\gamma \in I\!R$

$$e^{(\boldsymbol{i}_3\gamma)} = cos(\gamma) + \boldsymbol{i}_3 sin(\gamma).$$

Note that $\boldsymbol{i}_3$ commutes with every element of G_3. That means that for every multivector $\mathbf{A} \in G_3$

$$\begin{aligned} \mathbf{A}e^{(\boldsymbol{i}_3\gamma)} &= \mathbf{A}cos(\gamma) + \mathbf{A}\boldsymbol{i}_3 sin(\gamma) \\ &= cos(\gamma)\mathbf{A} + \boldsymbol{i}_3 sin(\gamma)\mathbf{A} \qquad (1) \\ &= e^{(\boldsymbol{i}_3\gamma)}\mathbf{A} \end{aligned}$$

The dual of the multivector $\mathbf{A}$ is defined as $-\mathbf{A}\boldsymbol{i}_3$. Thus the dual of a scalar is a trivector, the dual of a vector is a bivector $\boldsymbol{i}_3 a$ and vice versa. Therefore we name for example scalar and trivector a dual pair. The dual pairs are:

$$\begin{aligned} 1 &\leftrightarrow \boldsymbol{i}_3 \\ \mathbf{e}_1 &\leftrightarrow \mathbf{e}_2\mathbf{e}_3 \\ \mathbf{e}_2 &\leftrightarrow \mathbf{e}_3\mathbf{e}_1 \\ \mathbf{e}_3 &\leftrightarrow \mathbf{e}_1\mathbf{e}_2 \end{aligned}$$

The dual pairs give only the bases of the duals. The dual of $a_1\mathbf{e}_1$ for example is $-a_1\mathbf{e}_2\mathbf{e}_3$. As 1 and $\boldsymbol{i}_3$ are a dual pair, $\boldsymbol{i}_3$ is also called pseudoscalar.

2) 2D: Analog to the 3D case $\boldsymbol{i}_2$ is an abbreviation of $\mathbf{e}_1\mathbf{e}_2$. $(\boldsymbol{i}_2)^2 = -1$ and thus

$$S = \{\alpha + \boldsymbol{i}_2\beta | \alpha, \beta \in I\!R\} \subset G_2$$

is isomorph to the complex numbers C, too. S is the algebra of spinors which are used e.g. to describe rotations. For every $\gamma \in I\!R$

$$e^{(\boldsymbol{i}_2\gamma)} = cos(\gamma) + \boldsymbol{i}_2 sin(\gamma).$$

As $\boldsymbol{i}_2$ commutes with scalars and bivectors but anti-commutes with vectors, theorems in 2D involving exponential functions tend to be more difficult than their 3D counterparts. Therefore we will often divide a 2D multivector $A = \alpha + \mathbf{a} + \boldsymbol{i}_2\beta \in G_2$ into the vector $\mathbf{a}$ and the spinor $\alpha + \boldsymbol{i}_2\beta$. The dual of A is defined as $-A\boldsymbol{i}_2$. This time the dual of a scalar is a bivector. But the dual of a vector is again a vector. The dual pairs are:

$$\begin{aligned} 1 &\leftrightarrow \boldsymbol{i}_2 \\ \mathbf{e}_1 &\leftrightarrow \mathbf{e}_2 \end{aligned}$$

Here, $\boldsymbol{i}_2$ is the pseudoscalar. Again the dual pairs give only the bases of the duals and not the transformation rules itself.

E^2 can be identified with C by multiplying all elements in E^2 with $\mathbf{e}_1$ or $\mathbf{e}_2$ from any side [14]. For $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \in E^2$ we define:

$$\tilde{\mathbf{a}} = \mathbf{e}_1(a_1\mathbf{e}_1 + a_2\mathbf{e}_2) = a_1 + a_2\mathbf{e}_1\mathbf{e}_2$$

The inverse operation is again multiplication with $\mathbf{e}_1$ from the left. For $\mathbf{a}$ being spinor valued it is indicated with $\tilde{\mathbf{a}}$, too. Note that $\tilde{\tilde{\mathbf{a}}} = \mathbf{a}$.

IV. CONVOLUTION

Every linear and shift invariant filter (LSI filter) on a scalar field can be described as a convolution with a filter. A lot of filters for smoothing images and for edge detection are LSI filter. Thus, the convolution is an important operation in image processing [8], [9]. Convolution and correlation are closely related as every correlation can be computed as a convolution with a suitably permutated filter. The convolution theorem states that a convolution in image space is equal to a multiplication in frequency space. Therefore, convolution is used more often than correlation.

A. Convolution and correlation of scalar fields

In image processing, a filter is a map from one image to another. For a continuous signal $f : E^d \to C$, the convolution with the filter $h : E^d \to C$ is defined by

$$(h * f)(\mathbf{x}) = \int_{E^d} h(\mathbf{x'}) f(\mathbf{x} - \mathbf{x'}) d\mathbf{x'}$$

The spatial correlation is defined by

$$(h \star f)(\mathbf{x}) = \int_{E^d} h(\mathbf{x'}) f(\mathbf{x} + \mathbf{x'}) d\mathbf{x'}$$

A convolution is just a correlation with a filter that has been reflected at it's center.

B. Clifford convolution

Now let $\mathbf{F}$ be a multivector field and $\mathbf{H}$ a multivector filter. The Clifford convolution is defined as

$$(\mathbf{H} *_l \mathbf{F})(\mathbf{x}) = \int_{E^d} \mathbf{H}(\mathbf{x'}) \mathbf{F}(\mathbf{x} - \mathbf{x'}) |d\mathbf{x'}|.$$

$$(\mathbf{F} *_r \mathbf{H})(\mathbf{x}) = \int_{E^d} \mathbf{F}(\mathbf{x} - \mathbf{x'}) \mathbf{H}(\mathbf{x'}) |d\mathbf{x'}|.$$

using the Clifford product of multivectors. As Clifford multiplication is not commutative, we distinguish between application of the filter from left and right. For discrete multivector fields, convolution has to be discretized. For 3D uniform grids and for the application of the filter from left, the discretization is:

$$(\mathbf{H} *_l \mathbf{F})_{j,k,l} = \sum_{s=-r}^{r} \sum_{t=-r}^{r} \sum_{u=-r}^{r} \mathbf{H}_{s,t,u} \mathbf{F}_{j-s,k-t,l-u}$$

with $j,k,l,s,t,u \in Z$. r^3 is the dimension of the grid of the filter and the (j,k,l) are grid nodes.

Clifford convolution is an extension of the convolution on scalar fields, but it is also an extension of the scalar convolution defined by Heiberg et al. [6]. Because of

$$(\mathbf{h} *_s \mathbf{f})(\mathbf{x}) = \int_{E^d} < \mathbf{h}(\mathbf{x'}), \mathbf{f}(\mathbf{x} - \mathbf{x'}) > d\mathbf{x'}$$

we get

$$(\mathbf{h} *_s \mathbf{f})(\mathbf{x}) = < (\mathbf{h} *_l \mathbf{f}) >_0 = < (\mathbf{f} *_r \mathbf{h}) >_0$$

for vector fields $\mathbf{h}, \mathbf{f}$.

C. Clifford correlation

The spatial Clifford correlation is defined analog to the Clifford Convolution:

$$(\mathbf{H} \star_l \mathbf{F})(\mathbf{x}) = \int_{E^d} \mathbf{H}(\mathbf{x'}) \mathbf{F}(\mathbf{x} + \mathbf{x'}) |d\mathbf{x'}|.$$

$$(\mathbf{F} \star_r \mathbf{H})(\mathbf{x}) = \int_{E^d} \mathbf{F}(\mathbf{x} + \mathbf{x'}) \mathbf{H}(\mathbf{x'}) |d\mathbf{x'}|.$$

Thus, it is a Clifford convolution with a filter whose positions have been reflected at its center. Let $\dot{\mathbf{H}}$ denote a thus modified version of $\mathbf{H}$ with $\dot{\mathbf{H}}(\mathbf{x}) = \mathbf{H}(-\mathbf{x})$ Then

$$(\mathbf{H} \star_m \mathbf{F})(\mathbf{x}) = (\dot{\mathbf{H}} *_m \mathbf{F})(\mathbf{x}), \; m = l, r \tag{2}$$

Another idea could be to mirror not only the positions of the filter but the multivectors as well. This would need a special treatment for the different grades of the multivector as the scalar part stays the same but the vector part has to be negated, and so on. Furthermore, the relation between convolution and correlation would be much more difficult.

For pattern matching, one is actually interested in a correlation of pattern and dataset and not in a convolution. As the filter can be adapted easily for the computation of the correlation via convolution, the correlation itself will not be regarded in detail.

V. Clifford Fourier transform

The Fourier transform is a basis transform from image space to frequency space. Thus, images can be analyzed in frequency space where it is easier to describe features like orientation, phase, frequency and curvature. Filter responses can often be better analyzed in the frequency domain because of the convolution theorem. Transferring the Fourier transform to vector fields in combination with the Clifford convolution thus opens up a whole new approach for analyzing vector fields.

A. Fourier transform on scalar fields

For continuous signals $f, h : E^d \to C$, the Fourier transform of f is defined as

$$\mathscr{F}\{f\}(\mathbf{u}) = \int_{E^d} f(\mathbf{x}) e^{(-2\pi i < \mathbf{x}, \mathbf{u} >)} d\mathbf{x}$$

with $i^2 = -1$ provided the integral does exist. The inverse transform is

$$\mathscr{F}^{-1}\{f\}(\mathbf{u}) = \int_{E^d} f(\mathbf{x}) e^{(2\pi i < \mathbf{x}, \mathbf{u} >)} d\mathbf{x}$$

A discussion of the existence of $\mathscr{F}$, can be found by Bracewell [1]. $\mathscr{F}$ exists if

$$\int_{E^d} |f(\mathbf{x})| e^{(-2\pi i < \mathbf{x}, \mathbf{u} >)} d\mathbf{x} < \infty.$$

This is not always fulfilled though the integral of $\mathscr{F}$ might exist nonetheless. The convolution theorem is

$$\mathscr{F}\{h * f\}(\mathbf{u}) = \mathscr{F}\{h\}(\mathbf{u})\mathscr{F}\{f\}(\mathbf{u})$$

and the derivation theorems are

$$\mathscr{F}(\nabla f)(\mathbf{x}) = 2\pi i\mathbf{u}\mathscr{F}\{f\}(\mathbf{u})$$

and

$$\mathscr{F}(\Delta f)(\mathbf{x}) = -4\pi^2\mathbf{u}^2\mathscr{F}\{f\}(\mathbf{u}).$$

B. Clifford Fourier transform in 3D

We define the Clifford Fourier transform for a multivector valued function $\mathbf{F}: E^3 \to G_3$ and vectors $x, x', u \in E^3$ as

$$\mathscr{F}\{\mathbf{F}\}(\mathbf{u}) = \int_{E^3} \mathbf{F}(\mathbf{x}) e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)} |d\mathbf{x}|. \qquad (3)$$

The inverse transformation is

$$\mathscr{F}^{-1}\{\mathbf{F}\}(\mathbf{x}) = \int_{E^3} \mathbf{F}(\mathbf{u}) e^{(2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)} |d\mathbf{u}|. \qquad (4)$$

Note that the Clifford Fourier kernel $e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)}$ is multivector valued. To be more exact, it consists of a scalar and a pseudoscalar. The Clifford Fourier transform is a linear combination of several classical Fourier transforms:

A multivector field $\mathbf{F}: E^3 \to G_3$,

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_0 + \mathbf{F}_1\mathbf{e}_1 + \mathbf{F}_2\mathbf{e}_2 + \mathbf{F}_3\mathbf{e}_3 \\ &\quad + \mathbf{F}_{23}\mathbf{e}_{23} + \mathbf{F}_{31}\mathbf{e}_{31} + \mathbf{F}_{12}\mathbf{e}_{12} + \mathbf{F}_{123}\mathbf{e}_{123}\end{aligned}$$

can be regarded as four complex signals which are transformed separately with the usual complex Fourier transformation:

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= [\mathbf{F}_0(\mathbf{x}) + \mathbf{F}_{123}(\mathbf{x})\boldsymbol{i}_3]1 \\ &+ [\mathbf{F}_1(\mathbf{x}) + \mathbf{F}_{23}(\mathbf{x})\boldsymbol{i}_3]\mathbf{e}_1 \\ &+ [\mathbf{F}_2(\mathbf{x}) + \mathbf{F}_{31}(\mathbf{x})\boldsymbol{i}_3]\mathbf{e}_2 \\ &+ [\mathbf{F}_3(\mathbf{x}) + \mathbf{F}_{12}(\mathbf{x})\boldsymbol{i}_3]\mathbf{e}_3\end{aligned}$$

is understood as an element of C^4. With the linearity of the Clifford Fourier transform we get

$$\begin{aligned}\mathscr{F}\{\mathbf{F}\}(\mathbf{u}) &= [\mathscr{F}\{\mathbf{F}_0(\mathbf{x}) + \mathbf{F}_{123}(\mathbf{x})\boldsymbol{i}_3\}(\mathbf{u})]1 \\ &+ [\mathscr{F}\{\mathbf{F}_1(\mathbf{x}) + \mathbf{F}_{23}(\mathbf{x})\boldsymbol{i}_3\}(\mathbf{u})]\mathbf{e}_1 \\ &+ [\mathscr{F}\{\mathbf{F}_2(\mathbf{x}) + \mathbf{F}_{31}(\mathbf{x})\boldsymbol{i}_3\}(\mathbf{u})]\mathbf{e}_2 \\ &+ [\mathscr{F}\{\mathbf{F}_3(\mathbf{x}) + \mathbf{F}_{12}(\mathbf{x})\boldsymbol{i}_3\}(\mathbf{u})]\mathbf{e}_3\end{aligned}$$

Note that dual pairs form Fourier pairs. This is a basis transform as the classical Fourier transform is a basis transform and the multivector space is divided into four orthogonal spaces which are then transformed separately. Because of (1), the 3D Clifford Fourier kernel commutes with every multivector though Clifford multiplication is not generally commutative. All of the well-known theorems hold. Because of the non commutativity of the Clifford multiplication, we present theorems for the application of a filter from left and right.

Theorem 5.1 (Shift theorem): Let $\mathbf{F} : E^3 \to G_3$ be multivector valued and let $\mathscr{F}\{\mathbf{F}\}$ exist. Then

$$\mathscr{F}\{\mathbf{F}(\mathbf{x} - \mathbf{x'})\}(\mathbf{u}) = \mathscr{F}\{\mathbf{F}\}(\mathbf{u}) e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x'},\mathbf{u}>)}$$

Theorem 5.2 (Convolution theorem): Let $\mathbf{F}, \mathbf{H}: E^3 \to G_3$ be multivector valued and let $\mathscr{F}\{\mathbf{F}\}$ and $\mathscr{F}\{\mathbf{H}\}$ exist. Then

$$\mathscr{F}\{\mathbf{H} *_l \mathbf{F}\}(\mathbf{u}) = \mathscr{F}\{\mathbf{H}\}(\mathbf{u})\mathscr{F}\{\mathbf{F}\}(\mathbf{u})$$

$$\mathscr{F}\{\mathbf{F} *_r \mathbf{H}\}(\mathbf{u}) = \mathscr{F}\{\mathbf{F}\}(\mathbf{u})\mathscr{F}\{\mathbf{H}\}(\mathbf{u})$$

Theorem 5.3 (Derivation theorem): Let the preconditions be as in 5.2. Then

$$\begin{aligned}\mathscr{F}\{\nabla\mathbf{F}\}(\mathbf{u}) &= 2\pi\boldsymbol{i}_3\mathbf{u}\mathscr{F}\{\mathbf{F}\}(\mathbf{u}) \\ \mathscr{F}\{\Delta\mathbf{F}\}(\mathbf{u}) &= -4\pi^2\mathbf{u}^2\mathscr{F}\{\mathbf{F}\}(\mathbf{u}) \\ \mathscr{F}\{\mathbf{F}\nabla\}(\mathbf{u}) &= \mathscr{F}\{\mathbf{F}\}(\mathbf{u})2\pi\boldsymbol{i}_3\mathbf{u} \\ \mathscr{F}\{\mathbf{F}\Delta\}(\mathbf{u}) &= -4\pi^2\mathbf{u}^2\mathscr{F}\{\mathbf{F}\}(\mathbf{u})\end{aligned}$$

Theorem 5.4 (Parseval's theorem): Let the preconditions be as in 5.1. Then

$$||\mathbf{F}||_2 = ||\mathscr{F}\{\mathbf{F}\}||_2$$

This is also true for the different grades of the multivector valued function $\mathbf{F}$:

$$|| < \mathbf{F} >_j ||_2 = ||\mathscr{F}\{< \mathbf{F} >_j\}||_2, \; j = 0,..,3$$

C. Clifford Fourier transform in 2D

We define the Clifford Fourier transform for multivector valued functions $\mathbf{F}: E^2 \to G_2$ and vectors $\mathbf{x}, \mathbf{x'}, \mathbf{u} \in E^2$ as

$$\mathscr{F}\{\mathbf{F}\}(\mathbf{u}) = \int_{E^2} \mathbf{F}(\mathbf{x}) e^{(-2\pi \boldsymbol{i}_2 <\mathbf{x},\mathbf{u}>)} |d\mathbf{x}| \qquad (5)$$

The inverse transformation is

$$\mathscr{F}^{-1}\{\mathbf{F}\}(\mathbf{x}) = \int_{E^2} \mathbf{F}(\mathbf{u}) e^{(2\pi \boldsymbol{i}_2 <\mathbf{x},\mathbf{u}>)} |d\mathbf{u}| \qquad (6)$$

A multivector field $\mathbf{F}: E^2 \rightarrow G_2$,

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1\mathbf{e}_1 + \mathbf{F}_2\mathbf{e}_2 + \mathbf{F}_{12}\mathbf{e}_{12}$$

can be regarded as two complex signals which are transformed separately with the usual complex Fourier transformation:

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= 1[\mathbf{F}_0(\mathbf{x}) + \mathbf{F}_{12}(\mathbf{x})\boldsymbol{i}_2] \\ &+ \mathbf{e}_1[\mathbf{F}_1(\mathbf{x}) + \mathbf{F}_2(\mathbf{x})\boldsymbol{i}_2]\end{aligned}$$

is understood as an element of C^2. With the linearity of the Clifford Fourier transform we get

$$\begin{aligned}\mathscr{F}\{\mathbf{F}\}(\mathbf{u}) &= 1[\mathscr{F}\{\mathbf{F}_0(\mathbf{x}) + \mathbf{F}_{12}(\mathbf{x})\boldsymbol{i}_2\}(\mathbf{u})] \\ &+ \mathbf{e}_1[\mathscr{F}\{\mathbf{F}_1(\mathbf{x}) + \mathbf{F}_2(\mathbf{x})\boldsymbol{i}_2\}(\mathbf{u})]\end{aligned}$$

That means that the 2D Clifford Fourier transform is the linear combination of two classical Fourier transforms. Again, dual pairs form Fourier pairs.

Again all of the well-known theorems hold. This time, the Fourier kernel does not commute with every multivector but it commutes with the spinor part and anticommutes with the vector part. Therefore we present convolution theorems for vector and spinor valued fields separately. Note also that the multiplication of the Fourier kernel from the right is not quite the same as from the left as in 3D.

Theorem 5.5 (Shift theorem): Let $\mathbf{F}$ be multivector valued and let $\mathscr{F}\{\mathbf{F}\}$ exist. Then

$$\mathscr{F}\{\mathbf{F}(\mathbf{x}-\mathbf{x'})\}(\mathbf{u}) = \mathscr{F}\{\mathbf{F}\}(\mathbf{u})e^{(-2\pi\boldsymbol{i}_2 <\mathbf{x'},\mathbf{u}>)}$$

Theorem 5.6 (Convolution theorem): Let $\mathbf{F},\mathbf{H}$ be multivector valued, $\mathbf{f},\mathbf{h}$ be vector valued and $\boldsymbol{f},\boldsymbol{h}$ be spinor valued. Let $\mathscr{F}\{\mathbf{F}\}$, $\mathscr{F}\{\mathbf{H}\}$, $\mathscr{F}\{\mathbf{f}\}$, $\mathscr{F}\{\mathbf{h}\}$, $\mathscr{F}\{\boldsymbol{f}\}$ and $\mathscr{F}\{\boldsymbol{h}\}$ exist. Then

$$\mathscr{F}\{\mathbf{H} *_l \boldsymbol{f}\}(\mathbf{u}) = \mathscr{F}\{\mathbf{H}\}(\mathbf{u})\mathscr{F}\{\boldsymbol{f}\}(\mathbf{u})$$

$$\mathscr{F}\{\mathbf{H} *_l \mathbf{f}\}(\mathbf{u}) = \mathscr{F}\{\dot{\mathbf{H}}\}(\mathbf{u})\mathscr{F}\{\mathbf{f}\}(\mathbf{u})$$

$$\mathscr{F}\{\mathbf{F} *_r \boldsymbol{h}\}(\mathbf{u}) = \mathscr{F}\{\mathbf{F}\}(\mathbf{u})\mathscr{F}\{\boldsymbol{h}\}(\mathbf{u})$$

$$\mathscr{F}\{\mathbf{F} *_r \mathbf{h}\}(\mathbf{u}) = \mathscr{F}\{\dot{\mathbf{F}}\}(\mathbf{u})\mathscr{F}\{\dot{\mathbf{h}}\}(\mathbf{u})$$

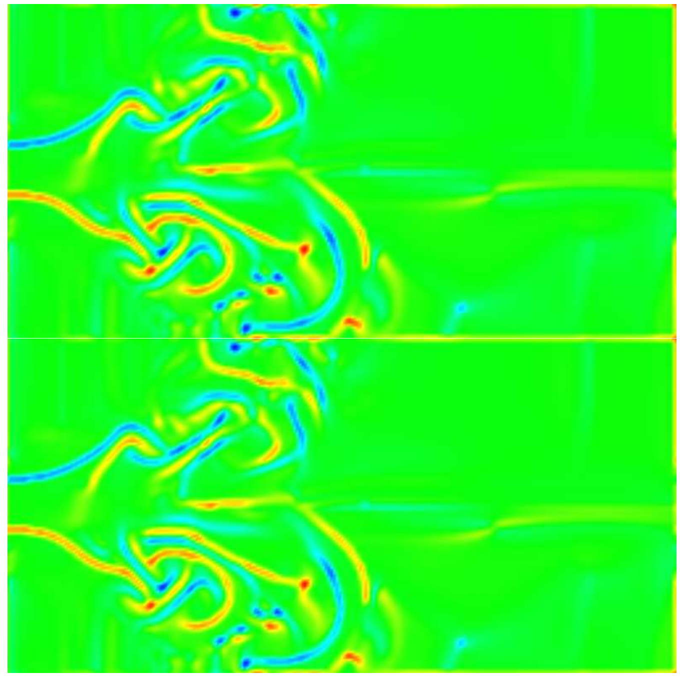

Fig. 3. Convolution of the swirling jet data set with a 5×5 rotational pattern. The data set has been normalized beforehand to emphasize the structures. Top: Clifford convolution in spatial domain. Bottom: Convolution computed in frequency domain, the result transformed back to spatial domain. Both: Color coding of the scalar part. Deep blue indicates high negative similarities and a left-handed rotation and deep red high positive similarities and a right-handed rotation.

Theorem 5.7 (Derivation theorem): Let the preconditions be as in 5.6. Then

$$\begin{aligned}\mathscr{F}\{\nabla\mathbf{f}\}(\mathbf{u}) &= -2\pi\boldsymbol{i}_2 u\mathscr{F}\{\mathbf{f}\}(\mathbf{u}) \\ \mathscr{F}\{\nabla\boldsymbol{f}\}(\mathbf{u}) &= 2\pi\boldsymbol{i}_2 u\mathscr{F}\{\boldsymbol{f}\}(\mathbf{u}) \\ \mathscr{F}\{\mathbf{F}\nabla\}(\mathbf{x}) &= \mathscr{F}\{\mathbf{F}\}(\mathbf{u}) - 2\pi\boldsymbol{i}_2 u \\ \mathscr{F}\{\Delta\mathbf{F}\}(\mathbf{u}) &= 4\pi^2u^2\mathscr{F}\{\mathbf{F}\}(\mathbf{u}) \\ \mathscr{F}\{\mathbf{F}\Delta\}(\mathbf{u}) &= 4\pi^2u^2\mathscr{F}\{\mathbf{F}\}(\mathbf{u})\end{aligned}$$

Theorem 5.8 (Parseval's theorem): Let the preconditions be as in 5.5. Then

$$||\mathbf{F}||_2 = ||\mathscr{F}\{\mathbf{F}\}||_2$$

D. Discrete Clifford Fourier transformation

As the Clifford Fourier transform is a linear combination of several classical Fourier transforms, properties of the classical Fourier transform like sampling theorem and issues regarding discretization and periodicity can be transfered easily:

Theorem 5.9 (Sampling theorem): Let $\mathbf{F}$ be a multi-vector valued function sampled on a uniform grid with spacings Δx_w. Is the bandwidth of $\mathbf{F}$ bandlimited, that is

$$\mathbf{F} = 0 \, \forall |k_w| \geq k_{nyq},$$

then it can be reconstructed without error provided that $\frac{1}{\Delta x_w} > 2k_{nyq}$, where k_{nyq} is the Nyquist rate.

E. Scalar convolution

As the scalar convolution on vector fields as given by Heiberg et al. [6] is part of the Clifford convolution, we can analyze it within this context, too. But the theorems for scalar convolution and correlation are not as simple as those for Clifford convolution and correlation. For example, we regard convolution in 3D. Let $\mathbf{f},\mathbf{h} : E^3 \to E^3 \subset G_3$ be two vector fields. As

$$(\mathbf{h} *_s \mathbf{f})(\mathbf{x}) = <(\mathbf{h} *_l \mathbf{f})>_0$$

and $<\mathbf{h} * \mathbf{f}>_3 = 0$, we get

$$\mathscr{F}\{(\mathbf{h} *_s \mathbf{f})\}(\mathbf{u}) = <\mathscr{F}\{\mathbf{h}\},\mathscr{F}\{\mathbf{f}\}> + <\mathscr{F}\{\mathbf{h}\}\mathscr{F}\{\mathbf{f}\}>_3$$

Since the Clifford Fourier transforms of 3D vector fields contain a vector and a bivector part, $<\mathscr{F}\{\mathbf{h}\}\mathscr{F}\{\mathbf{f}\}>_3$ is not generally zero. In figure 4, one can see that there are substantial vector and bivector parts, even for typical vector pattern.

F. Fast transform

One of the reasons the Fourier transform is so successful in image processing is the existence of fast Fourier transforms (FFT). Algorithms for the fast computation of the Fourier transform take a divide and conquer approach based on recursively dividing even and odd elements. The basic approach assumes that the dimensions of the images are of the form 2^k. There exist algorithms based on the decomposition into the prime factors of the dimensions of the images, too. As the Clifford Fourier Transform can be computed as a linear combination of several regular Fourier transforms, FFT algorithms can be applied directly for acceleration of the CFT. The Clifford Fourier transform of figures 1 and 3 has been computed using a fast Clifford Fourier transform. Thus, the computational time of the CFT of this data set was reduced from hours to mere seconds.

VI. Analysis of basic vector valued pattern

We created some patterns by using discretized half waves over the whole length of the patterns in one direction in the coordinates of the vector field. Thus, we got rotation, convergence, divergence and saddle points in 2D and rotation in one coordinate plane, convergence and divergence in 3D. Some of these patterns and their DCFT's are shown in figure 4 and 5. Note the discretization artifacts of the rotation in 3D. Their formulas are:

$$\text{Rotation (3D)} \quad : f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -sin(2\pi z) \\ sin(2\pi y) \end{pmatrix}$$

$$\text{Convergence (3D)} : f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -sin(2\pi x) \\ -sin(2\pi y) \\ -sin(2\pi z) \end{pmatrix}$$

$$\text{Rotation (2D)} \quad : f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} sin(2\pi y) \\ -sin(2\pi x) \end{pmatrix}$$

$$\text{Convergence (2D)} : f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -sin(2\pi x) \\ -sin(2\pi y) \end{pmatrix}$$

$$\text{Saddle point (2D)} : f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} sin(2\pi y) \\ sin(2\pi x) \end{pmatrix}$$

These patterns are all defined on $[-1,1]^2$ or $[-1,1]^3$ respectively. The same patterns can also be defined on $[-k,k]^2$ or $[-k,k]^3$ with these formulas by replacing x, y and z with x/k, y/k and z/k where the real number $2k+1$ is the overall size of the patterns in x,y,z-direction respectively. Subsequently, the patterns can been multiplied with a Gaussian window function to reduce the influence of values with the distance to the center of the patterns. Thus the values approach zero at the borders.

A swirl can be created by adding a homogeneous flow to the rotation in one coordinate plane in 3D. In frequency domain we get the summation of the DCFT's of homogeneous flow and rotation, too (figure 5). Thus, it is reasonable to use only rotation in one coordinate plane as a filter kernel for potential vortices. Source and sink are divergence from or convergence to a point, sometimes overlaid with a rotation. Thus, they can be build from the patterns above.

VII. Results

As a vector valued test data set we chose a turbulent swirling jet entering a fluid at rest. The simulation considers a cylinder and a planar cut along the axis of the

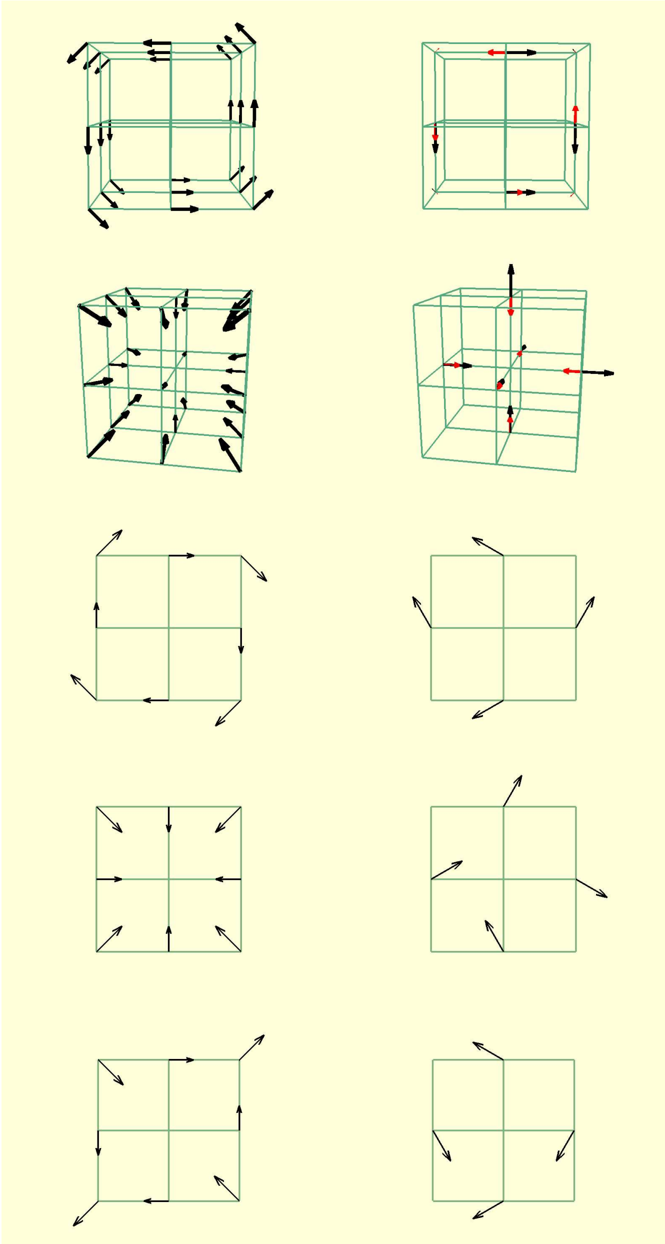

Fig. 4. Some patterns (left) and their DCFT (right). Top to bottom: $3\times3\times3$ rotation in one coordinate plane, $3\times3\times3$ convergence, 3×3 rotation, 3×3 convergence, 3×3 saddle point. The mean value of the DCFT is situated in the center of the field. Vectors are drawn in black and bivectors as normal vectors of the bivector planes in red. The results of the 2D DCFT looks a bit confusing at first as vectors are mapped to vectors. In 3D, the half waves forming the pattern can be seen easily in frequency domain.

cylinder was used as domain. The domain is discretized by a 124×101 rectilinear grid with smaller rectangles towards the axis of the cylinder. Since a lot of small and large scale vortices are present in the flow, a discrete numerical simulation (DNS) using a higher order finite difference scheme is used to solve the incompressible Navier-Stokes equations. For demonstration of a fast

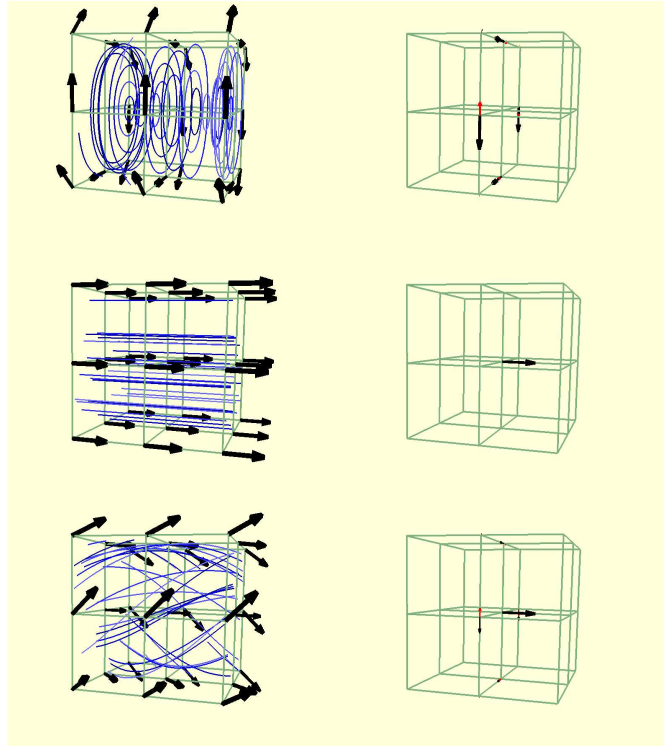

Fig. 5. Rotation in one coordinate plane and homogeneous flow sum to a swirl in space and frequency domain. Masks (left) and their DCFT (right). Top to bottom: $3\times3\times3$ rotation in one coordinate plane, $3\times3\times3$ homogeneous flow, $3\times3\times3$ swirl. The mean value of the DCFT is situated in the center of the field. Vectors are drawn in black and bivectors as normal vectors of the bivector planes in red. Streamlines are drawn in blue for further understanding of the flow of the patterns.

Clifford Fourier transform, we did a uniform resampling of the grid with new dimensions of $256\times128=2^8\times2^7$. The resampled data set is smaller in height than the original one. On the resampled field, a fast Clifford Fourier transformation was applied. Figure 1 shows the absolute values of the original and the transformed field where the zeroth Clifford Fourier coefficient is drawn in the center of the image. A 2D vector field transforms into a 2D vector field as it forms one complex signal. A 3D vector field transforms into a multivector field with only vector and bivector part unequal zero as vector and bivector form 3 complex signals.

The use of the fast CFT can also speed up the convolution computation of large data sets. Figure 3 shows pattern matching with a 5×5 rotational pattern of the normalized swirling jet data set, first computed directly in spatial domain and then using convolution theorem and fast CFT. The results of the matching were color coded. Red denotes high positive similarities and corresponds to right handed rotation and blue high negative similarity values and left handed rotations. Some timings of the convolution computation on a 1,6 GHz

Intel centrino:

computation	size of pattern	time (sec)
spatial domain	5×5	13
spatial domain	15×15	55
via fast CFT	5×5	28
via fast CFT	15×15	28

VIII. Conclusion

Convolution and Fourier transform are basic tools for image processing. Many robust and well known techniques for image analysis and feature extraction of scalar fields are based on them. It is a basic idea to apply those tools to vector fields from flow visualization, too. Therefore the Clifford convolution on (multi-)vector fields has been recently introduced [3]. In this article we took the next step of applying image processing to vector fields by extending the Fourier transform to multivector fields.

We presented the Clifford Fourier transform which is an extension of the classical Fourier transform on scalar fields to (multi-)vector fields in 2D and 3D. We proved convolution, correlation and derivation theorems for the Clifford convolution on multivector fields which has been introduced by us in an earlier paper [3]. The theorems extend those of the Fourier transform on scalar fields while still remaining reasonably simple. Furthermore, there exist fast algorithms for the computation of the Clifford Fourier transform. Thus, the computation of the Clifford convolution and related pattern matching algorithms can be accelerated.

We showed that typical vector valued pattern for pattern matching in flow fields like rotation, convergence, divergence and saddle points can be created by using half waves in the coordinates of the filter kernel. Therefore, it is even more reasonable to analyze vector fields in frequency domain.

IX. Future work

One question is how to best apply the CFT to data sets defined on irregular grids as these are common in flow visualization. Of course, the data sets could be resampled but this introduces interpolation artefacts and can substantially increase the amount of data. Therefore, other approaches should be tried, too. There exits algorithms for resampling of the field in Fourier domain, e.g. [16]. These should be tested on flow vis data.

Fourier transform is a global transformation with all of the drawbacks. So the next step will be to use Gabor filter on multivector fields as a localized Fourier transform. Gabor filter are located best in frequency and spatial domain at the same time. They are used for local frequency and orientation estimation on scalar fields. They might also provide a way to circumvent a global resampling of data sets defined on irregular grid.

The Clifford Fourier transform will be used for the analysis of discretization, sampling, measurement and interpolation errors. It will also be used to analyze the behavior of smoothing and differential operators, which play an important role in feature detection on flow fields.

Acknowledgments

First of all, we like to thank the members of the FAnToM development team at University of Kaiserslautern, especially Tom Bobach, Christoph Garth, David Gruys, Kai Hergenröther, Max Langbein, Michael Schlemmer and Xavier Tricoche, for their help with programming and production of the pictures. We further acknowledge the fruitful discussions in the whole computer graphics group at Kaiserslautern. Further thanks go to Prof. Krüger, FB Physik, Kaiserslautern for his help with Clifford algebra. We thank Prof. Kollmann, MAE department, University of California at Davis, for producing the swirling jet data set. Last but not least we thank the unknown reviewers for their suggestions and questions which enriched this work.

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Julia Ebling received her Master’s degree in computer science in 2003 from the University of Kaiserslautern. She worked as a reasearch assistant at the University of Kaiserslautern for about a year. Currently, she is working as a research assistant in the Computer Science Department of the University of Leipzig. Her research interests include image processing, Clifford algebra and scientifi c visualization.

Gerik Scheuermann received the BS and MS degrees in mathematics in 1995 from the University of Kaiserslautern. In 1999, he received a PhD degree in computer science, also from the University of Kaiserslautern. During 1995-1997, he conducted research at Arizona State University for about a year. He has worked as a postdoctoral researcher at the Center for Image Processing and Integrated Computing (CIPIC) at the University of California at Davis in 1999 and 2000. Between 2001 and 2004, he was an assistant professor for computer science at the University of Kaiserslautern. Currently, he is professor in the Computer Science Department of the University of Leipzig. His research topics include algebraic geometry, topology, Clifford algebra, image processing, graphics, and scientifi c visualization. Dr. Scheuermann is a member of ACM, IEEE and GI.

Appendix

A. Proofs (3D)

Shift Theorem (5.1):

$$\begin{aligned}
&\mathcal{F}\{\mathbf{F}(\mathbf{x}-\mathbf{x'})\}(\mathbf{u})\\
&= \int_{E^3} \mathbf{F}(\mathbf{x}-\mathbf{x'})e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|\\
&= \int_{E^3} \mathbf{F}(\mathbf{b})e^{(-2\pi \boldsymbol{i}_3 <\mathbf{b},\mathbf{u}>)}e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x'},\mathbf{u}>)}|d\mathbf{b}|\\
&= \int_{E^3} \mathbf{F}(\mathbf{b})e^{(-2\pi \boldsymbol{i}_3 <\mathbf{b},\mathbf{u}>)}|d\mathbf{b}|e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x'},\mathbf{u}>)}\\
&= \mathcal{F}\{\mathbf{F}\}(\mathbf{u})e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x'},\mathbf{u}>)}
\end{aligned}$$

Convolution Theorem (5.2):

$$\begin{aligned}
&\mathcal{F}\{\mathbf{H} *_l \mathbf{F}\}(\mathbf{u})\\
&= \int_{E^3}(\int_{E^3} \mathbf{H}(\mathbf{x'})\mathbf{F}(\mathbf{x}-\mathbf{x'})|d\mathbf{x'}|)e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|\\
&= \int_{E^3}(\int_{E^3} \mathbf{H}(\mathbf{x'})\mathbf{F}(\mathbf{x}-\mathbf{x'})e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)}|d\mathbf{x'}|)|d\mathbf{x}|\\
&= \int_{E^3}(\int_{E^3} \mathbf{H}(\mathbf{x'})\mathbf{F}(\mathbf{x}-\mathbf{x'})e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|)|d\mathbf{x'}|\\
&= \int_{E^3} \mathbf{H}(\mathbf{x'})(\int_{E^3} \mathbf{F}(\mathbf{x}-\mathbf{x'})e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|)|d\mathbf{x'}|\\
&= \int_{E^3} \mathbf{H}(\mathbf{x'})e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x'},\mathbf{u}>)}\mathcal{F}\{\mathbf{F}\}(\mathbf{u})|d\mathbf{x'}|\\
&= \int_{E^3} \mathbf{H}(\mathbf{x'})e^{(-2\pi \boldsymbol{i}_3 <\mathbf{x'},\mathbf{u}>)}|d\mathbf{x'}|\mathcal{F}\{\mathbf{F}\}(\mathbf{u})\\
&= \mathcal{F}\{\mathbf{H}\}(\mathbf{u})\mathcal{F}\{\mathbf{F}\}(\mathbf{u})
\end{aligned}$$

Because of the commutativity of the Clifford Fourier kernel (1), the proof of the other cases is analog to the one above.

Derivation theorem (5.3):
As

$$\begin{aligned}
&(\nabla f)(\mathbf{x})\\
&= \nabla \mathcal{F}^{-1}\{\mathcal{F}\{\mathbf{F}\}\}(\mathbf{x})\\
&= \nabla \int_{E^3} \mathcal{F}\{\mathbf{F}\}(\mathbf{u})e^{(2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|\\
&= \int_{E^3} \nabla(\mathcal{F}\{\mathbf{F}\}(\mathbf{u})e^{(2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)})|d\mathbf{x}|\\
&= \int_{E^3} \nabla(e^{(2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)})\mathcal{F}\{\mathbf{F}\}(\mathbf{u})|d\mathbf{x}|\\
&= \int_{E^3} 2\pi \boldsymbol{i}_3 \mathbf{u} e^{(2\pi \boldsymbol{i}_3 <\mathbf{x},\mathbf{u}>)}\mathcal{F}\{\mathbf{F}\}(\mathbf{u})|d\mathbf{x}|\\
&= \mathcal{F}^{-1}(2\pi \boldsymbol{i}_3 \mathbf{u}\mathcal{F}\{\mathbf{F}\}(\mathbf{u}))
\end{aligned}$$

we get

$$\mathcal{F}\{\nabla \mathbf{F}\}(\mathbf{u}) = 2\pi \boldsymbol{i}_3 \mathbf{u}\mathcal{F}\{\mathbf{F}\}(\mathbf{u})$$

and

$$\mathscr{F}\{\Delta\mathbf{F}\}(\mathbf{u}) = 2\pi\boldsymbol{i}_3\mathbf{u}\mathscr{F}\{\nabla\mathbf{F}\}(\mathbf{u}) = -4\pi^2\mathbf{u}^2\mathscr{F}\{\mathbf{F}\}(\mathbf{u})$$

The application of the derivation from the right can be proved analogous.

Parseval's theorem (5.4):
The theorem for the classical Fourier transform is $||f||_2 = ||\mathscr{F}\{f\}||_2$. The proof for the Clifford Fourier transform follows directly out of this one and the CFT being a linear combination of several classical Fourier transforms.

B. Proofs (2D)

Shift theorem (5.5) and Parseval's theorem (5.8) can be proved analog to the 3D case (5.1, 5.4). The proof of the convolution theorem (5.6) is:

Let $\boldsymbol{f}$ be spinor valued.

$$\begin{aligned}
&\mathscr{F}\{\mathbf{H} *_l \boldsymbol{f}\}(\mathbf{u})\\
&= \int_{E^2}(\int_{E^2}\mathbf{H}(\mathbf{x'})\boldsymbol{f}(\mathbf{x}-\mathbf{x'})|d\mathbf{x'}|)e^{(-2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|\\
&= \int_{E^2}(\int_{E^2}\mathbf{H}(\mathbf{x'})\boldsymbol{f}(\mathbf{x}-\mathbf{x'})e^{(-2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}|d\mathbf{x'}|)|d\mathbf{x}|\\
&= \int_{E^2}\mathbf{H}(\mathbf{x'})(\int_{E^2}\boldsymbol{f}(\mathbf{x}-\mathbf{x'})e^{(-2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|)|d\mathbf{x'}|\\
&= \int_{E^2}\mathbf{H}(\mathbf{x'})\mathscr{F}\{\boldsymbol{f}\}(\mathbf{u})e^{(-2\pi\boldsymbol{i}_2<\mathbf{x'},\mathbf{u}>)}|d\mathbf{x'}|\\
&= \int_{E^2}\mathbf{H}(\mathbf{x'})e^{(-2\pi\boldsymbol{i}_2<\mathbf{x'},\mathbf{u}>)}|d\mathbf{x'}|\mathscr{F}\{\boldsymbol{f}\}(\mathbf{u})\\
&= \mathscr{F}\{\mathbf{H}\}(\mathbf{u})\mathscr{F}\{\boldsymbol{f}\}(\mathbf{u})
\end{aligned}$$

Let $\mathbf{f}$ be vector valued. We get

$$\begin{aligned}
&\mathscr{F}\{\mathbf{H} \star_l \mathbf{f}\}(\mathbf{u})\\
&= \int_{E^2}(\int_{E^2}\mathbf{H}(\mathbf{x'})\mathbf{f}(\mathbf{x}+\mathbf{x'})|d\mathbf{x'}|)e^{(-2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|\\
&= \int_{E^2}(\int_{E^2}\mathbf{H}(\mathbf{x'})\mathbf{f}(\mathbf{x}+\mathbf{x'})e^{(-2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}|d\mathbf{x'}|)|d\mathbf{x}|\\
&= \int_{E^2}\mathbf{H}(\mathbf{x'})(\int_{E^2}\mathbf{f}(\mathbf{x}+\mathbf{x'})e^{(-2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|)|d\mathbf{x'}|\\
&= \int_{E^2}\mathbf{H}(\mathbf{x'})\mathscr{F}\{\mathbf{f}\}(\mathbf{u})e^{(2\pi\boldsymbol{i}_2<\mathbf{x'},\mathbf{u}>)}|d\mathbf{x'}|\\
&= \int_{E^2}\mathbf{H}(\mathbf{x'})e^{(-2\pi\boldsymbol{i}_2<\mathbf{x'},\mathbf{u}>)}\mathscr{F}\{\mathbf{f}\}(\mathbf{u})|d\mathbf{x'}|\\
&= \int_{E^2}\mathbf{H}(\mathbf{x'})e^{(-2\pi\boldsymbol{i}_2<\mathbf{x'},\mathbf{u}>)}|d\mathbf{x'}|\mathscr{F}\{\mathbf{f}\}(\mathbf{u})\\
&= \mathscr{F}\{\mathbf{H}\}(\mathbf{u})\mathscr{F}\{\mathbf{f}\}(\mathbf{u})
\end{aligned}$$

and therefore

$$\mathscr{F}\{\mathbf{H} *_l \mathbf{f}\}(\mathbf{u}) = \mathscr{F}\{\dot{\mathbf{H}}\}(\mathbf{u})\mathscr{F}\{\mathbf{f}\}(\mathbf{u})$$

The other cases of the convolution theorem can be prooved analogous.

The derivation theorem (5.7): For $\boldsymbol{f}$ being spinor valued the proof of the derivation theorem is analogous to the one of (5.3). Let $\mathbf{f}$ be vector valued. We get

$$\begin{aligned}
&(\nabla\mathbf{f})(\mathbf{x})\\
&= \nabla\mathscr{F}^{-1}\{\mathscr{F}\{\mathbf{f}\}\}(\mathbf{x})\\
&= \nabla\int_{E^2}\mathscr{F}\{\mathbf{f}\}(\mathbf{u})e^{(2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|\\
&= \int_{E^2}\nabla(e^{(-2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}\mathscr{F}\{\mathbf{f}\}(\mathbf{u}))|d\mathbf{x}|\\
&= \int_{E^2}\nabla(e^{(-2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)})\mathscr{F}\{\mathbf{f}\}(\mathbf{u})|d\mathbf{x}|\\
&= \int_{E^2}-2\pi\boldsymbol{i}_3ue^{(-2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}\mathscr{F}\{\mathbf{f}\}(\mathbf{u})|d\mathbf{x}|\\
&= \int_{E^2}-2\pi\boldsymbol{i}_3u\mathscr{F}\{\mathbf{f}\}(\mathbf{u})e^{(2\pi\boldsymbol{i}_2<\mathbf{x},\mathbf{u}>)}|d\mathbf{x}|\\
&= \mathscr{F}^{-1}(-2\pi\boldsymbol{i}_2u\mathscr{F}\{\mathbf{f}\}(\mathbf{u}))
\end{aligned}$$

and

$$\mathscr{F}\{\nabla\mathbf{f}\}(\mathbf{u}) = -2\pi\boldsymbol{i}_2u\mathscr{F}\{\mathbf{f}\}(\mathbf{u}).$$

As ∇ can be regarded as a vector and thus anticommutes with the 2D Clifford Fourier kernel, the derivation theorem for the application of the derivative from right can be prooved analogous. This time, we have

$$\mathscr{F}\{\Delta\mathbf{F}\}(\mathbf{u}) = \mathscr{F}\{\mathbf{F}\nabla\}(\mathbf{u}) = 4\pi^2u^2\mathscr{F}\{\boldsymbol{F}\}(\mathbf{u})$$

as the frequency u is vector valued and thus anticommutes with $\boldsymbol{i}_2$.

C. Proof (Sampling theorem)

The sampling theorem holds for f being complex valued. As $\mathbf{F}$ can be understood as a linear combination of two respectively four complex signals which are sampled on the same grid and the Clifford Fourier transform is a linear combination of two respectively four classical Fourier transforms, the sampling theorem for multivector valued functions follows directly out of the sampling theorem of complex valued functions.