TABLE I
NUMBER OF MULTIPLIERS AND NOISE GAIN, σ. (a) $H_1 = H_2 = A/D$. (b) $H_1 = A(A/D)$, $H_2 = 1/D$.

	Powell & Chau		$H_1 = A\,(A/D)$, $H_2 = 1/D$				
	$H_1 = H_2 = A/D$			A/D	A	$1/D$	$A(A/D)(1/D)$
example	multipliers	σ (dB)	multipliers	σ (dB)	σ (dB)	σ (dB)	σ (dB)
1	15	21.9	19	21.1	4.8	78.0	78.1
2	21	20.6	27	19.0	6.0	61.6	61.9
3	21	20.0	27	17.8	6.0	43.2	43.6
4	33	31.0	43	28.5	7.8	131.6	131.6
5	33	26.0	43	23.0	7.8	86.2	86.5
6	21	20.9	27	17.4	6.0	39.9	40.5
7	15	19.1	19	15.2	4.8	31.9	32.6
8	39	38.6	51	34.4	8.5	206.8	206.8
9	21	29.4	27	24.9	6.0	112.8	112.9

the complexity of the cascade realizations of $1/D(z)$ and $B(z)$ (a single block can be used for each second-order section [4]). Thus, the number of multiplier blocks of the new realization is approximately the same as that corresponding in [1], and the complexity of the new realization can be the same as in [1]. The number of multipliers in allpass sections can be also reduced; half of the multipliers can be implemented with a shifter and an adder or a shifter only [5].

Since $B(z)/D(z)$ is realized as two different filters [$B(z)$ and $1/D(z)$], the quantization noise due to multiplication is increased, as shown in Table I. The very high quantization noise of the filter $1/D(z)$ can be reduced by appropriate selection of transfer function [6]. In addition, by increasing the wordlength in the last section only, the quantization noise is reduced, and it can be made lower than the noise caused by truncation to L-sample segments.

III. CONCLUSION

In this correspondence, a new improvement to the realization of the linear-phase IIR filters is described. It is based on the rearrangement of the numerator polynomials of two IIR filter functions that are used in the real-time realizations in [1] and [3]. The new realization has better total harmonic distortion when sine input is used and smaller phase error due to finite section length. It enables shorter sample delay for the same phase error or lower phase error and THD improvement for the same sample delay. The considerable improvement in phase response and lower truncation noise are obtained at the expense of a slightly increased number of multipliers and increased wordlength.

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Generalized Digital Butterworth Filter Design

Ivan W. Selesnick and C. Sidney Burrus

***Abstract*—This correspondence introduces a new class of infinite impulse response (IIR) digital filters that unifies the classical digital Butterworth filter and the well-known maximally flat FIR filter. New closed-form expressions are provided, and a straightforward design technique is described. The new IIR digital filters have more zeros than poles (away from the origin), and their (monotonic) square magnitude frequency responses are maximally flat at $\omega = 0$ and at $\omega = \pi$. Another result of the correspondence is that for a specified cut-off frequency and a specified number of zeros, there is only one valid way in which to split the zeros between $z = -1$ and the passband. This technique also permits continuous variation of the cutoff frequency. IIR filters having more zeros than poles are of interest because often, to obtain a good tradeoff between performance and implementation complexity, just a few poles are best.**

I. INTRODUCTION

The best known and most commonly used method for the design of IIR digital filters is probably the bilinear transformation of the classical analog filters (the Butterworth, Chebyshev I and II, and Elliptic filters) [9]. One advantage of this technique is the existence of formulas for these filters. However, the numerator and denominator of such IIR filters have equal degree. It is sometimes desirable to be able to design filters having more zeros than poles (away from the origin) to obtain an improved compromise between performance and implementation complexity.

The new formulas introduced in this correspondence unify the classical digital Butterworth filter and the well-known maximally flat FIR filter described by Herrmann [3]. The new maximally flat lowpass IIR filters have an unequal number of zeros and poles and possess a specified half-magnitude frequency. It is worth noting that not all the zeros are restricted to lie on the unit circle, as is the case for some previous design techniques for filters having an unequal number of poles and zeros. The method consists of the use of a formula and polynomial root finding. No phase approximation is done; the approximation is in the magnitude squared, as are the classical IIR filter types.

Another result of the correspondence is that for a specified number of zeros and a specified half-magnitude frequency, there is only one valid way to divide the number of zeros between $z = -1$ and the

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TABLE I
NOTATION

Parameters	
$L+M$	total number of zeros
L	number of zeros at $z=-1$
M	number of zeros contributing to the passband
N	total number of poles
ω_o	half-magnitude frequency
x_o	$\frac{1}{2}(1-\cos\omega_o)$
Flatness	
$L+M+N$	total degrees of flatness
$M+N$	degree of flatness at $\omega=0$
L	degree of flatness at $\omega=\pi$

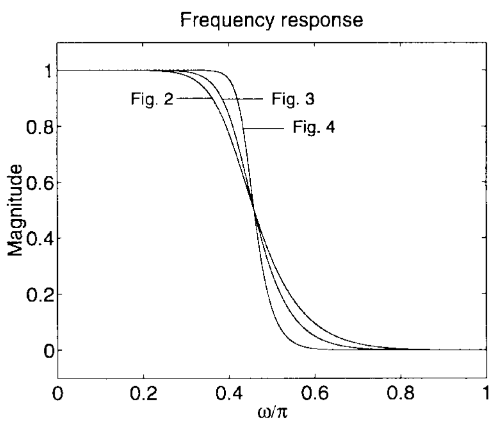

Fig. 1. Magnitudes of the three digital IIR filters shown in Figs. 2–4.

passband. The correspondence also describes how to construct a table from which it is simple to determine the correct way in which to split the zeros between these two bands.

Given a half-magnitude frequency ω_o, the filters produced by the formulas described below are optimal (maximally flat) in the following sense: The maximum number of derivatives at $\omega=0$ and $\omega=\pi$ are set to zero under the constraint that the filter possesses the half-magnitude frequency ω_o and a monotonic frequency response magnitude. The classical digital Butterworth filter and the well-known maximally flat FIR filter [3], [5], [6], [20], [23] are both special cases of the filters produced by the formulas given in this paper.

Several authors have addressed the design and the advantages of IIR filters with an unequal number of (nontrivial) poles and zeros. While [14] and [22] give formulas for IIR filters with Chebyshev stopbands having more zeros than poles, these methods require that all zeros lie on the unit circle. This restriction limits the range of achievable cutoff frequencies. In [4], Jackson notes that the use of just two poles "is often the most attractive compromise between computational complexity and other performance measures of interest." In [13], Saramäki discusses the tradeoffs between numerator and denominator order and describes an iterative algorithm in which zeros are not constrained to lie on the unit circle for the design of filters having Chebyshev stopbands. In [12] and [13], Saramäki finds that the classical Elliptic and Chebyshev filter types are seldom the best choice.

II. NOTATION

Let $H(z)=B(z)/A(z)$ denote the transfer function of a digital filter. Its frequency response magnitude is given by $|H(e^{j\omega})|$. Throughout this correspondence, the degree of $B(z)$ will be denoted by $L+M$, where L is the number of zeros at $z=-1$, and M is

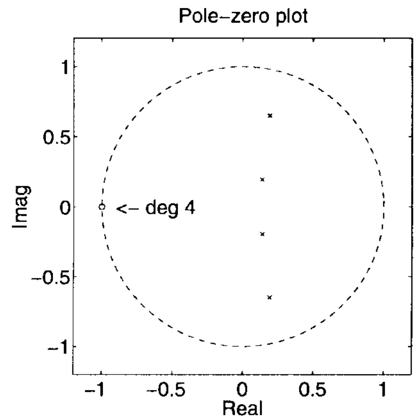

Fig. 2. $L=4$, $M=0$, $N=4$.

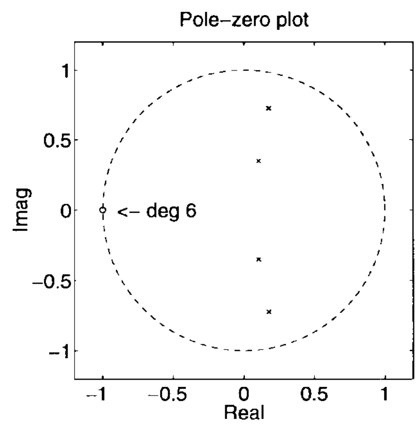

Fig. 3. $L=6$, $M=0$, $N=4$. The poles at the origin are not shown in the figure.

the number of remaining zeros. The zeros at $z=-1$ produce a flat behavior in the frequency response magnitude at $\omega=\pi$, whereas the remaining zeros, together with the poles, are used to produce a flat behavior at $\omega=0$. The half-magnitude frequency is that frequency at which the magnitude equals one half. Like the 3 dB point, it indicates the location of the transition band. The meanings of the parameters are shown in Table I. It should be noted that by "degree of flatness," we mean the number of derivatives that are made to match the desired response, including the zeroth derivative.

III. EXAMPLES

The classical digital Butterworth filters (defined by $L=N$ and $M=0$) are special cases of the filters discussed in this paper. Figs. 1 and 2 illustrate a classical digital Butterworth filter of order 4 ($L=4$, $M=0$, $N=4$). The first generalization of the classical digital Butterworth filter described below permits L to be greater than N, with $M=0$. Fig. 3 illustrates an IIR filter with $L=6$, $M=0$, $N=4$. It was designed to have the same half-magnitude frequency. It turns out that when $L>N$, the restriction that M equal zero limits the range of achievable half-magnitude frequencies, as will be elaborated upon below. This motivates the second generalization. In addition to permitting L to be greater than N, the second generalization permits M to be greater than zero: $L>N$ and $M>0$. Fig. 4 illustrates an IIR filter with $L=16$, $M=7$, $N=4$.

As mentioned above, for a specified half-magnitude frequency ω_o and specified numerator and denominator degrees, there is only one

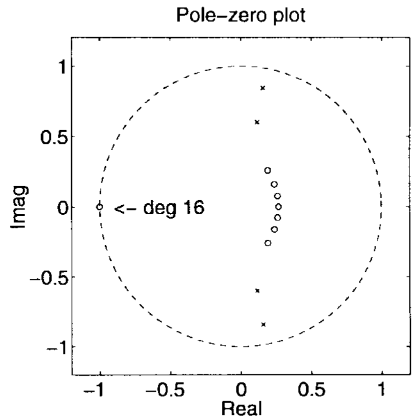

Fig. 4. $L = 16$, $M = 7$, $N = 4$. The poles at the origin are not shown in the figure.

TABLE II
FOR THE CHOICE L, M, AND N SHOWN IN THE TABLE, THE INTERVAL OF ACHIEVABLE HALF-MAGNITUDE FREQUENCIES ω_o IS GIVEN BY $[\omega_{\min}, \omega_{\max}]$. $L + M$ IS THE NUMERATOR DEGREE (NUMBER OF ZEROS), AND N IS THE DENOMINATOR DEGREE (NUMBER OF POLES)

$L+M$	L	M	N	ω_{min}/π	ω_{max}/π
4	4	0	4	0	1
5	5	0	4	0	0.5349
	4	1	4	0.5349	1
6	6	0	4	0	0.4620
	5	1	4	0.4620	0.6017
	4	2	4	0.6017	1
7	7	0	4	0	0.4140
	6	1	4	0.4140	0.5299
	5	2	4	0.5299	0.6446
	4	3	4	0.6446	1

correct way to split the zeros between $z = -1$ and the passband. To illustrate this property, it is helpful to construct a table that indicates the appropriate values for L, M, and N. When $N = 4$ and $L + M$ is varied from 4 to 7, Table II gives the values L and M required to achieve a desired half-magnitude frequency. As can be seen from the table, the intervals cover the interval (0,1) and do not overlap. This will be true, in general, as long as at least one pole is used. In the FIR case, the intervals cover an interval (a, b) with $a > 0$ and $b < 1$. (Neither the passband nor the stopband can be arbitrarily narrow). Notice that in the case of the classical Butterworth filter $(L + M = N)$, M equals zero, regardless of the specified half-magnitude frequency. As will be explained below, these intervals can be unambiguously computed by inspecting the roots of an appropriate set of polynomials.

To illustrate the tradeoffs that can be achieved with the generalized Butterworth filters described in this correspondence, it is useful to examine a set of filters all of which have the same half-magnitude frequency and the same total number of poles and zeros $(L+M+N)$. For example, when $L+M+N$ is fixed at 20 and the half-magnitude ω_o is fixed at 0.6π, the filters shown in Fig. 5 are obtained. The number of poles of the filters in this figure vary from 0 to 10 in steps of 2. It is interesting to measure the slope of the magnitude $|H(e^{j\omega})|$ at the half-magnitude frequency. The figure shows the negative reciprocal of the slope of $|H(e^{j\omega})|$ at ω_o—this indicates the approximate width of the transition band. Notice from Table III and Fig. 5 that for this example, as the number of poles and zeros become more equal, the slope of the magnitude at ω_o becomes more negative, and the transition region becomes sharper. However, it is interesting to note that the improvement in magnitude is greatest when the number of poles is increased from 0 to 2.

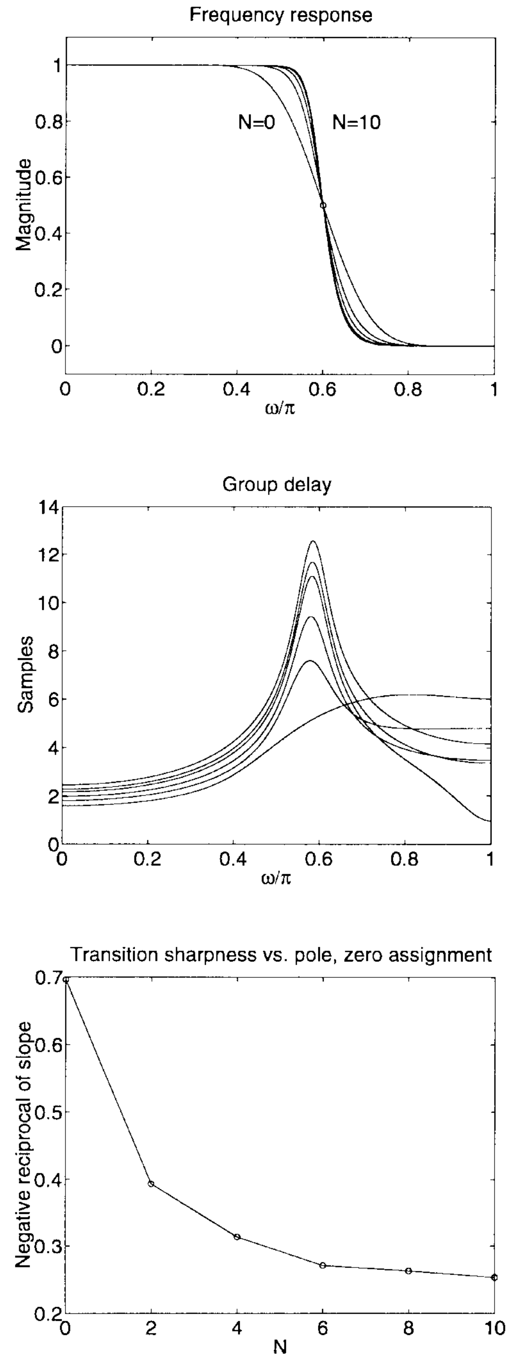

Fig. 5. Generalized Butterworth filters. $L + M + N = 20$, $\omega_o = 0.6\pi$. N is varied from 0 to 10 in increments of 2. $N = 10$ corresponds to the filter having the steepest transition and the most peaked group delay. The values of L, M, and N are shown in Table III.

IV. DESIGN FORMULAS

The approach described below uses the mapping $x = \frac{1}{2}(1 - \cos\omega)$ and provides formulas for two non-negative polynomials $P(x)$ and $Q(x)$. A stable IIR filter $B(z)/A(z)$ is obtained having a magnitude squared frequency response $|H(e^{j\omega})|^2$ given by

$$|H(e^{j\omega})|^2 = \frac{P\left(\frac{1}{2} - \frac{1}{2}\cos\omega\right)}{Q\left(\frac{1}{2} - \frac{1}{2}\cos\omega\right)}$$

TABLE III
FOR THE HALF-MAGNITUDE FREQUENCY $\omega_o = 0.6\pi$ AND $L + M + N = 20$, THE TABLE SHOWS THE CORRECT VALUES OF L AND M AND THE DERIVATIVE OF THE MAGNITUDE RESPONSE AT ω_o. THE FILTER RESPONSES ARE SHOWN IN FIG. 5

L	M	N	$\|H(e^{j\omega_o})\|'$
8	12	0	-1.4366
8	10	2	-2.5410
8	8	4	-3.1869
8	6	6	-3.6882
9	3	8	-3.8012
10	0	10	-3.9430

TABLE IV
PERMISSIBLE RANGES FOR c FOR THE FIRST GENERALIZATION

N even	$c \geq 0$
N odd	$c \geq \binom{L-1}{N}$

as in [3]. Accordingly, $F(x) = P(x)/Q(x)$ is designed to approximate a lowpass response over $x \in [0, 1]$. $B(z)$ and $A(z)$ are most conveniently found by first computing the roots of $P(x)$ and $Q(x)$ and by then mapping those roots to the z plane via

$$z = 1 - 2x \pm \sqrt{1 - 2x - 1}. \quad (1)$$

For stable minimum-phase solutions, take the sign of the square root yielding points inside the unit circle. We begin with the classical digital Butterworth filter. This establishes notation and makes the generalization more clear.

A. Classical Digital Butterworth Filter

Assume $L \leq N$ and $M = 0$; then, the rational function $F(x) = P(x)/Q(x)$ is given by

$$F(x) = \frac{(1-x)^L}{(1-x)^L + cx^N}. \quad (2)$$

The classical Butterworth filter is obtained when $N = L$. Note that $|H(e^{j\pi/2})|^2 = F(1/2) = 1/(1 + c \cdot 2^{L-N})$. Clearly, c should be chosen so that this value lies between 0 and 1. Therefore, c must be greater than zero.

To choose c to achieve a specified half-magnitude frequency is straightforward. The equation $|H(e^{j\omega_o})| = 1/2$ becomes $F(x_o) = 1/4$, where $x_o = \frac{1}{2}(1 - \cos\omega_o)$. Solving this equation for c, we get $c = 3(1 - x_o)^L/x_o^N$. Because this expression is positive for all $x_o \in (0, 1)$, any half-magnitude $\omega_o \in (0, \pi)$ is achievable when $L \leq N$ and $M = 0$.

B. First Generalization

For the first generalization, assume that $L > N$ and that $M = 0$. Then, introducing the notation $\mathcal{T}_N$ for polynomial truncation (discarding all terms beyond the Nth power), $F(x)$ can be written as

$$F(x) = \frac{(1-x)^L}{\mathcal{T}_N\{(1-x)^L\} + cx^N}. \quad (3)$$

The term c is the free parameter that, as in the classical case, can be chosen to achieve a specified half-magnitude frequency and must be chosen to lie within an appropriate range. The allowable ranges for c are given in Table IV. When c is chosen to lie in the ranges shown in the table, then $0 < F(x) < 1$ for $x \in (0, 1)$. See [16] for a proof.

To choose c to achieve a specified half-magnitude frequency ω_o, solve $F(x_o) = 1/4$ for c. This yields

$$c = \frac{4(1-x_o)^L - \mathcal{T}_N\{(1-x)^L\}(x_o)}{x_o^N}. \quad (4)$$

TABLE V
NUMBER AND LOCATIONS OF THE REAL ROOTS OF $\mathcal{T}_N\{(1-x)^L\} + cx^N - 4(1-x)^L$ FOR $L > N > 0$

	L even	L odd
N even $c = 0$	2 real roots: $x_1 \in (0,1), x_2 > 1$	1 real root: $x_1 \in (0,1)$
N odd $c = \binom{L-1}{N}$	2 real roots: $x_1 \in (0,1), x_2 = 1$	3 real roots: $x_1 \in (0,1), x_2 = 1, x_3 > 1$

The value this expression gives for c may or may not lie in the appropriate range, as shown in Table IV. If it does not, then the specified half-magnitude frequency is too high for the current choice of L and N. It should be noted that although the passband can be made arbitrarily narrow, it cannot be made arbitrarily wide for a fixed L and N (when $L > N$).

The greatest half-magnitude frequency achievable for a fixed L and N can be found by setting c equal to the appropriate value shown in Table IV and solving (4) for x_o. It is seen that x_o is a root of the polynomial

$$\mathcal{T}_N\{(1-x)^L\} + cx^N - 4(1-x)^L = 0. \quad (5)$$

Note that x_o should lie in $(0, 1)$. When $L > N > 0$, this polynomial has exactly one real root in $(0, 1)$; see [16] for a proof. The number and locations of the real roots of (5) are given in Table V.

Example: For $L = 6$ and $N = 4$, the boundary value for c from Table IV is 0 (N is even); therefore, the polynomial equation (5) becomes $\mathcal{T}_4\{(1-x)^6\} - 4(1-x)^6 = 0$. It roots are $3.9476, 0.3798 \pm 1.1659\jmath, 0.4262 \pm 0.3245\jmath, 0.4404$. Therefore, for this choice of L and N, x_o must lie in $(0, 0.4404]$ so that ω_o must lie in $(0, 0.4620\pi]$. To obtain filters having wider passbands with the same number of zeros and (nontrivial) poles, it is necessary to move at least one zero from $x = 1$ ($z = -1$) to the passband.

C. Second Generalization

For the second generalization, assume that $L > N$ and that $M > 0$. The zeros lying off the unit circle are used to obtain a higher degree of flatness at $\omega = 0$. Such a filter is shown in Fig. 4. In this case, $F(x)$ is given by

$$F(x) = \frac{(1-x)^L(R(x) + cT(x))}{\mathcal{T}_N\{(1-x)^L(R(x) + cT(x))\}} \quad (6)$$

where $R(x)$ and $T(x)$ are given in Table VI. Table VI also provides expressions for $(1-x)^L R(x)$ and $(1-x)^L T(x)$. These polynomials are such that the numerator of $F(x) - 1$ is divisible by x^{M+N}. Again, the free parameter c can be chosen to precisely position the location of the transition band. However, c must lie in the ranges shown in Table VII. (When N is even, for example, the positive endpoint of this interval is that point beyond which $F(x)$ is no longer monotonic—and the negative endpoint of this interval is that point beyond which $F(x)$ is no longer non-negative.)

To choose c to achieve a specified half-magnitude frequency, solve $F(x_o) = 1/4$ for c. This yields

$$c = \frac{4(1-x_o)^L R(x_o) - \mathcal{T}_N\{(1-x)^L R(x)\}(x_o)}{\mathcal{T}_N\{(1-x)^L T(x)\}(x_o) - 4(1-x_o)^L T(x_o)}. \quad (7)$$

The value this expression gives for c may or may not lie in the appropriate range given by Table VII. If it does not, then the specified half-magnitude frequency is either too high or too low for the current choice of L, M, and N—it is necessary to alter the distribution of zeros between $x = 1$ ($z = -1$) and the passband.

TABLE VI
AUXILIARY POLYNOMIALS. FOR NEGATIVE VALUES OF n, THE CONVENTION [11], $\binom{n+k-1}{k} = (-1)^k \binom{-n}{k}$ FOR $k \geq 0$ IS USED. IN ADDITION, NOTE THAT $\binom{n}{k} = 0$ FOR $n \geq 0,\ k < 0$ AND THAT $\binom{n}{k} = 0$ FOR $n \geq 0,\ k > n$

$R(x) = \sum_{k=0}^{M-1} \binom{M+N-k-1}{N} \binom{L-N+k-1}{k} x^k$
$T(x) = x \sum_{k=0}^{M-1} \binom{M+N-k-2}{N-1} \binom{L-N+k}{k} x^k$
$S(x) = \sum_{k=0}^{M} \binom{M+N-k}{N} \binom{L-N+k-1}{k} x^k$
$(1-x)^L R(x) = \sum_{k=0}^{L+M-1} \binom{M+N-k-1}{M-1} \binom{L+M-1}{k} (-x)^k$
$(1-x)^L T(x) = x \sum_{k=0}^{L+M-1} \binom{M+N-k-2}{M-1} \binom{L+M-1}{k} (-x)^k$
$(1-x)^L S(x) = \sum_{k=0}^{L+M} \binom{M+N-k}{M} \binom{L+M}{k} (-x)^k$

TABLE VII
PERMISSIBLE RANGES FOR c FOR THE SECOND GENERALIZATION

N even	$-1 \leq c \leq \frac{L-N}{M+N}$
N odd	$\frac{L-N}{N} \leq c$

For fixed L, M, and N, the minimum and maximum permissible values of the half-magnitude frequency ω_o can be computed by

i) setting c to the values in Table VII;
ii) solving (7) for x
iii) using $\omega = \arccos\,(1-2x)$.

When c is finite, it is seen that x is a root of the polynomial

$$\mathcal{T}_N\{(1-x)^L(R(x)+cT(x))\} - 4(1-x)^L(R(x)+cT(x)) = 0. \tag{8}$$

Note that when N is odd, c can be chosen to be arbitrarily large. Letting c approach infinity, we get, instead of (8), the polynomial

$$\mathcal{T}_N\{(1-x)^L T(x)\} - 4(1-x)^L T(x) = 0. \tag{9}$$

Therefore, for both even and odd N, the range of achievable half-magnitude frequencies can be found by computing the roots of appropriate polynomials. It was found that each relevant polynomial has exactly one real root in the interval (0,1); therefore, there is no ambiguity regarding root selection. A table similar to Table V indicating the number and the location of the real roots of the relevant polynomials is given in [16].

D. Special Values

For fixed values N and $L+M$, as the specified frequency ω_o is varied over $(0, \pi)$, the values of L and M must be varied according to a table such as Table II. For the boundary values of ω_o (for example, $\omega_o = 0.5349$ when $L+M = 5$ and $N = 4$), an extra degree of flatness is achieved when N is even. For those filters, the rational function $F(x)$ is given by

$$F(x) = \frac{(1-x)^L S(x)}{\mathcal{T}_N\{(1-x)^L S(x)\}} \tag{10}$$

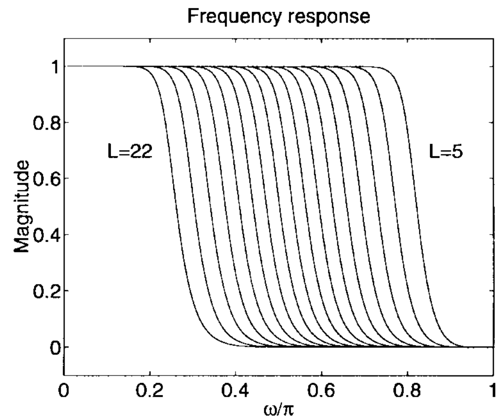

Fig. 6. Generalized Butterworth filters for special values of ω_o. $L+M = 22$, $N = 4$. L is varied from 5 to 21. The widest band filter corresponds to $L = 5$.

where $S(x)$ is given in Table VI. The exact location of the half-magnitude frequency is entirely determined by the parameters L, M, and N. Fixing $L+M = 22$ and $N = 4$, the frequency response magnitudes of the filters obtained using (10), as L is varied from 5 to 21, are shown in Fig. 6.

The FIR solution obtained, when $N = 0$, is a special case well established in the literature. When $N = 0$, the function (10) specializes to

$$F(x) = (1-x)^L \sum_{k=0}^{M} \binom{L+k-1}{k} x^k \tag{11}$$

which was given by Herrmann in [3] for the design of symmetric (Type 1) FIR filters. It is worth noting that recently, formulas for all four types of symmetric FIR filters have been given [1].

When $L = M+N+1$, with N even, the function (10) is useful in the design of IIR orthogonal wavelets with a maximal number of vanishing moments [2], [17]. In this case, the transfer function $H(z)$ obtained from (10) satisfies $H(z)H(1/z) + H(-z)H(-1/z) = 1$, which is an equation that is central to the design of orthogonal two-channel filter banks and orthogonal wavelet bases.

V. FURTHER REMARKS

To summarize, the design procedure described above requires three parameters.

- the denominator degree (N);
- the numerator degree $(L+M)$;
- the half-magnitude frequency (ω_o).

By making a table such as Table II, the way to split the number of zeros between $z = -1$ and the passband (L and M) can be determined. The corresponding formulas can then be used to compute $F(x)$. After polynomial root finding and the mapping (1), the filter coefficients can be obtained. To clarify the design process presented in this paper, we list the steps.

1) Specify the numerator and denominator degrees of $H(z)$ and the frequency ω_o.
2) Construct a specification table, like Table II, using the equations discussed above.
3) Locate ω_o in the specification table. This gives L and M individually—thereby indicating how to split the zeros between $z = -1$ and the passband.
4) Use formulas given above to construct the rational function $F(x) = P(x)/Q(x)$.
5) Compute roots of $P(x)$ and $Q(x)$.

TABLE VIII
EXPRESSION FOR $F(x)$ GIVES THE MAGNITUDE SQUARED FUNCTION IN THE x DOMAIN IN TERMS OF A CONSTANT c. WHEN c IS CHOSEN ACCORDING TO THE EXPRESSION GIVEN IN THE TABLE, $F(x_o)$ EQUALS $1/4$

$F(x)$	c	Case
$\dfrac{(1-x)^L}{(1-x)^L + cx^N}$	$3\dfrac{(1-x_o)^L}{x_o^N}$	$L \le N, M = 0$
$\dfrac{(1-x)^L}{\mathcal{T}_N\{(1-x)^L\} + cx^N}$	$\dfrac{4(1-x_o)^L - \mathcal{T}_N\{(1-x)^L\}(x_o)}{x_o^N}$	$L > N, M = 0$
$\dfrac{(1-x)^L(R(x)+cT(x))}{\mathcal{T}_N\{(1-x)^L(R(x)+cT(x))\}}$	$\dfrac{4(1-x_o)^L R(x_o) - \mathcal{T}_N\{(1-x)^L R(x)\}(x_o)}{\mathcal{T}_N\{(1-x)^L T(x)\}(x_o) - 4(1-x_o)^L T(x_o)}$	$L > N, M > 0$
$\dfrac{(1-x)^L S(x)}{\mathcal{T}_N\{(1-x)^L S(x)\}}$	no variation of ω_o	$L > N, M > 0, N$ even

6) Map roots to z-plane via (1).
7) Compute coefficients by forming polynomials from roots.

Using a specification table like Table II in conjunction with the formulas, the half-magnitude frequency ω_o can be varied continuously in the interval $(0, \pi)$. If desired, a frequency other than the half-magnitude frequency can be specified. To specify a frequency ω_o for which $|H(e^{j\omega_o})| = H_o$ is possible for any H_o, $0 < H_o < 1$. The resulting design formulas differ only in that they contain slightly different constants. In addition, note that, although the examples illustrate minimum-phase solutions, nonminimum-phase solutions can also be obtained by reflecting "passband" zeros about the unit circle. This is equivalent to using different signs of the square root in (1).

Note that when N is odd, one of the poles must lie on the real line. When there are zeros that lie off the unit circle, in the passband $(M > 0)$, it is expected that the pole lying on the real line does little to contribute to the performance of the frequency response. This is indeed true. In some situations, a pole and a zero will lie close together on the real line and, depending on the specified half-magnitude frequency, almost cancel. For this reason, it is expected that generalized digital Butterworth filters having an odd number of poles, and passband zeros will be of little interest—they have been presented in this paper for completeness.

It should be noted that for the classical Butterworth filter, explicit solutions for the locations of the poles are known [9]. For the generalized case, however, it appears that the roots of $P(x)$ and $Q(x)$ must be found numerically. It should also be realized that a filter formed by cascading i) a classical Butterworth digital filter and ii) a maximally flat FIR digital filter is not optimal in the maximally flat sense in general. To obtain a true maximally flat solution, all the degrees of freedom must be considered together.

It is also worth noting that the classical Butterworth filter can be realized as a parallel sum of two allpass filters [24], which is a structure that has received much attention recently. The approach taken in this correspondence did not attempt to preserve this property; however, it is possible to obtain a quite different generalization of the Butterworth filter by structurally imposing this property [15]. Finally, if phase linearity is important and a maximally flat response is desired, then it is more appropriate to use symmetric FIR filters [1], nearly symmetric FIR filters (with reduced delay) [16], [19], or approximately linear-phase IIR filters [15].

VI. CONCLUSION

By using appropriate formulas, by computing polynomial roots, and by employing a transformation (1), maximally flat IIR filters having more zeros than poles (away from the origin) can be easily designed and without the restriction that all zeros lie on the unit circle. The technique presented allows for the continuous variation of the half-magnitude frequency. In addition, for fixed numerator and denominator degrees and a fixed half-magnitude frequency, the formulas above give a direct way of finding the correct way to split the number of zeros between $z = -1$ and the passband.

The maximally flat FIR filter described by Herrmann [3] and the classical Butterworth filter are special cases of the filters given by the formulas described in this paper. Table VIII gives a summary of the filter design formulas. Table VI gives auxiliary polynomials. An earlier version of this paper is [18]. A more detailed description is given in [16]. Matlab programs are available on the World Wide Web at URL *http://www.dsp.rice.edu/.*

APPENDIX
CONNECTION TO A SERIES OF GAUSS

The polynomials $R(x)$, $T(x)$, and $S(x)$ given in Table VI are special cases of the Gauss hypergeometric series [7] $\mathcal{F}(a, b; c; z)$, given by

$$\mathcal{F}(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{12}$$

where the pochhammer symbol[1] $(a)_k$ denotes the rising factorial $(a)_k = (a) \cdot (a+1) \cdot (a+2) \cdots (a+k-1)$. When a or b is a negative integer, $\mathcal{F}(a, b; c; z)$ is a polynomial. The polynomials $R(x)$, $T(x)$, and $S(x)$ can be written as

$$S(x) = \frac{(M+1)_N}{N!} \cdot \mathcal{F}(-M, L-N; -M-N; x) \tag{13}$$

$$R(x) = \frac{(M)_N}{N!} \cdot \mathcal{F}(1-M, L-N; 1-M-N; x) \tag{14}$$

$$T(x) = \frac{(M)_{N-1}}{(N-1)!} \cdot x \cdot \mathcal{F}(2-M, L-N-1; 2-M-N; x). \tag{15}$$

There are many recurrence formulas for the hypergeometric series; with them, recursion formulas for $R(x)$, $S(x)$, and $T(x)$ can be obtained. Those relationships may also facilitate the computation of the roots of the polynomials, as suggested in [8] and [21].

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Design of IIR Eigenfilters in the Frequency Domain

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***Abstract*—The eigenfilter approach is an appealing way of designing digital filters, mainly because of the simplicity of its implementation. In this correspondence, a new method of applying the eigenfilter approach to the design of infinite impulse response (IIR) filters is described. The procedure works in the frequency domain and yields the coefficients of a causal rational transfer function having an arbitrary number of poles and zeros. Some examples of filter design are given to show the effectiveness of the method presented.**

I. Introduction

The eigenfilter approach is a simple and flexible way of designing digital filters. The method consists of expressing the error between a target and a digital filter response as a real, symmetric, positive-definite quadratic form in the filter coefficients. The error can be referred either to the time or the frequency domain or to both of them. The eigenvector corresponding to the minimum eigenvalue yields the optimum filter coefficients according to the chosen error measure.

This method was introduced for least-squares design of a variety of linear-phase finite impulse response (FIR) digital filters in [1]. It has been extended to the case of FIR Hilbert transformers and digital differentiators in [2] and [3]. In [4], the eigenfilter approach has been applied to the design of FIR filters with an arbitrary frequency response not having, in general, a linear phase.

The design of IIR eigenfilters in the time domain has been addressed in [5]. The filter coefficients are found by approximating a target impulse response. The transfer function has the form $H(z) = H_1(z) + H_1(z^{-1})$, where $H_1(z)$ is stable and causal so that a noncausal implementation of the system is necessary. If only the magnitude of the filter frequency response is of interest, a causal system is achieved by substituting the poles outside the unit circle with their inverse conjugate; therefore, stable poles must be double. Moreover, the error weighting function operates in the time domain, making a different consideration of the passbands and of the stopbands more complex.

In [6] and [7], the eigenfilter approach is applied to the design of allpass sections with a given phase response. The method may also be used to design IIR filters whose transfer function $H(z)$ is the sum of two allpass sections [7], [8]; the two sections must be designed to be in phase in the passband and out of phase in the stopband of the filter. The degrees of the numerator and of the denominator of $H(z)$, however, are related to the degrees of the allpass sections composing the system and cannot be completely arbitrary. Examples of design methods for IIR filters (having equiripple frequency responses) with an arbitrary number of poles and zeros are given in [9]–[12]. In [12], the solution of an eigenvalue problem yields the filter coefficients, even though the classical eigenfilter approach, based on the Rayleigh's principle [1] and on the search for the minimum eigenvalue of a positive-definite matrix, is not used.

In this correspondence, a new and simple method based on the eigenfilter approach to design causal IIR filters with an arbitrary

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