The analytic classification of plane branches

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Abstract
An effective solution of the problem of analytic classification of plane branches is given. A finite stratification of any given equisingularity class of plane branches is determined and normal forms for each stratum are exhibited in such a way that two branches in normal form are easily recognized to be analytically equivalent or not. In this way, we solve the main problems proposed by O. Zariski (Le problème des modules pour les branches planes. Course given at the Centre de Mathématiques de L’École Polytechnique, Paris, October–November 1973).

1. Introduction
After analysing several examples throughout [11], Zariski concluded that the problem of constructing the moduli space associated to an equisingularity class of plane branches is hard to solve, since the set of analytic classes is badly behaved.

The most significant nontrivial related result after [11] was obtained by Delorme in [2]: he described the generic component of the moduli in the simplest case of equisingularity classes determined by one Puiseux pair and computed its dimension.

Our strategy to attack the classification problem is to reduce its complexity by stratifying each equisingularity class by means of a ‘good numerical invariant’ that separates branches into finitely many strata making analytic equivalence manageable. We use the set Λ of values of Kähler differentials on a branch as such numerical invariant to stratify the open set of the affine space representing the parameter space of an equisingularity class corresponding to Puiseux parametrizations with fixed characteristic exponents. Then we exhibit a family of special representatives for each stratum in normal form in such a way that analytic equivalence becomes trivial, allowing us to describe the moduli space of each stratum.

In addition to the tools used by Zariski in [11], we introduce in the game two computational techniques. The first is a SAGBI algorithm, due to Robbiano and Sweedler [9], which we adapted in [5, 6] to describe distinguished bases of the local rings of plane branches and of their modules of Kähler differentials, as well, allowing us to compute the set of invariants Λ.

The second technique is the algorithm of Complete Transversal due to Bruce, Kirk and du Plessis [1] that permits to determine all normal forms of map germs under Mather’s group actions, but does not, in general, predict the final result. The strength of our method stems from the combination of these two algorithms which, by means of the existence of certain differentials, allows us to control each step of the Complete Transversal algorithm, giving explicitly all possible normal forms and necessary and sufficient conditions for the analytic equivalence of germs in normal form.

All our results are effective in the sense that there is an efficient algorithm that puts any plane branch into normal form; furthermore, it is easy to recognize whether or not two plane
immersed germs of curves in \((\mathbb{C},\mathbb{C})\) study. We will adopt the analytic point of view to have a geometric interpretation of the objects we study.

Let \(C \subset (\mathbb{C}^2,0)\) be an irreducible plane curve germ defined by an equation \(f = 0\), where \(f \in \mathcal{M}_2\). Such curve germ will be called a plane branch and represented by \((f)\).

Two plane branches are said equisingular, if they are topologically equivalent as complex immersed germs of curves in \((\mathbb{C}^2,0)\). The set of all plane branches which are equisingular to each other will be called an equisingularity class. When the above equivalence among two branches \(C_1\) and \(C_2\) is not only topological, but also analytic, we write \(C_1 \sim C_2\) and say shortly that the branches are equivalent. Our main concern in this work is the analytic classification of plane branches within a given equisingularity class.

Every branch \(C\) admits a parametrization \(\varphi : (\mathbb{C},0) \to C\) which is bijective. We only consider such parametrizations.

It is well known, and easy to prove, that given two plane branches \(C_1\) and \(C_2\) parametrized, respectively, by \(\varphi_1\) and \(\varphi_2\), then \(C_1 \sim C_2\) if and only if \(\varphi_1\) and \(\varphi_2\) are \(\mathcal{A}\)-equivalent, writing \(\varphi_1 \sim_{\mathcal{A}} \varphi_2\), where \(\mathcal{A}\)-equivalence means that there exist germs of analytic isomorphisms \(\sigma\) and \(\rho\) of \((\mathbb{C}^2,0)\) and \((\mathbb{C},0)\), respectively, such that \(\varphi_2 = \sigma \circ \varphi_1 \circ \rho^{-1}\).

So, analytic classification of plane branches reduces to \(\mathcal{A}\)-classification of parametrizations, which we are going to undertake in this paper.

Any plane branch is known to be equivalent to a branch with a Puiseux parametrization: \(\varphi(t) = (x(t),y(t)) = (t^{\beta_0}, \sum_{i > \beta_1} a_i t^i)\), with \(a_i \in \mathbb{C}\), \(a_{\beta_i} = 1\), \(\beta_0 < \beta_1\), \(\beta_0 \neq \beta_1\) (cf. \([11]\)). We denote by \(\beta_0, \beta_1, \ldots, \beta_\gamma\) the characteristic exponents of \(\varphi\), and call \(\gamma\) the genus of \(\varphi\). We define \(n_0 = 1\), \(e_0 = \beta_0\) and for \(1 \leq i \leq \gamma\), \(n_i = e_{i-1}/e_i\), where \(e_i = \gcd(\beta_0, \ldots, \beta_i)\). It is a classical result that the characteristic exponents determine completely the equisingularity classes of plane branches.

We denote by \(v_\varphi\) the function from \(\mathcal{O}_2 \setminus \ker \varphi^*\) to \(\mathbb{N}\), defined by \(v_\varphi(h) = \ord_1(\varphi^*(h))\), where \(\varphi^* : \mathcal{O}_2 \to \mathcal{O}_1\) is the natural morphism defined by \(\varphi^*(h) = h \circ \varphi(t)\). The semigroup of values of \(\varphi\) is the semigroup \(\Gamma_\varphi = v_\varphi(\mathcal{O}_2 \setminus \ker \varphi^*)\) of the naturals. This semigroup will be represented by its minimal set of generators in increasing order as \(\Gamma_\varphi = \langle v_0, v_1, \ldots, v_\gamma \rangle\), where \(v_0 = \beta_0\) and \(v_1 = \beta_1\). When \(\varphi\) is a Puiseux parametrization, the characteristic exponents of \(\varphi\) and \(\Gamma_\varphi\) determine each other (cf. \([11]\)).

The conductor of the semigroup \(\Gamma_\varphi\) will be denoted by \(c\) and the elements of the finite set \(\mathbb{N} \setminus \Gamma_\varphi\) are called the gaps of \(\Gamma_\varphi\). If \(\varphi(t) = (x(t),y(t))\) and \(\varphi_1(t_1) = (x_1(t_1), y_1(t_1))\) are two Puiseux parametrizations, then in order to have \(\varphi_1(t_1) = \sigma \circ \varphi \circ \rho^{-1}(t_1)\), it is necessary and sufficient that

\[
\sigma(X,Y) = (r^{v_0}X + p, r^{v_0}Y + q) \quad \text{and} \quad t_1 = \rho(t) = rt \sqrt{1 + \frac{\varphi^*_0(p)}{r^{v_0}t^{v_0}}},
\]

(2.1)

where \(r \in \mathbb{C}^*\) and \(p, q \in \mathcal{O}_2\), with \(v_\varphi(p) > v_0\) and \(v_\varphi(q) > v_1\).

Notation as above, we have that \((t^{v_0}, y(t)) \sim_{\mathcal{A}} (t_1^{v_0}, y_1(t_1))\) if and only if

\[
y_1(t_1) = r^{v_0}(y(\rho^{-1}(t_1))) + q(\rho^{-1}(t_1))^{v_0}, \quad y(\rho^{-1}(t_1))).
\]

(2.2)

Ebey and Zariski (cf. \([3, 11]\)) gave the following elimination criteria (EC) of parameters.
Let $\varphi = (t^{v_0}, t^{v_1} + \sum_{i \geq v_1} a_it^i)$ be a Puiseux parametrization, and let $j > v_1$ be an integer. If one of the following conditions holds:

(EC1) $j \in \varphi$, or (EC2) $j + v_0 - v_1 \in \varphi$,

then $\varphi$ is $A$-equivalent to a parametrization $(t^{v_0}, t^{v_1} + \sum_{i > v_1} a'_it^i)$, with $a'_i = a_i$, when $i < j$, and $a'_j = 0$.

So, any Puiseux parametrization $\varphi = (t^{v_0}, \sum_{v_1 \leq i \leq c} a_it^i)$ is $A$-equivalent to the parametrization $(t^{v_0}, \sum_{v_1 \leq i < c} a_it^i)$, where $c$ is the conductor of $\varphi$.

Let $\Sigma^\lambda$ denote the set of all parametrizations $\varphi$ of the form $(t^{v_0}, \sum_{v_1 \leq i < c} a_it^i)$ such that $\Gamma_\varphi$ is equal to a given $\Gamma$. This set can be identified with an open set (the complement of the union of the hyperplanes $a_{\beta_i} = 0$, $i = 2, \ldots, \gamma$) of an affine space, whose points are the ordered sets of the coefficients of $y(t)$ (those which are not necessarily zero), excluding the coefficient $a_{\beta_1}$, which is taken to be 1. Therefore, we are reduced to classify, modulo $A$-equivalence, the parametrizations in the set $\Sigma^\lambda$, in order to classify analytically plane branches.

Zariski noticed in [11] that, in order to get more Elimination Criteria than the above ones, it was necessary to consider the module of Kähler differentials over the local ring of the branch, introducing new numerical analytic invariants.

Let $\Omega_2 = \{dhX + gdY; \ g, h \in \Omega_2\}$ and $\Omega_1 = \{\xi dt; \ \xi \in \Omega_1\}$ denote, respectively, the $\Omega_2$-free modules of germs of differentials at $(C^2, 0)$ and the $\Omega_1$-free module of germs of differentials at $(C, 0)$.

A parametrization $\varphi: (C, 0) \to C \subset (C^2, 0)$, $t \mapsto (x(t), y(t))$, induces a natural $\Omega_2$-modules homomorphism (an extension of the map $\varphi^*: \Omega_2 \to \Omega_1$):

$$
\varphi^*: \Omega_2 \to \Omega_1,
$$

$$
hdX + gdY \mapsto (\varphi^*(h)x'(t) + \varphi^*(g)y'(t))dt,
$$

where $x'(t)$ and $y'(t)$ stand for the derivatives of $x(t)$ and $y(t)$, respectively. A parametrization $\varphi$ also induces a function $v_\varphi$ on $\Omega_2 \setminus \ker \varphi^*$ with values in the naturals, defined by

$$
v_\varphi(hdX + gdY) = \text{ord}_t(\varphi^*(h)x'(t) + \varphi^*(g)y'(t)) + 1,
$$

We now define

$$
\Lambda_\varphi = v_\varphi(\Omega_2 \setminus \ker \varphi^*).$

The set $\Lambda_\varphi$ is $A$-invariant, as we will observe later, and will play a key role in our solution of the classification problem.

Since for all $h \in \mathcal{M}_2$ we have that $v_\varphi(dh) = v_\varphi(h)$, it follows that $\Gamma_\varphi \setminus \{0\} \subset \Lambda_\varphi$; and because $\Gamma_\varphi$ has a conductor, we have that the set $\Lambda_\varphi \setminus \Gamma_\varphi$, as a subset of the set of gaps of $\Gamma_\varphi$, is finite.

In [10], Zariski showed that $\Lambda_\varphi \setminus \Gamma_\varphi = \emptyset$ if and only if $\varphi$ is $A$-equivalent to the parametrization $(t^{v_0}, t^{v_1})$, where $\text{GCD}(v_0, v_1) = 1$.

Since $\Lambda_\varphi$ and $\Gamma_\varphi$ are $A$-invariant, it follows that if $\Lambda_\varphi \setminus \Gamma_\varphi \neq \emptyset$, then $\lambda = \min(\Lambda_\varphi \setminus \Gamma_\varphi) - v_0$ is an invariant under the $A$-action, called the Zariski invariant of $\varphi$. It is known (see [11]) that such a $\varphi$ is $A$-equivalent to a Puiseux parametrization of the form

$$
\varphi = (t^{v_0}, t^{v_1} + t^j + \sum_{\lambda < i < c} a_it^i),
$$

and, in this case, $\lambda = v_\varphi(v_0XdY - v_1YdX) - v_0$.

Related to the invariant $\lambda$, Zariski in [11] proved the following extra elimination criterion.

(EC3) If $\varphi$ is as in (2.3) and $j - \lambda \in (v_0, v_1)$, then $\varphi$ is $A$-equivalent to a parametrization $(t^{v_0}, t^{v_1} + t^i + \sum_{\lambda < i < c} a'_it^i)$, with $a'_i = a_i$, when $i < j$, and $a'_j = 0$.

In the next theorem, our main result in this work, we will determine all possible such elimination criteria, which will lead us to what we call the normal forms for the Puiseux parametrizations.
Theorem 2.1 (The Normal Forms Theorem). Let \( \varphi \) be a Puiseux parametrization of a plane branch with semigroup of values \( \Gamma = \langle v_0, \ldots, v_n \rangle \). Then, either \( \varphi \) is \( \mathcal{A} \)-equivalent to the monomial parametrization \( (t^{v_0}, t^{v_1}) \), or it is \( \mathcal{A} \)-equivalent to a parametrization
\[
\left( t^{v_0}, t^{v_1} + t^\lambda + \sum_{i>\lambda, i \notin \Lambda-v_0} a_i t^i \right),
\]
where \( \lambda \) is its Zariski invariant and \( \Lambda = \Lambda_\varphi \) is the set of orders of differentials of the branch. Moreover, if \( \varphi \) and \( \varphi' \) (with coefficients \( a'_i \) instead of \( a_i \)) are parametrizations as in \( (2.4) \), representing two plane branches with same semigroup of values and same set of values of differentials, then \( \varphi \sim_\mathcal{A} \varphi' \) if and only if there is \( r \in \mathbb{C}^* \) such that \( r^{\lambda-v_1} = 1 \) and \( a_i = r^{i-v_1} a'_i \), for all \( i \).

Note that the \( \mathcal{A} \)-normal form in \( (2.4) \) is completely determined by the semigroup \( \Gamma \) and the set \( \Lambda \). So, once \( \Gamma \) is fixed, the number of \( \mathcal{A} \)-normal forms is finite, corresponding to all possible sets \( \Lambda \) in the equisingularity class determined by \( \Gamma \), which may be computed by the algorithm presented in [6].

The above theorem gives the ultimate elimination criterion \( \text{EC} \) that contains, as special cases, all the known criteria \( \text{EC}1, \text{EC}2 \) and \( \text{EC}3 \).

(\text{EC}) If \( \varphi \) is as in \( (2.3) \) and \( j + v_0 \in \Lambda_\varphi \), with \( j > \lambda \), then \( \varphi \) is \( \mathcal{A} \)-equivalent to a parametrization \( (t^{v_0}, t^{v_1} + t^\lambda + \sum_{\lambda<i<c} a'_i t^i) \), with \( a'_i = a_i \), when \( i < j \), and \( a'_j = 0 \).

The remainder of the paper is devoted to prove Theorem 2.1.

3. Orbits and their tangent spaces

We denote by \( j^h(h) \) the \( k \)-jet of an element \( h \). Let \( \text{Aut}(\mathbb{C}^n,0) \) denote the group of germs of analytic automorphisms of \((\mathbb{C}^n,0)\), and let \( \text{Aut}_1(\mathbb{C}^n,0) \) denote the subgroup of elements \( A \) such that \( j^1(A) = \text{Id} \). We also denote by \( \tilde{\text{Aut}}(\mathbb{C}^2,0) \) the subgroup of elements \( A \) such that \( j^1(A) = (X + \beta Y, Y) \), with \( \beta \in \mathbb{C} \).

Two Puiseux parametrizations \( \varphi_1 \) and \( \varphi_2 \) are said to be \( \mathcal{A}_1 \)-equivalent, or \( \tilde{\mathcal{A}} \)-equivalent, if \( \varphi_2 = \sigma \circ \varphi_1 \circ \rho^{-1} \) with \( \sigma \in \text{Aut}_1(\mathbb{C}^2,0) \), or \( \sigma \in \tilde{\text{Aut}}(\mathbb{C}^2,0) \), respectively, and \( \rho \in \text{Aut}_1(\mathbb{C},0) \).

We say that \( \varphi_1 \) and \( \varphi_2 \) in \( \Sigma_{\Gamma} \), where \( \Gamma = \langle v_0, v_1, \ldots, v_n \rangle \), are homothetic, or \( \mathcal{H} \)-equivalent if \( \varphi_2 = \sigma \circ \varphi_1 \circ \rho^{-1} \), with \( \rho(t) = at \) and \( \sigma(X,Y) = (\alpha^v X, \alpha^{v_1} Y) \), for some \( \alpha \in \mathbb{C}^* \).

So, the \( \mathcal{A} \)-action on the space of Puiseux parametrizations representing an equisingularity class may be obtained by the \( \tilde{\mathcal{A}} \)-action followed by the \( \mathcal{H} \)-action.

If \( G \) represents one of the actions \( \mathcal{A}, \mathcal{A}_1 \) or \( \tilde{\mathcal{A}} \), then \( G^K \) will represent the Lie group action of \( k \)-jets of corresponding automorphisms on the space \( \Sigma_{\Gamma}^K \) of \( k \)-jets of elements of \( \Sigma_{\Gamma} \).

Recall (a special case) of the Complete Transversal Theorem of [1].

The Complete Transversal Theorem. Let \( G \) be a Lie group acting on an affine space \( \mathbb{A} \) with underlying vector space \( V \), and let \( W \) be a subspace of \( V \) such that \( \forall g \in G, \forall v \in V \) and \( \forall w \in W \) one has \( g \cdot (v + w) = g \cdot v + w \). If \( W \subset T_v(G \cdot v) \), with \( v \in V \), and \( T_v(G \cdot v) \) is the tangent space at \( v \) to the orbit \( G \cdot v \), then for every \( w \in W \) one has \( G(v + w) = G(v) \).

Although we are mainly interested in the \( \mathcal{A} \)-equivalence, we will start analysing the unipotent \( \mathcal{A}_1 \)-action, passing to the \( \mathcal{A} \)-action and, finally, applying homotheties, to get to the \( \mathcal{A} \)-equivalence.

Let \( \mathbb{A} \) be the affine space where the set \( \Sigma_{\Gamma}^k \) lives, and let \( G = \mathcal{A}_{\Gamma}^k \). Then the initial hypothesis at the beginning of the Complete Transversal Theorem is fulfilled for \( W = \{ (0, bt^k) ; b \in \mathbb{C} \} \).
The tangent spaces to the orbits $A_t \cdot \varphi$ and $\tilde{A} \cdot \varphi$ at an element $\varphi = (x(t), y(t)) \in \Sigma^k_t$ are given by
\[
T_\varphi(A_t \cdot \varphi) = \{j^k((x'(t), y'(t))\epsilon + (\varphi^*(g), \varphi^*(h)) \mid \epsilon \in \mathcal{M}_1^2, g, h \in \mathcal{M}_2^2\}
\]
and
\[
T_\varphi(\tilde{A} \cdot \varphi) = \{j^k((x'(t), y'(t))\epsilon + (\varphi^*(g), \varphi^*(h)) \mid \epsilon \in \mathcal{M}_1^2, h \in \mathcal{M}_2^2, g \in \langle X^2, Y \rangle\}.
\]

The proof of the first expression may be found, for example, in [4], while the proof of the second one may be obtained in a similar way.

We show how, by using the Complete Transversal Theorem, one obtains all normal forms of Puiseux parametrizations, with respect to the $A_1$-equivalence, by eliminating terms in the expansion of $y(t)$, finding more elimination criteria than the general ones of Ebey and Zariski, adapted to a specific branch. The idea is to verify at each step if the $k$-jet of the parametrization is $A_t^k$-equivalent to its $(k-1)$-jet, which implies that the term of degree $k$ in $y(t)$ can be eliminated under the $A_1$-action. For this, according to the Complete Transversal Theorem, it is enough to verify if the vector $(0, bt^k)$ belongs to the tangent space to the $A_t^k$-orbit of the $k$-jet of the parametrization, and this fact may be expressed in terms of the existence of differentials in $(C^2, 0)$ of certain order with respect to the valuation determined by the parametrization, as we shall see soon. The procedure will stop after finitely many steps since all terms in $y(t)$ of order greater than or equal to the conductor $c$ of the semigroup of values of the branch are eliminable. Next, we find the normal forms under the $\tilde{A}$-action by analysing separately some few remaining cases. Finally, the normal forms under the $A$-action are obtained applying homotheties.

In order to apply this procedure, we need to describe more explicitly the tangent spaces to orbits in $\Sigma^k_t$.

The proof of the following result is straightforward and will be omitted.

**Lemma 3.1.** Let $k > v_1$ and $\varphi \in \Sigma^k_t$. For $b \neq 0$, we have that the vector $(0, bt^k)$ belongs to $T_\varphi(A_t^k \cdot \varphi)$ (respectively, to $T_\varphi(\tilde{A}^k \cdot \varphi)$), if and only if there exist $g, h \in \mathcal{M}_2^2$ (respectively, $g \in \langle X^2, Y \rangle$, $h \in \mathcal{M}_2^2$) such that
\[
k + v_0 - 1 = \text{ord}_i(\varphi^*(h)x'(t) - \varphi^*(g)y'(t)).
\]

Put $\mathcal{M}_2^2 = O_2$ and, for $i \in \mathbb{N}$, define $\Omega^{(i)}_2 = \{hdX + gdY \in \Omega_2 \mid g, h \in \mathcal{M}_2^2\}$. Given a parametrization $\varphi$, we also define $A^0_\varphi = v_\varphi(\Omega^{(i)}_2)$. Note that $A^0_\varphi = A_\varphi$. These sets, by functoriality, are clearly invariant under $A$-equivalence.

If we define $\Omega' = \{hdX + gdY \in \Omega_2 \mid g \in \langle X^2, Y \rangle$, $h \in \mathcal{M}_2^2\}$, and $A'_\varphi = v_\varphi(\Omega'_2)$, then Lemma 3.1 may be rephrased as follows.

**Proposition 3.2.** Let $k > v_1$ and $\varphi \in \Sigma^k_t$. For $b \neq 0$, we have that $(0, bt^k)$ belongs to $T_\varphi(A_t^k \cdot \varphi)$ (respectively, to $T_\varphi(\tilde{A}^k \cdot \varphi)$) if and only if
\[
k + v_0 \in A^2_\varphi \ (\text{respectively}, k + v_0 \in A'_\varphi).
\]

4. **Normal $A_1$-Forms**

In this section, we find the normal forms of Puiseux parametrizations in an equisingularity class with given semigroup of values $\Gamma$ under $A_1$-equivalence. This will be obtained after we prove the following three propositions. The first one will give us the recursion step.
PROPOSITION 4.1. Let \( \varphi = (t^{v_0}, t^{v_1} + \sum_{v_i < v < c} a_i t^i) \in \Sigma_\Gamma \), and let \( k \) be an integer such that \( k + v_0 \in \Lambda_\varphi^2 \). Then there exists \( \varphi_1 \in \Sigma_\Gamma \) such that \( \varphi_1 \sim_{A_1} \varphi \) and

\[
j^k(\varphi_1) = j^{k-1}(\varphi_1) = j^{k-1}(\varphi).
\]

Proof. From Proposition 3.2, we have that the vector \( (0, -a_k t^k) \in T_j^k(\varphi)(A \setminus j^k(\varphi)) \), and therefore by the Complete Transversal Theorem it follows that \( j^k(\varphi) \sim j^{k-1}(\varphi) \). Hence, there are appropriate germs of analytic isomorphisms \( \sigma \) and \( \rho \) such that \( \sigma \circ j^k(\varphi) \circ \rho^{-1} = j^{k-1}(\varphi) \). So, \( j^k(\sigma \circ \varphi \circ \rho^{-1}) = j^{k-1}(\varphi) \). Now the result follows putting \( \varphi_1 = \sigma \circ \varphi \circ \rho^{-1} \).

PROPOSITION 4.2. Let \( \varphi = (t^{v_0}, t^{v_1} + t^\lambda + \ldots) \) be a Puiseux parametrization with \( \Gamma_\varphi = \langle v_0, v_1, \ldots, v_\gamma \rangle \). If \( S = \{v_0, 2v_0, v_1, v_0 + v_1, 2v_1, v_0 + \lambda\} \), then one has

\[
S \subseteq \Lambda_\varphi \setminus \Lambda_\varphi^2 \subseteq S \cup \{v_1 + \lambda\},
\]

with equality on the top if and only if \( n_1 = 2 \) and \( \gamma \geq 2 \).

Proof. Clearly, \( n \in \Lambda_\varphi \setminus \Lambda_\varphi^2 \) if and only if \( n = v_\omega(\omega) \), where \( \omega \in \Omega_2 \setminus \Omega_2^0 \). Since \( v_\varphi(dX) = v_0 \), \( v_\varphi(dY) = v_2 \), \( v_\varphi(XdX) = 2v_0 \), \( v_\varphi(YdY) = v_0 + v_1 \), \( v_\varphi(YdY) = 2v_1 \), and \( v_\varphi(v_1 YdX - v_0 XdY) = v_0 + \lambda \), we have that \( S \subseteq \Lambda_\varphi \setminus \Lambda_\varphi^2 \).

Now, suppose \( v_\varphi(hdX + gdY) \notin S \), where \( h = \alpha X + \beta Y + h_2 \) and \( g = \alpha X + bY + g_2 \), with \( h_2, g_2 \in M_2^2 \). So, in this case, we must have \( v_\varphi(hdX + gdY) > v_\varphi(hdX) = v_\varphi(gdY) \). This implies \( v_\varphi(h) + v_0 = v_\varphi(g) + v_1 \), with \( v_\varphi(h), v_\varphi(g) \in \{v_0, v_1\} \).

If \( v_\varphi(h) = v_0 \), then we would have \( 2v_0 = v_\varphi(g) + v_1 \), which is not possible. Hence \( \alpha = 0 \).

We have \( v_\varphi(g) = v_0 \) if and only if \( v_\varphi(h) = v_1 \). Therefore, in this case, \( v_\varphi(hdX + gdY) = v_0 + \lambda \in S \), which is to be excluded. Hence, \( \alpha = \beta = 0 \).

The only remaining possibility is that \( v_\varphi(g) = v_1 \), in which case, \( b \neq 0 \) and \( a = \alpha = \beta = 0 \). So, we have \( v_\varphi(h) + v_0 = 2v_1 \), hence \( v_1 < v_\varphi(h) < 2v_1 \), which implies \( v_\varphi(h) = sv_1 + rv_0 \), with \( s = 0, 1 \). We have that \( s = 0 \), because, otherwise, we would have \( v_1 + (r + 1)v_0 = 2v_1 \), which would imply that \( v_0 \) divides \( v_1 \), a contradiction.

If the genus \( \gamma \) of \( \varphi \) is 1, then \( v_0 = 2 \), and in this case, because of EC1, we have that \( \varphi \) is \( A \)-equivalent to the parametrization \((t^2, t^{v_1})\), hence not satisfying the hypothesis of the proposition.

Therefore, \( \gamma \geq 2 \) and \( v_\varphi(h) = rv_0 \). Hence, \( (r + 1)v_0 = 2v_1 \), which implies \( n_1 = 2 \). Also, \( v_\varphi(hdX + gdY) > 2v_1 \), which in view of the expression of \( \varphi \) and the above equality implies \( v_\varphi(hdX + gdY) = v_1 + \lambda \).

Conversely, if \( \gamma \geq 2 \) and \( n_1 = 2 \), choosing \( r \) such that \( (r + 1)v_0 = 2v_1 \), then we have that \( v_\varphi(v_1 X^r dX - v_0 YdY) = v_1 + \lambda \).

PROPOSITION 4.3. Let \( \varphi \in \Sigma_\Gamma \) and set \( \Lambda = \Lambda_\varphi \). Suppose \( \Lambda \setminus \Gamma \neq \emptyset \), and let \( \lambda \) be the Zariski invariant of \( \varphi \). Then \( \varphi \) is \( A_1 \)-equivalent to a parametrization

\[
(t^{v_0}, t^{v_1} + t^\lambda + \sum_{i \geq 2, \lambda > v_0} a_i t^i).
\]

Proof. For \( l \) greater than or equal to the conductor \( c \) of \( \Gamma \) we have that \( l + v_0 \in \Lambda \), hence, by Proposition 4.2, the set \( \mathbb{N} \setminus (\Lambda^2 - v_0) \) is bounded from above by \( \max\{c - 1, v_1 + \lambda\} \).
Let \( \lambda_1, \ldots, \lambda_s \) be the elements in \( \Lambda^2 - v_0 \) in the interval \( (\lambda, c) \). From Proposition 4.1, there exists a Puiseux parametrization \( \varphi_1 \) with \( \varphi_1 \sim_{\Lambda_1} \varphi \) such that

\[
j^{\lambda_1}(\varphi_1) = j^{\lambda_1-1}(\varphi_1) = j^{\lambda_1-1}(\varphi).
\]

Next, do the same with \( \varphi_1 \) instead of \( \varphi \) and \( \lambda_2 \) instead of \( \Lambda_1 \), observing, by Proposition 3.2, that \( \Lambda_{v_1}^2 = \Lambda^2 \), and so on. \( \square \)

The next step will be to pass from the \( \Lambda_1 \)-equivalence to the \( \tilde{\Lambda} \)-equivalence.

5. Passage from the \( \Lambda_1 \)-equivalence to the \( \tilde{\Lambda} \)-equivalence

To get the normal forms of Theorem 2.1, in view of the result of Proposition 4.3, it suffices to show that the terms in \( y(t) \) of a Puiseux parametrization of order \( k \), such that \( k > \lambda \) and \( k \in (\Lambda \setminus \Lambda^2) - v_0 \), may be eliminated without changing the preceding terms. The terms of order \( k \in S - v_0 \), where \( S \) is as in Proposition 4.2, excepting \( k = \lambda \), may be eliminated by EC2. The only remaining possibilities are terms of order \( v_1 + \lambda - v_0 \), when \( \gamma \geq 2 \) and \( n_1 = 2 \) (cf. Proposition 4.2), which we show below how to eliminate without changing the preceding ones.

For this purpose, we need to analyse more closely the \( \tilde{\Lambda} \)-action on Puiseux parametrizations. Let \( \varphi(t) = (t^{v_0}, y(t)) \), where \( y(t) = \sum_i a_i t^i \), and let \( \sigma \) and \( \rho \) be as in (2.1), but with \( r = 1 \), \( p = \beta^\gamma + p_1 \), where \( \beta \in \mathbb{C} \) and \( p_1, q \in M_{2} \).

Now, considering the expression of \( \rho \) in (2.1), raising both sides to the power \( i \) and then applying the binomial expansion we get

\[
t_1^i = t^i \sum_{j=0}^{\infty} \binom{i/v_0}{j} \left( \frac{p(t)}{t^{v_0}} \right)^j,
\]

where \( p(t) = \varphi^*(p) \).

By using this in the expression \( y(t) = \sum_i a_i t^i \), we get

\[
y(t_1) = y(t) + \sum_i a_i t^i \frac{i}{v_0} \frac{p(t)}{t^{v_0}} + A(t),
\]

where

\[
A(t) = \sum_i a_i t^i \sum_{j=2}^{\infty} \binom{i/v_0}{j} \left( \frac{p(t)}{t^{v_0}} \right)^j.
\]

Now, from the expression of \( y_1(t_1) \) in (2.2) we get

\[
y_1(t_1) = y(t_1) + B(t),
\]

where, if we put \( q(t) = \varphi^*(q) \),

\[
B(t) = \frac{q(t)x'(t) - p(t)y'(t)}{x'(t)} - A(t).
\]

**Proposition 5.1.** Let \( \varphi(t_1) = (t_1^{v_0}, y(t_1)) \), where \( y(t_1) = t_1^{v_1} + t_1^\lambda + \sum_{i>\lambda} a_i t_1^i \), be a Puiseux parametrization such that the genus of \( \varphi \) is greater than 1, and \( n_1 = 2 \). Then there exists \( y_1(t_1) = t_1^{v_1} + t_1^\lambda + \sum_{i>\lambda} a_i' t_1^i \), with \( a_i' = a_i \), for \( i < v_1 + \lambda - v_0 \) and \( a_{v_1+\lambda-v_0}' = 0 \), such that \( (t_1^{v_0}, y_1(t_1)) \sim_{\tilde{\Lambda}} (t_1^{v_0}, y(t_1)) \).
Concerning the normal forms. If and only if they are conjugate under homothety, it will be sufficient to prove that two such \(\Gamma\), let us consider the linear space \(\mathbb{R}^n\). Now, to prove that two Puiseux parametrizations in the same normal form are \(A\)-equivalent if and only if they are conjugate under homothety, it will be sufficient to prove that two such parametrizations are \(A\)-equivalent if and only if they are equal. This is so, because the \(A\)-action decomposes into the \(A\)-action and the \(H\)-action.

Fixing a set \(\Lambda\) of values of differentials in the equisingularity class determined by a semigroup \(\Gamma\), let us consider the linear space

\[
N_{\Lambda} = \left\{ \left( t^{\alpha_0}, t^{\alpha_1} + t^{\lambda} + \sum_{\lambda < j < c} a_j t^j \right) \in \Sigma_\Gamma; \ a_j = 0, \ \text{for} \ j \in \Lambda - v_0 \right\}.
\]

If we denote by \(N_{\Lambda}^k\) the space \(j^k(N_{\Lambda})\), we have the following lemma.

**Lemma 5.2.** If \(\alpha \in N_{\Lambda}\), then, for all \(k > \lambda\), we have

\[
N_{\Lambda}^k \cap T_{j^k(\alpha)}(\tilde{A}^k : j^k(\alpha)) = \{ j^k(\alpha) \}.
\]

**Proof.** Suppose the assertion not true. Take \(k\) minimal with the following property:

\[
N_{\Lambda}^k \cap T_{j^k(\alpha)}(\tilde{A}^k : j^k(\alpha)) \not\equiv \{ j^k(\alpha) \}.
\]
So, there exists $\beta \in N^k_{\Lambda} \cap T_{j^k(\alpha)}(\tilde{A}^k \cdot j^k(\alpha))$ such that $\beta \neq j^k(\alpha)$ and $j^{k-1}(\beta) = j^{k-1}(\alpha)$. Therefore, there exists $b \in \mathbb{C}^*$ such that

$$\beta - j^k(\alpha) = (0, bt^k) \in T_{j^k(\alpha)}(\tilde{A}^k \cdot j^k(\alpha)).$$

Hence, from Proposition 3.2, it follows that $k \in \Lambda - v_0$. But, since $j^k(\alpha) \in N^k_{\Lambda}$, it follows that $j^k(\alpha) = j^{k-1}(\alpha)$. So, for some $b \neq 0$, we have

$$\beta = j^{k-1}(\alpha) + (0, bt^k).$$

But, since $\beta \in N^k_{\Lambda}$, one should have $b = 0$, which is a contradiction.

Now we proceed to prove the uniqueness of the $\tilde{A}$-normal forms.

Let $\varphi(t) = (t^{v_0}, t^{v_1} + t^{1} + \sum_{j \geq \lambda} a_j t^j) \in \Sigma_{\Gamma}$ be a Puiseux parametrization with $\Lambda_{\varphi} = \Lambda$. We denote by $\tilde{A}^{c-1} \cdot \varphi$ the orbit of $\varphi$ in $\Sigma_{\Gamma}$ with respect to the $\tilde{A}^{c-1}$-action. Recall also that $N_{\Lambda} = N^k_{\Lambda}$. We have

$$N_{\Lambda} \cap \tilde{A}^{c-1} \cdot \varphi = \{ \varphi \}.$$  

Indeed, if $N_{\Lambda} \cap \tilde{A}^{c-1} \cdot \varphi \neq \{ \varphi \}$, then take $\varphi_1 \in N_{\Lambda} \cap \tilde{A}^{c-1} \cdot \varphi$, with $\varphi_1 \neq \varphi$. Since $\tilde{A}^{c-1} \cdot \varphi$ is arcwise connected, there exists an arc in $\tilde{A}^{c-1} \cdot \varphi$ joining $\varphi$ to $\varphi_1$. Since reduction to the normal form is continuous, it follows that $\varphi$ would not be an isolated point in $N_{\Lambda} \cap \tilde{A}^{c-1} \cdot \varphi$. But this is a contradiction because of Lemma 5.2.

The methods we presented here, together with the algorithms developed in [6], show that the analytic classification of plane branches may be effectively performed. With those algorithms, the sets $\Lambda$ and the conditions on the coefficients that determine them, for a fixed equisingularity class, may be computed. The moduli problem of Zariski (the stratified moduli problem) is, in this way, solved, since it is the disjoint union of quotients of a finite number of semi-algebraic sets, by finite groups, corresponding to the quotients modulo the finite groups of homotheties of the normal form corresponding to a given $\Lambda$ under the $\tilde{A}$-equivalence. One of the sets $\Lambda$, denoted by $\Lambda_{\text{gen}}$, corresponds to the generic branch, easily recognized by the open conditions on the coefficients. Finally, the dimension of the component of the moduli corresponding to a given set $\Lambda$ is determined by the normal form and is at most equal to the number of gaps of $\Lambda$ greater than $\lambda$ (not necessarily equal since some of the coefficients of the parametrization may be fixed constants). The dimension of the generic component is exactly equal to the number of gaps of $\Lambda_{\text{gen}}$ greater that $\lambda$, since in this case, no coefficient in the corresponding normal form is constant.

As an application of the results presented here we refer to [7], where all branches up to multiplicity four are classified.

References


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