

# On the Turing Degrees of Weakly Computable Real Numbers

Xizhong Zheng  
Theoretische Informatik  
BTU Cottbus, 03044 Cottbus, Germany  
zheng@informatik.tu-cottbus.de

## Abstract

The Turing degree of a real number  $x$  is defined as the Turing degree of its binary expansion. This definition is quite natural and robust. In this paper we discuss some basic degree properties of semi-computable and weakly computable real numbers introduced by Weihrauch and Zheng [19]. Among others we show that, there are two real numbers of c.e. binary expansions such that their difference does not have an  $\omega$ .c.e. Turing degree.

## 1 Introduction

For simplicity, we consider only real numbers in the unit interval  $[0; 1]$ . For any such real number  $x \in [0; 1]$ , there is a set  $A \subseteq \mathbb{N}^+$  such that  $x = x_A := \sum_{i \in A} 2^{-i}$ . The set  $A$  consists of all 1-positions in the binary expansion of  $x$ . If we choose the finite set  $A$  to correspond to rational  $x$ , then this correspondence is even one-to-one. Naturally, the set  $A$  can be called a *binary set* of the number  $x_A$  and the real number  $x_A$  is called a *binary real number* of the set  $A$ . According to Turing [18], a real number  $x$  is computable if  $x$  has a computable decimal expansion, i.e.,  $x = \sum_{i \in \mathbb{N}} f(i) \cdot 10^{-i}$  for a computable function  $f : \mathbb{N} \rightarrow \{0, 1, 2 \dots, 9\}$ . The computability of real numbers is in fact independent of their representations as observed by Robinson [13]. In other words,  $x$  is computable, if and only if  $x$  has a computable Dedekind cut  $L_x := \{r \in \mathbb{Q} : r < x\}$ , if and only if the binary set of  $x$  is recursive and if and only if there is a computable sequence  $(x_s)$  of rational numbers which converges to  $x$  effectively in the sense that  $|x - x_s| \leq 2^{-s}$  for any  $s \in \mathbb{N}$ . A relativization of this observation implies immediately that the decimal expansion, the binary expansion, the effectively convergent Cauchy representation and the Dedekind cut of a real number are Turing equivalent. More precisely, we have the following proposition.

**Proposition 1.1** *For any real number  $x \in [0; 1]$ , the following functions are equivalent under the Turing reduction:*

1.  $f_1 : \mathbb{N} \rightarrow \{0, 1\}$  is the binary expansion of  $x$ , i.e.,  $x = \sum_{i=0}^{\infty} f_1(i) \cdot 2^{-i}$ ;
2.  $f_2^k : \mathbb{N} \rightarrow \{0, 1, \dots, k-1\}$  is the base- $k$  expansion of  $x$  for  $k \geq 2$ , i.e.,  $x = \sum_{i=0}^{\infty} f_2(i) \cdot k^{-i}$ ;
3.  $f_3 : \mathbb{N} \rightarrow \mathbb{Q}$  is a (fast) Cauchy representation of  $x$ , i.e.,  $\lim_{i \rightarrow \infty} f_3(i) = x$  and  $(\forall i \in \mathbb{N}) (|f_3(i) - f_3(i+1)| < 2^{-i})$ ;

4.  $f_4 : \mathbb{N} \rightarrow \mathbb{Q}$  is a nested interval sequence representation of  $x$  in the sense that  $(\forall i \in \mathbb{N})(f_4(2i) < x < f_4(2i + 1))$  and  $\lim_{i \rightarrow \infty} |f_4(2i) - f_4(2i + 1)| = 0$ .
5.  $f_6 : \mathbb{Q} \rightarrow \{0, 1\}$  is the characteristic function of the Dedekind cut of  $x$ , i.e.,  $f_6(r) = 1$  if  $r < x$  and  $f_6(r) = 0$  otherwise for any  $r \in \mathbb{Q}$ .

From this observation, we can naturally define the Turing reducibility on real numbers as follows. A real number  $x_A$  is *Turing reducible* to  $x_B$  (denoted by  $x_A \leq_T x_B$ ) if  $A \leq_T B$ .  $x$  is *Turing equivalent* to  $y$  (denoted by  $x \equiv_T y$ ) if  $x \leq_T y$  &  $y \leq_T x$ . Accordingly, the *Turing degree*  $\deg_T(x)$  of a real number  $x$  is the class of real numbers which are Turing equivalent to  $x$ , i.e.,  $\deg_T(x) := \{y \in \mathbb{R} : x \equiv_T y\}$  (see also [21, 5]). Because of the corresponding between the set  $A$  and the real number  $x_A$ , we do not distinguish the degrees  $\deg_T(x_A)$  and  $\deg_T(A)$  explicitly in this paper. This should not cause confusion from the context. Moreover, a real number  $x_A$  is called *B-computable* if  $A \leq_T B$ . If  $B$  is of the Turing degree  $\mathbf{b}$  (i.e.,  $B \in \mathbf{b}$ ), then  $x_A$  is also called  *$\mathbf{b}$ -computable*. In this way, all the notions and results in classical recursion theory about Turing degrees of subsets of natural numbers can be transferred to that of the real numbers straightforwardly. For example, by definition, a real number  $x$  is computable iff it has the computable degree, i.e.,  $\deg_T(x) = \mathbf{0}$ . Similar to the Limit Lemma of Shoenfield [15] for  $\Delta_2^0$  subsets of natural numbers, Ho [9] shows that, for any real number  $x$ ,  $x$  is  $\mathbf{0}'$ -computable if and only if there is a computable sequence  $(x_s)$  of rational numbers which converges to  $x$ .

In this paper we are interested mainly in the Turing degrees of the  $\mathbf{0}'$ -computable real numbers. By Ho's observation, a real number  $x$  is  $\mathbf{0}'$ -computable iff it is the limit of a computable sequence of rational numbers. For this reason,  $\mathbf{0}'$ -computable real numbers are called *computably approximable* (c.a. for short). A lot of subclasses of c.a. real numbers are investigated in literature [1, 12, 19, 20]. First, as mentioned above, a real number  $x$  is computable if there is a computable sequence  $(x_s)$  which converges to  $x$  effectively.  $x$  is called *left (right) computable* if there is an increasing (decreasing) computable sequence  $(x_s)$  of rational numbers which converges to  $x$ . Left and right computable real numbers are called *semi-computable*.  $x$  is called *weakly computable* if there are left computable real numbers  $y, z$  such that  $x = y - z$ . The classes of computable, left computable, right computable, semi-computable, weakly computable and computably approximable real numbers are denoted by **EC**, **LC**, **RC**, **SC**, **WC** and **CA**, respectively. We summarize the most important properties about these classes as the following theorem.

**Theorem 1.2 (Ambos-Spies, Weihrauch and Zheng [1] and Ho [9])**

For any real number  $x$ ,

1.  $x$  is left computable iff  $-x$  is right computable;
2.  $x$  is left computable iff the left Dedekind cut  $L_x := \{r \in \mathbb{Q} : r < x\}$  is a c.e. set.
3.  $x$  is weakly computable if and only if there is a computable sequence  $(x_s)$  of rational numbers which converges weakly effectively to  $x$  in the sense that  $\sum_{i \in \mathbb{N}} |x_s - x_{s+1}| \leq c$  for some constant  $c$ .
4.  $x$  is computably approximable iff  $x$  is  $\mathbf{0}'$ -computable;

5. The classes **EC**, **WC** and **CA** are closed under the arithmetical operations and hence are closed field;
6. The following relations hold:

$$\mathbf{EC} = \mathbf{LC} \cap \mathbf{RC} \subsetneq \frac{\mathbf{LC}}{\mathbf{RC}} \subsetneq \mathbf{SC} = \mathbf{LC} \cap \mathbf{RC} \subsetneq \mathbf{WC} \subsetneq \mathbf{CA}$$

Because of the item 2 of Theorem 1.2, left computable real numbers are also called computably enumerable by some authors (e.g., [4, 2]). In the following, we will discuss the degree properties of real numbers from above classes. Especially, we are interested in the computable enumerability and  $\omega$ -computable enumerability of real numbers. Let us recall the definition of the  $\omega$ -computable enumerability of a subset of  $\mathbb{N}$  at first.

For any finite set  $E \subset \mathbb{N}$ , we define its *canonical index*  $i$  by  $i := \sum_{j \in E} 2^{-j}$ . A finite set with canonical index  $i$  is denoted by  $D_i$ . A sequence  $(E_n)$  of finite subsets of  $\mathbb{N}$  is called *computable* if there is a computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $E_n = D_{f(n)}$  for all  $n \in \mathbb{N}$ . In the following,  $A \Delta B := (A \setminus B) \cup (B \setminus A)$  is the symmetrical difference of sets  $A$  and  $B$ .

**Definition 1.3 (Putnam [11], Gold [8] and Ershov [7])** Let  $h : \mathbb{N} \rightarrow \mathbb{N}$  be any function. A set  $A \subseteq \mathbb{N}$  is called *h.c.e.* if there is a computable sequence  $(A_s)$  of finite subsets of  $\mathbb{N}$  such that  $A_0 = \emptyset$ ,  $(\forall n \in \mathbb{N})(|\{s \in \mathbb{N} : n \in A_{s+1} \Delta A_s\}| \leq h(n))$  and  $A = \lim_{s \rightarrow \infty} A_s$ . The sequence  $(A_s)$  is called an *effective h-enumeration* of  $A$ .

For the constant function  $h(n) \equiv k$  and the computable function  $h$ , *h.c.e.* set is called *k.c.e.* and  $\omega$ .c.e., respectively.

So, in particular, the empty set  $\emptyset$  is the only 0.c.e. set and the 1.c.e. sets are exactly the c.e. sets. But for convenience, computable sets are also called 0.c.e. The 2.c.e. sets are usually called *d.c.e.* because they are the differences of two c.e. sets. Namely,  $A$  is 2.c.e. iff  $A = B \setminus C$  for some c.e. sets  $B$  and  $C$ . Similarly,  $A$  is  $(k+1)$ .c.e. iff  $A = B \setminus C$  for some c.e. set  $B$  and  $k$ .c.e. set  $C$ . A Turing degree is called *k.c.e.* (or  $\omega$ .c.e.) if it contains at least one  $k$ .c.e. (or  $\omega$ .c.e.) set.

In recursion theory, c.e. degrees and  $\omega$ .c.e. degrees have been widely discussed ([17, 3, 10]). For the real numbers of c.e. degrees, Dunlop and Pour-El [5] have shown an interesting characterization as follows.

**Theorem 1.4** *A real number  $x$  has a c.e. degree  $\mathbf{a}$  iff there is a computable sequence  $(x_s)$  of rational numbers which converges to  $x$  with an  $\mathbf{a}$ -computable modulus  $m : \mathbb{N} \rightarrow \mathbb{N}$  such that  $(\forall s, n \in \mathbb{N})(s \geq m(n) \implies |x - x_s| \leq 2^{-n})$ .*

As a consequence, if a  $\mathbf{0}'$ -computable real number  $x$  has a non-c.e. degree  $\mathbf{a}$ , then any computable sequence  $(x_s)$  of rational numbers which converges to  $x$  has only the modulus whose degree is strictly bigger than  $\mathbf{a}$ . On the other hand, the following proposition hold obviously.

**Proposition 1.5** 1. *Any semi-computable real number has a c.e. degree.*

2. *Any c.e. degree contains at least one left computable real number.*

**Proof.** 1. Let  $x$  be a left-computable real number. Then there is an increasing computable sequence  $(x_s)$  of rational numbers which converges to  $x$ . The c.e. set  $L \subseteq \mathbb{Q}$  defined by  $L := \{r \in \mathbb{Q} : (\exists n \in \mathbb{N})(r < x_n)\}$  is in fact the left Dedekind cut of  $x$ . By Proposition 1.1,  $x$  has a c.e. degree  $\deg_T(L)$ . Similarly, any right computable real number has a co-c.e. left Dedekind cut and has a c.e. degree too.

2. Let  $\mathbf{a}$  be a c.e. degree. There is a c.e. set  $A \in \mathbf{a}$ . Obviously,  $x_A$  is a left computable real number.  $\square$

Notice that, if we consider the binary expansion of a real number, the item 1 of Proposition 1.5 seems not so trivial, because Jockusch (see [16]) has observed that not every left computable real number has a c.e. binary representation. And in general, it can only be shown that, if  $x_A$  is a left computable real number, then  $A$  is a h.c.e. set for  $h(n) := 2^{n+1}$  (see [16]). On the other hand, by Cooper [3], there are even d.c.e. set which does not have a c.e. degree and thus this does not imply directly that  $A$  is of a c.e. degree.

However, the binary expansions of real numbers will be very useful to answer the question whether each real number of c.e. Turing degree is semi-computable or weakly computable. In this case we apply the following interesting observation about the binary expansions of semi-computable and weakly computable real numbers. Let  $\overline{C}$  denote the complement of the set  $C$  and “ $A \oplus B$ ” the join of subsets  $A, B \subseteq \mathbb{N}$  defined by  $A \oplus B := \{2n : n \in A\} \cup \{2n + 1 : n \in B\}$ .

**Theorem 1.6 (Ambos-Spies, Weihrauch and Zheng [1])**

1. If  $B, C \subseteq \mathbb{N}$  are Turing incomparable c.e. sets, i.e.,  $B \not\leq_T C$  &  $C \not\leq_T B$ , then the real number  $x_{B \oplus \overline{C}}$  is not semi-computable.
2. For any set  $A \subseteq \mathbb{N}$ , if the real number  $x_{A \oplus \emptyset}$  is weakly computable, then  $A$  is h.c.e. for  $h(n) = 2^{3n}$ .

Notice that every c.e. degree contains a semi-computable real number by Proposition 1.5.2. But the next theorem shows that any non-computable c.e. degree contains also a non-semi-computable real number. Therefore, every non-computable c.e. degree contains both semi-computable and non-semi-computable real numbers.

**Theorem 1.7** For any non-computable c.e. Turing degree  $\mathbf{a}$ , there is a set  $A \in \mathbf{a}$  such that  $x_A$  is weakly computable but not semi-computable.

**Proof.** Let  $\mathbf{a}$  be a non-computable c.e. degree. By Sacks’ Splitting Theorem [14] there exist two incomparable c.e. degrees  $\mathbf{b}_0, \mathbf{b}_1$  such that  $\mathbf{a} = \mathbf{b}_0 \cup \mathbf{b}_1$ . Choose two c.e. sets  $B_0 \in \mathbf{b}_0$  and  $B_1 \in \mathbf{b}_1$  and define set  $A := B_0 \oplus \overline{B_1}$ . Then,  $\deg_T(A) = \deg_T(B_0 \oplus B_1) = \deg_T(B_0) \cup \deg_T(B_1) = \mathbf{b}_0 \cup \mathbf{b}_1 = \mathbf{a}$ . Furthermore,  $A$  is a d.r.e set, because  $A = C_0 \setminus C_1$  for  $C_0 := 2B_0 \cup (2\mathbb{N} + 1)$  and  $C_1 := 2B_1 + 1$ . This implies that  $x_A = x_{C_0} - x_{C_1}$  and hence  $x_A$  is weakly computable of c.e. degree  $\mathbf{a}$ . On the other hand,  $x_A$  is not semi-computable by Theorem 1.6.1, since  $B_0$  and  $B_1$  are Turing incomparable.  $\square$

For weakly computable real numbers, we can show that not every real number of c.e. degree is weakly computable.

**Theorem 1.8** There is a real number of c.e. degree which is not weakly computable.

**Proof.** By Theorem 1.6.2, it suffices to show that there is a subset  $A \subseteq \mathbb{N}$  of c.e. degree such that  $A$  is not  $\omega$ .c.e.

By Hierarchy Theorem of Ershov [6], there is a  $\Delta_2^0$ -set  $B \subseteq \mathbb{N}$  which is not  $\omega$ .c.e. Let  $A := B \oplus K$  where  $K$  is the halting problem. Then  $A$  is obviously not  $\omega$ .c.e. too. Since  $A \oplus \emptyset \equiv_T A \equiv_T K$ , the real number  $x_{A \oplus \emptyset}$  has the c.e. degree  $\mathbf{0}'$ . However, by Theorem 1.6.2  $x_{A \oplus \emptyset}$  is not weakly computable.  $\square$

## 2 Main Result

In this section we will show our main result that not every weakly computable real number is of c.e. Turing degree. In fact we show a stronger result that there are real numbers  $x, y$  of c.e. binary expansions such that their difference  $x - y$  does not have an  $\omega$ .c.e. degree. The proof is a sophisticated finite injury priority construction. In this proof, we will use the following kinds of the restriction of a subset of  $\mathbb{N}$ :

$$\begin{aligned} A \upharpoonright n &:= \{x \in A : x < n\}; & A \downharpoonright n &:= \{x \in A : x > n\}; \\ A \upharpoonright (n; m) &:= \{x \in A : n < x < m\}; & A \upharpoonright [n; m) &:= \{x \in A : n \leq x < m\}. \end{aligned}$$

The proof of our main theorem applies the following technical lemma which is not difficult to prove.

**Lemma 2.1** *Let  $A, B, C \subset \mathbb{N}$  be finite sets such that  $x_A = x_B - x_C$  and  $n, m$  and  $y$  be any natural numbers. Then the following hold.*

1.  $\max A \leq \max(B \cup C)$ ;
2. If  $B \downharpoonright n = C \downharpoonright n$ , then  $\max A \leq n$  and  $n \in A \iff n \in B \setminus C$ ,
3. If  $n, m \in B \setminus C$ ,  $n < y < m$  and  $(B \upharpoonright (n; m)) \setminus \{y\} = (C \upharpoonright (n; m)) \setminus \{y\}$ , then  $n \notin A \iff y \in C \setminus B$ ;
4. If  $x_{A_1} = x_{B_1} - x_{C_1}$ ,  $(B \cup C) \upharpoonright (n+1) = (B_1 \cup C_1) \upharpoonright (n+1)$  and  $n \in B \setminus C$ , then  $A \upharpoonright n = A_1 \upharpoonright n$ .
5. Suppose that  $m < y < n$ ,  $n \in A$ .
  - (5.a) If  $m \in A$ ,  $x_{A_1} = x_A - 2^{-y}$  and  $A \upharpoonright (m; n) = \emptyset$ , then  $A_1 := (A \setminus \{m\}) \cup \{m+1, m+2, \dots, y\}$ .
  - (5.b) If  $m \notin A$ ,  $x_{A_1} = x_A + 2^{-y}$  and  $A \upharpoonright (m; n) = \{m+1, m+2, \dots, y\}$ , then  $A_1 := (A \cup \{m\}) \setminus \{m+1, \dots, y\}$ .

Let's now recall some standard notations of recursion theory (see [17, 10] for details). Let  $(M_e^A)_{e \in \mathbb{N}}$  be an effective enumeration of all Turing machines (with oracle  $A$ ) and  $(\varphi_e^A)_{e \in \mathbb{N}}$  be the corresponding effective enumeration of all  $A$ -computable functions from  $\mathbb{N}$  to  $\mathbb{N}$  where  $\varphi_e^A : \subseteq \mathbb{N} \rightarrow \mathbb{N}$  is the function computed by the  $e$ -th Turing machine  $M_e^A$  with oracle  $A$ .  $\varphi_{e,s}^A$  is the  $s$ -th approximation of  $\varphi_e^A$  defined by

$$\varphi_{e,s}^A(x) := \begin{cases} y & \text{if } e, x, y < s \text{ and } M_e^A(x) \text{ outputs } y \text{ in } s \text{ steps;} \\ \uparrow & \text{otherwise.} \end{cases}$$

The use-function  $u_{e,s}^A(x)$  of the computation  $M_e^A(x)$  is defined by

$$u_{e,s}^A(x) := \begin{cases} \text{minimal } t & \text{if } \varphi_{e,s}^A(x) \downarrow \text{ and only part } A \upharpoonright t \text{ of the} \\ & \text{oracle is used in the computation;} \\ 0 & \text{otherwise.} \end{cases}$$

The use-function records the length of the initial segment of the oracle which is really used in a computation. It is important to note that, if  $u_{e,s}^A(x) = t$ , and  $A \upharpoonright t = B \upharpoonright t$ , then the computations of  $M_e^A(x)$  and  $M_e^B(x)$  are completely the same. Obviously, we have always  $u_{e,s}^A(x) \leq s$ . This is a very useful estimation because we need only to preserve simply the initial segment  $A \upharpoonright s$  so that the computation  $M_{e,s}^A(x)$  will not be destroyed. As the partial computable functionals,  $M_e^A(x)$  is often denoted by uppercase Greek letters  $\Gamma, \Delta, \Lambda$ , etc. and the corresponding lowercase Greek letters  $\gamma, \delta, \lambda$  are their use-functions, respectively. To simplify the notation, instead of pointing out the subscript  $s$ , we will often use the expression like  $\Gamma^W(x) \upharpoonright \gamma(x)[s]$  to denote the current value of this expression  $\Gamma^W(x) \upharpoonright \gamma(x)$  at the stage  $s$ . We usually do not distinguish between a subset of  $\mathbb{N}$  and its characteristic function. That is, we have always that  $x \in A \iff A(x) = 1$  and  $x \notin A \iff A(x) = 0$ .

**Theorem 2.2** *There are c.e. sets  $B, C \subseteq \mathbb{N}$  such that the set  $A$  which satisfies  $x_A = x_B - x_C$  does not have an  $\omega$ .c.e. Turing degree.*

**Proof.** We will construct the computable sequences  $(A_s)$ ,  $(B_s)$  and  $(C_s)$  of finite subsets of  $\mathbb{N}$  which satisfy the following conditions:

1.  $x_{A_n} = x_{B_n} - x_{C_n}$  for all  $n \in \mathbb{N}$ ;
2. The limits  $A := \lim_{n \rightarrow \infty} A_n$ ,  $B := \lim_{n \rightarrow \infty} B_n$  and  $C := \lim_{n \rightarrow \infty} C_n$  exist and  $x_A = x_B - x_C$  holds;
3.  $(B_s)$  and  $(C_s)$  are the computable enumerations of the c.e. sets  $B$  and  $C$ , respectively. Hence  $x_A$  is weakly computable.
4. The degree  $\text{deg}_T(A)$  is not  $\omega$ .c.e.

To satisfy the third condition, we will define the finite sets  $B_{s+1}$  and  $C_{s+1}$  at stage  $s+1$  in such a way that  $B_s \subseteq B_{s+1}$  and  $C_s \subseteq C_{s+1}$  and define  $A_{s+1}$  by  $x_{A_{s+1}} = x_{B_{s+1}} - x_{C_{s+1}}$ . This satisfies automatically the first and second conditions too. For the fourth condition, it suffices to satisfy, for all  $\omega$ .c.e. set  $V$ , partial computable functionals  $\Gamma$  and  $\Delta$ , the following requirements:

$$R_{V, \Gamma, \Delta} : A \neq \Gamma^V \text{ or } V \neq \Delta^A.$$

From an effective enumerations of all  $\omega$ .c.e. sets<sup>1</sup> and all partial computable functionals, these requirements can also be effectively enumerated as  $(R_e)$ . All

---

<sup>1</sup>We do not really have an effective enumeration of all  $\omega$ .c.e. sets. For our proof it suffices to consider the effective enumeration  $((V_{i,s})_{s \in \mathbb{N}}, \varphi_j)_{i,j \in \mathbb{N}}$  of all computable sequences  $(V_{i,s})_{s \in \mathbb{N}}$  of finite subsets  $V_{i,s} \subseteq \mathbb{N}$  and partial computable functions  $\varphi_j$ . As long as the condition  $|\{t \leq s : n \in V_{i,t} \Delta V_{i,t+1}\}| \leq \varphi_{j,s}$  is not destroyed at stage  $s$ , we treat  $(V_{i,s})_{s \in \mathbb{N}}$  as an  $\omega$ -enumeration. Otherwise, we can simply ignore the pair  $((V_{i,s})_{s \in \mathbb{N}}, \varphi_j)$ . However, the technical details related to this are omitted in our proof for the simplicity.

requirements are given different priorities according to this enumeration. That is,  $R_i$  is of the higher priority than  $R_j$  if and only if  $i < j$ .

The strategy for satisfying a single requirement  $R_{V,\Gamma,\Delta}$  is as follows:

We choose an  $x \in \mathbb{N}$  and a  $y > \max\{x, \delta\gamma(x)[s]\}$ . By putting  $y$  into  $B$  or  $C$  it is possible to force  $x$  to enter or leave  $A$  because of  $x_A = x_B - x_C$ . The purpose of  $x$  is to witness the requirement  $R_e$  in the way that  $A(x) \neq \Gamma^V(x)$  or  $V(x) \neq \Delta^A(x)$ . To this end, we consider the following phases.

**Phase 1:**  $x, y \notin A \cup B \cup C$ . We wait for the stage  $s$  such that:

$$A_s(x) = \Gamma^V(x)[s] \ \& \ V \upharpoonright \gamma(x)[s] = \Delta^A \upharpoonright \gamma(x)[s]. \quad (1)$$

(if this never happens, then  $x$  witnesses the requirement  $R_{V,\Gamma,\Delta}$  already.) Define  $B_{s+1} := B_s \cup \{x\}$ ,  $C_{s+1} := C_s$  and then enter phase 2. It can be shown that  $A_{s+1} = A_s \cup \{x\}$ . This implies that  $A_{s+1}(x) \neq A_s(x) = \Gamma^V(x)[s]$ .

**Phase 2:**  $x \in A$  and  $y \notin B \cup C$ . We wait for some new stage  $s' > s$  such that

$$A_{s'}(x) = \Gamma^{V \upharpoonright \gamma(x)}(x)[s'] \ \& \ V \upharpoonright \gamma(x)[s'] = \Delta^{A \upharpoonright \delta\gamma(x)} \upharpoonright \gamma(x)[s']. \quad (2)$$

(if this never happens, then  $x$  is also a witness of the requirement  $R_{V,\Gamma,\Delta}$ .) In this case, we hope to remove  $x$  from  $A$  to force the initial segment  $V \upharpoonright \gamma(x)[s']$  to be changed if this condition is satisfied later again. This can be achieved by putting  $y$  into  $C$ , i.e., define  $B_{s'+1} := B_{s'}$ ,  $C_{s'+1} := C_{s'} \cup \{y\}$ . Then go into phase 3.

**Phase 3:**  $x \notin A$  and  $y \in C \setminus B$ . We wait for some new stage  $s'' > s'$  such that

$$A_{s''}(x) = \Gamma^{V \upharpoonright \gamma(x)}(x)[s''] \ \& \ V \upharpoonright \gamma(x)[s''] = \Delta^{A \upharpoonright \delta\gamma(x)} \upharpoonright \gamma(x)[s'']. \quad (3)$$

(if this never happens, then  $x$  is a witness of  $R_{V,\Gamma,\Delta}$  again.) In this case we hope to put  $x$  into  $A$  again. To achieve this, we put  $y$  into  $B$ . Namely, we define  $B_{s''+1} := B_{s''} \cup \{y\}$ ,  $C_{s''+1} := C_{s''}$ . Now the supplementary element  $y$  is used two times and  $y \in B \cup C$ . We choose a new supplementary element  $y := y + 1$  (this new  $y$  is not in  $B \cup C$ ) and go to phase 2.

Notice that, if we go from phase  $X$  to phase  $Y$ , then  $A(x)$ , hence also  $V \upharpoonright \gamma(x)$  has to be changed. But if we go from phase  $Y$  back to phase  $X$  later,  $A \upharpoonright \delta\gamma(x)$  is recovered to that of last appearance of phase  $X$ . So, the initial segment  $\Delta^{A \upharpoonright \delta\gamma(x)} \upharpoonright \gamma(x)$ , hence also the initial segment  $V \upharpoonright \gamma(x)$ , is recovered. This can happen at most finitely often, because  $V$  is an  $\omega$ .c.e. set. Therefore, after some stages, (2) or (3) will never hold again. Thus the requirement  $R_{V,\Gamma,\Delta}$  is finally satisfied by the witness  $x$ .

To satisfy all the requirements, we apply the finite injury priority construction. A witness  $x_e$  and a supplementary element  $y_e (> x_e)$  are appointed to the requirement  $R_e$  at any stage. Whenever an action for  $R_e$  appears, all requirements  $R_i$  with  $i > e$  will be *initialized* by redefining the witnesses  $x_i$  and supplementary elements  $y_i$  of  $R_i$  such that they are bigger than all elements enumerated into  $B$  and  $C$  so far and also bigger than the  $\delta\gamma(x_e)$  to preserve the computations of  $\Delta^{A \upharpoonright \delta\gamma(x)} \upharpoonright \gamma(x_e)$  from the injury by lower priority requirements. Furthermore, we build a “firewall” between the supplementary elements  $y_e$  of  $R_e$  and the witness  $x_i$  of  $R_i$  for  $i > e$ . Namely, we choose a second supplementary element  $z_e$  such that  $(\forall i > e) (y_e < z_e < x_i, y_i)$  and put it into  $B \setminus C$ .

Here is the formal construction:

Stage 0: Let  $A[0] = B[0] = C[0] := \emptyset$ ,  $x_e[0] := 3e + 1$ ,  $y_e[0] := 3e + 2$  and  $z_e[0] := 3e + 3$  for any  $e \in \mathbb{N}$ . In this case, we say that all requirements  $R_e$  is initialized.

Stage  $s + 1$ : Given  $A[s], B[s], C[s], x_e[s], y_e[s]$  and  $z_e[s]$  for all  $e \in \mathbb{N}$ . For some  $\omega$ .c.e. set  $V$ , and partial computable functionals  $\Gamma$  and  $\Delta$ , the requirement  $R_e (= R_{V, \Gamma, \Delta})$  *requires attention* if the following condition holds:

$$A(x_e)[s] = \Gamma^V \upharpoonright \gamma(x_e)(x_e)[s] \ \& \ V \upharpoonright \gamma(x_e)[s] = \Delta^{A \upharpoonright \delta\gamma(x_e)} \upharpoonright \gamma(x_e)[s]. \quad (4)$$

If no requirement requires attention, then go directly to the next stage. Otherwise, let  $R_e (= R_{V, \Gamma, \Delta})$  be the requirement of highest priority which requires attention. Then we consider the following cases where in each case we define always  $x_e[s + 1] := x_e[s]$  and define  $A[s + 1]$  by  $x_{A[s+1]} = x_{B[s+1]} - x_{C[s+1]}$ .

**Case 1.**  $x_e[s] \notin B[s]$ .

This means that  $R_e$  received no attention after it was initialized last time at, say, stage  $s_0$ . Between stage  $s_0$  and stage  $s$ , only the requirements, if any, with lower priority than  $R_e$  could have received attention. Hence only the elements bigger than  $z_e$  can be changed their memberships to  $A, B$  and  $C$ . Thus, we have in this case that  $x_e[s] \notin A_s \cup B_s \cup C_s$ . Now we hope to put  $x_e[s]$  into  $A$  and at the same time remove all elements bigger than  $x_e[s]$  from  $A$  so that we can control the membership of  $x_e[s]$  to  $A$  by putting some (bigger) supplementary elements into  $C$  or  $B$ . For this purpose, we put  $x_e[s]$  into  $B$  (but not into  $C$ ) and all elements of  $B[s] \cup C[s]$  which are bigger than  $x_e[s]$  into both  $B$  and  $C$ . Namely, we define

$$\begin{cases} B[s + 1] & := B[s] \cup \{x_e[s], z_e[s + 1]\} \cup (C[s] \upharpoonright x_e[s]); \\ C[s + 1] & := C[s] \cup (B[s] \upharpoonright x_e[s]). \end{cases} \quad (5)$$

where the supplementary elements  $y_e[s + 1]$  and  $z_e[s + 1]$  are defined by

$$\begin{cases} y_e[s + 1] & := \max\{x_e[s], \delta\gamma(x_e)[s], \max(B[s] \cup C[s])\} + 1; \\ z_e[s + 1] & := y_e[s + 1] + g(x_e)[s] + 1 \end{cases} \quad (6)$$

and  $g(x_e)[s] := \sum_{i=0}^{\gamma(x_e)[s]} h(i)$  for the computable function  $h$  such that  $V$  is *h.c.e.* Then we have  $x_e < y_e < z_e$ ,  $x_e \in A$ ,  $z_e \in B \setminus C$  and  $y_e \notin B \cup C$ . Later, we can switch the membership of  $x_e$  to  $A$  by putting the supplementary element  $y_e$  into  $C$  or into  $B$  while preserve the segment  $(A \upharpoonright \delta\gamma(x_e)) \setminus \{x_e\}$ . Every such supplementary element  $y_e$  can be used at most two times, i.e., put it into  $C$  at some stage and put it into  $B$  later. After that, a new supplementary element has to be defined, i.e.,  $y_e := y_e + 1$ . Since the initial segment  $V \upharpoonright \gamma(x_e)[s]$  can be recovered at most  $g(x_e)[s]$  times, it suffices to reserve the interval  $[y_e[s + 1]; z_e[s + 1])$  for the possible new candidates of  $y_e$ .

To prevent the action for  $R_e$  being injured by other requirements of lower priority, all requirements  $R_i$  for  $i < e$  are initialized by

$$\begin{cases} x_i[s + 1] & := z_e[s + 1] + 3i + 1; \\ y_i[s + 1] & := z_e[s + 1] + 3i + 2; \\ z_i[s + 1] & := z_e[s + 1] + 3i + 3. \end{cases} \quad (7)$$

In summary, we have now  $x_e[s], z_e[s + 1] \in B[s + 1]$ ,  $x_e[s] \in A[s + 1]$  and

$$\begin{aligned} x_e[s] = x_e[s + 1] &< \max\{\delta\gamma(x_e)[s], \max(B \cup C[s])\} < y_e[s + 1] \\ &< y_e[s + 1] + g(x_e)[s] < z_e[s + 1] < x_i[s + 1] < y_i[s + 1] < z_i[s + 1], \end{aligned}$$

for all  $i > e$ . (This case corresponds to the first phase above.)

**Case 2.**  $x_e[s] \in B[s]$ .

This means that  $R_e$  received already attention at least once after it was initialized last time. We consider the following four possibilities:

**Case 2.1.**  $x_e[s] \in A[s] \ \& \ y_e[s] \notin B[s]$ .

In this case, the current supplementary element  $y_e[s]$  is not yet used and hence not in  $(B \cup C)[s]$ . Then we put  $y_e$  into  $C$ , i.e., we define

$$B[s+1] := B[s] \ \& \ C[s+1] := C[s] \cup \{y_e[s]\}. \quad (8)$$

Thus we have  $x_e[s] \notin A[s]$  again. This case corresponds to the second phase.

**Case 2.2.**  $x_e[s] \in A[s] \ \& \ y_e[s] \in C[s]$ .

Here  $x_e[s] \in A_s$  means that if a supplementary element  $y_e$  is put into  $C$ , then it must be put into  $B$  too. In this situation, a new supplementary element is defined which is still not in  $C$  (see case 2.3.) So this case can not happen in fact.

**Case 2.3.**  $x_e[s] \notin A[s] \ \& \ y_e[s] \in C[s]$ .

In this case,  $y_e[s]$  is already put into  $C$  but not yet enumerated into  $B$ . We need only put  $y_e[s]$  into  $B$  to force  $x_e$  enter  $A$  again. Since  $y_e \in B \cup C$  now, we have to define a new supplementary element  $y_e$ . That is, we define

$$\begin{cases} B[s+1] := B[s] \cup \{y_e[s]\} \ \& \ C[s+1] := C[s]; \\ y_e[s+1] := y_e[s] + 1 \ \& \ z_e[s+1] := z_e[s]. \end{cases} \quad (9)$$

This corresponds to the third phase.

**Case 2.4.**  $x_e[s] \notin A[s] \ \& \ y_e[s] \notin C[s]$ .

This can in fact never happen. So we do nothing in this case.

In all these cases, any requirement  $R_i$  of lower priority than  $R_e$  (i.e.,  $i > e$ ) are *initialized* by the definition (7). Any other parameters which not mentioned above remain the same as that of stage  $s$ . If the requirements  $R_i$  ( $i > e$ ) has received attention before stage  $s+1$  and was not yet been initialized thereafter, then it is *injured* at this stage by the requirement  $R_e$  and the requirement  $R_e$  *receives attention* at this stage.

This ends of the construction. Now we show that our construction succeeds by proving the following sublemmas.

**Sublemma 2.2.1** *The limits  $B := \lim_{s \rightarrow \infty} B[s]$  and  $C := \lim_{s \rightarrow \infty} C[s]$  exist and the sets  $B$  and  $C$  are c.e.*

**Proof.** In the construction we only put some elements into  $B$  or  $C$  and never remove them. That is  $B[s] \subseteq B[s+1]$  and  $C[s] \subseteq C[s+1]$  hold for all  $s \in \mathbb{N}$ . So the limits  $B := \bigcup_{s \in \mathbb{N}} B[s] = \lim_{s \rightarrow \infty} B[s]$  and  $C := \bigcup_{s \in \mathbb{N}} C[s] = \lim_{s \rightarrow \infty} C[s]$  exist and they are c.e. sets.

**Sublemma 2.2.2** *For any  $e \in \mathbb{N}$ , the requirement  $R_e$  requires and receives attention finitely often.*

**Proof.** We prove the assertion by induction on  $e \in \mathbb{N}$ . Assume by the induction hypothesis that the assertion is true for all  $i < e$ . Then there is a minimal  $s_0$  such that all requirements  $R_i$  ( $i < e$ ) require no attention any more after stage  $s_0$ . By the minimality of  $s_0$ , all requirements  $R_i$  are initialized at the stage  $s_0$  and the requirement  $R_e$  will never be initialized again after stage  $s_0$ .

We call a stage  $s$  *e-stage* if  $R_e$  receives attention at the stage  $s$ . Denote by  $S_e$  the set of all *e-stages* after stage  $s_0$ , i.e.,

$$S_e := \{s > s_0 : R_e \text{ receives attention at stage } s\}.$$

Suppose that  $S_e$  is enumerated increasingly as  $S_e = \{s_1, s_2, s_3, \dots\}$ . We will show that  $S_e$  is finite. To this end, we prove the following four claims.

**Claim 1.** For any  $s \in \mathbb{N}$ , the following hold:

- (a)  $x_i[s] < y_i[s] < z_i[s] < x_{i+1}[s]$  and  $x_i[s], y_i[s], z_i[s]$  are all nondecreasing for  $s$ , for any  $i \in \mathbb{N}$ ;
- (b)  $s \geq s_0 \implies x_e[s] = x_e[s_0]$ ;
- (c)  $s \in [s_0; s_1) \implies x_e[s], y_e[s], z_e[s] \notin (A \cup B \cup C)[s]$ ;
- (d)  $s \geq s_1 \implies z_e[s] = z_e[s_1] \ \& \ x_e[s], z_e[s] \in (B \setminus C)[s]$ ;

Then, we can denote the limits simply by  $x_e := \lim_{s \rightarrow \infty} x_e[s] = x[s_0]$  and  $z_e := \lim_{s \rightarrow \infty} z_e[s] = z[s_0]$ .

*Proof of Claim 1.* Part (a) follows immediately from the construction.

Part (b) is easy to prove. A witness  $x_i$  will be redefined only if the corresponding requirement  $R_i$  is initialized.  $R_e$  will never be initialized after stage  $s_0$ , so we have  $(\forall s \geq s_0)(x_e[s] = x_e[s_0])$ .

For (c), suppose that  $s \in [s_0; s_1)$ . By the choice of  $s_0$ ,  $R_e$  is initialized at stage  $s_0$  and  $x_e[s_0], y_e[s_0], z_e[s_0]$  are defined according to (7) and hence  $x_e[s_0], y_e[s_0], z_e[s_0] \notin (A \cup B \cup C)[s_0]$ . By (a), we have  $(\forall t \geq s_0)(\forall i > e)(x_e[s_0] < y_e[s_0] < z_e[s_0] < x_i[t] < y_i[t] < z_i[t])$ . Since  $R_e$  receives no attention and only  $R_i$  with  $i > e$  may receive attention at stage  $s$ , this implies that  $(A \cup B \cup C) \upharpoonright (z_e + 1)[s] = (A \cup B \cup C) \upharpoonright (z_e + 1)[s_0]$  and that  $x_e[s] = x_e[s_0] \ \& \ y_e[s] = y_e[s_0]$  and  $z_e[s] = z_e[s_0]$  for any  $s \in [s_0; s_1)$ . Hence  $x_e[s], y_e[s], z_e[s] \notin (A \cup B \cup C)[s]$  holds for any  $s \in [s_0; s_1)$ .

For assertion (d), suppose that  $s \geq s_1$ . At stage  $s_1$ ,  $R_e$  receives attention corresponding to the case 1 because  $x_e \notin B[s_1 - 1]$  by (c). We can separate the actions in this case into two steps. At step one, according to the definition (5), all elements in  $(B \cup C)[s_1 - 1]$  which are bigger than  $x_e[s_1 - 1]$  are enumerated both into  $B_1[s_1]$  and  $C_1[s_1]$ , i.e., we have  $B_1 \upharpoonright x_e[s_1] = C_1 \upharpoonright x_e[s_1]$ . It follows from Lemma 2.1.2 that  $\max A_1[s_1] \leq x_e[s_1]$ . In other words, all elements which are bigger than  $x_e[s_1]$ , if any, are removed from  $A$ . At step two, based on  $B_1[s_1]$  and  $C_1[s_1]$ , the elements  $x_e[s_1]$  and  $z_e[s_1]$  are put into  $B[s_1]$ , hence into  $A[s_1]$  by the Lemma 2.1.2, while the supplementary element  $y_e[s_1]$ , which satisfies that  $x_e[s_1] < y_e[s_1] < z_e[s_1]$ , remains out of  $B[s_1]$  and  $C[s_1]$ . This concludes at first the  $x_e[s], z_e[s] \in B[s]$  for all  $s \geq s_1$ , because we never remove any element from the set  $B$  in the construction. Since the requirement  $R_e$  receives no attention corresponding case 1 (because  $x_e[s] \in B[s]$ ) and it is never initialized after stage  $s_1$ , we have  $z_e[s] = z_e[s_1]$  for any  $s \geq s_1$ . Besides, we never put  $x_e, z_e$  into  $C$ , so we have  $x_e, z_e \in (B \setminus C)[s]$ .

**Claim 2.** Suppose that  $k < g(x_e)[s_1]$ . Then the following hold for any  $s \geq s_1$ .

- (a) If  $s_{2k+1} \leq s < s_{2k+2}$ , then
  - (a1)  $A[s] \upharpoonright [x_e, z_e] = \{x_e\}$ ;

- (a2)  $\exists z' \geq z_e (z' \in A[s] \ \& \ A[s] \upharpoonright [z_e; z'] = \emptyset)$ ;
- (a3)  $y_e[s] \notin (B \cup C)[s]$ ;
- (a4)  $(B \cup C)[s] \upharpoonright (y_e[s]; z_e) = \emptyset$ ;
- (a5)  $y_e[s] \leq y_e[s_1] + k$ .

(b) If  $s_{2k+2} \leq s < s_{2k+3}$ , then

- (b1)  $A[s] \upharpoonright [x_e, z_e) = \{x_e + 1, x_e + 2, \dots, y_e[s]\}$ ;
- (b2)  $\exists z' \geq z_e (z' \in A[s] \ \& \ A[s] \upharpoonright [z_e; z'] = \emptyset)$ ;
- (b3)  $y_e[s] \in (C \setminus B)[s]$ ;
- (b4)  $(B \cup C)[s] \upharpoonright (y_e[s]; z_e) = \emptyset$ ;
- (b5)  $y_e[s] \leq y_e[s_1] + k + 1$ .

*Proof of Claim 2.* We prove assertions (a) and (b) of the claim by induction on  $s \geq s_1$  simultaneously.

For  $s = s_1$ ,  $s$  is the first  $e$ -stage after  $s_0$ . Since  $x_e \notin B[s_1 - 1]$  by (c) of the Claim 1, the requirement  $R_e$  receives attention at stage  $s$  according to the case 1. Then the assertion (a1) – (a4) follows from the definitions (5) and (6) immediately.

For any  $s > s_1$ , assume by the induction hypothesis that the claims (a) and (b) hold for any  $t$  with  $s_1 \leq t < s$ . We consider the following cases:

(I)  $s = s_{2k+1}$ , ( $k > 0$ ).

The stage  $s$  is an  $e$ -stage and the requirement  $R_e$  receives attention at this stage. Since  $x_e \in B[s - 1]$  (by (d) of the Claim 1), this corresponds to the case 2. By the induction hypothesis for  $s - 1 \in [s_{2(k-1)+2}; s_{2(k-1)+3})$ , (b1) – (b5) hold for  $s - 1$  and  $k - 1$ . From (b1) (i.e.,  $x_e \notin A[s - 1]$ ) and (b3) (i.e.,  $y_e[s - 1] \in C[s - 1]$ ), the case 2.3. applies at this stage. Namely, we put the supplementary element  $y_e[s - 1]$  into  $B[s]$  and redefine a new one by  $y_e[s] := y_e[s - 1] + 1$ . Then we the assertions (a3), (a4) and (a5) follows easily from the old (a3), (b4) and (b5) (for  $s - 1$  and  $k - 1$ ).

Suppose that  $z'$  is the number, whose existence is guaranteed by (b2), which satisfies the following condition:

$$z' \geq z_e \ \& \ z' \in A[s - 1] \ \& \ A[s - 1] \upharpoonright [z_e; z'] = \emptyset. \quad (10)$$

From the construction that  $B[s] := B[s - 1] \cup \{y_e[s - 1]\}$  and  $C[s] := C[s - 1]$ , we know that  $x_{A[s]} = x_{A[s-1]} + 2^{y_e[s-1]}$ . By (b1) and (10), we have also that  $A[s - 1] \upharpoonright (x_e; z') = \{x_e + 1, x_e + 2, \dots, y_e[s - 1]\}$ . It follows from (5.b) of the Lemma 2.1 that

$$A[s] = (A[s - 1] \cup \{x_e\}) \setminus \{x_e + 1, x_e + 2, \dots, y_e[s - 1]\}. \quad (11)$$

This implies the assertion (a1) immediately. Obviously, this  $z'$  satisfies the assertion (a2) too. Then all (a1) – (a5) hold for this  $s$ .

(II)  $s_{2k+1} < s < s_{2k+2}$ .

Now  $s$  is not an  $e$ -stage and the requirement  $R_e$  receives no attention at this stage. If on requirement  $R_i$  receives attention at this stage, then everything remains unchanged. So (a1) – (a5) hold directly by the induction hypothesis for  $s - 1 \in [s_{2k+1}; s_{2k+2})$ .

Suppose that there is an  $i > e$  such that  $R_i$  receives attention at this stage. From (a) of the Claim 1. and the definitions (5) – (9), this means that some elements

which are bigger than  $z_e$  may be put into  $B[s]$  or  $C[s]$  by the construction. Because  $z_e \in B[s] \setminus C[s]$  for any  $s \geq s_1$  by (d) of the Claim 1, we have always that

$$2^{-z_e+1} > \sum \{2^{-i} : i \geq z_e \ \& \ i \in B[s]\} - \sum \{2^{-i} : i \geq z_e \ \& \ i \in C[s]\} > 0.$$

i.e.,  $2^{-z_e+1} > x_{B[s] \setminus \{z_e\}} - x_{C[s] \setminus \{z_e\}} > 0$ . There is a non-empty finite subset  $E \subseteq \mathbb{N}$  such that  $x_E = x_{B[s] \setminus \{z_e\}} - x_{C[s] \setminus \{z_e\}}$ . Let  $z'$  be the minimal element of  $E$ . It is easy to see that  $z'$  satisfies (a2). Because of  $(A \cup B \cup C)[s] \upharpoonright z_e = (A \cup B \cup C)[s-1] \upharpoonright z_e$  and  $y_e[s] = y_e[s-1]$  by the construction, the assertions (a1) and (a3) – (a5) holds obviously by the induction hypothesis.

(III)  $s = s_{2k+2}$ .

The stage  $s$  is again an  $e$ -stage and the requirement  $R_e$  receives attention at this stage. By (d) of the Claim 1, this corresponds also to the case 2.

By the induction hypothesis on  $s-1 \in [s_{2k+1}; s_{2k+2})$ , the assertions (a1) – (a5) hold for  $s-1$ . Because of (a1) (i.e.,  $x_e \in A[s-1]$ ) and (a3) (i.e.,  $y_e \notin C[s-1]$ ), the case 2.1 is applicable at stage  $s$ . That is, we put the supplementary element  $y_e[s-1]$  into  $C[s]$  and do not change anything else. So (b3) – (b5) follow immediately from this action and the induction hypothesis of (a3) – (a5).

Let  $z'$  be the number which satisfies (a2), i.e., the condition (10) is satisfied correspondingly. Obviously,  $z'$  satisfies (b2) too. Furthermore, because  $y_e[s] = y_e[s-1]$ ,  $B[s] = B[s-1]$ , and  $C[s] = C[s-1] \cup \{y_e[s]\}$  by the construction, we have  $x_{A[s]} = x_{A[s-1]} - 2^{-y_e[s]}$ . By (a1) of the induction hypothesis and the condition (10), we have also  $A[s-1] \upharpoonright [x_e; z'] = \{x_e\}$ . It follows from (5.a) of the Lemma 2.1 that

$$A[s] = (A[s-1] \setminus \{x_e\}) \cup \{x_e + 1, x_e + 2, \dots, y_e[s]\}.$$

Thus (b1) is satisfied, hence all (b1) – (b5) hold.

(IV)  $s_{2k+2} < s < s_{2k+3}$ .

Similar to the case of (II), we can show that all (b1) – (b5) are satisfied in this situation.

From the above proof, we see that the case 2.2 and case 2.4 are indeed not applicable.

**Claim 3.** Let  $t_k := s_k - 1$  for any  $k \in \mathbb{N}$  and  $m = \delta\gamma(x_e)[t_1]$ . Then the following hold for all  $s \geq s_1$  and  $k \in \mathbb{N}$ .

- (a)  $A \upharpoonright x_e[s] = A \upharpoonright x_e[t_1]$ ;
- (b)  $A[s_{2k+1}] \upharpoonright m = A[t_1] \upharpoonright m$ ;
- (c)  $A[s_{2k+2}] \upharpoonright m = A[t_2] \upharpoonright m$ .

*Proof of Claim 3.* For (a). By (d) of the Claim 1,  $x_e \in (B \setminus C)[s]$  for any  $s \geq s_1$  and  $x_e$  is put into  $B$  at the stage  $s_1$ . After stage  $s_1$ , only the elements bigger than  $x_e$  can be put into  $B$  or  $C$ . This can change only the membership of  $x_e$  to  $A$  but not of smaller elements. So we have always that  $A \upharpoonright x_e[s] = A \upharpoonright x_e[t_1]$ . (A more detail proof can be carried out by induction on  $s \geq t_1$ .)

(b) and (c) follow from (a) of the claim and (a1) and (b1) of the Claim 2, respectively.

**Claim 4.** Let  $t_i := s_i - 1$  for any  $i \geq 1$ . Then the following hold.

- (a)  $V \upharpoonright \gamma(x_e)[t_i] \neq V \upharpoonright \gamma(x_e)[t_{i+1}]$  for any  $i \geq 1$ ;
- (b)  $\gamma(x_e)[t_{2k+1}] = \gamma(x_e)[t_1]$  for any  $k \in \mathbb{N}$ ;
- (c)  $\delta\gamma(x_e)[t_{2k+1}] = \delta\gamma(x_e)[t_1]$  for any  $k \in \mathbb{N}$ ;
- (d)  $V \upharpoonright \gamma(x_e)[t_{2k+1}] = V \upharpoonright \gamma(x_e)[t_1]$  for any  $k \in \mathbb{N}$ .

*Proof of Claim 4.* (a) If  $i = 2k$  with  $k > 0$ . Because  $R_e$  receives attention at stage  $t_i + 1 (= s_{2k})$  and at stage  $t_{i+1} + 1 (= s_{2k+1})$ , we have

$$\begin{aligned} \Gamma^{V \upharpoonright \gamma(x_e)}(x_e)[t_i] &= \Gamma^{V \upharpoonright \gamma(x_e)}(x_e)[s_{2k} - 1] = A(x_e)[s_{2k} - 1] = 0 \\ &\neq 1 = A(x_e)[s_{2k+1} - 1] = \Gamma^{V \upharpoonright \gamma(x_e)}(x_e)[s_{2k+1} - 1] = \Gamma^{V \upharpoonright \gamma(x_e)}(x_e)[t_{i+1}]. \end{aligned}$$

by (4), (a1) and (b1) of the Claim 2. This implies obviously that  $V \upharpoonright \gamma(x_e)[t_i] \neq V \upharpoonright \gamma(x_e)[t_{i+1}]$ .

For  $i = 2k + 1$ , the proof is similar.

Now we prove (b) – (d) simultaneously by induction on  $k \in \mathbb{N}$ ,

The case for  $k = 0$  is trivial.

Assume by the induction hypothesis that (b) – (d) hold for  $k$ . Consider the case of  $k + 1$ .

At stage  $t_{2k+3} + 1$ ,  $R_e$  receives attention. By the condition (4) and (b1) of the Claim 2, we have

$$\Gamma^{V \upharpoonright \gamma(x_e)}(x_e)[t_{2k+3}] = A(x_e)[t_{2k+3}] = 0 \quad (12)$$

$$V \upharpoonright \gamma(x_e)[t_{2k+3}] = \Delta^{A \upharpoonright \delta\gamma(x_e)} \upharpoonright \gamma(x_e)[t_{2k+3}] \quad (13)$$

This implies further

$$\begin{aligned} &V[t_{2k+3}] \upharpoonright \gamma(x_e)[t_1] \\ &= (V \upharpoonright \gamma(x_e)[t_{2k+3}]) \upharpoonright \gamma(x_e)[t_1] && \text{(by } \gamma(x_e)[t_1] \leq \gamma(x_e)[t_{2k+3}]) \\ &= (\Delta^{A \upharpoonright \delta\gamma(x_e)} \upharpoonright \gamma(x_e)[t_{2k+3}]) \upharpoonright \gamma(x_e)[t_1] && \text{(by (13))} \\ &= (\Delta^{A \upharpoonright \delta\gamma(x_e)[t_{2k+3}]} \upharpoonright \gamma(x_e)[t_1]) && \text{(by } \gamma(x_e)[t_1] \leq \gamma(x_e)[t_{2k+3}]) \end{aligned}$$

By (b) of Claim 3. and the fact that  $\delta\gamma(x_e)[t_1] \leq \delta\gamma(x_e)[t_{2k+3}]$ , we obtain that  $A \upharpoonright \delta\gamma(x_e)[t_1] = (A \upharpoonright \delta\gamma(x_e)[t_{2k+3}]) \upharpoonright \delta\gamma(x_e)[t_1]$ . At stage  $t_1$ ,  $R_e$  requires attention, so the condition (4) is satisfied for  $t_1$ . Then  $\Delta^{A \upharpoonright \delta\gamma(x_e)}(x)[t_1] \downarrow$  for all  $x \leq \gamma(x_e)[t_1]$ . This means that the initial segment  $A[t_{2k+3}] \upharpoonright \delta\gamma(x_e)[t_1]$  of the above oracle  $A[t_{2k+3}] \upharpoonright \delta\gamma(x_e)[t_{2k+3}]$  suffices for all the computations  $\Delta^{A \upharpoonright \delta\gamma(x_e)[t_{2k+3}]}(x)$ , for  $x \leq \gamma(x_e)[t_1]$ . Then they are in fact same as the computations of  $\Delta^{A \upharpoonright \delta\gamma(x_e)[t_1]}(x)$ , respectively. From this we have continuously that:

$$\begin{aligned} &(\Delta^{A \upharpoonright \delta\gamma(x_e)[t_{2k+3}]} \upharpoonright \gamma(x_e)[t_1]) \\ &= \Delta^{A[t_{2k+3}] \upharpoonright \delta\gamma(x_e)[t_1]} \upharpoonright \gamma(x_e)[t_1] \\ &= \Delta^{A \upharpoonright \delta\gamma(x_e)} \upharpoonright \gamma(x_e)[t_1] && \text{(by (b) of Claim 3.)} \\ &= V \upharpoonright \gamma(x_e)[t_1]. && \text{(by (4) for } t_1) \end{aligned}$$

in short, we have  $V[t_{2k+3}] \upharpoonright \gamma(x_e)[t_1] = V \upharpoonright \gamma(x_e)[t_1]$ . This implies that the computations  $\Gamma^{V \upharpoonright \gamma(x_e)}(x_e)[t_{2k+3}]$  and  $\Gamma^{V \upharpoonright \gamma(x_e)}(x_e)[t_1]$  are in fact the same, because the

initial segment  $V \upharpoonright \gamma(x_e)[t_1]$  of the oracle  $V[t_{2k+3}]$  suffices for the first computation and on this initial segment both computations behave completely the same. Then their corresponding use function should be equal, i.e.,  $\gamma(x_e)[t_{2k+3}] = \gamma(x_e)[t_1]$ . So (b) holds for  $k + 1$ . Similarly, the computations  $\Delta^{A \upharpoonright \delta\gamma(x_e)[t_{2k+3}]}(x)$  and  $\Delta^{A \upharpoonright \delta\gamma(x_e)[t_1]}(x)$  are also completely the same for any  $x \leq \gamma(x_e)[t_{2k+1}]$ , respectively. This concludes that  $\delta\gamma[t_{2k+3}] = \delta\gamma[t_1]$  and  $V \upharpoonright \gamma(x_e)[t_{2k+3}] = V \upharpoonright \gamma(x_e)[t_1]$  immediately. Therefore, (b) – (c) hold for  $k + 1$  too, hence the claim 4 is proved.

Now we can complete the proof of Sublemma 2.2.2. By Claim 4. we know that, from the stage  $s_{2k+1} - 1$  to stage  $s_{2k+2} - 1$ , the initial segment  $V \upharpoonright \delta\gamma(x_e)$  changes and then it recovers to that of stage  $s_{2k+1} - 1$  at stage  $s_{2k+3} - 1$ . Because  $V$  is a *h.c.e.* set. There are at most  $g(x_e)$  such kind of recoveries. This means that,  $S_e$  can not be infinite, i.e., after some stage,  $R_e$  will never requires attention any more. So  $R_e$  requires and receives attention at most finitely often.

**Sublemma 2.2.3** *The limit  $A := \lim_{s \rightarrow \infty} A_s$  exists and  $x_A$  is a weakly computable real number.*

**Proof.** By the construction, if there is only finite many requirements receive attention, then the sets  $B$  and  $C$  are all finite. There is an  $s$  such that  $B_s = B$  and  $C_s = C$ . So  $A = A_t$  for all  $t \geq s$ . The claim is true in this case.

Suppose that there are infinitely many requirements receives attention. For any  $n \in \mathbb{N}$ , there is an  $x_e > n$  such that  $x_e$  is put into  $B \setminus C$  at some stage  $s$  by the action for the requirement  $R_e$ . By Sublemma 2.2.2, we can assume that  $R_e$  is never initialized after stage  $s$ . Then by the Claim 3. of the Sublemma 2.2.2, we have  $A \upharpoonright x_e[t] = A \upharpoonright x_e[s]$ . So  $n$  will never change its membership to  $A$  after stage  $s$  again, hence  $A := \lim_{s \rightarrow \infty} A_s$  exists. The second part of the Sublemma follows from the fact that  $x_{A_s} = x_{B_s} - x_{C_s}$  for all  $s \in \mathbb{N}$  and Sublemma 2.2.1.

**Sublemma 2.2.4** *For any  $e \in \mathbb{N}$ , the requirement  $R_e$  is satisfied eventually.*

**Proof.** By Sublemma 2.2.2,  $R_e$  requires and receives attention at most finitely often. So the condition (4) will never be satisfied again after some stage, hence the requirement  $R_e$  is eventually satisfied.

This complete the proof of the theorem. □

**Corollary 2.3** *1. There is a weakly computable real number which does not have an  $\omega$ .c.e. Turing degree.*

*2. The class of real numbers which have the c.e. degrees is not closed under the addition and subtraction.*

## References

- [1] K. Ambos-Spies, K. Weihrauch, and X. Zheng. Weakly computable real numbers. *Journal of Complexity*, 16(4):676–690, 2000.
- [2] C. S. Calude. A characterization of c.e. random reals. *Theoretical Computer Science*, 217:3–14, 2002.

- [3] B. S. Cooper. Degrees of unsolvability. Ph.d thesis, Leicester University, Leicester, England, 1971.
- [4] R. G. Downey. Some computability-theoretical aspects of real and randomness. Preprint, September 2001.
- [5] A. J. Dunlop and M. B. Pour-El. The degree of unsolvability of a real number. In J. Blanck, V. Brattka, and P. Hertling, editors, *Computability and Complexity in Analysis*, volume 2064 of *LNCS*, pages 16–29, Berlin, 2001. Springer. CCA 2000, Swansea, UK, September 2000.
- [6] Y. L. Ershov. A certain hierarchy of sets. i, ii, iii. (Russian). *Algebra i Logika*, 7(1):47–73, 1968; 7(4):15–47, 1968; 9:34–51, 1970.
- [7] Y. L. Ershov. A certain hierarchy of sets. i. (Russian). *Algebra i Logika*, 7(1):47–73, 1968.
- [8] E. M. Gold. Limiting recursion. *J. of Symbolic Logic*, 30:28–48, 1965.
- [9] C.-K. Ho. Relatively recursive reals and real functions. *Theoretical Computer Science*, 210:99–120, 1999.
- [10] P. Odifreddi. *Classical Recursion Theory*, volume 129 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1989.
- [11] H. Putnam. Trial and error predicates and solution to a problem of Mostowski. *Journal of Symbolic Logic*, 30:49–57, 1965.
- [12] R. Rettinger, X. Zheng, R. Gengler, and B. von Braunmühl. Monotonically computable real numbers. *Math. Log. Quart.*, 48(3):459–479, 2002.
- [13] R. Robinson. Review of “Peter, R., Rekursive Funktionen”. *The Journal of Symbolic Logic*, 16:280–282, 1951.
- [14] G. E. Sacks. On the degrees less than  $\mathbf{0}'$ . *Ann. of Math.*, 77:211–231, 1963.
- [15] J. R. Shoenfield. On degrees of unsolvability. *Ann. of Math. (2)*, 69:644–653, 1959.
- [16] R. Soare. Cohesive sets and recursively enumerable Dedekind cuts. *Pacific J. Math.*, 31:215–231, 1969.
- [17] R. I. Soare. *Recursively enumerable sets and degrees. A study of computable functions and computably generated sets*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987.
- [18] A. M. Turing. On computable numbers, with an application to the “Entscheidungsproblem”. *Proceedings of the London Mathematical Society*, 42(2):230–265, 1936.
- [19] K. Weihrauch and X. Zheng. A finite hierarchy of the recursively enumerable real numbers. In *Proceedings of MFCS'98, Brno, Czech Republic, August, 1998*, volume 1450 of *LNCS*, pages 798–806. Springer, 1998.

- [20] X. Zheng. Recursive approximability of real numbers. *Mathematical Logic Quarterly*, 48(Suppl. 1):131–156, 2002.
- [21] X. Zheng, D. Ding, and Z. Sun. On the definition of degrees of unsolvability for reals. *Chinese Science Bulletin*, 38(3):172–175, 1993. (in Chinese).