Combinatorial Games, Theory and Applications

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Abstract

Combinatorial games are two-person perfect information zero-sum games, and can in theory be analyzed completely. However, in most cases a complete analysis calls for a tremendous amount of computations. Fortunately, some theory has been developed that enables us to find good strategies. Furthermore, it has been shown that the whole class of impartial games fairly easy can be analyzed completely.

In this thesis we will give an introduction to the theory of combinatorial games, along with some applications on other fields of mathematics. In particular, we will examine the theory of impartial games, and the articles proving how such games can be solved completely. We will also go into the theory of Thermography, a theory that is helpful for finding good strategies in games. Finally, we will take a look at lexicodes that are error correcting codes derived from game theory.

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Chapter 1

Introduction

Combinatorial game theory is not only an interesting theory on its own, it links to many other fields of mathematics, making it an exciting vertex. In this thesis we will give an account of some of the theory that exists for combinatorial games. We will also present some applications where this game theory provides a linking to other parts of mathematics, e.g. to algebra and coding theory.

Combinatorial games are two-person perfect information zero-sum games, and can in theory be analyzed completely. There are, however, still a lot of unsolved problems left, R.K. Guy has made a list of some of them [2, pages 183–189]. In chapter 2 we will give a more thorough description of what a combinatorial game is.

A subset of the games are the so called *impartial* ones. These can fairly easy be solved completely, which was proved by Sprague and Grundy (independently) in the 1930's. In chapter 3 we will take a closer look at their articles.

All other games may be solved combinatorially, but this usually calls for some extremely long and tedious calculations. Fortunately, we can develop some tools that give us a very good estimate of the best move from a given position. This tool is called Thermographs, and we will be assigning temperatures to games, measuring how eager the (perfect) players are to move in them.

It may not be surprising that we can map games on all integers, but this can also be done with all reals; and we can extend our field of numbers to include *surreal* [4] numbers like $\sqrt[3]{\infty}$, by defining numbers as games.

Another surprise is that we can link combinatorial games to error correcting codes. This we will explore in chapter 7.

Usually, combinatorial game theory only treats games without draws. However, A. Fraenkel has extended the theory to include games with draws as well [2, e.g. in pages 111–153], but we will not go further into this.

We will limit the number of examples of different games in this thesis, since this could easily draw the attention away from the theory and general applications.

Combinatorial game theory has sometimes been called *Conway's theory of games*. And, indeed, J.H.Conway has been the key figure in the development of this theory - as we may notice throughout the thesis. Especially the book *Winning Ways for your Mathematical Plays* [1] by Conway, Guy and Berlekamp has become the principal work, and this is where all later work about combinatorial game theory spring from.

Chapter 2

Basic Concepts

Webster's encyclopedia [12] defines a game in the following way

> A competitive activity involving skill, chance or endurance on the part of two or more persons who play according to a set of rules, usually for their own amusement or that of spectators.

In this paper we will be concerned with a specific kind of games we call combinatorial games. The most significant restriction on combinatorial games is that stochastic moves are not allowed, i.e. games with dice or other probability generators are not combinatorial; the games are deterministic and the players have perfect information. The games are all two player games, and the players move alternately. We will name the two players *Left* and *Right* and sympathize with Left, in accordance with the tradition. For ease of reference, we will let Left be male and Right be female, so whenever we refer to "she", we will mean Right.

We will demand that the games terminate, such that we can determine the outcome of the game. Furthermore, we do not consider games with draws. Usually, the rules define the winner as the player who moves last, such that whoever is unable to move loses.

Let us list the conditions the games we call combinatorial are subject to.

- There are exactly two players
- The players move alternatingly
- The moves are subject to some well defined rules
- The players have perfect information

- The games must terminate, and there cannot be a draw [1]
- In the normal play convention the one winner is the player who moves last - whoever cannot move, loses (some games have a so called misère form, where the player who moves last loses, but we will not be concerned with these).

Games can be defined formally in many ways. The definition we want to pursue is due to Conway [3] (John Horton Conway 1937–). It leans upon the fact that a game is a sequence of moves through certain allowed positions. The positions in the game are somehow games themselves, because from a position the players have a sequence of moves through certain allowed positions as well. To clarify what we mean, let us look at the well-known game checkers (notice that checkers is normally not considered a combinatorial game, because it allows draws).

This drawing should represent a position in checkers. It is not the whole game of checkers, but just a snapshot of it. If instead we showed all the many possible sequences of moves from this position, we would have the whole game of checkers. However, writing down all the possible sequences of play does not seem feasible, since there are so incredibly many of them. But we could supply the information of what positions can be reached in one move. Each of these positions are as much a game as the position we started with, so for each of them we could in the same way supply the information of the possible next moves, etc. After a finite number of steps (checkers is a finite game), we would indeed have the whole game of checkers. This way we can define the game recursively.

[1] A. Fraenkel has successfully done some game theory without this condition e.g [2, pages 111–153], but we will not go further into this.

The game where neither player has any legal moves is also a game, the sequence is empty, but we can still view it as a sequence. We call this the *Endgame*, and it will be the basis of our recursion.

Now let us turn to the formal definition of a combinatorial game as given by J.H. Conway in his book "On Numbers and Games" [3]. The definition is very important to us, and will be used intensively throughout the paper. A game is defined recursively over all the positions the two players are allowed to move to, if it is their turn. This is written as a divided set with the set (denoted G^L) of positions Left can move to on the left, and the set (denoted G^R) of positions Right can move to on the right:

Definition 1 (A Combinatorial Game) *:*
A game is defined recursively as a divided set

$$G = \{G^L | G^R\}$$

where G^L and G^R are sets of games. The base case is the game $\{\emptyset|\emptyset\}$, called the Endgame.

The line | is just a partition line, and has no connection to the similar line in set theory.

We view the game as a position from where both players can move. To get a better understanding of this definition, let us draw the game tree of some simple games. The nodes are positions and the branches are moves. Branches going left are player Left's possible moves, and branches going right are Right's moves.

Example 1 :
The simplest game is the Endgame. We draw the tree corresponding to the Endgame $\{\emptyset|\emptyset\}$ as a root without any branches:

In this game neither player has any moves, so the game consists of but one position.

Example 2 :
Using the Endgame as the Left option G^L, and the empty set as the Right option G^R, we get the game $\{\{\emptyset|\emptyset\} \mid \emptyset\}$. In this game Left has one move, namely to the Endgame, and Right has no moves. The corresponding tree must have one branch going left and none going right:

```
      o
     /
    o
```

Left will always win this game: If Right starts, she loses (because she has no legal moves). If Left starts, he can move to the Endgame; and Right, who is next, loses.

Example 3 :
Let us now try to find the game corresponding to the following game tree:

```
         o  A
        / \
   B  o     o  D
     /
 C  o
```

We have labelled the nodes, such that we can refer to them.

If Right starts, she can move to the Endgame (D), and Left, who is next, loses. If Left starts, he can move to the position that we have labelled (B). From there Right, who is then next, has no moves; whereas Left could move to the Endgame (C). The game A is thus $\{B|D\}$, where the game B is $\{C|\emptyset\}$, and both C and D are $\{\emptyset|\emptyset\}$. So writing it all out with empty sets as the definition prescribes, we get $A = \Big\{\{\{\emptyset|\emptyset\}|\emptyset\}|\{\emptyset|\emptyset\}\Big\}$.

The game can be described much easier as $\{1|0\}$, when we have introduced the notion of numbers, which we will do in chapter 4.

The term *game* is used both for the whole game and for the position in a game. As we can see, the definition justifies this double use.

We usually omit the braces around G^L and G^R for ease of notation. Instead we insert multiple partition lines, where we need to specify the precedence. Thus, we write $\{2 \mid\mid 1|0\}$ to mean $\Big\{ \ \{2\} \mid \{ \ \{1\}|\{0\} \ \}\Big\}$. We will return to the meaning of the numbers in section 4.1.

2.1 The Game of *Domineering*

The connection between our definition of a game and what we usually think of as a game, may be a bit hard to find. Let us therefore look at the game called Domineering as a concrete example. Domineering is played on a checkerboard (not necessarily 8x8, but always finite), where the two players take

Figure 2.1: A position in Domineering

turns in placing a domino, such that is covers exactly 2 squares. Left places his dominos in the North-South direction, and Right places hers in the East-West direction. Since the board is finite, the game will terminate with a player being unable to place a domino in the desired direction, and this player loses.

In Domineering it pays to look at the empty blocks. A 0x0 or 1x1 block is the Endgame, since neither player can place a domino there. In a 2x1 block that is oriented North-South Left can place one domino, whereas Right cannot place any and will lose whenever she gets the turn. In a 1x2 block oriented East-West Right can place one domino and Left can place none. A 3x1 block can only hold one domino, so either Left or Right can place one domino, depending on the orientation, whereas the other player loses instantly.

An oblong 4x1 block can hold two dominoes from one player and none from the other. The player who can place dominoes here even has a space to spare, or use later in the game if there are more blocks.

In a 3x2 block like the one on figure 2.1 Left can place his first domino in four positions, but these are all symmetric, so we need only mention one of them. Right can place her domino in three positions, two of them being symmetric.

Domineering is a partial game with some positions being numbers as we will see in chapter 4. However, not all the positions are numbers, and this is what makes the game interesting.

Figure 2.2: The game Domineering on a 3x2 board

2.2 Good, Bad and Fuzzy

Some games are clearly more advantageous for one player than other games. For example, Left likes the game $\{\ \{\emptyset|\emptyset\}\ |\ \emptyset\}$ with the game tree

```
    o
   /        A
  o
```

much better than the game $\{\emptyset\ |\ \{\emptyset|\emptyset\}\}$ with the game tree

```
  o
   \       B
    o
```

In the game A Left can move to the Endgame, whereas Right will simply lose, so Left will always win in this game. We call this game $+1$, because it is a victory for Left (who has our sympathy). In the game B it is opposite, Left will lose whenever he moves in it, whereas Right can move to the Endgame, such that Left loses. We call this game -1. We will explain these and more numbers in chapter 4.

Sometimes the outcome depends on who starts the game, and other times a game is a win for one player whatever the other player does, and whoever starts. With this in mind we can classify games in four outcome classes depending on who has a winning strategy in the game.

		Left starts	
		Left wins	Right wins
Right starts	Left wins	POSITIVE	ZERO
	Right wins	FUZZY	NEGATIVE

If Left wins no matter who starts, we say that the game is positive (our sympathy is with Left), and if Right wins no matter who starts, we say that the game is negative. But if the winner depends on who starts, it is not necessarily either a good or a bad game for us. Instead we say that the game is zero if the second player wins. This way the Endgame $\{\emptyset|\emptyset\}$ is zero, because whoever moves in it first loses. If the game is won by the first player, we call it fuzzy with zero, meaning incomparable to zero. More about fuzzy games in chapter 6.

2.2.1 $\mathcal{P}$- and $\mathcal{N}$-positions

In the so called impartial games (chapter 3 will be devoted to impartial games) both players have exactly the same possible moves (both players may move all the pieces), and there will only be two outcome classes: First player wins and second player wins, also called $\mathcal{N}$-positions and $\mathcal{P}$-positions. $\mathcal{N}$-positions are *Next player wins*-positions, meaning whoever moves next (first) from that position has a winning strategy. $\mathcal{P}$-positions are *Previous player wins*-positions, meaning whoever moves next (first) from that position does not have a winning strategy. All positions in impartial games without draws are either $\mathcal{N}$- or $\mathcal{P}$-positions. When a player moves in a $\mathcal{N}$-position, he always leaves a $\mathcal{P}$-position for his opponent; whereas if he moves in a $\mathcal{P}$-position it is possible for him to leave a $\mathcal{N}$-position for his opponent. The Endgame is obviously a $\mathcal{P}$-position, since whoever must move in it loses, and leaves the victory to the previous player.

Example 4 Let us analyze the positions in the impartial game with the following game tree:

```
    o
   / \
  o   o
 /   / \
o   o   o
   / \   \
  o   o   o
```

Since the game is impartial, both players are allowed the moves represented by the left and right branches.

All the leaves are $\mathcal{P}$-positions, because they are the Endgame. The positions that are followed by at least one $\mathcal{P}$-position are $\mathcal{N}$-positions, because the player who has the turn has a move to a $\mathcal{P}$-position (a good move for the player executing it). Positions that are not followed by any $\mathcal{P}$-positions must instead be $\mathcal{P}$-positions themselves, because the player who has the turn can only move to $\mathcal{N}$-positions, and it is thus the opponent who has a winning strategy. Applying this to the game, we have:

```
    No
    / \
  No  Po
  /   / \
Po  No  No
    / \   \
  Po  Po  Po
```

We realize that the game is a $\mathcal{N}$-position, so the players are eager to move first in it.

From our reasoning of which positions are $\mathcal{P}$-positions and $\mathcal{N}$-positions in the example, we can write the following definition for impartial games, i.e. games where the Left and Right options are identical:

Definition 2 *Let $G = \{G^X | G^X\}$ be an impartial game.*

$$G \in \mathcal{N} \iff \exists g^X \in G^X : g^X \in \mathcal{P}$$

$$G \in \mathcal{P} \iff \nexists g^X \in G^X : g^X \in \mathcal{P}$$

A $\mathcal{N}$-position is a position from which it is possible to move to a $\mathcal{P}$-position in one move. A $\mathcal{P}$-position is a position from which it is *not* possible to move to a $\mathcal{P}$-position in one move. The Endgame is a $\mathcal{P}$-position, and with that as base case we can determine all other positions from this definition.

2.3 Structure on Games

Let us return to games that are not necessarily impartial. In section 2.2 we saw that some games can be described as being positive and other as being negative, depending on who has a winning strategy. Now we will differentiate games a bit more, and introduce an ordering of (some) games. With this ordering we will be able to compare games, and decide which game a player should choose to play in. Furthermore we will need the structures we define in this section, when we define numbers as games in chapter 4.

Before we define any relations amongst games, let us define a concept called the *birthday* of a game; we will need this for some proofs by induction.

The birthday of a game is the number of iterations it takes to produce it from definition 1. If each of Left's options of a game can be produced in, say, n or less steps; and each of Right's options of the same game can be produced in, say, m or less steps, then the options have all been created after n or m iterations, whichever is larger. The game itself it then produced in the next iteration: $1 + max\{n, m\}$

Definition 3 (The Birthday of a Game) *:*
The birthday of a game $G = \{G^L|G^R\}$ is defined recursively as

$$1 + max\{Birthday(G^L), Birthday(G^R)\}$$

The base case is the game $\{\emptyset|\emptyset\}$ born on day 0.

Remark 1 *The birthday of a game is the maximum depth of the corresponding game tree. It is also the maximum number of moves left in the game.*

Example 5 Let G be a game with the following game tree:

```
          o
         / \
        o   o
       / \   \
      o   o   o
     /
    o
```

It consists of a left part G^L and a right part G^R:

```
            L
       o   G                   R
      / \                  o  G
     o   o                  \
    /                        o
   o
```

So we have to find the birthdays of these first. G^L consists of a left part G^{L^L} and a right part which is the Endgame. G^R does only have one option, and that is the Endgame. The Endgame is born on day 0, so G^R is born on day

$1 + max\{0\} = 1$. Similarly we see that G^{L^L} is born on day 1, so G^L is born on day $1 + max\{1, 0\} = 2$.

Now that we know the game G^L is born on day 2 and G^R is born on day 1, we conclude that the game $G = \{G^L|G^R\}$ is born on day $1+max\{2, 1\} = 3$.

We notice that the maximum depth of G's game tree is 3 too.

Now we are ready to introduce relations on games. The first relation we chose to define is *greater than or equal*. With this definition we can define other relations easier afterwards. When we say that a game, say G, is greater than or equal to another game, say H, it means that G is "at least as good' for Left as H is, and H is at least as good for Right as G is (we let Left be positive). So when G is greater than or equal to H, then Right cannot have any options in G that are better for her than H is (in other words, $\nexists g^R \in G^R : H \geq g^R$), and Left cannot have any options in H that are better for him than G is (in other words, $\nexists h^L \in H^L : h^L \geq G$). This way the relation *greater than or equal* is defined recursively. The Endgame is the base case, since there are no elements in the empty set and therefore $\nexists x \in \emptyset : Y$ is vacuously true for any condition Y - we will use this consideration a lot in the following proofs.

Definition 4 (greater than or equal) *:*
Let G and H be games, $G = \{G^L|G^R\}$ and $H = \{H^L|H^R\}$. We say that

$$\begin{aligned} G \geq H \quad \Longleftrightarrow \quad & \nexists g^R \in G^R : \ H \geq g^R \\ \textit{and} \quad & \nexists h^L \in H^L : \ h^L \geq G \end{aligned} \tag{2.1}$$

All the elements g^R and h^L are games themselves (since they are elements of sets of games), therefore they can be measured against the games G and H.

Example 6 Let us look at the games G and H with the following game trees again:

```
        o   G
       /  \              o     H
     o      o             \
    /                       o
  o
```

To test if $G \geq H$, we need to examine the relation between $H = \{\emptyset \mid \{\emptyset|\emptyset\}\}$ and all the elements of G^R, plus the relation between G and all elements of H^L.
Fortunately, $H^L = \emptyset$, so we immediately have $\nexists h^L \in H^L : \ h^L \geq G$ (1),

since there are no elements in H^L.
In the other case G^R, we only have the Endgame $\{\emptyset|\emptyset\}$ to test against H. We want to prove that $H \geq \{\emptyset|\emptyset\}$ is false.
We see that the element $\{\emptyset|\emptyset\} \in H^R$. Obviously, $\{\emptyset|\emptyset\} \geq \{\emptyset|\emptyset\}$ (we will show this shortly in theorem 1), so we cannot have $H \geq \{\emptyset|\emptyset\}$.
Therefore we also have $\nexists g^R \in G^R : \; H \geq g^R$ (2). The statements (1) and (2) imply that $G \geq H$. This is in accordance with our intuition that Left will rather play the game G than the game H (where he always loses).

From the definition of $\geq$ we can easily define more relations on games:

Definition 5 (other relations) *:*

$$
\begin{array}{lcl}
G \leq H & \iff & H \geq G \\
G \not\leq H & \iff & G \leq H \textit{ is false.} \\
G \not\geq H & \iff & G \geq H \textit{ is false.} \\
G = H & \iff & G \geq H \textit{ and } H \geq G \\
G > H & \iff & G \geq H \textit{ and } H \not\geq G \\
G < H & \iff & H > G
\end{array}
$$

The following statements are quite intuitive, but we should prove them anyway, since we will be using them intensely.

Theorem 1 *For all games $G = \{G^L|G^R\}$ we have*

1. $G \not\geq G^R$
2. $G^L \not\geq G$
3. $G \geq G$
4. $G = G$

By writing $G \not\geq G^R$ we mean that no element g^R in G^R can be less than or equal to G. This is not the same as stating $G < G^R$ (the games may be fuzzy).

Proof of theorem 1 We will prove all four statements together by induction over the birthday of G. As the base case we take G to have birthday 0, i.e. let first $G = \{\emptyset|\emptyset\}$:

1. $\{\emptyset|\emptyset\} \not\geq \emptyset$ because $\forall g^R \in \emptyset : \; \emptyset \geq g^R$ (no elements in the empty set)

2. Likewise, $\emptyset \not\geq \{\emptyset|\emptyset\}$ because $\forall g^L \in \emptyset : \; g^L \geq \emptyset$

3. $\{\emptyset|\emptyset\} \geq \{\emptyset|\emptyset\}$ because $\nexists g^R \in \emptyset : \; g^R \leq \{\emptyset|\emptyset\}$ and $\nexists g^L \in \emptyset : \quad \{\emptyset|\emptyset\} \leq g^L$ (from statement 1 and 2).

4. Statement 3 with the roles of G and G inverted, gives us $G \leq G$ too, so $G = G$.

Now to the induction step: Assume 1-4 hold for games with birthdays strictly smaller than the birthday of $G = \{G^L|G^R\}$. In particular 1-4 hold for G^L and G^R.

1. By hypothesis 3: $G^R \geq G^R$ so $\exists g^R \in G^R : \; g^R \leq G^R$, which implies $G \not\geq G^R$

2. Likewise, $G^L \geq G^L$ by hypothesis 3, so $\exists g^L \in G^L : \; G^L \leq g^L$, which implies $G^L \not\geq G$

3. Through statement 1 we have that $\nexists g^R \in G^R : \; g^R \leq G$, and through statement 2 we have that $\nexists g^L \in G^L : \; G \leq g^R$. So by the definition of the relation "$\geq$: $G \geq G$

4. $G \geq G$ and (with the roles inverted) $G \geq G$ imply $G = G$.

Example 7 Two games with different game trees may be equal to one another. Let us compare the Endgame `o` with the game that has the following tree:

```
       A
       o
     /   \
 B o       o C
     \   /
     D o o E
```

If Left starts, he will move to the game (B), and Right will answer by moving to (D); then Left has no moves and loses. If Right starts, she will move to the game (C), from where Left will move to (E). Now Right has no moves, and she loses. We can conclude that who ever starts, loses - just like in the Endgame; so we expect the game A to be equal to $\{\emptyset|\emptyset\}$.

Formally, we want to prove *1)* $\{\emptyset|\emptyset\} \geq A$ and *2)* $A \geq \{\emptyset|\emptyset\}$.

Re 1 By definition,

$$\{\emptyset|\emptyset\} \geq A \iff \nexists g^R \in \emptyset : \cdots \text{(OK)},$$
$$\text{and} \quad \nexists a^L \in A^L : \ a^L \geq \{\emptyset|\emptyset\}.$$

The only element in A^L is $\{\emptyset \mid \{\emptyset|\emptyset\}\ \}$, so let us prove that this game cannot be greater than or equal to $\{\emptyset|\emptyset\}$. That is easy, since there is an element, namely $\{\emptyset|\emptyset\} \in \{\emptyset \mid \{\emptyset|\emptyset\}\ \}^R$, such that $\{\emptyset|\emptyset\} \geq \{\emptyset|\emptyset\}$ (true by theorem 1). So $\{\emptyset|\emptyset\} \geq A$

Re 2 Similarly,

$$A \geq \{\emptyset|\emptyset\} \iff \nexists g^L \in \emptyset : \cdots \text{(OK)},$$
$$\text{and} \quad \nexists a^R \in A^R : \ \{\emptyset|\emptyset\} \geq a^R.$$

There is only one element in A^R, namely $\{\ \{\emptyset|\emptyset\} \mid \emptyset\}$, so we want to prove that $\{\emptyset|\emptyset\}$ cannot be greater than or equal to this game. But that is also easy, since there is an element, namely $\{\emptyset|\emptyset\} \in \{\ \{\emptyset|\emptyset\} \mid \emptyset\}^L$, such that $\{\emptyset|\emptyset\} \geq \{\emptyset|\emptyset\}$ (theorem 1). So $A \geq \{\emptyset|\emptyset\}$.

Now that we have $\{\emptyset|\emptyset\} \geq A$ and $A \geq \{\emptyset|\emptyset\}$, we can conclude that $A = \{\emptyset|\emptyset\}$, as we expected.
The conclusion is that games with different positions in them, may have the same outcomes as functions of who starts, even quantitatively. Not all elements in an outcome class are equal.

Now we can decide whether a game is better for Left than another game, or perhaps equal to it. Furthermore these relations can sometimes be used to simplify games, making the analysis easier for us. The following section is devoted to exploring two ways of simplifying games.

2.4 Dominated and Reversible Options

When a game contains options from two certain categories it is possible to make some simplifications. These categories are called *dominated options* and *reversible options*.

An option is said to be dominated if the player has a better one, i.e. a Left option is dominated if Left has a greater option, and a Right option is dominated if Right has a smaller one.

Definition 6 *Let $G = \{A, B, \ldots, C|D, E, \ldots, F\}$ be a game.*
If for the Left options $A \leq B$ then we say that A is dominated by B.
If for the Right options $D \geq E$ then we say that D is dominated by E.

Figure 2.3: The option D in G is bypassed in H

Notice the different inequalities for the Left and Right options.

Our definition of dominated options is quite similar to the definition of dominated options in another kind of game theory known from economics. And here too, we may delete the dominated options:

Theorem 2 *If a game contains dominated options these may be deleted, provided we retain the options that dominates them. The value of the game will not be changed by this operation.*

We will not attempt to prove this before we have introduced addition of games in chapter 5.

The other kind of simplification is called bypassing reversible options. A Right option, D, can be bypassed if it has a Left option D^L that is better for Left than G is; i.e. if $D^L \geq G$. We bypass D by replacing it in G with all the Right options, say $X, Y, \ldots, Z$ of D^L (see figure 2.3).

Left's reversible options can be bypassed as well. For any Left option, A, of G it is a Right option A^R that must be better for Right than G was, i.e. $A^R \leq G$ (notice the reversed inequality; Right is playing for the game to be negative).

Bypassing a reversible move does not change the value of the game. I.e. the game $G = \{A, B, \ldots, C | D, E, \ldots, F\}$ is equal to the game $H =$

$\{A, B, \ldots, C | X, Y, \ldots, Z, E, \ldots, F\}$, where we replaced D by the Right options of D^L (assuming $D^L \geq G$). This we will prove later. Let us write the statements about bypassing reversible moves as a theorem:

Theorem 3 *Let $G = \{A, B, \ldots, C | D, E, \ldots, F\}$ be a game.*
If any Right option D has a Left option D^L such that $D^L \geq G$, then we can simplify G by replacing D with all the Right options $X, Y, \ldots, Z$ of D^L.
Similarly, if any Left option A has a Right option A^R such that $A^R \leq G$, then we can simplify G by replacing A with all the Left options of A^R.
Such replacements do not change the value of the game.

We postpone the proof to section 5.3.

Now we have established two ways of simplifying games,

- Deleting dominated options
- Bypassing reversible options.

These tools prove useful when analyzing games.

Chapter 3

Impartial Games

Impartial games are games where both players have the same possible moves, i.e. games of the form $G = \{A, B, \cdots | A, B, \cdots\}$. Such games are much easier to analyze than partial ones, where the allowed moves are different for the two players. In this chapter we will introduce the celebrated game called Nim together with its analysis that is due to C.L. Bouton. We will also take a look at the works of R.P. Sprague and P.M. Grundy, who (independently) showed that the analysis of Nim implicitly contains the additive theory of all impartial games! This means that we will be able to analyze any impartial game by the end of this chapter.

3.1 The Game of Nim

Nim is played with sticks (or other tokens) placed in piles. The two players alternately take any number ($\in \mathbb{N}$) of sticks from any one pile. The game ends when there are no more sticks left to take. The winner is the player who took the last stick(s) - the opponent cannot move, and loses.

Example 8 An example of a position in Nim:

I

$I \quad I$

$I \quad I \quad I \quad I$

$I \quad I$

We have a pile of one stick, two piles of two, and a pile of four stick.

3.1.1 Nimbers

The analysis of Nim is quite cunning. But first we need to define *Nimbers* together with a special kind of addition called *Nim addition*.

A Nimber is simply the number of sticks in a pile in Nim (notice the combination of Nim and number). They are often represented as a number preceded with a star. To add Nimbers we have to convert them to binary numbers, and then add them without carrying (still in binary). The result (written in base 10) is called the Nim sum.

Let us look at three simple examples:

Example 9 $*2 + *4 = *6$, $*2 + *3 = *1$ and $*3 + *3 = *0$ because

```
*4 = 100                *2 = 10               *3 = 11
*2 = 010                *3 = 11               *3 = 11
     ---                     --                    --
     110   = *6              01 = *1               00 = *0
```

Nimbers have some interesting properties. The most significant one is shown in the third part of the example: they are their own inverse. In other words $*n + *n = *0$ for all Nimbers.

Nimbers, the numbers of Nim, form a field, as does the numbers of any other game, as we will see later in section 5.

3.1.2 The Analysis of Nim

Once we know Nimbers and Nim sums, we can quickly find the $\mathcal{P}$-positions in Nim. The $\mathcal{P}$-positions are namely exactly the ones with Nim sum zero. I.e. a winning move is a move to a position with Nim sum zero (and vice versa).

Example 10 Let us analyze the game in example 3.1:

Sticks	*# of sticks*	*in binary*	*Nim sum*
I	1	1	
II	2	10	
IIII	4	100	
II	2	10	
		101	= 5

From this position we want to move to a position with Nim sum 0. We can obtain this by removing three sticks from the third pile:

Sticks	*# of sticks*	*in binary*	*Nim sum*
I	1	1	
II	2	10	
I	1	1	
II	2	10	
		00	$= 0$

This is a $\mathcal{P}$-position, so whatever our opponent does, we will have a move to a $\mathcal{P}$-position again in our next move. This way we will eventually win, because we will at some point be able to take the last sticks (zero sticks is a $\mathcal{P}$-position).

Some positions are particularly easy to find good moves in, namely the positions with a high degree of symmetry, like the one above. Each Nimber is repeated an even number of times (We have two piles of one stick, two piles of two sticks and zero piles with other Nimbers). In such a position we can win by mirroring whatever our opponent does. If she takes one stick from the second pile, we take one stick from the fourth pile (that also contained two sticks). If she takes one from the first pile, we take one from the third pile. This is called the Tweedledum and Tweedledoo Strategy [1, pp 5], or the copy cat principle.

Sometimes it is not possible to move to a winning position using Nim sums - namely if we are faced with a game that has already got the Nim sum zero. In that case there is nothing mathematical left to do. We may choose to play randomly, or we can postpone our defeat as long as possible.

3.1.3 Bouton's Article

C.L.Bouton was the first to give an analysis with proofs of the game Nim. This he does in his article "Nim, a Game with a Complete Mathematical Theory" from 1902 [6]. His main concern is Nim with only three piles, but later in his article he generalizes his theorems to include any amount of piles. He does not give a proof of this generalization, because "The induction proof is so direct that it seems unnecessary to give it" [6, pp 39]. We will prove it later anyway.

Bouton states that his purpose is to prove that if one player leaves a certain kind of combination, and after that plays without mistakes, then the other player cannot win. He call such combinations *safe positions*, and we know them as $\mathcal{P}$-positions. He then states how to identify whether a position is safe or not, using what we recognize as Nim sums.

It is obvious, he argues, that if the size of two of the piles in a safe position are known, then the size of the third one is determined too. We can understand why, because we know that a Nimber is its own inverse. Let us call the sizes of the two known piles $*a$ and $*b$, and the size of the unknown one $*c$. The position is safe iff the Nim sum is zero: $*a + *b + *c = *0$. This implies $*a + *b = *c$. So $*c$ is indeed determined by $*a$ and $*b$ for a safe combination.

Bouton's first theorem states that *if player A leaves a safe combination, say* $(*a, *b, *c)$*, then player B cannot move to a safe position in one move.* This he proves with the fact that one move only alters one pile, say $*c$ is altered to $*c' \neq *c$, and leaves the two other piles ($*a$ and $*b$) unaltered. But if $(*a, *b, *c)$ is safe, then the position $(*a, *b, *c')$ cannot be safe too, because $*c$ is uniquely determined by $*a$ and $*b$ in a safe position.

Bouton's second theorem states that *if B is faced with a safe position* $(*a, *b, *c)$*, then no matter how she moves, A can regain a safe position in his next move.* Player B diminishes one pile, say $*c$ is altered to $*c'$ where $*c' < *c$. When comparing the two binary numbers $*c$ and $*c'$ digit-wise from left to right, the first change we will encounter is a 1 in c that has been altered to a 0 in c' (the number has been diminished). Only one of $*a$ and $*b$ has a 1 in the same place (there were an even number of 1s in the place before B's move), and this is the pile player A should alter. This digit is to be changed to a zero, leaving the more significant digits unaltered (they were OK); now we know that the move is to a smaller pile. The less significant digits are decided by the other two piles for the new position to be safe.

Example 11 Player A leaves the following safe Nim position for his opponent:

Sticks	*Nimber*	
II	010	
IIIII	101	
IIIIIII	111	
Nim sum	000	$= *0$

From the first theorem we know that player B cannot move to a safe position. Instead she moves to the following position:

Sticks	*Nimber*	
II	010	
IIIII	101	
IIII	100	
Nim sum	011	$= *3$

Now we know how player A can move to a safe position. The most significant digit that has been changed is the second. We see that the only pile that now has a 1 in this place is the first pile, therefore we want to change that. Taking the Nim sum of the other two piles, gives us the (unique) size we can change it to, to obtain a safe position: $*5 + *4 = *1$. We take a stick from the first pile, leaving the following position to player B:

Sticks	*Nimber*
I	001
IIIII	101
IIII	100
Nim sum	000

We recognize the theorems from our discussion of $\mathcal{N}$- and $\mathcal{P}$-positions, because safe positions are $\mathcal{P}$-positions.

Another thing Bouton does in his article is to calculate the chance of beginning the game with a safe position, if the tokens laid out is a random number between zero and 2^n. He also analyses the Misère version of Nim, where whoever takes the last stick loses. But we consider his analysis of the normal version of Nim most important.

As mentioned, we can generalize the analysis of Nim to include any number of piles. Bouton remarks that it can easily be proved by induction, but we will give a direct proof.

First we need to assert our notion of a safe combination.

Definition 7 *A safe combination is a set of numbers, such that when written in binary (in the scheme for addition) the sum of each column is even, i.e. congruent with zero (mod 2).*

Then we can write Bouton's Theorems generalized to any number of piles.

Theorem 4 (Bouton's First Theorem, generalized) *:*
If a player leaves a safe combination, $(*a, *b, \ldots, *c)$*, then the opponent cannot move to a safe position in one move.*

Proof of theorem 4 A move alters exactly one pile. The binary representation of this pile will change at least one digit, and this will be the only digit in that column to change. So the sum of the column cannot stay even; the new position cannot be safe.

Theorem 5 (Bouton's Second Theorem, generalized) *:*
If a player is faced with a safe position $(*a, *b, \ldots, *c)$*, then no matter how she moves, the opponent can regain a safe position in his next move.*

Proof of theorem 5 Let player B be the one faced with the safe position $(*a, *b, \ldots, *c)$. Now that it is her turn, she will have to reduce a pile, say $*b$ to $*b'$, where $*b' < *b$. We compare the binary representations of $*b$ and $*b'$ digit wise from left to right. The first difference we encounter is a digit where $*b$ has a 1 and $*b'$ has a 0 (because $*b' < *b$). There must be an odd number of piles that have a 1 in the same place in the new position (before B's move the number was even), and the opponent should alter any one of them. He should change this digit to a zero, leaving the more significant digits unaltered (we know they are OK), thus his move diminishes the pile as dictated by the rules. How much he should diminish the pile by, is decided by the less significant digits, since we want the sum of each column to be zero. This way he can regain a safe position. After a finite number of moves the safe position reached will be the Endgame, and we know the player who must move in that, loses.

Now we can analyze any game of Nim. In the next section we will learn to apply this analysis to include any impartial game. But first, let us look at a Nim game in disguise.

3.1.4 Turning Turtles

Instead of using turtles for this game, coins are used. A number of coins are laid out in a row. A legal move is to turn one or two coins, with the restriction that the rightmost coin must be turned from heads to tails. This game terminates since eventually all coins are tails.

Example 12 In the game

```
H T T H T H T
```

the first player can choose between three coins for his move from heads to tails. If he does not choose the leftmost coin, he has the option of turning a coin left of it as well; this may either go from heads to tails or vice versa. A legal move is to the position

```
H H T T T H T
```

but there are ten other legal moves to consider as well.

Turning Turtles is in fact Nim in disguise. We number the coins from left to right, and let a tail in the $n'th$ place correspond to a pile of n sticks in Nim, i.e. the Nimber $*n$.

In the example above the corresponding move in Nim is from

```
I
IIII
IIIIII
```

to

```
I
II
IIIIII
```

which is not very good, since it is a move to a $\mathcal{N}$-position. Using our knowledge of Nim, we know it would had been better to move to the $\mathcal{P}$-position

```
I
IIII
IIIII
```

or, in the Turning Turtles game:

```
H T T H H T T
```

This is a legal move, we have turned the sixth coin from heads to tails, and have chosen to turn the fifth coin (which is left of the sixth one) as well.

Now that we have a simple correspondence between positions in Turning Turtles and positions in Nim, we know how to analyze this new game as well.

3.2 Grundy's and Sprague's conclusions

Any impartial game can be regarded as a variant of Nim. This was the conclusion in Sprague's article "Über mathematische Kampfspiele" from 1935 [8] and independently hereof Grundy's article "Mathematics and Games" from 1939 [7].

Grundy and Sprague both partition the positions of a game in two classes: Winning and losing positions. These correspond to our $\mathcal{P}$- and $\mathcal{N}$-positions. Sprague refers to E. Lasker, who calls the two classes Gewinnstellungen G und Verluststellungen V. He defines Gewinnstellungen as the positions from where the player who has the move can enforce the victory, and the Verluststellungen as the positions where the other player can.

> In Gewinnstellungen, kurz: "G", kann *der* Spieler den Sieg erzwingen, der am Zuge ist, in Verluststellungen, "V", der andere. [8, pp 438]

He remarks that whoever causes a V-position can win, because in the subsequent moves he will always be able to present his opponent with a V-position.

Grundy begins by explaining how either player A or player B must have a winning strategy [1]. He then introduces the game Nim and Nim addition, before he defines winning positions W as positions with Nim sum zero, and losing positions L as the rest.

His first theorem states that from a L-position it is always possible to move to a W-position (if there are more moves left in the game), and that from a W-position it is impossible to move to another W-position (in one move).

Grundy remarks that a player, who has once moved to a W-position, can continue to do so in his subsequent turns, and thus win the game. This remark is comparable to Spragues remark "whoever causes a V-position can win ...". But the remarks are dissimilar in that Sprague is concerned with the class of the position *we present our opponent with* - we want to give her a *losing* position, whereas Grundy is concerned with the class of the position *we move to* - we want to move to a *winning* position. This is why the definitions are apparently contradictionary. To avoid the confusion of whether you should move in a winning position or hand over a losing position, the common notation has become $\mathcal{P}$ and $\mathcal{N}$ as we have seen, $\mathcal{P}$-positions being *Previous player winning* positions. In this terminology the winning moves are from $\mathcal{N}$-positions to $\mathcal{P}$-positions. We can conclude that V-positions = W-positions = $\mathcal{P}$-positions.

Example 13 Let us look at a game with the following game tree. In this tree either player can choose the left and right branches (it is an impartial game). The game ends, as usual, when a player reaches a leaf.

```
      o  A
     / \
 B  o    o  C
   /    / \
J o    oF  o  D
      / \    \
  H o    oG   o E
   / \     \
I o    oL   o M
   \
    o K
```

[1]or a drawing strategy, but we have excluded games with draws in our analysis

If we start, we can move to B or to C. If we moved to B, our opponent would move to J in her next turn, and thus win. So we move to C.

From C our opponent can either move to F or to D. Moving to D would give us the chance of moving to E and win in our next move, so she moves to F.

From F we can move to H or to G, but from either position our opponent can win in her next move. So the game A is a lose for the player who moves first: It is a $\mathcal{P}$-position.

Sprague and Grundy both introduce an important function (now sometimes called the Sprague-Grundy function, other times just the Grundy function) on a game. But first Sprague defines two more concepts: "Nachfolger" and "Grad". The Nachfolger, or follower as it means, is a well-known concept namely the descendant (in graph theory): A position T is a Nachfolger of S, if there exists a move from S to T. The Grad, or degree as it means, of a position is the maximum number of moves left in the game, counting from the position.

Example 14

```
            o  A
           / \            The Nachfolgers of A are B and C.
       B  o   o  C
         /                The Grad of A is 2.
     D  o
```

Now Sprague and Grundy are ready to introduce their function, which Sprague calls *Rang* and Grundy calls Ω.

Sprague states that we can assign a non-negative integer called "Rang" with the following properties to the positions of any finite game.

A) No position has any followers with the same Rang.

B) Every position with Rang $R > 0$ has for every integer P with $0 \leq P < R$ at least one Nachfolger (descendant) with Rang P.

Grundy states that for games in which the last player to have a legal move wins, the classification of positions can be made easier (than recursing W- and L-positions) by means of a function $\Omega(X)$ of the position X, where $\Omega(X)$ is a non-negative integer determined by the following properties:

1. Any one move changes the value of $\Omega(X)$.
2. For any integer ω with $0 \leq \omega < \Omega(X)$, the value of Ω can be decreased to ω in one move.
3. $\Omega(X) = 0$ when X is terminal (i.e. the Endgame).

Grundy does not prove the theorem, but concludes that a position X is a winning position iff $\Omega(X) = 0$. Sprague proves the theorem by induction over the Grad:

Proof *As the base case we have the positions with Grad 0 or 1. The Grad 0 positions are the Endgames, and they are assigned Rang 0. Properties A) and B) are trivially satisfied (no Nachfolgers). The Grad 1 positions have the Endgames as Nachfolger, and they are assigned Rang 1. All their Nachfolgers (at least one) have Rang $0 \neq 1$, and the properties A) and B) are satisfied. We now assume that the theorem holds for all games with Grad g or less. Then the theorem also holds for games of Grad g+1, as we assign the Rang as the least positive integer that is not the Rang of any of the Nachfolgers.*

The Rang is uniquely determined through A) and B). Property B) further assures that the Endgame can only have the Rang 0. With this comment we see that Sprague's and Grundy's theorems are equivalent.

The function Sprague and Grundy defines is also called the *mex*-value (**m**inimal **ex**cluded value) of the followers.

Example 15 $mex\{0,1\} = 2$, $mex\{0,1,2,4,7\} = 3$, $mex\{5\} = 0$.

The function *mex* is applied recursively to the options of a position in the sense that for the game $G = \{a, b, \ldots | a, b, \ldots\}$

$$\Omega(G) = mex(\Omega(a, b, \ldots))$$

The recursion starts with the Endgame $\emptyset$, because $\Omega(\emptyset) = 0$. So the function Ω on a game is mex of the function Ω applied to the options of the game.

Example 16 Consider the game with the following tree:

```
      A o
       / \
    B o   o C
     / \
  D o   o E
   /
F o
```

We know that Ω on the leaves is 0, so

$$\Omega(F) = \Omega(E) = \Omega(C) = 0$$

The position D has only one option, namely F, so

$$\Omega(D) = mex(\Omega(F)) = mex\{0\} = 1$$

The position B has two options, namely D and E, so

$$\Omega(B) = mex(\Omega(D), \Omega(E)) = mex\{1, 0\} = 2$$

The position A has the two options B and C, so

$$\Omega(A) = mex(\Omega(B), \Omega(C)) = mex\{2, 0\} = 1$$

We have now found the Ω-value for the game A to be 1. We will show that this means that the game is equivalent with the game of one Nim stick.

Sprague's next theorem states that a position in a games is either a G-position or a V-position, depending on whether the Rang of the piles in the corresponding Nim-game yields a G- or a V-position. He then remarks that this theorem implies that any (impartial) game can be seen as a game of Nim:

> Mit Rücksicht auf diesen Satz erscheint jedes Gesamtspiel als ein verallgemeinertes Nim. [8, pp 441]

He now gives a variant of Bouton's theorem for finding the Rang of Nim-positions: Writing the Rangs of the piles in Nim in a scheme like the one used for normal addition, we can obtain a binary number R, where the digits are either 0 or 1, depending on whether there is an even or an odd number of 1's in the column. The value of R is the Rang of the Nim-game. This is just another wording for the analysis of Nim, we gave above.

Grundy, on the other hand, simply states that from the definition of the function $\Omega(X)$, it may be deduced that the Ω-function of several components is the Nim-sum of the Ω-function of the components

$$\Omega(XY \ldots) = \Omega(X) +_N \Omega(Y) +_N \ldots$$

where $+_N$ denotes Nim-sum. The components of a game are the parts it can be split in - like the piles in Nim - such that a move only alters one component.

With this statement Grundy reaches the fact that any (impartial) game can be translated (by the function Ω) into numbers that add like nimbers, i.e. the game can be regarded as a Nim variant.

In some games it may be possible to make a move that corresponds to putting Nim sticks instead of taking them. We must, however, assume that the stock of Nim sticks is limited, for the game to be finite. So if any player uses his move to puts down sticks, his opponent can reverse (cf section 2.4) his move by taking the same number of sticks.

Grundy and Sprague have both established a function that translates positions of any (impartial) game to a Nimber, so we can analyze the game as a Nim game.

Chapter 4

Numbers

4.1 Games as Reals and Surreals

In this section we will see that we can define all numbers as games, and that games in fact define more than the reals, such that we can extend our field of numbers to what has been called the surreals.

The definition of numbers is very close to the definition of games. The only difference is that we demand all Left options be strictly smaller than all Right options.

Definition 8 (Numbers) *:*
Numbers are defined recursively by $\{G^L|G^R\}$, where G^L and G^R are sets of numbers with the restriction that no member of G^L is greater than or equal to any member of G^R, i.e.

$$\forall g^L \in G^L \text{ and } \forall g^R \in G^R : \quad g^L < g^R$$

The basis of the induction is $\{\emptyset|\emptyset\}$.

The restriction differs from theorem 1 by being a positive assertion. In theorem 1 we have $G^L \not\geq G^R$, whereas we have $G^L < G^R$ here (exactly for numbers it is the same).

The definition starts off with the Endgame, where neither player has any legal moves: $\{\emptyset|\emptyset\}$. The Endgame satisfies the definition of being a number, since a set of numbers can be empty, and no member of $G^L = \emptyset$ is greater than or equal to any member of $G^R = \emptyset$ (there are no members). This number is called 0, and it behaves like the zero we know. We will later show that $-0 = 0 + 0 = 0$, but for now we will just verify that $0 \geq 0$ and $0 = 0$:

Example 17 $0 = 0$:
If $G = \{\emptyset|\emptyset\}$ and $H = \{\emptyset|\emptyset\}$, then G^R and H^L are empty, and vacuously,

$$\nexists g^R \in G^R : \; g^R \leq 0 \text{ and } \; \nexists h^L \in H^L : \; 0 \leq h^L$$

So $G \geq H$ by definition 4. Similarly we get that $H \geq G$, so $G = H$.

4.1.1 Numbers Born on Day 1

With 0 and $\emptyset$ we can generate three new games on day one (recall definition 3): $\{\{0\}|\emptyset\}$, $\{\emptyset|\{0\}\}$ and $\{\{0\}|\{0\}\}$. Only two of these are numbers:
The game $\{\{0\}|\emptyset\}$ is a number, because $\forall x \in \emptyset : \quad 0 < x$
Likewise is $\{\emptyset|\{0\}\}$ a number, because $\forall x \in \emptyset : \quad x < 0$
The game $\{\{0\}|\{0\}\}$, however, is not a number because $0 < 0$ is false.

The number $\{\{0\}|\emptyset\}$ is called 1, and the number $\{\emptyset|\{0\}\}$ is called -1. The game $\{\{0\}|\{0\}\}$ is called $*$, and we will return to this game later. Fortunately, we can show that the numbers relate in the ordinary way, $-1 < 0 < 1$, $-1 = -1$, etc. Let us just show $-1 < 0$ and $0 < 1$:

Example 18 $[-1 < 0]$:
Let $G = \{\emptyset|\{0\}\}$ and $H = \{\emptyset|\emptyset\}$. We want to show that $G \ngeq H$ and $H \geq G$.
We have $G \ngeq H$, because $0 \in G^R$ and $0 \leq H = 0$ (definition 4)
and $H \geq G$, because both H^R and G^L are empty, so
$\nexists h^R \in H^R : \; h^R \leq -1$ and $\nexists g^L \in G^L : \; 0 \leq g^L \Rightarrow H \geq G$
Therefore $G < H$, q.e.d.

Example 19 $[0 < 1]$:
Let $H = \{\emptyset|\emptyset\}$ and $K = \{\{0\}|\emptyset\}$. We want to show that $H \ngeq K$ and $K \geq H$.
$H \ngeq K$, because $0 \in K^L$ and $0 = H \leq 0$.
$K \geq H$, because both K^R and H^L are empty, so
$\nexists k^R \in K^R : \; k^R \leq 0$ and $\nexists h^L \in H^L : \; -1 \leq h^L \Rightarrow K \geq H$
Therefore $H < K$, q.e.d.

In domineering:

4.1.2 Numbers Born on Day 2

With the three numbers $-1, 0$ and 1 and the empty set we can generate 64 games, including 20 numbers. Many of these, however, are multiplets or copies of games that have already been generated. For example the games $\{-1|\emptyset\}$, $\{\emptyset|1\}$ and $\{-1|1\}$ are all equal to the number 0 that was generated on the 0'th day.

We recall from example 2.3 that numbers do not define games uniquely, (the mapping from games to numbers is surjective but not injective).

A total of four new numbers are created, namely

$$\{-1|0\} = -\frac{1}{2}, \qquad \{0|1\} = \frac{1}{2}, \qquad \{\emptyset| -1\} = -2 \quad \text{and} \quad \{1|\emptyset\} = 2$$

In domineering:

All four numbers are generated in more than one way, e.g.:
$\{\{1\}|\emptyset\} = \{\{0,1\}|\emptyset\} = \{\{-1,1\}|\emptyset\} = \{\{-1,0,1\}|\emptyset\} = 2.$

However, we only need the greatest element of the set G^L, and the least element of the set G^R; because these are the options in question when we compare games (the other options are dominated anyhow). So it is fair to represent our four new numbers in the above way, with only one element in each set.

Going through all the relations of these numbers to justify the names, will quickly become tedious. Fortunately the *simplicity theorem* tells us what number a game is.

Theorem 6 (The Simplicity Theorem) *:*
Suppose for the game $G = \{G^L|G^R\}$ some number H satisfies the equation

$$g^L \ngeq H \ngeq g^R \qquad \forall g^R \in G^R \quad \forall g^L \in G^L$$

but no *number H' with birthday before H satisfies the same equation.*
Then $H = G$.

This theorem tells us that if a game is equal to a number, then it equal to the simplest (i.e. born earliest) number that fits strictly between all the Left and all the Right option. Thus, the game $\{\frac{1}{2}|2\frac{1}{2}\}$ turnes out to be the number 1 rather than the average of the Left and Right options $1\frac{1}{2}$.

Proof of theorem 6 [The Simplicity Theorem]:
Let $G = \{G^L|G^R\}$ be a game and let $H = \{H^L|H^R\}$ be a number satisfying:

1. $g^L \ngeq H \ngeq g^R, \quad \forall g^L \in G^L, \quad \forall g^R \in G^R$, and

2. No number H' with birthday before H satisfies **1**.

To show the equality $G = H$ we want to show that $G \geq H$ and $G \leq H$. This we will do by contradiction.

"$\geq$" Suppose first that $G \ngeq H$, then by definition 4 either

a. $\exists g^R \in G^R : \quad g^R \leq H$ or

b. $\exists h^L \in H^L : \quad G \leq h^L$

Re a. False by hypothesis **1**, because $g^R \nleq H, \quad \forall g^R \in G^R$

Re b. All h^L are born before H (otherwise H could not be born), so given a Left option $h^L \in H^L$ we have assumed in **2** that $g^L \ngeq h^L \ngeq g^R$ is *not* true, i.e. (for some $g^L \in G^L, g^R \in G^R$) either

c. $g^L \geq h^L$ or

d. $h^L \geq g^R$

Re c. Assuming $g^L \geq h^L$ and combining this with hypothesis **b**, we get $g^L \geq h^L \geq G \Rightarrow g^L \geq G$ in contradiction with theorem 1.

Re d. Let us instead assume $h^L \geq g^R$. However, H is a number, so $H > h^L$, and we get $H > h^L \geq g^R \Rightarrow H > g^R$ in contradiction with hypothesis **1**.

Now we have established that $G \leq h^L$ leads to contradictions, and that $g^R \leq H$ is false as well, so we cannot have $G \ngeq H$, and therefore we must have $G \geq H$.

"$\leq$" We can prove $G \leq H$ similarly to $G \geq H$:
Suppose that $G \nleq H$, then either

e. $\exists h^R \in H^R : \quad G \geq H^R$ or

f. $\exists g^L \in G^L : \quad G^L \geq H$

Re f. False by hypothesis **1**.

Re e. All h^R are born before H, so given a Right option $h^R \in H^R$ we have assumed in **2** that $g^L \ngeq h^R \ngeq g^R$ is *not* true, i.e. (for some $g^L \in G^L, g^R \in G^R$) either

g. $g^L \geq h^R$ or

h. $h^R \geq g^R$

Re g. Assuming $g^L \geq h^R$. H is a number, so $h^R > H$, implying $g^L \geq h^R > H \Rightarrow g^L > H$, which is false by hypothesis **1**.

Re h. Assuming $g^R \geq h^R$ and combining this with hypothesis **e**, we get $G \geq G^R$ in contradiction with theorem 1.

Now we have established that $G \geq h^R$ leads to contradictions, and that $g^L \geq H$ is false as well, so we cannot have $G \nleq H$, and therefore we must have $G \leq H$.

All in all we have proved that $G = H$, q.e.d.

4.2 More Numbers

From the numbers on day 0, 1 and 2, we can begin to see a pattern of how the numbers are generated simplest. The numbers $+n$ and $-n$ are both generated on the n'th day by $n-1$ and $\emptyset$. For the next step, we can write:

$$-n-1 = \{\emptyset | -n\} \quad \text{and} \quad n+1 = \{n|\emptyset\}$$

This resembles Cantors and von Neumanns system of ordeal numbers, where $n+1 = \{0, 1, \ldots, n\}$, whereas we can write $n+1 = \{\{0, 1, \ldots, n\}|\emptyset\}$. Even for transfinite numbers this similarity holds: Von Neumann has $\alpha = \{\beta : \beta < \alpha\}$, and we could prove (by induction) that $\alpha = \{\{\beta < \alpha\}|\emptyset\}$.

For all the numbers between the integers, we turn to Dedekind (or the author of the fifth book of Euclid), who constructed the reals from the rationals by means of sections. The method looks very much like our definition of numbers, because it defines a real as a number strictly between two rationals (c.f. the Simplicity Theorem).

Fractions of powers of 2 are easily generated, namely as:

$$\frac{1}{2^n} = \{0 \mid \frac{1}{2^{n-1}}\}, \text{ and } -\frac{1}{2^n} = \{-\frac{1}{2^{n-1}} \mid 0\}.$$

These fractions combined with the integers gives us the so called dyadic rationals (i.e. numbers of the form $\frac{m}{2^n}$ where $m, n \in \mathbb{Z}$), that are generated by:

$$\frac{2n+1}{2^{m+1}} = \{\frac{n}{2^m}|\frac{n+1}{2^m}\}$$

In fact the dyadic rationals generate all the remaining rationals and all the reals by the so called Dedekind sections.

If we have a series that converge upwards and one that converge downwards towards a number, the two series generate the number as a game. For example, $\sum_{i=1}^{\infty} \frac{1}{2^{2i}} = \frac{1}{3}$ and $\frac{1}{2} - \sum_{i=1}^{\infty} \frac{1}{2^{2i+1}} = \frac{1}{3}$. Thus, we have:

$$\frac{1}{3} = \{\frac{1}{4}, \frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{1}{16} + \frac{1}{64}, \ldots | \frac{1}{2}, \frac{1}{2} - \frac{1}{8}, \frac{1}{2} - \frac{1}{8} - \frac{1}{32}, \ldots\}$$

In definition 12 we will give the formula for the multiplicative inverse, i.e. for $\frac{1}{n}$. Indeed, the expression we get then as the multiplicative inverse of 3 is the same as above.

The number $\frac{1}{3}$ is born on the $\aleph_0$th day. So is infinity: $\omega = \{0, 1, 2, 3, \ldots |\}$, $-\omega = \{|0, -1, -2, -3, \ldots\}$, and $\epsilon = \frac{1}{\omega} = \{0|1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$, fortunately. It is possible to prove that $\epsilon \cdot \omega = 1$, but we will not endeavour to do that.

After creating these numbers, we may use them to create new interesting numbers. For example, $\frac{\omega}{2} = \{0, 1, 2, 3, \ldots |\omega, \omega - 1, \omega - 2, \ldots\}$. And another interesting surreal number: $\sqrt{\omega} = \{0, 1, 2, 3, \ldots |\omega, \frac{\omega}{2}, \frac{\omega}{4}, \frac{\omega}{8}, \ldots\}$. In fact we can create any number we can think of, as long as we can find sequences approaching it.

However, not all games are numbers. We have already encountered Nimbers, a representation of some games that are not numbers. In subsection 4.1.1 we realized that the game $\{0|0\}$ is not a number. Instead we call it $*$. Likewise, we have the games $1 + * = \{1|1\}$ called $1*$, and $n + * = \{n|n\}$ called $n*$. In theorem 6.1 we will learn why $n + * = \{n|n\}$ for any number n. With these games we can create myriads of more games, some of which have been named, e.g. $\{0|*\}$ is called $\uparrow$. We will return to games that are not numbers in the chapter 6.

Chapter 5

Algebraic Structure

The reals as we know them, form a totally ordered field. In this chapter we will show that this is true for the numbers we have defined too. We will start by expanding our view to all (combinatorial) games, and find structures there as well. Most of the propositions are proved in [3].

5.1 Games Form an Additive Group

Let us begin by defining a rule for addition of games. This definition, that is due to Conway [3], is a natural one: The player makes a normal move in one of the compounds, leaving the other compounds unaltered. Many games can be viewed as sums of disjunctive compounds. In the game Domineering, for example, the whole game can be viewed as a sum of all the empty blocks on the board.

Definition 9 (Sum of games) *:*
The sum of the two games G and H is defined recursively as,

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$$

The basis is the empty set.

By $G^L + H$ we mean the set that comes from adding H to each element of G^L, and likewise for other sums.

We will do some examples of addition right after defining the negative of a game, that is the game we obtain if we interchange the roles of Left and Right.

Definition 10 (The negative game) *:*
The negative of a game G is (defined recursively as):

$$-G = \{-G^R | -G^L\}$$

(with basis $\emptyset$)

This definition is due to Conway as well.

As examples of how games are added (and subtracted), let us do $-0 = 0 + 0 = 0$, as promised, together with $(-1) + (-1)$ and $\frac{1}{2} + \frac{1}{2}$:

Example 20 [0+0]:
It is easy to show that $0+0 = 0$, since 0 added to each element of the empty set is the empty set, in other words,

$$0 + 0 = \{\emptyset + 0, 0 + \emptyset | \emptyset + 0, 0 + \emptyset\} = \{\emptyset|\emptyset\} = 0$$

In terms of games we understand that $0 + 0 = 0$, because no matter how many copies of the Endgame we have, it is still not possible to find any legal moves.

Likewise, all elements of the empty set negated is still the empty set, so $0 = -0$. If Left and Right change seats in the Endgame, there are still no legal moves.

Example 21 $[-1 + (-1)]$:
To prove that $-1 + (-1) = -2$, we simply use the definition of addition, recalling that $-1 = \{\emptyset|0\}$:

$$(-1) + (-1) = \{\emptyset|0\} + \{\emptyset|0\}$$

By the definition, the left hand side becomes all elements of the empty set added to (-1), which is the empty set; and -1 plus all elements of the empty set, which also is the empty set.
On the right hand side we get $0 + (-1)$ and $-1 + 0$, which both equal -1:

$$0 + (-1) = \{\emptyset + (-1), 0 + \emptyset | \emptyset - 1, 0 + 0\} = \{\emptyset|0\} = -1$$

$$-1 + 0 = \{\emptyset + 0, -1 + \emptyset | 0 + 0, -1 + \emptyset\} = \{\emptyset|0\} = -1$$

All in all we get

$$(-1) + (-1) = \{\emptyset| - 1\} = -2$$

as expected.

Example 22 $[\frac{1}{2} + \frac{1}{2}]$:
We generated $\frac{1}{2}$ as $\{0|1\}$, so $\frac{1}{2} + \frac{1}{2} = \{0|1\} + \{0|1\}$.

By our definition of addition, this equals: $\{0+\frac{1}{2}, \frac{1}{2}+0|1+\frac{1}{2}, \frac{1}{2}+1\}$ Assuming addition is associative, which we will prove shortly, this is equal to $\{0+\frac{1}{2}|1+\frac{1}{2}\}$. By the simplicity theorem, $\frac{1}{2}+\frac{1}{2}$ is now equal to the simplest number strictly between $0+\frac{1}{2}$ and $1+\frac{1}{2}$, i.e. 1.

In the group $(\mathbb{R}, +)$ we know that the neutral element is 0, so we expect that to be the case in our definition of the numbers too. It turns out that $0 = \{\emptyset|\emptyset\}$ is the neutral element with respect to addition, not just for numbers, but for all games.

Proposition 1 (0 is the neutral element for +) *:*
For all games $G = \{G^L|G^R\}$ we have:

$$G + 0 \ = \ G$$

Proof Let us prove this by induction over the birthday of the game G. For the base case $G = 0$, and we have $0 + 0 = 0$, which we have already proved to be true.
Now assume that $G^L + 0 = G^L$ and $G^R + 0 = G^R$ (both G^L and G^R are born before G).
Using this together with the definition of addition, we can get:

$$G + 0 = \{G^L + 0, G + \emptyset|G^R + 0, G + \emptyset\} = \{G^L|G^R\} = G$$

We have now shown for an arbitrary game G that $G + 0 = G$. The equation $0 + G = G$ is shown similarly, and we have established that 0 is the neutral element for addition.

To prove that the game $-G$ is in fact the additive inverse of G, we will need the following lemma:

Lemma 1
1. *If $G^L + (-G^L) \leq 0$ then $G^L + (-G) \ngeq 0$*
2. *If $G^R + (-G^R) \leq 0$ then $G + (-G^R) \ngeq 0$*
3. *If $G^R + (-G^R) \geq 0$ then $G^R + (-G) \nleq 0$*
4. *If $G^L + (-G^L) \geq 0$ then $G + (-G^L) \nleq 0$*

Proof This proof is straight forward, we do one statement at a time.

1. Assume $G^L + (-G^L) \leq 0$.
   We have that
   $$G^L + (-G) = \{G^{L^L} + (-G), G^L + (-G^R)|G^{L^R} + (-G), G^L + (-G^L)\}$$
   So there is an element, namely $G^L + (-G^L)$ in $(G^L + (-G))^R$, with $G^L + (-G^L) \leq 0$. By definition of $\geq$, we now know that we cannot have $G^L + (-G) \geq 0$.

2. Assume $G^R + (-G^R) \leq 0$. We have that

$$G - G^R = \{G^L + (-G^R), G + (-G^{R^R}) | G^R + (-G^R), G + (-G^{R^L})\}$$

So there is an element, namely $G^R + (-G^R)$ in $(G + (-G^R))^R$, with $G^R + (-G^R) \leq 0$. By definition of $\geq$, we now know that we cannot have $G + (-G^R) \geq 0$.

3. Assume $G^R + (-G^R) \geq 0$. We have that

$$G^R + (-G) = \{G^{R^L} + (-G), G^R + (-G^R) | G^{R^R} + (-G), G^R + (-G^L)\}$$

So there is an element, namely $G^R + (-G^R)$ in $(G^R + (-G))^L$, with $0 \leq G^R + (-G^R)$. By definition of $\geq$, we now know that we cannot have $0 \geq G^R + (-G)$.

4. Assume $G^L + (-G^L) \geq 0$. We have that

$$G + (-G^L) = \{G^L + (-G^L), G + (-G^{L^R}) | G^R + (-G^L), G + (-G^{L^L})\}$$

So there is an element, namely $G^L + (-G^L)$ in $(G + (-G^L))^L$, with $0 \leq G^L + (-G^L)$. By definition of $\geq$, we now know that we cannot have $0 \geq G + (-G^L)$.

The negative of a game is its inverse, and in fact any game has an inverse game with respect to addition:

Proposition 2 (The inverse of a game) *:*
For all games $G = \{G^L | G^R\}$ we have: $G + (-G) \ = \ 0$

Proof We will prove this using induction over the birthday n of G.
For the base case we have $0 + (-0) = 0 + 0 = 0$ (from the example).
Assume now that $G^R + (-G^R) = 0$ and $G^L + (-G^L) = 0$ (both games G^R and G^L have birthdays before G). In particular $G^L + (-G^L) \leq 0$, $G^R + (-G^R) \leq 0$, $G^R + (-G^R) \geq 0$, and $G^L + (-G^L) \geq 0$, so we have the four prerequisites from lemma 1, yielding to four inequalities $G^L + (-G) \ngeq 0$, $G + (-G^R) \ngeq 0$, $G^R + (-G) \nleq 0$ and $G + (-G^L) \nleq 0$.

Using the definition of negative and addition, we have:

$$\begin{aligned} G + (-G) &= \{G^L | G^R\} + \{-G^R | -G^L\} \\ &= \{G^L + (-G),\ G + (-G^R) \mid G^R + (-G),\ G + (-G^L)\} \end{aligned}$$

Let us call $G + (-G)$ for $K = \{K^L | K^R\}$. We want to show that $K \geq 0$ and $0 \geq K$.

By definition of $\geq$, $K \geq 0 \iff \nexists k^R \in K^R : 0 \geq k^R$ and $\nexists h^L \in \emptyset : h^L \geq K$. The second statement is trivially true, since there are no elements in the empty set. The first statement is true because both parts of K^R are $\not\leq 0$. So $K \geq 0$.
Similarly, $0 \geq K \iff \nexists g^R \in \emptyset \ldots$, true; and $\nexists k^L \in K^L : k^L \geq 0$, true because both parts of K^L are $\not\geq 0$.
All in all we have $K = 0$, q.e.d.

Addition of games is associative:

Proposition 3 (The associative law for addition of games) *:*
For all games G, H, K :

$$(G + H)\ + K\ =\ G +\ (H + K)$$

Proof Let $G = \{G^L | G^R\}$, $H = \{H^L | H^R\}$ and $K = \{K^L | K^R\}$ all be games. Let n be the sum of their birthdays. We want to prove the associative law by induction over n.
As the base case we take $n = 0$, i.e. $G = H = K = \{\emptyset | \emptyset\} = 0$:
We have already proved that $0 + 0 = 0$, so $(0 + 0)\ + 0\ =\ 0 + 0 = 0$ as we wanted.
Now, assume the associative law holds $n - 1$. Using the definition of addition twice, we get:

$$\begin{aligned} (G + H) + K &= \{G^L + H, G + H^L \mid G^R + H, G + H^R\} + K \\ &= \{(G^L + H) + K,\ (G + H^L) + K,\ (G + H) + K^L \mid \\ &\quad (G^R + H) + K,\ (G + H^R) + K,\ (G + H) + K^R\} \end{aligned}$$

Each of these six parts of the game $(G + H) + K$ are born on day $n - 1$, so by the induction hypothesis they obey the associative law; and we have:

$$\begin{aligned} &= \{G^L + (H + K), G + (H^L + K), G + (H + K^L) \mid \\ &\quad G^R + (H + K), G + (H^R + K), G + (H + K^R)\} \\ &= G + \{H^L + K, H + K^L \mid H^R + K, H + K^R\} \\ &= G + (H + K), \qquad \textit{q.e.d.} \end{aligned}$$

Games commute under the act of addition:

Proposition 4 (Addition of games is commutative) *:*
For all games G *and* H : $\quad G + H = H + G$

Proof Let $G = \{G^L|G^R\}$ and $H = \{H^L|H^R\}$ be games. Let n be the sum of their birthdays. We want to prove the commutative law by induction over n.
As the base case we take $n = 0$, i.e. $G = H = \{\emptyset|\emptyset\} = 0$:
$0 + 0 = 0 = 0 + 0$, OK.
Now, let us assume that the commutative law holds for $n - 1$.
We have: $G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$
Since each of the four parts of the game $G + H$ is born on day $n - 1$, they commute by hypothesis. We also know that the order of a set is indifferent, so:
$\{G^L + H, G + H^L \mid G^R + H, G + H^R\} = \{H^L + G, H + G^L \mid H^R + G, H + G^R\}$, which by definition is $H + G$, q.e.d.

We have now shown that games form an Abelian group under addition!

5.2 Numbers form a Field

Multiplying games is not as simple as adding them. If we expect the definition to be as simple as the one for addition, we will get disappointed when trying it on examples. Instead of trying that, let us turn directly to the definition of multiplication as given by Conway [3]:

Definition 11 (Product of games) *:*
The product of the two numbers G and H is defined recursively as:

$$\begin{aligned} G \cdot H \;=\; & \{G^L H + G H^L - G^L H^L \,,\; G^R H + G H^R - G^R H^R \mid \\ & G^L H + G H^R - G^L H^R \,,\; G^R H + G H^L - G^R H^L\} \end{aligned}$$

With $\emptyset$ as basis.

He explains the definition [3, page 6] by looking at the fact that for numbers $G - G^L > 0$ and $H - H^L > 0$, deducing that the product $(G - G^L)(H - H^L) > 0$, so $G \cdot H > G^L H + G H^L - G^L H^L$, and similarly for the three other parts of the product $G \cdot H$.

By $G^L H$ we mean the list of all elements of G^L each multiplied by H, and likewise for the other parts of the product.

Multiplication has a neutral element, but before we show that, we need the following:

Lemma 2 *For all numbers G, H:*

$$G \cdot 0 = 0 \;\; \textit{and} \;\; 0 \cdot H = 0$$

Proof

$$\begin{aligned} G\cdot 0 &= \{G^L\cdot 0+G\cdot\emptyset-G^L\cdot\emptyset,\ G^R\cdot 0+G\cdot\emptyset-G^R\cdot\emptyset| \\ &\quad G^L\cdot 0+G\cdot\emptyset-G^L\cdot\emptyset,\ G^R\cdot 0+G\cdot\emptyset-G^R\cdot\emptyset\} \\ &= \{\emptyset|\emptyset\}=0 \end{aligned}$$

Because the empty set remains empty, even when multiplied or added (there are no elements to add anything to). And similarly:

$$\begin{aligned} 0\cdot H &= \{\emptyset\cdot H+0\cdot H^L-\emptyset\cdot H^L,\emptyset\cdot H+0\cdot H^R-\emptyset\cdot H^R| \\ &\quad \emptyset\cdot H+0\cdot H^R-\emptyset\cdot H^R,\emptyset\cdot H+0\cdot H^L-\emptyset\cdot H^L\} \\ &= \{\emptyset|\emptyset\}=0 \end{aligned}$$

The neutral element for multiplication is 1:

Proposition 5 (1 is neutral in multiplication) *:*
For all numbers G and H:

$$G\cdot 1=G \ \ \textit{and}\ \ 1\cdot H=H$$

Proof Let us prove the proposition by induction over the birthday of the game. For the base case we have $0\cdot 1=0$ and $1\cdot 0=0$, which we know is true by lemma 2.

Now assume that the first part of proposition 5 holds for all games born before G (in particular, it holds for G^L and G^R then), and let us prove it holds for G as well:

$$\begin{aligned} G\cdot 1 &= \{G^L\cdot 1+G\cdot 0-G^L\cdot 0,G^R\cdot 1+G\cdot\emptyset-G^R\cdot\emptyset\ | \\ &\quad G^L\cdot 1+G\cdot\emptyset-G^L\cdot\emptyset,G^R\cdot 1+G\cdot 0-G^R\cdot 0\} \\ &= \{G^L|G^R\} \\ &= G \end{aligned}$$

Where we have used lemma 2 again.
Likewise for the second part of the proposition. We assume it holds for all games born before H, and prove it then holds for H as well, using the lemma:

$$\begin{aligned} 1\cdot H &= \{0\cdot H+1\cdot H^L-0\cdot H^L,\emptyset\cdot H+1\cdot H^R-\emptyset\cdot H^R\ | \\ &\quad 0\cdot H+1\cdot H^R-0\cdot H^R,\emptyset\cdot H+1\cdot H^L-\emptyset\cdot H^L\} \\ &= \{H^L|H^R\} \\ &= H \end{aligned}$$

Now we have proved that for any games $G\cdot 1=G$ and $1\cdot H=H$, i.e. 1 is the neutral element for multiplication, as we expected.

Multiplication of games is commutative:

Proposition 6 (Multiplication of games is commutative) *:*
For all numbers G and H we have:

$$G \cdot H = H \cdot G$$

Proof Let $G = \{G^L|G^R\}$ and $H = \{H^L|H^R\}$ be numbers.
Let n be the birthday of the game $G \cdot H$. We want to prove the commutation by induction over n.
Lemma 2 gives us the base case.
Assume now that games with product of birthdays $\leq n-1$ commute. In particular the options of G and H commute with G and H (e.g. $G^L \cdot H = H \cdot G^L$)

$$\begin{aligned} G \cdot H \;\; = \;\; &\{G^LH + GH^L - G^LH^L \;,\; G^RH + GH^R - G^RH^R \;| \\ &G^LH + GH^R - G^LH^R \;,\; G^RH + GH^L - G^RH^L\} \end{aligned}$$

Using the induction hypothesis:

$$\begin{aligned} = \;\; &\{HG^L + H^LG - H^LG^L \;,\; HG^R + H^RG - H^RG^R \;| \\ &HG^L + H^RG - H^RG^L \;,\; HG^R + H^LG - H^LG^R\} \end{aligned}$$

Addition of games is commutative (proposition 4):

$$\begin{aligned} = \;\; &\{H^LG + HG^L - H^LG^L \;,\; H^RG + HG^R - H^RG^R \;| \\ &H^RG + HG^L - H^RG^L \;,\; H^LG + HG^R - H^LG^R\} \end{aligned}$$

Sets are not ordered, so we can permute the elements within the list:

$$\begin{aligned} = \;\; &\{H^LG + HG^L - H^LG^L \;,\; H^RG + HG^R - H^RG^R \;| \\ &H^LG + HG^R - H^LG^R \;,\; H^RG + HG^L - H^RG^L\} \\ = \;\; &H \cdot G \end{aligned}$$

Now we have shown that numbers commute under the act of multiplication, as expected.

Proposition 7 (The distributive law) *:*
For all numbers G, H, K, we have:

$$(G + H)K = GK + HK$$

Proof This law can be proved by induction over the birthday of the game $(G+H)K$.
Let G, H and K be numbers, and let n be the birthday of $(G+H)K$.
For the basis we have a game born on day zero. This is the case is either $G+H=0$ or $K=0$. In the first case we have $0 \cdot K = 0$ on the left hand side, and $0 \cdot K + 0 \cdot K = 0 + 0 = 0$ on the right hand side; so the sides are equal. In the second case $G+H$ is a game as well, so we can apply lemma 2 on the left hand side: $(G+H) \cdot 0 = 0$. On the right hand side we have $G \cdot 0 + H \cdot 0 = 0 + 0 = 0$. Here the two sides are equal as well, so the proposition holds in the base case.

Assume the distributive law holds for birthdays up to $n-1$ (inclusive). Using the definitions of sum and product (definitions 9 and 11), we can write the right hand side as:

$$\begin{aligned}
GK + HK \quad &= \quad \{G^LK + GK^L - G^LK^L\ ,\ G^RK + GK^R - G^RK^R\ | \\
&\quad\quad G^LK + GK^R - G^LK^R\ ,\ G^RK + GK^L - G^RK^L\}\ + \\
&\quad\quad \{H^LK + HK^L - H^LK^L\ ,\ H^RK + HK^R - H^RK^R\ | \\
&\quad\quad H^LK + HK^R - H^LK^R\ ,\ H^RK + HK^L - H^RK^L\} \\
&= \quad \{G^LK + GK^L - G^LK^L + HK\ ,\ G^RK + GK^R - G^RK^R + HK\ , \\
&\quad\quad GK + H^LK + HK^L - H^LK^L\ ,\ GK + H^RK + HK^R - H^RK^R\ | \\
&\quad\quad G^LK + GK^R - G^LK^R + HK\ ,\ G^RK + GK^L - G^RK^L + HK\ , \\
&\quad\quad GK + H^LK + HK^R - H^LK^R\ ,\ GK + H^RK + HK^L - H^RK^L\}
\end{aligned}$$

On the left hand side we have (by the same definitions):

$$\begin{aligned}
&(G+H)K = \\
&\quad \{((G^L+H)K, (G+H^L)K) + ((G+H)K^L) - ((G^L+H)K^L, (G+H^L)K^L)\ , \\
&\quad\ ((G^R+H)K, (G+H^R)K) + ((G+H)K^R) - ((G^R+H)K^R, (G+H^R)K^R) | \\
&\quad\ ((G^L+H)K, (G+H^L)K) + ((G+H)K^R) - ((G^L+H)K^R, (G+H^L)K^R)\ , \\
&\quad ((G^R+H)K, ((G+H^R)K) + ((G+H)K^L) - ((G^R+H)K^L, (G+H^R)K^L)\ \}
\end{aligned}$$

By the induction hypothesis (remembering that sets are added like vectors):

$$\begin{aligned}
= \quad & \{(G^LK + HK + GK^L + HK^L - G^LK^L - HK^L\ , \\
& GK + H^LK + GK^L + HK^L - GK^L - H^LK^L)\ , \\
& (G^RK + HK + GK^R + HK^R - G^RK^R - HK^R\ , \\
& GK + H^RK + GK^R + HK^R - GK^R - H^RK^R)\ |
\end{aligned}$$

$$
\begin{aligned}
&(G^LK + HK + GK^R + HK^R - G^LK^R - HK^R\ ,\\
&GK + H^LK + GK^R + HK^R - GK^R - H^LK^R)\ ,\\
&(G^RK + HK + GK^L + HK^L - G^RK^L - HK^L\ ,\\
&GK + H^RK + GK^L + HK^L - GK^L - H^RK^L)\}
\end{aligned}
$$

We know that $G - G = 0$ so, this simplifies to:

$$
\begin{aligned}
= \;& \{(G^LK + HK + GK^L - G^LK^L),\ (GK + H^LK + HK^L - H^LK^L),\\
& (G^RK + HK + GK^R - G^RK^R),\ (GK + H^RK + HK^R - H^RK^R) \mid\\
& (G^LK + HK + GK^R - G^LK^R),\ (GK + H^LK + HK^R - H^LK^R),\\
& (G^RK + HK + GK^L - G^RK^L),\ (GK + H^RK + HK^L - H^RK^L)\}
\end{aligned}
$$

Addition of games is commutative.

$$
\begin{aligned}
= \;& \{(G^LK + GK^L - G^LK^L + HK),\ (GK + H^LK + HK^L - H^LK^L),\\
& (HK + G^RK + GK^R - G^RK^R),\ (GK + H^RK + HK^R - H^RK^R) \mid\\
& (G^LK + GK^R - G^LK^R + HK),\ (GK + H^LK + HK^R - H^LK^R),\\
& (G^RK + GK^L - G^RK^L + HK),\ (GK + H^RK + HK^L - H^RK^L)\}\\
= \;& GK + HK
\end{aligned}
$$

We have now shown that the distributive law holds for any numbers, q.e.d.

Proposition 8 (Associative law for multiplication) *:*
For all numbers G, H and K:

$$(GH)K = G(HK)$$

To show this proposition, we could just apply the distributive law with $G = GH$, $H = 0$ and $K = K$. But let us prove it rigorously like the rest of the propositions.

Proof Let us do this proof by induction over the birthday n of the number $(GH)K$. For the base case we have a game born on day zero. This is the case if one of the factors is 0. But then $(G \cdot H) \cdot K = 0$, and also $G \cdot (H \cdot K) = 0$. The proposition holds for the base case.

Let us assume the associative law holds for all games with product of birthdays $\leq n - 1$. From the definition of multiplication, we know that

$$
\begin{aligned}
GH \;=\; & \{G^LH + GH^L - G^LH^L\ ,\ G^RH + GH^R - G^RH^R \mid\\
& G^LH + GH^R - G^LH^R\ ,\ G^RH + GH^L - G^RH^L\}
\end{aligned}
$$

and

$$
\begin{aligned}
HK \;=\; & \{H^LK + HK^L - H^LK^L\ ,\ H^RK + HK^R - H^RK^R \mid \\
& H^LK + HK^R - H^LK^R\ ,\ H^RK + HK^L - H^RK^L\}
\end{aligned}
$$

Multiplying GH with K for the left hand side, we get
$(GH)K =$

$$
\begin{aligned}
&\{((G^LH + GH^L - G^LH^L)K, (G^RH + GH^R - G^RH^R)K) + (GH)K^L - \\
&\qquad ((G^LH + GH^L - G^LH^L)K^L, (G^RH + GH^R - G^RH^R)K^L), \\
&((G^LH + GH^R - G^LH^R)K, (G^RH + GH^L - G^RH^L)K) + (GH)K^R - \\
&\qquad ((G^LH + GH^R - G^LH^R)K^R, G^RH + GH^L - G^RH^L)K^R) \mid \\
&((G^LH + GH^L - G^LH^L)K, (G^RH + GH^R - G^RH^R)K) + (GH)K^R - \\
&\qquad ((G^LH + GH^L - G^LH^L)K^R, (G^RH + GH^R - G^RH^R)K^R), \\
&((G^LH + GH^R - G^LH^R)K, (G^RH + GH^L - G^RH^L) + (GH)K^L - \\
&\qquad ((G^LH + GH^R - G^LH^R)K^L, (G^RH + GH^L - G^RH^L)K^L)\}
\end{aligned}
$$

Applying the distributive law and the hypothesis:

$$
\begin{aligned}
= \;& \{G^LHK + GH^LK - G^LH^LK + GHK^L - G^LHK^L - GH^LK^L + G^LH^LK^L, \\
& G^RHK + GH^RK - G^RH^RK + GHK^L - G^RHK^L - GH^RK^L + G^RH^RK^L, \\
& G^LHK + GH^RK - G^LH^RK + GHK^R - G^LHK^R - GH^RK^R + G^LH^RK^R, \\
& G^RHK + GH^LK - G^RH^LK + GHK^R - G^RHK^R - GH^LK^R + G^RH^LK^R \mid \\
& G^LHK + GH^LK - G^LH^LK + GHK^R - G^LHK^R - GH^LK^R + G^LH^LK^R, \\
& G^RHK + GH^RK - G^RH^RK + GHK^R - G^RHK^R - GH^RK^R + G^RH^RK^R, \\
& G^LHK + GH^RK - G^LH^RK + GHK^L - G^LHK^L - GH^RK^L + G^LH^RK^L, \\
& G^RHK + GH^LK - G^RH^LK + GHK^L - G^RHK^L - GH^LK^L + G^RH^LK^L\}
\end{aligned} \tag{5.1}
$$

Multiplying G with HK for the right hand side, we get
$G(HK) =$

$$
\begin{aligned}
&\{G^L(HK) + (G(H^LK + HK^L - H^LK^L), G(H^RK + HK^R - H^RK^R)) - \\
&\qquad (G^L(H^LK + HK^L - H^LK^L), G^L(H^RK + HK^R - H^RK^R)), \\
&G^R(HK) + (G(H^L + HK^R - H^LK^R), G(H^RK + HK^L - H^RK^L)) - \\
&\qquad (G^R(H^LK + HK^R - H^LK^R), G^R(H^RK + HK^L - H^RK^L)) \mid \\
&G^L(HK) + (G(H^LK + HK^R - H^LK^R), G(H^RK + HK^L - H^RK^L)) - \\
&\qquad (G^L(H^LK + HK^R - H^LK^R), G^L(H^RK + HK^L - H^RK^L)) \mid \\
&G^R(HK) + (G(H^LK + HK^L - H^LK^L), G(H^RK + HK^R - H^RK^R)) - \\
&\qquad (G^R(H^LK + HK^L - H^LK^L), G^R(H^RK + HK^R - H^RK^R))\}
\end{aligned}
$$

Applying the distributive law and the hypothesis again:

$$\begin{aligned} = \quad & \{G^LHK + GH^LK + GHK^L - GH^LK^L - G^LH^LK - G^LHK^L + G^LH^LK^L, \\ & G^LHK + GH^RK + GHK^R - GH^RK^R - G^LH^RK - G^LHK^R + G^LH^RK^R, \\ & G^RHK + GH^LK + GHK^R - GH^LK^R - G^RH^LK - G^RHK^R + G^RH^LK^R, \\ & G^RHK + GH^RK + GHK^L - GH^RK^L - G^RH^RK - G^RHK^L + G^RH^RK^L | \\ & G^LHK + GH^LK + GHK^R - GH^LK^R - G^LH^LK - G^LHK^R + G^LH^LK^R, \\ & G^LHK + GH^RK + GHK^L - GH^RK^L - G^LH^RK - G^LHK^L + G^LH^RK^L, \\ & G^RHK + GH^LK + GHK^L - GH^LK^L - G^RH^LK - G^RHK^L + G^RH^LK^L, \\ & G^RHK + GH^RK + GHK^R - GH^RK^R - G^RH^RK - G^RHK^R + G^RH^RK^R\} \end{aligned}$$

This is equal to 5.1, because addition commutes and the sets are unordered. So indeed $(GH)K = G(HK)$, q.e.d.

The last thing we need in our assertion that numbers form a field, is that every number has a multiplicative inverse. We will give a formula for finding the multiplicative inverse, but we will skip the proof. Instead we refer the reader to [3, pages 20–22].

It suffices to show this for all positive games, because games commute: $-G \cdot H = 1 \iff G \cdot (-H) = 1$.

We will use a special form of the games in this section, namely the one defined in the following lemma:

Lemma 3 *Each positive game G has a form where one of the Left options is 0, and every other Left option is positive.*

Proof Let G be a positive game. By negating the definition of $\geq$ we get: $0 \not\geq G \iff \exists g^L \in G^L : 0 \leq g^L$ (or $\exists h^L \in \emptyset$, which is not true, since there are not elements in the empty set). Therefore G has at least one non-negative Left option.
The non-negative option dominates all the negative ones in G^L, and they can therefore be deleted (theorem 2).
Inserting 0 as a Left option will not change the value of G, since Left already have non-negative options that are at least as good for him.

Now we have proved that inserting 0 as a Left option and deleting all negative options in G^L will not change the value of G, q.e.d.

In the rest of this section we will write $G = \{0, G^L | G^R\}$, and use the symbol G^L for the positive Left options. We also notice that all the Right options G^R are positive.

For the reciprocal of a game G we need a highly recursive definition.

Definition 12 *Let $G = \{G^L | G^R\}$ be a positive number. The reciprocal game H to G is defined recursively as:*

$$H = \{0 \ , \ \frac{1 + (G^R - G)H^L}{G^R} \ , \ \frac{1 + (G^L - G)H^R}{G^L} \mid \frac{1 + (G^L - G)H^L}{G^L} \ , \ \frac{1 + (G^R - G)H^R}{G^R}\}$$

$G^L = 0$ starts the recursion.

This definition calls for an explanation. The Left and Right options of G should be straight forward, but the presence of H^L and H^R in the Left and Right options of H is confusing seems impossible - therefore we will work our way through some examples soon. The solution comes from the Left option $H^L = 0$; that can be inserted in two of the formulae for options, and we gain a Right option[1]. This Right option can now be inserted in the two other formulae for options, etc. ad infinitum. Let us do some examples:

Example 23 This is the example from [3, p. 21]. Let $G = \{0, 2 | \emptyset\} = 3$ Here there is no G^R. The Left option is 2, so $G^L - G = 2 - 3 = -1$. The formula for H yields:

$$H = \{0 \ , \ \frac{1 + (-1)H^R}{2} \mid \frac{1 + (-1)H^L}{2}\}$$

Inserting $H_0^L = 0$, we get:

$$H = \{0 \ , \ \frac{1 + (-1)H^R}{2} \mid \frac{1}{2} \ , \ \frac{1 + (-1)H^L}{2}\}$$

And inserting $H_1^R = \frac{1}{2}$, we get:

$$H = \{0 \ , \ \frac{1}{4} \ , \ \frac{1 + (-1)H^R}{2} \mid \frac{1}{2} \ , \ \frac{1 + (-1)H^L}{2}\}$$

After six iterations we have:

$$H = \{0 \ , \ \frac{1}{4}, \ , \ \frac{5}{16} \ , \ \frac{21}{64} \ , \ \frac{1 + (-1)H^R}{2} \mid \frac{1}{2} \ , \ \frac{3}{8} \ , \ \frac{11}{32} \ , \ \frac{1 + (-1)H^L}{2}\}$$

and we can see a pattern emerging: it looks like $\frac{1}{3}$; and indeed, if we continued to infinity, that is what we would get. So the reciprocal of 3 is $\frac{1}{3}$, as expected.

[1] And a Left one, unless $G^R = \emptyset$. We have assumed G positive, so $G^L \neq \emptyset$, and we will always get the Right option of H.

Example 24 Let $G = \{0, 1|\emptyset\} = 2$
Here there is no G^R. The Left option is 1, so $G^L - G = 1 - 2 = -1$. The formula for H yields:

$$H = \{0\ ,\ \frac{1 + (-1)H^R}{1} \mid \frac{1 + (-1)H^L}{1}\}$$

Inserting $H_0^L = 0$, we get:

$$H = \{0\ ,\ 1 - H^R \mid 1\ ,\ 1 - H^L\}$$

And inserting $H_1^R = 1$, we get:

$$H = \{0\ ,\ 0\ ,\ 1 - H^R \mid 1\ ,\ 1 - H^L\}$$

We quickly see the pattern that all the Left options become 0, and all the Right ones become 1. Therefore $H = \{0|1\} = \frac{1}{2}$. The reciprocal of 2 is $\frac{1}{2}$ as expected.

Example 25 Let $G = \{0, 1|2\} = \frac{3}{2}$
Now there is a G^R, namely 2, so $G^R - G = \frac{1}{2}$. The Left option is 1, again, so $G^L - G = \frac{-1}{2}$. The formula for H becomes:

$$\begin{aligned} H &= \{0, \frac{1 + \frac{1}{2}H^L}{2}, \frac{1 - \frac{1}{2}H^R}{1} | \frac{1 - \frac{1}{2}H^L}{1}, \frac{1 + \frac{1}{2}H^R}{2}\} \\ &= \{0, \frac{1}{2} + \frac{1}{4}H^L, 1 - \frac{1}{2}H^R | 1 - \frac{1}{2}H^L, \frac{1}{2} + \frac{1}{4}H^R\} \end{aligned} \quad (5.2)$$

Inserting $H_0^L = 0$, we get a Left option of 1 and a Right option of $\frac{1}{2}$:

$$H = \{0, 1, \frac{1}{2} + \frac{1}{4}H^L, 1 - \frac{1}{2}H^R | \frac{1}{2}, 1 - \frac{1}{2}H^L, \frac{1}{2} + \frac{1}{4}H^R\}$$

When we now insert $H_1^R = \frac{1}{2}$ and $H_1^L = 1$, we get two Left options: $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and $1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$, and we get two Right options: $1 - \frac{1}{2} = \frac{1}{2}$, which we already have, and $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$. The expression for H is now:

$$H = \{0, 1, \frac{3}{4}, \frac{1}{2} + \frac{1}{4}H^L, 1 - \frac{1}{2}H^R | \frac{1}{2}, \frac{5}{8}, 1 - \frac{1}{2}H^L, \frac{1}{2} + \frac{1}{4}H^R\}$$

After three more iteration, we have:

$$H = \{0, 1, \frac{3}{4}, \frac{11}{16}, \frac{43}{64}, \frac{171}{256}, \frac{1}{2} + \frac{1}{4}H^L, 1 - \frac{1}{2}H^R | \frac{1}{2}, \frac{5}{8}, \frac{21}{32}, \frac{85}{128}, \frac{341}{512}, 1 - \frac{1}{2}H^L, \frac{1}{2} + \frac{1}{4}H^R\}$$

Both sides converge to $\frac{2}{3}$, which would be the value of H after infinitely many steps. So the reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$, as expected.

The definition is thus not impossible, because the tricky factors H^L and H^R in the options are iterations of earlier generated options, starting with 0.

Now we could show that the reciprocal of a game is indeed the multiplicative inverse: $G \cdot H = 1$. But we will skip the complicated proof, and refer the reader to [3, pages 21 – 22], where it can be found.

Instead we will conclude our presentation of the algebra of games with the proof promised in section 2.4 that reversible moves can be bypassed.

5.3 More on Reversible Moves

Recalling theorem 3:

Let $G = \{A, B, \ldots, C|D, E, \ldots, F\}$ be a game.
If any Right option D has a Left option D^L such that $D^L \geq G$, then we can simplify G by replacing D with all the Right options $X, Y, \ldots, Z$ of D^L.
Similarly, if any Left option A has a Right option A^R such that $A^R \leq G$, then we can simplify G by replacing A with all the Left options of A^R.
Such replacements do not change the value of the game.

We will only prove the theorem for Right options, since all the arguments easily can be carried over to Left options.

Let us repeat the figure of a reversible option (figure 2.3):

Proof of theorem 6 Let $G = \{A, B, \ldots, C|D, E, \ldots, F\}$ be a game, let the Right option D be reversible (i.e. it has a Left option D^L with $D^L \geq G$), and let $H = \{A, B, \ldots, C|X, Y, \ldots, Z, E, \ldots, F\}$ be the game where we have bypassed D. We want to prove that $G = H$. This we will do by looking at the game

$$G - H = \{A, B, \ldots, C|D, E, \ldots, F\} + \{-X, -Y, \ldots, -Z, -E, \ldots, -F|-A, -B, \ldots, -C\}$$

There are three different scenarios to consider

1. Left moves in $-H$ to either $-X$, $-Y$, ..., or $-Z$

2. Right moves to D in the game G

3. Either player moves to any other position

Re 1 If Left moves to $-X$ in the game $-H$, he will leave the position $G - X$ for Right to move in. The position $G - X$ will be worse than or equal to $D^L - X$ for him, since $D^L \geq G$. But Right can win the game $D^L - X$ by moving from D^L to X, leaving $X - X$, and then copying whatever Left does; Left will soon have to move in the Endgame, i.e. $D^L - X \leq 0$ (the equality occurs in games where Right loses as well, if she starts). Thus $G - X \leq 0$, and Left will not want to make this losing move. Similarly for $-Y$, ... and $-Z$.

Re 2 If Right moves to D in G, Left can reverse the move by moving to D^L. This will leave the position $D^L - H = \{U, V, \ldots, W|X, Y, \ldots, Z\} + \{-X, -Y, \ldots, -Z, -E, \ldots, -F|-A, -B, \ldots, -C\}$. Right now has two possibilities

a. She can move to X, Y, ..., or Z in D^L, or

b. She can move to $-A$, $-B$, ..., or $-C$ in $-H$.

Re a. If Right moves to either X, Y, ..., or Z in D^L, Left can counter by moving in $-H$ to the inverse, either $-X$, $-Y$, ..., or $-Z$. This will leave Right with a game, where Left can win by copying any move she makes - i.e it is a losing move for her, and she would rather not make it.

Re b. Alternatively, Right could move to either $-A$, $-B$, ..., or $-C$ in $-H$. This would leave the total position either $D^L - A$, $D^L - B$, ..., or $D^L - C$; but since we let $D^L \geq G$, these games must be at least as bad for Right as the games $G - A$, $G - B$, ..., or $G - C$, which Left can win by moving to A, B, ..., or C in the game G, and from there copy Right's moves. We conclude that this is also a losing move for her, and she would rather not make this move either.

Re 3 If Left chooses to move to either A, B, $\ldots$, or C or if Right moves to E, $\ldots$, or F in the game G, the opponent can easily counter the move by choosing the inverse $-A$, $-B$, $\ldots$, $-C$, $-E$, $\ldots$, or $-F$ in $-H$. Similarly, the moves to $-A$, $-B$, $\ldots$, $-C$, $-E$, $\ldots$, or $-F$ in $-H$ can be countered by the corresponding moves in G. Now the opponent can win by copying the player's moves. We realize that whoever moves in this game loses, i.e. it is a zero game: $G - H = 0$.

Although neither player wants to move in the zero game, the other possibility is worse or equal for them, and it is thus dominated by 3rd option, yielding $G - H = 0$. The equality $G - H = 0$ implies $G = H$, q.e.d.

We have seen that the numbers we have defined behave like the reals we know. Not only have we established a way of generating all reals as games, we have even expanded our horizon. We will now move on to games that are not necessarily numbers; returning to a more concrete side of the game theory.

Chapter 6

Temperature and Thermographs

In chapter 3 we saw how any impartial combinatorial game can be solved fully. For most partial ones an analysis would call for some very long and tedious calculations. The *Thermostatic Strategy* provides us with a very good estimate of the best move, and takes just a little calculation. Therefore we will devote this chapter to *Thermographs*, and the Thermostatic Strategy that springs from it. But before we can draw any Thermographs, we need to introduce the temperature of a game.

6.1 Hot Games and Confusion

Hot games are more interesting than cold games. A game is hot if having the move in it is an advantage. So a game where $G^L >> G^R$ for some options is a very hot game. On the other hand, numbers are very cold, because $G^L < G < G^R$ for all the options, and moving in a number will only make our position worse; as the Number Avoidance Theorem states:

Theorem 7 (Number Avoidance Theorem) *:*
"Don't move in a Number, unless there's Nothing else to do." **??***page 144]WW.*

Proof of theorem 7 Let x be a number and $G = \{G^L|G^R\}$ be a game that is not a number. Then $G \neq x$.
We will be concerned with the sum game $G + x = \{G^L + x, G + x^L \mid G^R + x, G + x^R\}$. We want to prove that $G^L + x > G + x^L$, and $G^R + x < G + x^R$. However, we will only prove it for Left, since the proof can be done similarly for Right.

Now, either

1. $0 \not\geq G + x$ or

2. $0 > G + x$, (since $G \neq x$).

Re 1. Let $0 \not\geq G + x$. By applying the definition of $\geq$ negated, we get:

$$\begin{aligned} 0 \not\geq G + x \quad &\iff \quad \exists q \in \emptyset \ldots, \text{ false, otherwise not empty} \\ &\quad\text{or} \quad \exists h^L \in (G + x)^L : \; h^L \geq 0 \end{aligned}$$

The options $(G + x)^L$ consist of $G^L + x$ and $G + x^L$, so either

a. $G^L + x \geq 0$ or

b. $G + x^L \geq 0$

Let us assume that $G^L + x \not\geq 0$ for all Left options of G, and try to obtain a contradiction.

Let $x > x^L > x^{L^L} > \ldots$ be the finite sequence of successive Left options of x. The sequence is decreasing since x is a number, and so are the options of x (by the definition of a number). Let y be the smallest number in the sequence for which $G + y \geq 0$ is true. Since G is not a number, we have $G \neq y$, so $G + y > 0$. This implies (using definition 4 negated) that $G^L + y \geq 0$, since $G + y^L \not\geq 0$ (y was the smallest number in the sequence such that $G + y \geq 0$). Combining this with the fact that $x \geq y$ (by definition of y), we have $G^L + x \geq G^L + y \geq 0$, implying $G^L + x \geq 0$, in contradiction with our hypothesis. Therefore we must have a Left option with $G^L + x \geq 0$

Re 2. Now, let $0 > G + x$. We want to show that $G^L + x > G + x^L$.
The game x is a number, so we have $x > x^L$, implying $x - x^L > 0$. Writing $G + x$ as $G + x^L + (x - x^L)$, and using $G + x^L + (x - x^L) > G + x^L$, we get $G + x > G + x^L$.
The game G is not a number, so $G \not> G^L$. Therefore $G^L + x > G + x^L$, q.e.d.

A generalization of the Number Avoidance Theorem is the Translation Principle:

Theorem 8 (Translation Principle) *:*
If x is a number and the game $G = \{G^L | G^R\}$ is not a number, then

$$\{G^L | G^R\} + x = \{G^L + x | G^R + x\}.$$

Proof of theorem 8 [Translation Principle]:
Let x be a number and $G = \{G^L|G^R\}$ a game that is not a number.

From the rule of addition we have:

$$G + x = \{G^L + x, G + x^L \mid G^R + x, G + x^R\}.$$

By the Number Avoidance Theorem, Left should move in G to G^L+x rather than in x to $G+x^L$, so we have $G^L+x \geq G+x^L$. I.e. $G+x^L$ is dominated by G^L+x as a Left option in G, and can thus be deleted (theorem 2). Similarly, $G + x^R$ is dominated by $G^R + x$ as a Right option, and can also be deleted. This way $G + x = \{G^L + x, G + x^L \mid G^R + x, G + x^R\}$ reduces to $G + x = \{G^L + x|G^R + x\}$, q.e.d.

We realize that the numerical options are dominated; and that is the reasoning behind the Number Avoidance Theorem.

Since numbers are not interesting to play, sensible players will just stop the game and sum up the score, when all the components of the game have become numbers. Using this stopping criterion, we can define the Left and Right stops of the game G:

Definition 13 (The Stops of a Game) *:*
The value that the game G will stop at, if Left moves first, is called the Left stop of G, and is denoted $L(G)$. Similarly, the Right stop $R(G)$ is the value that the game will stop at, if Right moves first.

The result of a game lies between the two stops, with the players fighting to pull it in their direction. The interval of the possible outcomes is called the confusion interval of the game.

Definition 14 (The Confusion Interval) *:*
The interval between the Left and the Right stops contains the possible[1] *outcomes of the game and is called the confusion interval.*

[1] Assuming that both players play well

Example

The larger the confusion interval, the hotter the game, and the more eager the players are to move in it.

Thus the game with the confusion interval:

is hotter than the game with the confusion interval:

We want to decrease the confusion in order to find the mean value of a hot game; so we cool the game down to a number.

To cool the game, we will lower the tension by imposing a tax t on moves. Left, who wants the game to be as great (positive) as possible, will be less anxious to move the smaller his options are, so we subtract a positive number t from all the Left options. Right wants the game to be as negative as possible, so we add the positive number t to all her options, making her less anxious to move.

When the tax we impose is large enough to cool the game down infinitely close to a number, then neither player has any incentive to move in it anymore, and there is no need for a further taxation.

With this in mind, we define the cooling of a game by increasing t as follows:

Definition 15 (Cooling of a Game) *:*
The game G cooled by the temperature $t \in \mathbb{R}$ is:

$$G_t = \begin{cases} \{G_t^L - t \mid G_t^R + t\}, & \textit{if } \forall t' < t \; G_{t'} - \epsilon \textit{ is not a number} \\ \mu, & \textit{if } \exists t' < t: \; G_{t'} - \epsilon = \mu, \textit{ is a number.} \end{cases}$$

Where ϵ is an arbitrary infinitesimal. So the first line of the definition is in effect, as long as G_t is not infinitely close to a number.

Another way of viewing taxation is the following: In a hot game the players are eager to move, therefore the players are interested in paying for the privilege of moving. The players are willing to pay t (victory points), so this is the price.

Example 26 Let us look at the game $G = \{4| - 3\}$ as an example. This is not a number, since $4 \not< -3$.

If we cool this game by one degree, we get:

$$G_1 = \{4 - 1| - 3 + 1\} = \{3| - 2\}$$

This game is not infinitely close to a number, so we impose some more tax. If we cool the game down another degree, we get

$$G_2 = \{4 - 2| - 3 + 2\} = \{2| - 1\}$$

which is still not close to a number, and we cool it another degree:

$$G_3 = \{4 - 3| - 3 + 3\} = \{1|0\}$$

still not a number, but now we only need to cool $\frac{1}{2}$ degree to get close enough to a number:

$$G_{3\frac{1}{2}} = \{4 - 3\frac{1}{2}| - 3 + 3\frac{1}{2}\} = \{\frac{1}{2}|\frac{1}{2}\} = \frac{1}{2}*$$

This is infinitely close to the number $\frac{1}{2}$; and imposing more tax will still yield $\frac{1}{2}$.

We conclude that the game $G = \{4| - 3\}$ can be cooled down to $\frac{1}{2}$, the mean value of the game.

Definition 15 is essential when drawing Thermographs, and in fact Thermographs are simply graphical representations of G_t.

When cooling a game, there will always be a taxation for which the game G_t is a number.

Definition 16 (The Temperature of a Game) *:*
The temperature T of a game G is the smallest taxation t, for which the game G_t is infinitely close to a number.

The temperature of the game in the example above is $3\frac{1}{2}$.

6.2 Drawing Thermographs

Now that we have defined the cooled game G_t, we are ready to draw graphical representations of of it, i.e. Thermographs, for various games.

The coordinate system used in thermography is rotated by $\frac{\pi}{2}$ about the origin with respect to the usual one. This implies that the tax level t (the independent variable) is on the vertical axis, and the game value G_t is on the (reversed) horizontal one. The reason behind this rotated coordinate system is our definition that Left plays for positive values and Right plays for negative ones. This way Left's options are mapped to the left, and Right's options to the right.

Let us begin with the simple case of $G = \{G^L | G^R\}$, where G^L and G^R are both numbers. In general we will then obtain a Thermograph of the following form:

To understand how we get this, let us look at a concrete example.

Example 27 Let us reuse the game $G = \{4| - 3\}$ from example 6.1. To draw its Thermograph we begin by marking the uncooled ($t = 0$) Left and Right options on the horizontal axis, $G^L = 4$, $G^R = -3$. Then we plot the game values that we calculated in the example: $G_1 = \{3|-2\}$, $G_2 = \{2|-1\}$, $G_3 = \{1|0\}$, and $G_{3\frac{1}{2}} = \{\frac{1}{2}|\frac{1}{2}\}$.

For $t > 3\frac{1}{2}$ we do not impose more tax, so $G_{t \geq 3\frac{1}{2}} = \frac{1}{2}$. If we calculated G_t for all taxes $t \in [0, 3\frac{1}{2}]$, we would get the following graph:

This is the Thermograph of the game $G = \{4| - 3\}$. The temperature $T = 3\frac{1}{2}$ is the point where the two legs of the graph meet and become a mast; the mean value $\mu = \frac{1}{2}$ is the corresponding value of G_t. The mast continues ad infinitum (c.f. definition 15) which we sometimes indicate by letting it end in an arrow. Thus the drawn length of the mast on a Themograph is not significant.

Example 28 The Thermograph of a number is trivially just a mast, but we will draw one to illustrate that numbers behave as we expect. Consider the game $G = \{-3|-1\}$. When we begin calculating G_t, we find that $G_0 = \{-3|-1\} = -2$ is a number, so $G_{t \geq 0} = -1$, and thus the Thermograph becomes a vertical line:

The temperature of a number is zero, as expected.

When we introduced the cooling of a game, we may have implied that the taxation began at zero, and that the Thermographs would therefore start at the axis. But the Thermographs we have drawn, have all been extending slightly below. The reason for this will be clear when we look at the Thermograph for $*$. Both options of the game $*$ are zero; but the tax imposed on Left's options pull the graph slightly in one direction, whereas the tax imposed on Right's options pull it in the other direction:

Since the game $*$ is infinitely close the number 0, this taxation has no effect above the axis.

Extending the graph slightly (ideally infinitesimally) below the axis, enables us to indicate whether a game is precisely equal to the number, slightly positive or slightly negative, or if it is fuzzy with the number, like $*$ is with 0.

Example 29 For the game $\downarrow = \{*|0\}$ we have:

For the general case with $a \geq b$ being numbers, we have the following Thermographs:

When the Thermograph changes slope at the axis, we call the line below $t = 0$ a *toenail* .

If the line below is vertical, the players will have made an equal number of moves when the game stops; whereas if the line slants, the first player makes the last move too. Therefore the confusion interval includes endpoints on slanting lines (e.g. $*$), but excludes endpoints on vertical lines (e.g. numbers) - a convincing proof of this can be found in [3, pages 105–106]. In the example right above, both endpoints are included in the confusion interval of G_1, whereas they are both excluded from the confusion interval of G_4, so we know how G_4 compares with the endpoints. Notice, the confusion is determined by the slope of the line under the axis.

Example 30 In the game with the Thermograph below, we see that $G > b$ and $G \leq a$, because there is a toenail at b but not at a. So G is confused with the number b, but not with the number a.

Until now, the only games we have drawn Thermographs for have had the form $\{a|b\}$, with a and b being numbers (or numbers and stars), and all except one (the number) have also had $a \geq b$. Such games are called switches.

Now we will extend our notion of Thermographs to include games where the Left player's option is a switch and Right's option a number. Such games have the form $G = \{\{b|a\} \mid c\}$, where a, b, c numbers $(a < b)$.

The game G is a combination of the games $\{b|a\}$ and c, so we expect the Thermograph of G to be a combination of the Thermographs of $\{b|a\}$ and c, that have the following Thermographs:

To find the compound Thermograph, we impose a tax of size t for each tax level on the Left and Right options, as before. This time, however, we can add the tax directly to the Thermograph:

At the tax level t we subtract t from the (Right boundary[2] of the) Left option, and add t to the (Left boundary of the) Right option; unless the boundaries of the resulting Thermograph has already coincided for a smaller t, in which case we just continue the vertical mast.

The resulting Thermograph is:

Example 31 [The Thermograph of a $\{switch \mid number\}$ game]:
As a concrete example, let us look at $G = \{\ \{7|3\} \mid -1\}$

[2]For reasons that will become clear, when we look at games with several options.

At $t = 0$ we do not impose any tax.
At $t = \frac{1}{2}$ we impose a tax of $-\frac{1}{2}$ on Left and $+\frac{1}{2}$ on Right, yielding

$$G^L_{\frac{1}{2}} = 3\frac{1}{2} - \frac{1}{2} = 3 \text{ and } G^R_{\frac{1}{2}} = -1 + \frac{1}{2} = -\frac{1}{2}.$$

$$\ldots$$

At $t = 3$ the boundaries finally coincide, because

$$G^L_3 = 5 - 3 = 2 \text{ and } G^R_3 = -1 + 3 = 2.$$

We see that G has the temperature $T = 3$, and the mean $\mu = 2$.

Obviously, it works the same way when Right has the switch and Left has the number as options.

Example 32 [The Thermograph of a $\{number \mid switch\}$ game]:
Let us take the example $G = \{-1 \mid \{-4| - 8\}\}$.
We impose tax in the same way as above, and find the following Thermograph (the thick one; the thin ones are just auxiliary lines representing the Thermographs of the options):

This game has the temperature $T = 2.5$, and the mean value $\mu = -3.5$.

The options need not be switches, and there may be several possible options as well. We are now ready to extend our theory of Thermographs to games with several options. When a player has more than one option, we need a way to decide which one to use for our compound Thermograph. Our choice may depend on the tax level, and the optimal move in a game at one tax level may not be the same at another tax level.

When Left has to choose between his options in a game, he must consider that after he has moved, it will be his opponent's turn to move. Therefore he should choose the Left option with the most positive Right options, such that Right gets the least attractive possible move for her turn. This analysis should be done for each tax level, since options may cross each other. We write this principle as a proposition:

Proposition 9 (The Option to Choose) *:*
When Left has several options at a given tax level t, the optimal one is whichever has the **leftmost Right boundary** *at that tax level.*
Similarly, the optimal option for Right is, whichever of her options has the **rightmost Left boundary** *at t.*

The principle we use here is actually von Neumann's Minimax Theorem for 2-player-zero-sum games from 1928. It states that the upper value of a game is $\bar{u} = max_i \, min_j \, u_{i,j}$, where $u_{i,j}$ is the payoff to player I, if player I chooses strategy i and player II chooses strategy j. The reasoning behind it is that when player I chooses the strategy i, he cannot prevent player II from imposing $min_j \, u_{i,j}$. Player I will therefore try to maximize the minimum payoff imposed by player II, by choosing $max_i \, min_j \, u_{i,j}$.

Returning to the combinatorial games, let us analyze which option Left should choose at various temperatures in the game with the following Thermographs:

When the game is colder than t_1 Left should choose the option c, because the Right boundary of c (which is c) is further left than b (the Right boundary of $\{a|b\}$). At the tax level t_1 the graphs cross, and at higher tax levels Left should choose the option $\{a|b\}$, because b is now further left than c.

Let us look at a concrete examples:

Example 33 [If Several Options are available]:
Let us take the game $G = \{\, \{5|1\},\ 2 \,|\, \{-2|-4\},\ -2\frac{1}{2}\}$ as our example.

For $t \in [0, \frac{1}{2}]$ Right will choose the option $-2\frac{1}{2}$, because its Left option (itself) is further right than the Left option of $\{-2|-4\}$. At $t = \frac{1}{2}$ the Right options cross, and for $t > \frac{1}{2}$, Right will choose $\{-2|-4\}$.

For $t \in [0, 1]$ Left will choose his option 2, but at $t = 1$ the options cross, so for $t > 1$ he will choose $\{5|1\}$.

The thick lines are the Thermograph of the game $G = \{\, \{5|1\},\ 2 \,|\, \{-2|-4\},\ -2\frac{1}{2}\}$; the thin lines are just auxiliary lines.

The options need not be switches; and now we know how to generate Thermographs of complicated games with multiple options, from the Thermographs of the Left and Right options recursively.

6.3 Strategies

We are now able to find Thermographs of almost any game we might wish. Sometimes, however, we are faced with games that are sums of several compounds, and we will need a strategy for this.

6.3.1 Thermographs for Sums of Games

In the last section we saw that Thermography supports combination, i.e. if we know the Thermographs for the games A and B, we know how to obtain the Thermograph for the game $\{A|B\}$. If instead we wanted to add the games A and B, we might run into problems when drawing the Thermograph, as the following example will show.

Example 34 [Problems when Adding Thermographs]:
Let A be the game $\{1|-1\}$, and B be the game $\{1, 1+A|-1, -1+A\}$. Drawing the Thermographs of these two games, we see that the resulting graphs are the same:

Let us do the Thermograph of B in details:

$$\begin{aligned} B &= \{1, 1+A|-1, -1+A\} \\ &= \Big\{1,\ 1+\{1|-1\}||-1,\ -1+\{1|-1\}\Big\} \\ &= \Big\{1,\ \{2|0\}||-1,\ \{0|-2\}\Big\} \end{aligned}$$

Left chooses the Left option with the leftmost Right boundary, i.e. 1 rather than $\{2|0\}$. Similarly Right chooses -1 rather than $\{0|-2\}$. So the players choose the same options as in the game A. Imposing tax on the options yields the promised Thermograph.

Now that the games A and B have the same Thermographs, we might expect the Thermographs of $A+A$ and $A+B$ to be similar as well. Let us try to calculate $A+A$ and $A+B$ by the addition rule:

First we calculate $A+A$ by the rule of addition:

$$\begin{aligned} A+A &= \{1|-1\}\ +\ \{1|-1\} \\ &= \Big\{1+\{1|-1\},\ 1+\{1|-1\}\ ||\ -1+\{1|-1\},\ -1+\{1|-1\}\Big\} \\ &= \Big\{\{1|-1\}+1\ ||\ \{1|-1\}-1\Big\} \\ &= \Big\{\{2|0\}\ ||\ \{0|-2\}\Big\} \end{aligned}$$

Now we draw the Thermograph of $A+A$:

The game $A + B$ is calculated similarly, but the calculations are longer, so we take it in smaller steps:

$$A + B = \{1| - 1\} \ + \ \{1, 1 + A| - 1, -1 + A\}$$

We get three Left options and three Right ones:

The first Left option of $A + B$:

$$\begin{aligned} 1 + \{1, 1 + A| - 1, -1 + A\} &= 1 + \Big\{1, \ \{2|0\}|| - 1, \ \{0| - 2\}\Big\} \\ &= \Big\{2, \{3|1\}||0, \{1| - 1\}\Big\} \end{aligned}$$

The second Left option:

$$\{1| - 1\} + 1 \ = \ \{2|0\}$$

The third Left option:

$$\begin{aligned} \{1|-1\}+\{2|0\} &= \Big\{\{3|1\},\{3|1\}||\{1|-1\},\{1|-1\}\Big\} \\ &= \Big\{\{3|1\}||\{1|-1\}\Big\} \end{aligned}$$

The first Right option of $A+B$:

$$\begin{aligned} -1+\{1,1+A|-1,-1+A\} &= -1+\Big\{1,\ \{2|0\}||-1,\ \{0|-2\}\Big\} \\ &= \Big\{0,\{1|-1\}||-2,\{-1|-3\}\Big\} \end{aligned}$$

The second Right option:

$$\{1|-1\}-1 = \{0|-2\}$$

The third Right option:

$$
\begin{aligned}
\{1|-1\}+\{0|-2\} &= \Big\{\{1|-1\},\{1|-1\}||\{-1|-3\},\{-1|-3\}\Big\} \\
&= \Big\{\{1|-1\}||\{-1|-3\}\Big\}
\end{aligned}
$$

From these six expressions we can draw the Thermograph of $A+B$:

Comparing the Thermographs for $A+A$ and $A+B$ we see that they are very different, and that Thermographs do not add easily. In fact, they do not contain enough information to add at all.

6.3.2 The Thermostatic Strategy

We cannot simply add Thermographs. However, if all we want is a good suggestion of where to make our move, then Thermography will provide us with an alternative to the tedious work of adding up the games. The Thermostatic Strategy (ThermoStrat for short) that we are about to describe makes for such an alternative. Other strategies have been suggested by Berlekamp [11, pages 390–391] too, namely *Hotstrat* (move in the hottest component) and *Sentestrat* (if your opponent have increased the local temperature, move in the same component). These two strategies do not perform as well as Thermostrat, and we will not go into further detail with them.

Instead, let us turn to Thermostrat. Without loss of generality we will assume the role of Left, as we have done throughout this paper (the same reasoning works for Right).

Given a game G that is sum of several components $A, B, \cdots, C$, we begin our Thermostratic analysis by drawing the individual Thermographs of $A, B, \cdots, C$.

If we refrained completely from moving in the game (in a taxed game moving can be viewed as a privilege that costs t), our opponent would win each component, and thus gain the sum $A_t^R + B_t^R + \cdots + C_t^R$, at the given tax level (she has to pay the tax to move). Therefore, at each tax level, the Right boundary R_t (which is the optimal value of the game for Right) in the compound game $A + B + \cdots + C$ is the sum of the Right boundaries of the components at that tax level:

$$R_t^{A+B+\cdots+C}(G) = R_t^A(G) + \cdots + R_t^C(G)$$

The width W_t of the Thermograph, i.e. the distance between the Left and the Right boundaries, at each tax level, is the difference between what we can at most get out of playing in that component, and what our opponent could get. When choosing our move we naturally wish to optimize this value. So, for each t we choose the greatest width among the widths of the components, to be the width of our compound Thermograph:

$$W_t^{A+B+\cdots+C} = max_{G^t}\{W_t^A, W_t^B, \cdots, W_t^C\}.$$

This way the Left boundary is <u>not</u> just the sum of the Left boundaries of the components. Instead, the resulting Left boundary is the sum of the value this game has for our opponent, and the greatest difference in value we can obtain by our moves, i.e. the greatest game value we can reach:

$$L_t^{A+B+\cdots+C}(G) = R_t^{A+B+\cdots+C}(G) + W_t^{A+B+\cdots+C}(G).$$

The leftmost value on this Left boundary is our best bargain; and as not to complicate the game for ourselves, we choose the lowest tax level that gives this value. The best move is in the component that is widest (and thus yielded the width) at this tax level, i.e.

$$min_t \, max_{G^t} \, L_t^{A+B+\cdots+C}(G).$$

Now it is time for some examples.

Example 35 [A Simple Example of Thermostrat]:
Let us look at a sum of two games A and B, where $A = \{10|4\}$ and $B = \{6|2|| - 2| - 6\}$.

They have the respective Thermographs (the thin lines are just auxiliary lines for determine the Thermograph for B):

First, we find the Right boundary of the compound Thermograph, by adding the Right boundaries of A and B at each tax level.

t	$R_t(G)$
0	$4 - 2 = 2$
1	$5 - 2 = 3$
2	$6 - 2 = 4$
3	$7 - 1 = 6$
4	$7 - 0 = 7$
5	$7 - 0 = 7$

Then we find the width of the compound Thermograph, by taking the maximum width of A and B at each tax level. We calculate some values; and by adding the width to the Right boundary on the graph, we get the

Left boundary, and can now draw the compound Thermograph.

t	$W_t(A)$	$W_t(B)$	$max = W$
0	6	4	6
1	4	4	4
2	2	4	4
3	0	2	2
4	0	0	0
5	0	0	0

We see that the maximum value of G_t is 8, and that this is obtained for $t = 0$ and for $t \in [2, 3]$. The minimal t is thus 0. The component that is widest at $t = 0$ is A, so Left should move in component A.

Example 36 [Another Example of Thermostrat]:
Let us assume that we have found the following three Thermographs for A, B and C, respectively:

We want to play in the sum of A, B and C, so we employ Thermostrat. So again, we begin by finding the Right boundary of the compound Thermograph by adding the Right boundaries of the compounds:

t	$R_t(G)$
0	$-44 - 24 - 6 = -74$
2	$-44 - 24 - 4 = -72$
4	$-44 - 24 - 2 = -70$
6	$-44 - 24 - 2 = -70$
8	$-44 - 22 - 2 = -68$
10	$-44 - 22 - 2 = -68$
12	$-44 - 20 - 2 = -66$
14	$-40 - 18 - 2 = -60$
16	$-40 - 18 - 2 = -60$
18	$-40 - 18 - 0 = -58$
20	$-40 - 18 + 2 = -56$
21	$-39 - 18 + 3 = -54$
22	$-38 - 18 + 3 = -53$

Then we find the width as the maximum width at each tax level, and draw the compound Thermograph:

t	$W_t(A)$	$W_t(B)$	$W_t(C)$	$max = W$
0	18	16	12	18
2	14	14	12	14
4	10	14	12	14
6	10	14	12	14
8	10	10	12	12
10	10	8	12	12
12	10	4	12	12
14	10	0	4	10
16	10	0	4	10
18	6	0	4	6
20	2	0	4	4
21	0	0	2	2
22	0	0	0	0

We find that the maximum value of G_t is -50, and that it is assumed for $t \in [14, 16]$. The minimum t for which G_t is maximum is thus 14. The component that is widest at $t = 14$ is A, so we should play in A.

We already know that we should not play in a number (if there are other options), but let us see what Thermostrat says in an example.

Example 37 [Thermostrat and Number Avoidance]:
Let us take A to be the number 9, B to be $\{7|3\}$, and C to be the number 2. So the respective Thermographs are:

First, we find the Right boundary of the sum:

t	$R_t(G)$
0	$9+3+2=14$
1	$9+4+2=15$
2	$9+5+2=16$
3	$9+5+2=16$

And then the widths, so we can draw the compound Thermograph:

t	$W_t(A)$	$W_t(B)$	$W_t(C)$	$max = W$
1	0	4	0	4
2	0	2	0	2
3	0	0	0	0
4	0	0	0	0

Here G_t has a maximum value of 18, and that it is only assumed for $t = 0$, so that is the minimum. So, we should play in B, which was the only option that was not a number.

Looking at the example, we may be able to deduce why Thermostrat will never advise us to play in a number (if there are other possibilities). The width of a number is namely 0 (for all $t \geq 0$), and the number can therefore not have the maximum width, if there are other options.

Another theorem that is illustrated here is the Translation Principle. We see that $B = \{7|3\}$ has been translated by $A + C = 9 + 2 = 11$ to become

$$\{B^L|B^R\} + (A + C) = \{7|3\} + 11 = \{18|14\} = (A + B + C)$$

Now that we have seen examples of how Thermostrat agrees with our theories, it may be time to look at how well it performs.

6.3.3 How Well Does Thermostrat Work?

To prove the efficiency of Thermostrat, we need two lemmas.

Lemma 4 *For any tax level $t \geq 0$ in the game G, we have*

$$R_t(G^R) + W_t(G^R) \geq R_t(G) - t$$

for all Right's options G^R of G.

Proof
Let T be the temperature of G and let t be a given tax level.
When we introduced Thermostrat, we argued that $L_t(G) = R_t(G) + W_t(G)$; so particularly we have

$$L_t(G^R) = R_t(G^R) + W_t(G^R).$$

If $t \leq T$ then the cooled game is[3]: $G_t = \{R_t(G^L) - t \mid L_t(G^R) + t\}$.
In particular we have the Left option, $G_t^L = L_t(G^R) + t$.
So

$$G_t^L - t = L_t(G^R) + t - t = L_t(G^R) = R_t(G^R) + W_t(G^R),$$

as we wished.
If $t > T$ then the tax exemption is in effect, and $R_t(G) < L_t(G^R) + t$, because if we had still charged tax at $+t$, $R_t(G)$ would have been greater (less attractive for Right).
Now we have

$$R_t(G) - t < L_t(G^R) + t - t = L_t(G^R) = R_t(G^R) + W_t(G^R),$$

as we wanted.

Lemma 5 *If $t \geq 0$ is a tax level that does not exceed the temperature T of the game G, we have*

$$R_t(G^L) - t = R_t(G) + W_t(G)$$

for all Left's options G^L of G.

Proof Let t be a tax level smaller than the Temperature of G. Then we have

$$R_t(G^L) - t = L_t(G),$$

cf. case 1 in previous proof.
We also have

$$L_t(G) = R_t(G) + W_t(G),$$

as before.
Now we conclude

$$R_t(G^L) - t = L_t(G) = R_t(G) + W_t(G),$$

as we wanted.

Theorem 9 *In a sum of games, the difference between the optimal strategy and Thermostrat is bounded by the largest Temperature of the components.*

[3]The notation differs slightly from our original definition, because we want to emphasize what boundary we tax

Proof of theorem 9 [Thermostrats Efficiency]:
Given a temperature $T \geq 0$ and the game $A + B + \cdots + C$, we wish to prove that

i) If Right starts, Left can guarantee: $R_T(A) + R_T(B) + \cdots + R_T(C) - T$

ii) If Left starts, Left can guarantee: $R_T(A) + R_T(B) + \cdots + R_T(C) + W_T$

We will do the proof by induction over the number of moves that are left in the game (we are only dealing with games that terminate); so if n is the number of moves left in A, then $n - 1$ is the number of moves in an option A^R or A^L from the set of Right and Left options.

The induction starts with the empty set. From the empty set, Left cannot gain or lose anything (because there is no further play). So i) is OK: Left gets at least $0 - T \leq 0$, because he gets 0. Furthermore, the width of the empty game is not positive for any $T \geq 0$, so ii) holds too.

Now, let n be the number of moves left in the game $A + B + \cdots + C$, and assume that i) and ii) hold for $n - 1$.

If Right moves first:
Right moves from the game $A+B+\cdots+C$ to the game, say, $A^R+B+\cdots+C$ (since we have not assumed anything about the components, the analysis would be identical, if Right had moved in any other component).

In the game $A^R + B + \cdots + C$, Left is guaranteed $R_T(A^R) + R_T(B) + \cdots + R_T(C) + W_T(A^R)$ by hypothesis ii)

Lemma 4 with $t = T$ gives us:

$$R_T(A^R)+R_T(B)+\cdots+R_T(C)+W_T(A^R) \geq R_T(A)+R_T(B)+\cdots+R_T(C)-T,$$

as desired (Left gets the guaranteed amount, or more).

If Left moves first:
We consider two cases:

Case I:

Suppose there is a component in the game with a Temperature of at least T. Let us say that the component B is widest at T (we could go through a similar analysis if any other component had been wider). Then we know that the temperature of B is at least T, because games of lower Temperature have width 0 at T. Thermostrat states that Left should move in B at this Temperature, i.e. from $A+B+\cdots+C$ to $A+B^L+\cdots+C$. Now it is Right's turn, so by ii) Left is guaranteed $R_T(A) + R_T(B^L) + \cdots + R_T(C) - T$.

Lemma 5 with $t = T$ gives us:

$$R_T(A) + R_T(B^L) + \cdots + R_T(C) - T \;=\; R_T(A) + R_T(B) + \cdots + R_T(C) + W_T,$$

as desired.

Case II: If no component has temperature at least T, then "...Left should reset his thermostat to T_0, the largest temperature of any component (or possibly to an even cooler temperature) before continuing." [1, page 181]. Lowering the "setting of the thermostat" to T_0 (i.e. picking a smaller T, and calling it T_0), will not *reduce* the value of $R_T(A)+R_T(B)+\cdots+R_T(C)+W_T$, since $R_T = R_{T_0}$ for all components, if none of them are hotter than T_0. Lowering it even more to $T_1 < T_0$ will increase the value of $R_T(A)+R_T(B)+\cdots+R_T(C)+W_T$, since $W_{T_1} - W_{T_0} \geq R_{T_1} - R_{T_0}$.

So Left gets at least the same as he would in case I.

This concludes our proof of the efficiency of Thermostrat, and also our journey in Thermography. Now we move on to a completely different application of the game theory, namely to coding theory.

Chapter 7

Error Correcting Codes from Game Theory

Coding theory falls in two main branches: Error-correcting codes and cryptography. In short, the purpose of error-correcting-codes is to ensure the correctness of the transmitted message, whereas cryptography is concerned with encoding messages in order to keep them secret. We will be concerned with the former in this chapter, because this kind of coding theory has successfully been linked to combinatorial game theory. Again, J. H. Conway is one of the key figures in the development of a link between combinatorial game theory and another field of mathematics. In their article [9] Conway and Sloane introduce lexicodes and prove them equal to certain impartial games.

Before we go deeper into the work of Conway and Sloane, let us introduce some basic concepts of coding theory.

7.1 Concepts from Error Correcting Codes

Error correcting codes were invented to lower the frequency of errors due to noisy channels in communication. They have many important applications, for instance computer technology rely heavily on the use of error correcting codes. Nowadays it may be fluctuating currents that cause the problem, but errors have occurred throughout the history of computers. In the days of punch cards, the computer tended to crash if some dirt or a bug had gotten stuck in a hole of a punch card (this is in fact the reason why errors in computer programs are called bugs). The punch card had to be debugged and the whole program rerun.

Suppose we have a message of k characters say $u = u_1 u_2 \cdots u_k$ we want to transmit though a noisy channel. These characters may be binary numbers, or numbers in any other base B. We want to ensure the user gets the right message, or at least is able to detect whether the received message has been altered by the transmission. This can be done by adding some check digits (for example by appending the sum of the digits in the message) or in other ways encoding the message. Thus, we encode the message $u = u_1 u_2 \cdots u_k$ to a codeword $x = x_1 x_2 \ldots x_n$, where $n \geq k$, and we can assume that it is the last $n - k$ symbols that are check symbols. The user will then decode the received message $y = y_1 y_2 \ldots y_n$, and see whether it is a legal code or it has been altered. In some cases the user will even be able to correct possible errors and guess the correct message.

The errors due to the channel can be viewed as an error vector $e = e_1 e_2 \ldots e_n$ added to the codeword x such that the received message is $y = x+e$

Two important concepts in codes are the length and the distance. The *length* of a codeword is simply the dimension of the vector, denoted by n above. The *distance* between two codewords is the number of places they differ in. For instance, the codeword (00111) has length 5 and its distance to the codeword (00888) is 3. The minimal distance of a code is the minimal distance between any two words in it.

A very well known error correcting code is the Hamming code (Richard Hamming 1915-97). This binary code is not only able to detect an error in a codeword, it can even correct it (as long as there is only one per word). The Hamming codewords are precisely the vectors in the null space of a parity check matrix H. For message words of length 4 we need codewords of length 7, and a legal parity check matrix for this is:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Notice that the columns of H are the binary numbers 1 to 7. Any word ζ in this code must obey the parity equation $H \cdot \zeta^T = \underline{0} \pmod 2$, i.e. the parity check matrix multiplied (mod 2) by any of these words transposed yield the zero vector.

If a received word Y has one error in it, the parity equation will not be satisfied. Instead $H \cdot Y^T$ will yield a 3-vector (transposed) called the syndrome. The syndrome is the binary representation of the position where the error is!

In the end of the following section we will show that this amazing code is in fact a lexicode.

7.2 Lexicodes and Games

This section will be devoted to the so called *lexicodes* defined by Conway and Sloane [9]. The peculiar thing about lexicodes is that the codewords correspond to $\mathcal{P}$-positions in certain impartial games.

The class of games we will be concerned with in this chapter are the impartial games, like the game Turning Turtles.

An impartial game has the form

$$G = \{P_a + P_b + P_c + \ldots | P_a + P_b + P_c + \ldots\},$$

where P_i denotes the number of i-heaps. For base 2 $P_i = 1$ denotes one heap of i objects, or rather that there is an odd number of heaps with i objects, cf. $*i$ in the game Nim. In the general case for base B we can view these games as Turning Turtles, where the "turtles" have B sides.

A legal move is to "turn a turtle to zero" (make $P_i = 0$), and, depending on the rules of the particular game in the class, turn some more "turtles" of lower i-value, see figure 7.2. With our notation this corresponds to replacing a term P_h by the sum $P_i + P_j + \ldots + P_k$, where $h > i > j \ldots > k$. We call the set of these numbers the *turning set* for the game.

Definition 17 (Turning sets) *We say that the finite set* $\{h, i, j, \ldots, k\}$ *is a turning set for a game if and only if the move that replaces the term* P_h *by the sum* $P_i + P_j + \cdots + P_k$ *is a legal move in that game.*

As usual, the first player who has no legal move loses.

Example 38 In his article [7] Grundy describe a heap game, now known as Grundy's game, together with its analysis as an example of his theory. In this game the family of turning sets consists of all sets of three distinct elements $\{h, i, j\}$ with $0 < i < h$, $0 < j < h$, $h = i + j$ and $i \neq j$. In other words, a legal move is to split any heap in two heaps of distinct sizes ($\in \mathbf{N}$).

We quickly realize that a heap of three tokens is the smallest one that can be split, since neither heaps of one nor of two tokens can be divided satisfactorily. The turning set for a heap with three tokens is $\{3, 2, 1\}$, because the only legal move is to split the heap into a heap of one and a heap of two. A heap of four can only be split into a heap of three and a heap of one, since

Figure 7.1: Moving from $G = \{5+0+2|5+0+2\}$ to $G' = \{0+6+4|0+6+4\}$, that is, we replace $P_3 = 5$ by $P_2 = 6$ plus $P_1 = 2$.

splitting equally is not permitted. The corresponding turning set is $\{4, 3, 1\}$. A heap of five can be split in two ways, either in four and one, or in three and two. The corresponding turning sets are $\{5, 4, 1\}$ and $\{5, 3, 2\}$.

We can apply the mex-function invented by Sprague and Grundy in order to find winning positions in this game. This will prove useful when we in the following show that the winning positions correspond to codewords in a lexicode.

Using turning sets we can define a code called the lexicode. The code is defined in such a way that the legal codewords have a certain distance determined by the turning sets, enabling the receiver to distinguish a legal codeword from illegal ones with a certain precision.

The possible words all have the form $(\ldots\zeta_3\zeta_2\zeta_1)$, where $0 \leq \zeta_i < B$, i.e. they are base B vectors. These are regarded in *lexicographic order*, determined by the corresponding number

$$N = \sum_i \zeta_i B^i$$

i.e. by the numerical value of the number they form. For instance, the codeword (10101) in base 2 has $N = 21$. This number N corresponds to a position uniquely, and may be used in place of it.

When comparing two possible words N and N', we may want to indicate the places the differ in, i.e. the set of i for which $\zeta_i \neq \zeta'_i$. This set is denoted $\Delta(N, N')$. For the codewords N =(0123) and N' =(1103) the set of differences $\Delta(N, N')$ is $\{4, 2\}$.

For each family of turning sets and each base B we define the lexicode by the following greedy algorithm:

Definition 18 (Lexicodes) *All possible words are regarded in lexicographic order. A word is rejected from the code if there exists an earlier word* $N' = (\ldots \zeta_3' \zeta_2' \zeta_1')$ *with* $N' < N$ *in the code for which the set of differences* $\Delta(N, N')$ *is a turning set. Words that are not rejected are placed in the code, thus building the lexicode.*

Example 39 Let the set $\{2, 1\}$ belong to the family of turning sets, and let the word $N' = (001)$ be accepted in the code. The word $N = (010)$ will be rejected from the code, because it differs from the codeword (001) in places 1 and 2 (such that $\Delta(N, N') = \{2, 1\}$), and the set $\{2, 1\}$ is a turning set.

We have a very important theorem linking lexicodes to impartial games:

Theorem 10 *For any family of turning sets and any base B the winning moves in the game are to positions that correspond to codewords in the lexicode.*

We will prove this by induction over N, the position.

Proof of theorem 10 Let the base B and a family of turning sets be given. As base case we look at the codeword $(\ldots 000)$. This corresponds to the Endgame, and a move to this position is trivially a winning move.
Now, assume all words with $N' < N$ in the code correspond to winning moves. For the next word N we either have

1. N is not in the lexicode, or
2. N is in the lexicode.

If the word N is not in the lexicode, we want to show that the corresponding move to N is not a winning one, i.e. N is not a $\mathcal{P}$-position. On the other hand, if N is in the lexicode, we want to show that the corresponding move to N is a winning one, i.e. that N is a $\mathcal{P}$-position.

Re 1. Let us assume that the word N is not in the lexicode. Then there exists a word N' in the code with $N' < N$ such that $\Delta(N, N')$ is a turning set. From the hypothesis we know that N' is a $\mathcal{P}$-position, and since $\Delta(N, N')$ is a turning set, there exists a legal move from N to N'. Now we have established the existence of a move from N to a $\mathcal{P}$-position, therefore N cannot be a $\mathcal{P}$-position (Bouton's first theorem).

Re 2. Now, let us assume that the word N is in the lexicode. If there exists a legal move from N to N', then $\Delta(N, N')$ is a turning set and $N' < N$. But since we accept N in the code, we must have rejected all such N' from the code when constructing it. Therefore no such N' is a $\mathcal{P}$-position, and we can conclude that N is a $\mathcal{P}$-position (Bouton's second theorem).

Now that we have established the connection between lexicodes and impartial games, we can apply the theory developed in chapter 3 on the codes to reveal some interesting properties.

- The lexicodes are linear when the base $B = 2$.
- The lexicodes are closed under Nim addition for base $B = 2^n$.
- The lexicodes are closed under Nim multiplication for base $B = 2^{2^n}$

In chapter 3 we established that the $\mathcal{P}$-positions in impartial binary games all have Nim sum zero. Combining this linear condition with theorem 10, we obtain that the winning code for a heap game is a linear code over $GF(2)$:

Theorem 11 *When the base $B = 2$ the lexicode defined by any family of turning sets is a binary linear code.*

This theorem can be generalized to bases that are powers of 2:

Theorem 12 *When the base is a power of 2, i.e. $B = 2^n$ ($n \in \mathbb{N}$), then the lexicode defined by any family of turning sets is closed under Nim addition (componentwise).*

Proof of theorem 12 Let us begin with the case $B = 8$, and then realize that the general case can be done similarly.

Given a family of turning sets for a game with positions in base 8. We can convert the position vectors into binary vectors by replacing each digit by the three digits that constitute its binary representation ($0_8 = 000_2, 1_8 = 001_2, 2_8 = 010_2, \ldots, 7_8 = 111_2$). Formally, we replace ζ_i by $\zeta_{3i+2}\zeta_{3i+1}\zeta_{3i}$ (here the places i run from 0). This way we can convert any octal game to a binary game. In the binary game T is a turning set iff $\{[\frac{i}{3}] : i \in T\}$ was a turning set in the original game (here $[x]$ denotes the integer part of x) - see the example below.

We can now apply theorem 11 to the binary game, so we conclude that it (and thus the original game) is closed under componentwise Nim addition.

The proof above can be executed for any base $B = 2^n$ ($n \in \mathbb{N}$): We convert the position vectors to binary vectors by replacing each digit in each vector by its binary representation. This way we obtain a binary game, where T is a turning set iff $\{[\frac{i}{n}] : i \in T\}$ was a turning set in the original game. We can then apply theorem 11 to the binary game.

If, conversely, B is not a power of 2, we cannot conclude that the lexicode is closed under any reasonably defined kind of addition. As an example, let $B = 3$ and the turning sets be the sets of cardinality 1 (i.e. the code has minimal distance 2), then the lexicode begins: 0000, 0011, 0022, 0101, 0110, 0202, ... If we consider adding the third and fourth word, we see that the sum cannot reasonably be said to belong to the code.

Example 40 Let $B = 8$ and let the family of turning sets be all sets of size 1 or 2 (legal codewords will differ in at least three places). Applying our definition of the lexicode, we can find legal codewords (column 1 below). Theorem 12 states that the Nim sum of codewords is also in the code. For example, the Nim sum 0111 + 1012 = 1103 is also in the code.

Codewords	Binary codeword
0000	000000000000
0111	000001001001
0222	000010010010
0333	000011011011
...	...
0777	000111111111
1012	001000001010
1103	001001000011
...	...

These codewords can be converted into binary codewords (column 2). The three rightmost digits correspond to the zeroth digit in the octal lexicode, the next three correspond to the first digit, ..., digit $3i+2, 3i+1, 3i$ correspond to digit i in the octal code (the integer part of the position in the binary code divided by 3). This way the set $\{8, 7, 6, 2, 1, 0\}$ belongs to the family of turning sets in the binary game, because the set $\{2, 0\}$ is a turning set in the octal game.

We can define a componentwise Nim multiplication, and prove that any lexicode with $B = 2^{2^n}$ ($n \in \mathbb{N}$) is closed under this operation by numbers $a : 0 \leq a < B$. That is, the lexicode is a linear code over the field $GF(2^{2^n})$. But we will not go further into this.

It has been found that several well known codes turn out to be lexicodes, among them Hamming codes and some Golay codes.

Example 41 The Hamming code of length 7 is a lexicode. Let the family of turning sets include all sets of one or two elements (the code distance d is 3), and the base B be 2. We want to generate the lexicode of length 7 for this binary game. We begin with the code with the zero word (0000000). The next possible word (0000001) is obviously not in the code, since $\{1\}$ is a turning set. The next five words (0000010), (0000011), (0000100), (0000101) and (0000110) are rejected too, because $\{2\}$, $\{2,1\}$, $\{3\}$, $\{3,1\}$ and $\{3,2\}$ are turning sets. The next word (0000111) is allowed in the code since it differs from the previously accepted codeword in the last three places $i = 1,2,3$, and $\{3,2,1\}$ is not a turning set. Now we have two words in the code to check all candidates up against. Once we have gone through all the 2^7 possible words, we will have generated a lexicode composed of the following 16 words:

(0000000)	(0000111)	(0011001)	(0011110)
(0101010)	(0101101)	(0110011)	(0110100)
(1001011)	(1001100)	(1010010)	(1010101)
(1100001)	(1100110)	(1111000)	(1111111)

These are all words in the Hamming code, because the Hamming Matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

multiplied (mod 2) with each of these words transposed yield the zero vector:

$$H \cdot N^T = \underline{0} \pmod 2$$

For instance for the codeword $N = (0110100)$ we get:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For all 16 codewords we get the zero vector, which proves that this lexicode is in fact the binary Hamming code of length 7.

Similarly, we could prove that all length $2^n - 1$ lexicodes with $B = 2$ and families of turning sets composed of all sets with exactly one or two elements are Hamming codes. Other lengths of lexicodes with the same properties are shortened Hamming Codes [9]. The extended Hamming codes are obtained by letting $B = 2$ and the family of turning sets contain all sets of exactly one, two or three elements. Other well-known codes mentioned by [9] as lexicodes include: Extended Binary Codes, The Extended Quadratic Residue Code of Length 18, The Extended Golay Code and The Tetracode. Being lexicodes we know that the code words correspond to winning positions in certain impartial games. This means that once we have identified the corresponding impartial game, we can solve it, and thus find the legal code words.

A. Fraenkel has extended the link between error correcting codes and game theory by extending lexicodes to so called anncodes [10], but these rely on annihilation games that allow draws, and therefore falls outside the scope of this thesis.

Chapter 8

Conclusion

In this thesis we have given an introduction to the theory of Combinatorial games. We have looked at applications of the theory, and especially how it links to some other branches of mathematics.

We have seen that all real numbers can be defined as games, and that games even extend our field of numbers to the so called surreals.

We have taken a profound look at impartial games, and seen that all such games can be solved by simple means, namely as the game Nim.

For partial games we have introduced the Temperature as a measure of how eager the players are to be in the move. We have then introduced Thermographs as a graphical way of representing games at different Temperatures, and realized that we can calculate the Thermograph for any game $G = \{G^L|G^R\}$, provided we know the Thermographs of G^L and G^R, which we can find recursively. Even in sums of games, Themography can be applied. Here the strategy called Thermostrat gives us a pretty good estimate of the best move. Thus, Thermography provides a powerful tool to analyze combinatorial games.

Finally, we have seen how games yield some important error correcting codes called Lexicodes; the well known Hamming codes are examples of these.

As mentioned in the beginning of this thesis, Guy has made a list of unsolved problems in combinatorial games, so there are still lots of work to be done in this field.

Furthermore, we could had taken a journey into the world of games with draws. As mentioned, Fraenkel has done some work on such games. He has analyzed the complexity of many games, which is very important when solving these combinatorial games on computers.

Another road we could had taken was to compare this kind of game theory to other kinds of game theory, e.g. to the economical game theory that sprung from the work of von Neumann and Morgenstern.

However, we have chosen to prepare this thesis, hoping to inspire readers to take a closer look at the wonders of combinatorial games.

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Chapter 9

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