

PACKING DIRECTED CIRCUITS

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ABSTRACT

We prove a conjecture of Younger, that for every integer $n \geq 0$ there exists an integer $t \geq 0$ such that for every digraph G , either G has n vertex-disjoint directed circuits, or G can be made acyclic by deleting at most t vertices.

1. INTRODUCTION

A well-known theorem of Erdős and Pósa [1] states the following.

(1.1) *For any integer $n \geq 0$ there exists an integer $t \geq 0$ such that for every graph G , either G has n circuits that are pairwise vertex-disjoint, or there exists $T \subseteq V(G)$ with $|T| \leq t$ such that T meets every circuit of G .*

(All graphs and digraphs in this paper are finite, and may have loops or multiple edges. We denote the vertex- and edge-sets of a graph or digraph G by $V(G)$ and $E(G)$.)

In 1973, Younger [8] conjectured that there was an analogue of (1.1) for directed circuits in digraphs. (The question probably occurred to other workers before then - for instance, it seems that Fulkerson worked on it in the 1950's, but put nothing in print.) For $n = 0$ and 1 this is obvious. For $n = 2$ it was conjectured independently by Gallai [4] (who also considered the $n > 2$ case without publishing it), and eventually was answered by McCuaig [5] in the following strong form.

(1.2) *For every digraph G , either G has two vertex-disjoint directed circuits, or there exists $T \subseteq V(G)$ with $|T| \leq 3$ such that T meets every directed circuit of G .*

But Younger's conjecture in general (and even for $n = 3$) has remained open until the present. In this paper we prove the conjecture.

For a digraph G we denote by $\nu(G)$ the maximum n such that G has n directed circuits, pairwise vertex-disjoint, and by $\tau(G)$ the minimum t such that there exists $T \subseteq V(G)$ with $|T| = t$ meeting all directed circuits. Evidently $\nu(G) \leq \tau(G)$, and Younger's conjecture is that $\tau(G)$ is at most $f(\nu(G))$ for some function f independent of G . Thus, the main result of this paper is the following.

(1.3) For every integer $n \geq 0$ there exists a (minimum) integer $t_n \geq 0$ such that for every digraph G , either $\nu(G) \geq n$ or $\tau(G) \leq t_n$.

Throughout the paper, t_n is defined as in (1.3); we shall prove by induction on n that t_n exists for all n . Obviously $t_0 = t_1 = 0$, and from McCuaig's theorem $t_2 = 3$.

We shall (implicitly) prove an upper bound on t_n , but it is very large (a multiply iterated exponential, where the number of iterations is also a multiply iterated exponential) and probably far from best possible. The best *lower* bound on t_n that we know is $O(n \log(n))$, due to Alon (unpublished).

The paper is organized as follows. In section 2 we apply Ramsey's theorem to deduce a lemma with several consequences. One of them, not needed here but perhaps of some independent interest, is that for any $n \geq 0$ there exists $t \geq 0$ so that for every digraph G , either G has a list of n directed circuits with each vertex in at most two of them, or $\tau(G) \leq t$. (A weak form of this was proved in [7].) Also, the lemma implies that to prove (1.3) in general, it suffices to prove it for digraphs in which each vertex has indegree and outdegree 2, and with a number of other properties.

In section 3 we prove a second lemma that says roughly that if G has as a subdigraph a kind of grid together with some additional paths, then $\nu(G)$ is large. Then in section 4, we prove that digraphs with the properties implied by the lemma of section 2 do contain such grids. Finally, in section 5 we use (1.3) to obtain a polynomial-time algorithm for the problem.

Let us establish some notation and terminology. A *path* (all paths in this paper are "directed paths") has at least one vertex, and no "repeated" vertices. If a and b are its first and last vertices we call it an (a, b) -*path* or a *path from a to b* . If x, y are vertices of a path P , and there is a subpath of P from x to y , we denote this subpath by $P[x, y]$. A *linkage* L in a digraph G is a subdigraph consisting of vertex-disjoint paths, called

the *components* of L . If a linkage L has components P_1, \dots, P_k and P_i is an (a_i, b_i) -path ($1 \leq i \leq k$) we say that L *links* (a_1, \dots, a_k) to (b_1, \dots, b_k) , and if $A, B \subseteq V(G)$ with $a_1, \dots, a_k \in A$ and $b_1, \dots, b_k \in B$ we say the linkage is *from* A *to* B . We call k the *size* of L .

A *separation* in a digraph G is a pair (X, Y) of subsets of $V(G)$ with $X \cup Y = V(G)$, so that no edge of G has tail in $X - Y$ and head in $Y - X$. Its *order* is $|X \cap Y|$. We shall frequently need the following version of Menger's theorem for digraphs:

(1.4) *Let G be a digraph, let $k \geq 0$ be an integer, and let $A, B \subseteq V(G)$. Then exactly one of the following holds:*

- (i) *there is a linkage from A to B of size k*
- (ii) *there is a separation (X, Y) of G of order $< k$ with $A \subseteq X$ and $B \subseteq Y$.*

We shall also need the following result of Erdős and Szekeres [2].

(1.5) *Let $s, t \geq 1$ be integers, and let $n = (s - 1)(t - 1) + 1$. Let a_1, \dots, a_n be distinct integers. Then either*

- (i) *there exist $1 \leq i_1 < \dots < i_s \leq n$ so that $a_{i_1} < a_{i_2} < \dots < a_{i_s}$, or*
- (ii) *there exist $1 \leq i_1 < \dots < i_t \leq n$ so that $a_{i_1} > a_{i_2} > \dots > a_{i_t}$.*

The third standard combinatorial result we use is Ramsey's theorem [6], the following.

(1.6) *For all integers $q, l, r \geq 1$ there exists a (minimum) integer $R_l(r; q) \geq 0$ so that the following holds. Let Z be a set with $|Z| \geq R_l(r; q)$, let Q be a set with $|Q| = q$, and for each $X \subseteq Z$ with $|X| = l$ let $f(X) \in Q$. Then there exists $S \subseteq Z$ with $|S| = r$ and there*

exists $x \in Q$ so that $f(X) = x$ for all $X \subseteq S$ with $|X| = l$.

2. AN APPLICATION OF RAMSEY'S THEOREM

In this section we prove that (1.3) is true in general if it is true for digraphs with the property that for every vertex v , the outdegree and indegree of v are equal and are at most 2. We begin with the following. If G is a digraph and $X \subseteq V(G)$, the digraph obtained by deleting X is denoted by $G \setminus X$.

(2.1) *Let $n \geq 1$ be an integer such that t_{n-1} exists. Let G be a digraph with $\nu(G) < n$, and let $T \subseteq V(G)$ with $|T| = \tau(G)$, meeting all directed circuits of G . Let $A, B \subseteq T$ be disjoint with $|A| = |B| = r$ say, where $r \geq 2t_{n-1}$. Then there is a linkage in G from A to B of size r with no vertex in $T - (A \cup B)$.*

Proof. Suppose not, and let $Z = T - (A \cup B)$. By (1.4) applied to $G \setminus Z$ there is a separation (X, Y) of G with $A \subseteq X, B \subseteq Y, Z \subseteq X \cap Y$ and with $|X \cap Y - Z| < r$. Let $W = X \cap Y - Z$; then $|W| < |B|$. Let $T_1 = A \cup (X \cap Y)$. Since

$$|T_1| \leq |A - Y| + |Z| + |W| < |A - Y| + |Z| + |B| \leq |T| = \tau(G)$$

there is a directed circuit C_1 of G with $T_1 \cap V(C_1) = \emptyset$. Since $T \cap V(C_1) \neq \emptyset$ it follows that $B \cap V(C_1) \neq \emptyset$ and hence $Y \cap V(C_1) \neq \emptyset$. Since (X, Y) is a separation and $X \cap Y \cap V(C_1) = \emptyset$, it follows that $V(C_1)$ does not meet X , and so $V(C_1) \subseteq Y - X$. Similarly there is a directed circuit C_2 with $V(C_2) \subseteq X - Y$.

Let $G_1 = G \setminus Y$. Since $\nu(G) < n$ and $V(C_1) \cap V(G_1) = \emptyset$, it follows that $\nu(G_1) < n - 1$, and consequently $\tau(G_1) \leq t_{n-1}$. Similarly $\tau(G_2) \leq t_{n-1}$, where $G_2 = G \setminus X$. But every directed circuit of G that is not a circuit of G_1 or G_2 meets $X \cap Y$, and so

$$\tau(G) \leq \tau(G_1) + \tau(G_2) + |X \cap Y| < 2t_{n-1} + |Z| + r$$

$$= 2t_{n-1} + |T| - r = 2t_{n-1} + \tau(G) - r$$

and so $r < 2t_{n-1}$, contrary to hypothesis. The result follows. \blacksquare

The main result of this section is the following. It was proved in joint work with Noga Alon.

(2.2) *Let $n \geq 1$ be an integer such that t_{n-1} exists, and let $k \geq 1$ be an integer. Then there exists an integer $t \geq 0$ so that the following holds. Let G be a digraph with $\nu(G) < n$ and $\tau(G) \geq t$. Then there are distinct vertices $a_1, \dots, a_k, b_1, \dots, b_k$ of G and two linkages L_1, L_2 of G , both of size k , so that*

- (i) L_1 links (a_1, \dots, a_k) to (b_1, \dots, b_k)
- (ii) L_2 links (b_1, \dots, b_k) to one of $(a_1, \dots, a_k), (a_k, \dots, a_1)$
- (iii) every directed circuit of $L_1 \cup L_2$ meets $\{a_1, \dots, a_k, b_1, \dots, b_k\}$.

Proof. Let $l = (k-1)^2 + 1$, $r = \max\{2t_{n-1}, (k+1)l\}$, and $q = (l+1)^2$. Let $t = R_l(r; q) + l$, where $R_l(r; q)$ is as in (1.6). Then $r \geq l$ and $t \geq 2r$, since trivially $R_l(r; q) \geq 2r - 1$. We claim that t satisfies the theorem. For let G be a digraph with $\nu(G) < n$ and $\tau(G) \geq t$. Choose $T \subseteq V(G)$ with $|T| = \tau(G)$, meeting all directed circuits of G . Choose $A \subseteq T$ with $|A| = l$, and let $Z = T - A$. Thus, $|Z| \geq R_l(r; q)$. Let $Z = \{z_i : 1 \leq i \leq |Z|\}$.

For each $X \subseteq Z$ with $|X| = x$ say, let $X = \{z_{i_1}, \dots, z_{i_x}\}$ where $i_1 < \dots < i_x$; we denote the x -tuple $(z_{i_1}, \dots, z_{i_x})$ by \bar{X} , and for $1 \leq h \leq x$ we denote z_{i_h} by $\bar{X}(h)$.

Let $X \subseteq Z$ with $|X| = l$. If there exists a linkage $L_1(X)$ say in G from A to X with no vertex in $Z - X$, then there is a sequence (a_1, a_2, \dots, a_l) with $\{a_1, a_2, \dots, a_l\} = A$ so that $L_1(X)$ links (a_1, a_2, \dots, a_l) to \bar{X} , and we define $p_1(X)$ to be (a_1, a_2, \dots, a_l) ; if no such linkage exists we define $p_1(X)$ to be \emptyset . Similarly, if there exists a linkage $L_2(X)$ say

in G from X to A with no vertex in $Z - X$, we define $p_2(X)$ to be (b_1, b_2, \dots, b_l) , where $\{b_1, b_2, \dots, b_l\} = A$ and $L_2(X)$ links \bar{X} to (b_1, b_2, \dots, b_l) ; if no such linkage exists we define $p_2(X)$ to be \emptyset . We define $f(X)$ to be the pair $(p_1(X), p_2(X))$. Let Q be the set of all pairs (a, b) such that each of a, b is either the empty set, or a sequence (x_1, x_2, \dots, x_l) with $\{x_1, x_2, \dots, x_l\} = A$. Then $|Q| = (l! + 1)^2$, and $f(X) \in Q$ for each $X \subseteq Z$ with $|X| = l$. By (1.6), there exist $S \subseteq Z$ with $|S| = r$, and $(a, b) \in Q$, so that $f(X) = (a, b)$ for all $X \subseteq S$ with $|X| = l$. We claim that $a \neq \emptyset$ and $b \neq \emptyset$. Indeed, suppose for a contradiction that $a = \emptyset$, and choose a set A' with $A \subseteq A' \subseteq T - S$ and $|A'| = r$. By (2.1) there is a linkage L' in G from A' to S of size r with no vertex in $T - (A' \cup S)$. The linkage L' includes a linkage from A to some $X \subseteq S$ with no vertex in $T - (A \cup X)$. Thus $a = p_1(X) \neq \emptyset$, and similarly $b \neq \emptyset$. This proves the claim.

Let $a = (a_1, a_2, \dots, a_l)$ and $b = (b_1, b_2, \dots, b_l)$. Consequently, for all $X \subseteq S$ with $|X| = l$, $L_1(X)$ links (a_1, \dots, a_l) to \bar{X} and $L_2(X)$ links \bar{X} to (b_1, \dots, b_l) .

For $1 \leq i \leq l$, choose j with $1 \leq j \leq l$ so that $a_i = b_j$, and define $j_i = j$. Then j_1, \dots, j_l are distinct integers. By (1.5), since $l = (k-1)^2 + 1$, there exist $1 \leq i_1 < i_2 < \dots < i_k \leq l$ so that either

$$j_{i_1} < j_{i_2} < \dots < j_{i_k}$$

or

$$j_{i_1} > j_{i_2} > \dots > j_{i_k}.$$

Define (i'_1, \dots, i'_k) to be $(j_{i_1}, \dots, j_{i_k})$ in the first case, and $(j_{i_k}, \dots, j_{i_1})$ in the second. Hence $i'_1 < \dots < i'_k$.

Let $D = \{\bar{S}(l), \bar{S}(2l), \dots, \bar{S}(kl)\}$. Choose $X \subseteq S$ with $|X| = l$ so that $\bar{S}(hl) = \bar{X}(i_h)$ for $1 \leq h \leq k$. (This is possible since $i_1 < \dots < i_k$ and there are $\geq l-1$ terms in \bar{S} between any two selected to be in D .) The linkage $L_1(X)$ links (a_1, \dots, a_l) to \bar{X} , and consequently it includes a linkage L_1 linking $(a_{i_1}, \dots, a_{i_k})$ to \bar{D} . Moreover, the only vertices of T in L_1 belong to $A \cup D$.

Similarly choose $Y \subseteq S$ with $|Y| = l$ so that for $1 \leq h \leq k$, $\bar{S}(hl) = \bar{Y}(i'_h)$. Since $L_2(Y)$ links \bar{Y} to (b_1, \dots, b_l) , it includes a linkage L_2 linking \bar{D} to $(b_{i'_1}, \dots, b_{i'_k})$. Now $(b_{i'_1}, \dots, b_{i'_k})$ is either $(b_{j_{i_1}}, \dots, b_{j_{i_k}}) = (a_{i_1}, \dots, a_{i_k})$ or its reverse. Moreover, every directed circuit in $L_1 \cup L_2$ meets T , and the only vertices of T in $V(L_1 \cup L_2)$ are a_{i_1}, \dots, a_{i_k} and $\bar{S}(l), \dots, \bar{S}(kl)$; and so L_1, L_2 satisfy the theorem. \blacksquare

Theorem (2.2) is quite powerful. For instance, define $\nu_2(T)$ to be the maximum number of directed circuits of G (with repetition allowed) so that every vertex is in at most two of them. Evidently

$$2\nu(G) \leq \nu_2(G) \leq 2\tau(G)$$

and since we are trying to bound $\tau(G)$ above by a function of $\nu(G)$, it should be easier to bound it above by a function of $\nu_2(G)$. This is indeed the case, and such a bound follows from (2.2), because the two linkages L_1 and L_2 imply that $\nu_2(G) \geq \frac{1}{2}k$. (Make a circuit from the i th and $(k+1-i)$ th component of each linkage, for $1 \leq i \leq \frac{1}{2}k$.) Of course, (2.2) assumes that t_{n-1} exists, but that can be replaced by the hypothesis that if $\nu_2(G) < n-1$ then $\tau(G)$ is bounded above by a function of $\nu_2(G)$, modifying (2.1) accordingly. We omit the details since we are going to prove a stronger result.

We say a digraph is *divalent* if every vertex has indegree 2 and outdegree 2, or indegree 1 and outdegree 1. In sections 3 and 4 we shall prove the following:

(2.3) *For every integer $n \geq 1$ there exists an integer $k \geq 0$ such that for every divalent digraph G , if there are two linkages L_1, L_2 in G , both of size $\geq k$, so that each component of L_1 meets each component of L_2 and $L_1 \cup L_2$ has no directed circuits, then $\nu(G) \geq n$.*

We shall assume (2.3) for the moment; in the remainder of this section we show how to combine (2.2) and (2.3) to prove the main result (1.3). First, we show:

(2.4) For every integer $n \geq 1$ there exists $k \geq 0$ so that the following holds. Let G be a digraph, and let $a_1, \dots, a_k, b_1, \dots, b_k$ be distinct vertices of G . Let L_1, L_2 be linkages in G linking (a_1, \dots, a_k) to (b_1, \dots, b_k) , and (b_1, \dots, b_k) to one of $(a_1, \dots, a_k), (a_k, \dots, a_1)$ respectively. Let every directed circuit of $L_1 \cup L_2$ meet $\{a_1, \dots, a_k, b_1, \dots, b_k\}$. Then $\nu(G) \geq n$.

Proof, assuming (2.3). Given $n \geq 1$, choose $k' \geq 0$ so that n, k' satisfy (2.3) (with k replaced by k'); we may assume that $6k' \geq n$, by increasing k' . Let $k = 2R_2(6k'; 9)$, defined as in (1.6). We claim that n and k satisfy (2.4). For let $G, a_1, \dots, a_k, b_1, \dots, b_k, L_1, L_2$ be as in (2.4). We show by induction on $|E(G)| + |V(G)|$ that $\nu(G) \geq n$. If $L_1 \cup L_2 \neq G$ the result follows immediately by induction, so we may assume that $L_1 \cup L_2 = G$. If some edge e belongs to L_1 and to L_2 , the result follows from the inductive hypothesis by contracting e . (For let this operation result in a digraph G' . Since e belongs to a component of L_1 and to a component of L_2 , and since $L_1 \cup L_2 = G$, the head of e is the head of no other edge of G , and so $\nu(G') = \nu(G)$. Moreover, there are clearly linkages in G' satisfying the hypotheses of (2.4).)

We therefore may assume that every edge of G belongs to exactly one of L_1, L_2 . In particular, G is divalent.

For $1 \leq i \leq k$, let P_i be the component of L_1 with first vertex a_i , and let Q_i be the component of L_2 with first vertex b_i .

(1) If there exist $I, J \subseteq \{1, \dots, k\}$ with $|I| = |J| = 3k'$ so that P_i meets Q_j for all $i \in I$ and $j \in J$, then $\nu(G) \geq n$.

Subproof. Choose $I' \subseteq I$ with $|I'| = k'$. There are $2k'$ vertices that are ends of paths $P_i (i \in I')$, and each of them is an end-vertex of at most one $Q_j (j \in J)$. Since $|J| = 3k'$

there exists $J' \subseteq J$ with $|J'| = k'$ so that P_i and Q_j have no common end-vertex for $i \in I'$ and $j \in J'$. Let $L'_1 \subseteq L_1$ be the union of the components $P_i (i \in I')$, and define $L'_2 \subseteq L_2$ similarly. Now every directed circuit in $L_1 \cup L_2$ meets $\{a_1, \dots, a_k, b_1, \dots, b_k\}$, and each of $a_1, \dots, a_k, b_1, \dots, b_k$ is incident with at most one edge of $L'_1 \cup L'_2$ (since P_i and Q_j have no common end-vertex for $i \in I'$ and $j \in J'$) and so $L'_1 \cup L'_2$ has no directed circuits. From (2.3), $\nu(G) \geq n$. This proves (1).

For $1 \leq h < i \leq \frac{1}{2}k$, define $f(\{h, i\})$ as follows. If $P_i \cup P_{k+1-i}$ is disjoint from $Q_h \cup Q_{k+1-h}$ and $Q_i \cup Q_{k+1-i}$ is disjoint from $P_h \cup P_{k+1-h}$, let $f(\{h, i\}) = 0$. Otherwise, at least one of the eight digraphs

$$\begin{aligned}
& P_i \cap Q_h \\
& P_i \cap Q_{k+1-h} \\
& P_{k+1-i} \cap Q_h \\
& P_{k+1-i} \cap Q_{k+1-h} \\
& Q_i \cap P_h \\
& Q_i \cap P_{k+1-h} \\
& Q_{k+1-i} \cap P_h \\
& Q_{k+1-i} \cap P_{k+1-h}
\end{aligned}$$

is non-null. Number them $1, \dots, 8$ in order; we define $f(\{h, i\})$ to be the number of the first which is non-null. Thus, for all h, i with $1 \leq h < i \leq \frac{1}{2}k$, $f(\{h, i\}) \in \{0, 1, \dots, 8\}$.

Since $k = 2R_2(6k'; 9)$ there exists $S \subseteq \{1, \dots, \frac{1}{2}k\}$ with $|S| = 6k'$ and x with $0 \leq x \leq 8$ such that $f(\{h, i\}) = x$ for all $h, i \in S$ with $h < i$. Let $S = I \cup J$, where I and J both have cardinality $3k'$ and $i < j$ for all $i \in I$ and $j \in J$. If $x \neq 0$ then $\nu(G) \geq n$ by (1) (using I and J). If $x = 0$, then the subdigraphs

$$G_i = P_i \cup P_{k+1-i} \cup Q_i \cup Q_{k+1-i}$$

are pairwise vertex-disjoint for all $i \in S$. But P_i, P_{k+1-i} are (a_i, b_i) - and (a_{k+1-i}, b_{k+1-i}) -paths, and either Q_i, Q_{k+1-i} are (b_i, a_i) - and (b_{k+1-i}, a_{k+1-i}) -paths, or they are (b_i, a_{k+1-i}) - and (b_{k+1-i}, a_i) -paths. In either case each G_i has a directed circuit, and so

$$\nu(G) \geq |S| = 6k' \geq n$$

as required. ■

Proof of (1.3), assuming (2.3).

We prove (1.3) by induction on n ; we therefore assume that $n \geq 1$ and t_{n-1} exists, and we show that t_n exists. Let k be as in (2.4), and let t be as in (2.2). We claim that there is no digraph G with $\nu(G) < n$ and $\tau(G) \geq t$. For suppose that G is such a digraph. By (2.2), there exist $a_1, \dots, a_k, b_1, \dots, b_k$ and L_1, L_2 as in (2.2), and so $\nu(G) \geq n$ by (2.4), a contradiction. Thus there is no such G , and consequently t_n exists and $t_n < t$. ■

3. USING A GRID

In this section we show that, if a digraph G contains a kind of grid, with some additional paths, then $\nu(G)$ is large. This is a lemma that will be used in section 4 to prove (2.3).

Let $p, q \geq 1$ be integers. A (p, q) -fence in a digraph G is a sequence $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ with the following properties:

- (i) P_1, \dots, P_{2p} are mutually disjoint paths of G , and so are Q_1, \dots, Q_q
- (ii) for $1 \leq i \leq 2p$ and $1 \leq j \leq q$, $P_i \cap Q_j$ is a path (and therefore non-null)
- (iii) for $1 \leq j \leq q$, the paths $P_1 \cap Q_j, \dots, P_{2p} \cap Q_j$ are in order in Q_j , and the first vertex of Q_j is in $V(P_1)$ and its last vertex is in $V(P_{2p})$

(iv) for $1 \leq i \leq 2p$, if i is odd then $P_i \cap Q_1, \dots, P_i \cap Q_q$ are in order in P_i , and if i is even then $P_i \cap Q_q, \dots, P_i \cap Q_1$ are in order in P_i .

Thus, each (p, q) -fence is a planar digraph with no directed circuits. Let Q_j be an (a_j, b_j) -path ($1 \leq j \leq q$); we call $\{a_1, \dots, a_q\}$ the *top* of the fence, and $\{b_1, \dots, b_q\}$ its *bottom*.

The main result of this section is the following.

(3.1) *For every integer $n \geq 1$, there are integers $p, r \geq 1$ with the following property. For any $q \geq 1$, let $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ be a (p, q) -fence in a digraph G , and let there be r disjoint paths in G from the bottom of the fence to the top. Then $\nu(G) \geq n$.*

To prove (3.1) we need some preliminary lemmas. The first is:

(3.2) *Let $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ be a (p, q) -fence in a digraph G , with top A and bottom B . Let $A' \subseteq A$ and $B' \subseteq B$ with $|A'| = |B'| = r$ say, where $r \leq p$. Then there are paths Q'_1, \dots, Q'_r in $P_1 \cup \dots \cup P_{2p} \cup Q_1 \cup \dots \cup Q_q$ so that $(P_1, \dots, P_{2p}, Q'_1, \dots, Q'_r)$ is a (p, r) -fence with top A' and bottom B' .*

Proof. Let $G' = P_1 \cup \dots \cup P_{2p} \cup Q_1 \cup \dots \cup Q_q$. For $1 \leq j \leq q$, let Q_j be an (a_j, b_j) -path. Suppose that (X, Y) is a separation of G' of order $< r$ with $A' \subseteq X$ and $B' \subseteq Y$. Since $|A'| = r$ there exists j_1 with $1 \leq j_1 \leq q$ so that $a_{j_1} \in A'$ and $V(Q_{j_1}) \cap X \cap Y = \emptyset$. Similarly there exists j_2 with $1 \leq j_2 \leq q$ so that $b_{j_2} \in B'$ and $V(Q_{j_2}) \cap X \cap Y = \emptyset$. Without loss of generality (by reversing the direction of all edges and considering the fence $(P_{2p}, \dots, P_1, Q_q, \dots, Q_1)$) we may assume that $j_1 \leq j_2$. Since $p \geq r$, there exists i with $1 \leq i \leq p$ such that $V(P_{2i-1}) \cap X \cap Y = \emptyset$. But $Q_{j_1} \cup P_{2i-1} \cup Q_{j_2}$ contains a path from A' to B' with no vertex in $X \cap Y$, a contradiction.

Hence there is no such (X, Y) . By (1.4) there are r disjoint paths of G' from A' to B' , and then the result follows easily from the fact that G' is planar and has no directed circuits. ■

From (3.2) we deduce:

(3.3) *Let $n \geq 1$ be an integer, and let $p \geq 2n$ and $N \geq 2n^2 - 3n + 2$ be integers. For some integer $q \geq 1$ let $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ be a (p, q) -fence in a digraph G . Let R_1, \dots, R_N be disjoint paths of G from the bottom of the fence to the top, so that each R_k has no vertex or edge in*

$$P_1 \cup \dots \cup P_{2p} \cup Q_1 \cup \dots \cup Q_q$$

except its end-vertices. Then $\nu(G) \geq n$.

Proof. For $1 \leq j \leq q$, let Q_j be an (a_j, b_j) -path. For $1 \leq k \leq N$, let $a_{s(k)}$ be the last vertex of R_k , and let $b_{t(k)}$ be the first vertex of R_k . By renumbering R_1, \dots, R_N we may assume that $s(1) < s(2) < \dots < s(N)$. Let

$$G' = P_1 \cup \dots \cup P_{2p} \cup Q_1 \cup \dots \cup Q_q.$$

By (1.5), either there exist $1 \leq k_1 < k_2 < \dots < k_n \leq N$ so that

$$t(k_1) < t(k_2) < \dots < t(k_n)$$

or there exist $1 \leq k_1 < k_2 < \dots < k_{2n} \leq N$ so that

$$t(k_1) > t(k_2) > \dots > t(k_{2n}).$$

In the first case, by (3.2) there exist $Q'_1, \dots, Q'_n \subseteq G'$ so that $(P_1, \dots, P_{2p}, Q'_1, \dots, Q'_n)$ is a (p, n) -fence with top $\{a_{s(k_1)}, \dots, a_{s(k_n)}\}$ and bottom $\{b_{t(k_1)}, \dots, b_{t(k_n)}\}$; but then for $1 \leq j \leq n$, Q'_j is an $(a_{s(k_j)}, b_{t(k_j)})$ -path and so $Q'_1 \cup R_{k_1}, \dots, Q'_n \cup R_{k_n}$ are n vertex-disjoint

circuits, and $\nu(G) \geq n$ as required. In the second case, by (3.2) there exist $Q'_1, \dots, Q'_{2n} \subseteq G'$ so that $(P_1, \dots, P_{2p}, Q'_1, \dots, Q'_{2n})$ is a $(p, 2n)$ -fence with top $\{a_{s(k_1)}, \dots, a_{s(k_{2n})}\}$ and bottom $\{b_{t(k_1)}, \dots, b_{t(k_{2n})}\}$. Since $k_1 > k_2 > \dots > k_{2n}$ it follows that for $1 \leq j \leq 2n$, Q'_j is an $(a_{s(k_j)}, b_{t(k_{2n+1-j})})$ -path, and therefore

$$Q'_j \cup R_{k_{2n+1-j}} \cup Q'_{2n+1-j} \cup R_{k_j} \quad (1 \leq j \leq n)$$

are n vertex-disjoint directed circuits of G , and again $\nu(G) \geq n$. ■

Proof of (3.1)

We proceed by induction on n . If $n = 1$ the result is easy (taking $p = r = 1$) and so we may assume that $n \geq 2$. Let n' be the least integer with $2n' \geq n$; then $1 \leq n' < n$, and so there exist $p', r' \geq 1$ so that the theorem holds (with n, p, r replaced by n', p', r'). Let

$$\begin{aligned} N &= 2n^2 - 3n + 2 \\ p'' &= p' + 2N + 1 \\ p &= 16n(r' + N) + 2n + 2p'' \\ r &= 8(r' + N). \end{aligned}$$

We claim that n, p, r satisfy the theorem.

For let $(P_1, \dots, P_{2p}, Q_1, \dots, Q_q)$ be a (p, q) -fence in a digraph G , with top A and bottom B . Let R_1, \dots, R_r be disjoint paths in G from B to A . From (3.2) we may assume that $q = r$. We may also assume (by replacing P_{2i-1} by a subpath) that for $1 \leq i \leq p$, P_{2i-1} has first vertex in $V(Q_1)$ and last vertex in $V(Q_r)$, and similarly P_{2i} has first vertex in $V(Q_r)$ and last vertex in $V(Q_1)$. For $1 \leq j \leq r$ let Q_j be an (a_j, b_j) -path.

For $1 \leq j \leq r$, the paths $P_1 \cap Q_j, P_2 \cap Q_j, \dots, P_{2p} \cap Q_j$ are disjoint, and in order in Q_j . Hence there are p disjoint subpaths Q_{1j}, \dots, Q_{pj} of Q_j so that

- (i) $V(Q_{1j}) \cup \dots \cup V(Q_{pj}) = V(Q_j)$, and
- (ii) for $1 \leq i \leq p$, $(P_{2i-1} \cup P_{2i}) \cap Q_j \subseteq Q_{ij}$.

Choose such paths Q_{1j}, \dots, Q_{pj} for each value of j . For $1 \leq i \leq p$ we define

$$D_i = P_{2i-1} \cup P_{2i} \cup \bigcup (Q_{ij} : 1 \leq j \leq r).$$

Then D_1, \dots, D_p are mutually disjoint, and

$$V(D_1 \cup \dots \cup D_p) = V(P_1 \cup \dots \cup P_{2p} \cup Q_1 \cup \dots \cup Q_r).$$

(1) *There is a linkage L_1 in G of size $\frac{1}{2}r$, so that*

- (i) *each component of L_1 has first vertex in B and last vertex in $V(D_i)$ for some i with $1 \leq i \leq p - p''$, and*
- (ii) *for $1 \leq i \leq p - p''$, at most one component of L_1 has last vertex in $V(D_i)$.*

Subproof. Let G_0 be obtained from G by adding, for $1 \leq i \leq p - p''$, a new vertex y_i and an edge from v to y_i for each $v \in V(D_i)$.

Suppose there is a separation (X, Y) of G_0 of order $< \frac{1}{2}r$, with $B \subseteq X$ and $y_1, \dots, y_{p-p''} \in Y$. Since $p - p'' \geq \frac{1}{2}r$, there exists i with $1 \leq i \leq p - p''$ such that $y_i \notin X \cap Y$, and hence $y_i \in Y - X$. Consequently $V(D_i) \subseteq Y$, and so Q_1, \dots, Q_r each have a vertex in Y . For $1 \leq j \leq r$ let R_j have last vertex $a_{s(j)}$. Since each vertex in $X \cap Y$ belongs to at most one of Q_1, \dots, Q_r and at most one of R_1, \dots, R_r , and since $2|X \cap Y| < r$, there exists j with $1 \leq j \leq r$ such that $R_j \cup Q_{s(j)}$ has no vertex in $X \cap Y$. But $R_j \cup Q_{s(j)}$ contains a path from the first vertex of R_j (which is in B and hence in X) to each vertex of $Q_{s(j)}$; and since some vertex of $Q_{s(j)}$ is in Y , this is impossible. Thus there is no such separation (X, Y) . By (1.4) there are $\frac{1}{2}r$ disjoint paths in G_0 from B to $\{y_1, \dots, y_{p-p''}\}$. This proves (1).

(2) *There exists L_1 as in (1) so that $V(L_1 \cap D_i) \neq \emptyset$ for only $\frac{1}{2}r$ values of i with $1 \leq i \leq p - p''$.*

Subproof. Choose a minimal linkage L_1 satisfying (1). If $1 \leq i \leq p - p''$ and $L_1 \cap D_i$ is non-null, let P be a component of L_1 that meets D_i . If no component of L_1 has last vertex in $V(D_i)$ we can replace P by a subpath from B to $V(D_i)$, contradicting the minimality of L_1 . Thus some component of L_1 has last vertex in $V(D_i)$. Hence there are at most $\frac{1}{2}r$ such values of i since L_1 has size $\frac{1}{2}r$. This proves (2).

By symmetry, we deduce

(3) *There is a linkage L_2 in G of size $\frac{1}{2}r$, so that*

(i) *each component of L_2 has last vertex in A and first vertex in $V(D_i)$ for some i with $p'' + 1 \leq i \leq p$)*

(ii) *for $p'' + 1 \leq i \leq p$, at most one component of L_2 has first vertex in $V(D_i)$*

(iii) *there are only $\frac{1}{2}r$ values of i with $p'' + 1 \leq i \leq p$ such that $V(L_2 \cap D_i) \neq \emptyset$.*

There are $p - 2p''$ values of i with $p'' + 1 \leq i \leq p - p''$, and $D_i \cap (L_1 \cup L_2)$ is non-null for $\leq r$ of them. Since $p - 2p'' \geq 2n(r + 1)$, there exists i_1 with $p'' + 1 \leq i_1 \leq p - p'' - 2n + 1$ so that $D_{i_1}, D_{i_1+1}, \dots, D_{i_1+2n-1}$ are all disjoint from $L_1 \cup L_2$. Let $i_2 = i_1 + 2n - 1$.

For $1 \leq j \leq r$ let a'_j be the first vertex of Q_j in P_{2i_1-1} , let b'_j be the last vertex of Q_j in P_{2i_2} , and let $Q'_j = Q_j[a'_j, b'_j]$. Let $A' = \{a'_1, \dots, a'_r\}$ and $B' = \{b'_1, \dots, b'_r\}$. Let

$$G' = P_1 \cup \dots \cup P_{2p} \cup Q_1 \cup \dots \cup Q_r \cup L_1 \cup L_2,$$

$$G_2 = P_{2i_1-1} \cup P_{2i_1} \cup \dots \cup P_{2i_2} \cup Q'_1 \cup \dots \cup Q'_r.$$

(4) *If in G' there are N disjoint paths from B' to A' then $\nu(G) \geq n$.*

Subproof. Let R be any minimal path in G' from B' to A' . We claim that no vertices or edges of R belong to G_2 except the end-vertices of R . For certainly $R \not\subseteq G_2$ since there is no path of G_2 from $V(P_{2i_2})$ to $V(P_{2i_1-1})$. But every edge e of G' not in $E(G_2)$ but with tail or head in $V(G_2)$ belongs to some $E(Q_j)$, since $G_2 \cap (L_1 \cup L_2)$ is null and each P_i is contained in or disjoint from G_2 ; and consequently every such edge has tail in B' or head in A' . Since no internal vertex of R is in $A' \cup B'$ from the minimality of R it follows that the only such edges in R are the first and the last; and this proves our claim that R is disjoint from G_2 except for its end-vertices.

Now suppose that there are N disjoint paths of G' from B' to A' . We may assume they are all minimal, and therefore each has no vertex or edge in G_2 except its end-vertices. From (3.3) applied to the $(2n, r)$ -fence

$$(P_{2i_1-1}, P_{2i_1}, \dots, P_{2i_2}, Q'_1, \dots, Q'_r)$$

(which has top A' and bottom B') we deduce that $\nu(G) \geq n$. This proves (4).

From (4) and (1.4) we may assume that there is a separation (X, Y) of G' of order $< N$ with $B' \subseteq X$ and $A' \subseteq Y$. Let $H = G' \setminus Y$. We shall show that $\nu(H) \geq n'$.

Now L_1 has $\frac{1}{2}r$ components, each with first vertex in $B = \{b_1, \dots, b_r\}$; let these components be S_j ($j \in J$) where $J \subseteq \{1, \dots, r\}$, $|J| = \frac{1}{2}r$, and for $j \in J$, S_j has first vertex b_j . For $j \in J$, let the last vertex of S_j belong to $V(D_{f(j)})$. Thus, $1 \leq f(j) \leq p - p''$. For $j \in J$ we define H_j to be the subdigraph of G' induced on the vertex set

$$V(Q_j \cup S_j \cup D_{f(j)} \cup D_{f(j)+1}).$$

(5) *For each $v \in V(G)$ there are at most four values of $j \in J$ so that $v \in V(H_j)$.*

Subproof. Any vertex of G belongs to at most one of Q_1, \dots, Q_r , at most one of $S_j (j \in J)$, at most one of $D_{f(j)} (j \in J)$, and at one most of $D_{f(j)+1} (j \in J)$. Hence it belongs to H_j for at most four values of j . This proves (5).

Let I_0 be the set of all i with

$$p - p' - 2N + 1 \leq i \leq p$$

such that $V(D_{i-1} \cup D_i) \cap X \cap Y = \emptyset$. Since there are $p' + 2N$ possibilities for i , and D_i meets $X \cap Y$ for at most N of them, and D_{i-1} meets $X \cap Y$ for at most N of them, it follows that $|I_0| \geq p'$. Choose $I \subseteq I_0$ with $|I| = p'$.

(6) $D_i \subseteq H$ for each $i \in I$.

Subproof. Since $r > |X \cap Y|$, there exists j with $1 \leq j \leq r$ so that Q_j does not meet $X \cap Y$. Since $b'_j \in B' \subseteq X$ it follows that $Q_j[b'_j, b_j] \subseteq H$. Let $i \in I$. Since b'_j belongs to P_{2i_2} , and

$$i_2 = i_1 + 2n - 1 \leq p - p'' = p - p' - 2N - 1 \leq i - 2,$$

it follows that $Q_j[b'_j, b_j]$ meets P_{2i-3} . Hence there exists $u \in X \cap V(P_{2i-3})$. Since $V(D_{i-1} \cup D_i)$ does not meet $X \cap Y$, every path of $D_{i-1} \cup D_i$ with first vertex u is a path of H . Since D_i is contained in the union of all such paths it follows that $D_i \subseteq H$. This proves (6).

Let i_3 and i_4 be the minimum and maximum members of I . For $1 \leq j \leq r$ let a''_j be the first vertex of Q_j in P_{2i_3-1} , and let b''_j be the last vertex of Q_j in P_{2i_4} .

(7) For each $j \in J$ there is a path $T_j \subseteq H_j$ from b''_j to a''_j .

Subproof. Certainly there is a path from b''_j to b_j , namely $Q_j[b''_j, b_j]$, and one from b_j to some vertex u of $D_{f(j)}$, namely S_j . Let v be a vertex of Q_j in $D_{f(j)+1}$; then there is

a path of G' with vertex set in $V(D_{f(j)} \cup D_{f(j)+1})$ from u to v . We claim there is a path in Q_j from v to a_j'' . To show this, we observe that

$$f(j) + 1 \leq p - p'' + 1 = p - p' - 2N < i_3,$$

and so v occurs in Q_j before a_j'' . This proves (7).

By (5), H_j meets $X \cap Y$ for at most $4|X \cap Y| < 4N$ values of $j \in J$, and so there exists $J' \subseteq J$ with $|J'| \geq \frac{1}{2}r - 4N$ so that $V(H_j) \cap X \cap Y = \emptyset$ for all $j \in J'$. Since by (6), T_j has first vertex in X and does not meet $X \cap Y$, it follows that $T_j \subseteq H$ for all $j \in J'$. Let

$$\begin{aligned} A'' &= \{a_j'' : j \in J'\} \\ B'' &= \{b_j'' : j \in J'\}. \end{aligned}$$

(8) *There are at least r' disjoint paths in H from B'' to A'' .*

Subproof. Let (X', Y') be a separation of H with $B'' \subseteq X'$ and $A'' \subseteq Y'$. Since for all $j \in J'$, $T_j \subseteq H$ and has first vertex in X' and last vertex in Y' , it follows that T_j meets $X' \cap Y'$. By (5),

$$|X' \cap Y'| \geq \frac{1}{4}|J'| \geq \frac{1}{8}r - N \geq r'.$$

Hence (1.4) implies (8).

Now the paths P_{2i-1} and P_{2i} ($i \in I$), and $Q_j[a_j'', b_j'']$ ($j \in J'$) define a $(p', |J'|)$ -fence in H , by (6) and the definition of J' ; and this fence has top A'' and bottom B'' . Hence from (8) and the choice of p', r' , it follows that $\nu(H) \geq n'$.

Thus, $\nu(G \setminus Y) \geq n'$. Similarly, $\nu(G \setminus X) \geq n'$, and so $\nu(G) \geq 2n' \geq n$, as required. \blacksquare

4. FINDING THE GRID

In this section we complete the proof of (1.3), by deriving (2.3) from (3.1).

Let $p, q \geq 1$ be integers. A (p, q) -web in a digraph G is a sequence $(P_1, \dots, P_p, Q_1, \dots, Q_q)$ such that

- (i) P_1, \dots, P_p are mutually disjoint paths of G , and so are Q_1, \dots, Q_q ,
- (ii) for $1 \leq i \leq p$ and $1 \leq j \leq q$, P_i meets Q_j , and
- (iii) $P_1 \cup \dots \cup P_p \cup Q_1 \cup \dots \cup Q_q$ has no directed circuits.

A (p, q) -web $(P_1, \dots, P_p, Q_1, \dots, Q_q)$ is a (p, q) -mesh if for $1 \leq i \leq p$ and for all j, j' with $1 \leq j < j' \leq q$, if $v \in V(P_i \cap Q_j)$ and $v' \in V(P_i \cap Q_{j'})$ then v occurs before v' in P_i . The first main result of this section is the following.

(4.1) *For every integer $p \geq 1$ there is an integer $p' \geq 1$ such that every digraph with a (p', p') -web has a (p, p) -mesh.*

To prove (4.1) we need a more complicated object, as follows. A (p, q) -web $(P_1, \dots, P_p, Q_1, \dots, Q_q)$ is called a (p, q, r, s) -sheaf if

- (i) $r, s \geq 0$ are integers and $r + s \leq q$,
- (ii) for $1 \leq j \leq r$ and for $1 \leq i, i' \leq p$ with $i \neq i'$, if $v_1, v_2 \in V(P_i \cap Q_j)$ and $v' \in V(P_{i'} \cap Q_j)$ then v' does not lie in Q_j between v_1 and v_2 ,
- (iii) for $r + 1 \leq j < j' \leq r + s$ and for $1 \leq i \leq p$, if $v \in V(P_i \cap Q_j)$ and $v' \in V(P_i \cap Q_{j'})$ then v occurs before v' in P_i .

We shall need the following four simple observations.

- (4.2) (i) *If G has a (p, q) -web then it has a $(p, q, 0, 0)$ -sheaf.*
- (ii) *If G has a $(p, q, r, 0)$ -sheaf with $r < q$ then it has a $(p, q, r, 1)$ -sheaf.*

- (iii) If G has a $(p, q, p!p, 0)$ -sheaf then it has a (p, p) -mesh.
- (iv) If G has a $(p, q, 0, p)$ -sheaf then it has a (p, p) -mesh.

Proof. Every (p, q) -web is a $(p, q, 0, 0)$ -sheaf, so (i) follows. If $r < q$, every $(p, q, r, 0)$ -sheaf is a $(p, q, r, 1)$ -sheaf, so (ii) follows. For (iii), let $(P_1, \dots, P_p, Q_1, \dots, Q_q)$ be a $(p, q, p!p, 0)$ -sheaf. For $1 \leq j \leq p!p$ there is a permutation π of $\{1, \dots, p\}$ so that for $1 \leq i < i' \leq p$, every vertex of $P_{\pi(i)} \cap Q_j$ occurs in Q_j before every vertex of $P_{\pi(i')} \cap Q_j$. Since there are only $p!$ such permutations, there are at least p values of j that yield the same permutation π , say j_1, \dots, j_p . Then

$$(Q_{j_1}, \dots, Q_{j_p}, P_{\pi(1)}, \dots, P_{\pi(p)})$$

is a (p, p) -mesh. This proves (iii). For (iv), let $(P_1, \dots, P_p, Q_1, \dots, Q_q)$ be a $(p, q, 0, p)$ -sheaf. Then $(P_1, \dots, P_p, Q_1, \dots, Q_p)$ is a (p, p) -mesh. ■

The following is somewhat less trivial.

- (4.3) If G has a (p^2, q, r, s) -sheaf where $s \geq 1$ and $r + s < q$, then it has either a $(p, q, r, s + 1)$ -sheaf or a $(p, q, r + 1, s - 1)$ -sheaf.

Proof. Let $(P_1, \dots, P_{p^2}, Q_1, \dots, Q_q)$ be a (p^2, q, r, s) -sheaf. For $1 \leq i \leq p^2$ let F_i be the minimal path of Q_{r+s} that includes $V(P_i \cap Q_{r+s})$.

Suppose first that some edge e of Q_{r+s} belongs to F_i for at least p values of i , say i_1, \dots, i_p . Let Q, Q' be the two paths obtained from Q_{r+s} by deleting e , where Q contains the tail of e . For $i = i_1, \dots, i_p$, P_i meets both Q and Q' , and every vertex in $P_i \cap Q$ occurs in P_i before every vertex of $P_i \cap Q'$ since $P_1 \cup \dots \cup P_{p^2} \cup Q_1 \cup \dots \cup Q_q$ has no directed circuits. Hence

$$(P_{i_1}, \dots, P_{i_p}, Q_1, \dots, Q_{r+s-1}, Q, Q', Q_{r+s+1}, \dots, Q_{q-1})$$

is a $(p, q, r, s + 1)$ -sheaf, as required.

We may assume then that every edge of Q_{r+s} occurs in F_i for fewer than p values of i . Consequently, there are p values of i , say i_1, \dots, i_p , so that either F_{i_1}, \dots, F_{i_p} each have only one vertex, or F_{i_1}, \dots, F_{i_p} each have at least two vertices and are mutually edge-disjoint. Then in either case, F_{i_1}, \dots, F_{i_p} are pairwise vertex-disjoint, and

$$(P_{i_1}, \dots, P_{i_p}, Q_1, \dots, Q_r, Q_{r+s}, Q_{r+1}, Q_{r+2}, \dots, Q_{r+s-1}, Q_{r+s+1}, \dots, Q_q)$$

is a $(p, q, r + 1, s - 1)$ -sheaf. ■

We deduce the following, which immediately implies (4.1).

(4.4) *Let $p \geq 1$ be an integer, and let $q = 2p!p + p$. If G has a (p^{2^q}, q) -web then it has a (p, p) -mesh.*

Proof. Let G have a (p^{2^q}, q) -web. By (4.2)(i), it has a $(p^{2^q}, q, 0, 0)$ -sheaf. Choose $r, s \geq 0$ and $n = 2r + s$, so that $n \leq q$ and G has a $(p^{2^{q-n}}, q, r, s)$ -sheaf, with n maximum. If $r \geq p!p$ then G has a $(p, q, p!p, 0)$ -sheaf and hence a (p, p) -mesh by (4.2)(iii). If $s \geq p$ then G has a $(p, q, 0, p)$ -sheaf and hence a (p, p) -mesh by (4.2)(iv). We may suppose then (for a contradiction) that $s < p$ and $r < p!p$, and so $n < q$. By (4.2)(ii), $s \neq 0$; and so by (4.3) G has either a $(p^{2^{q-n-1}}, q, r, s + 1)$ -sheaf or a $(p^{2^{q-n-1}}, q, r + 1, s - 1)$ -sheaf. In either case this contradicts the maximality of n . The result follows. ■

A (p, q) -web $(P_1, \dots, P_p, Q_1, \dots, Q_q)$ is a (p, q) -grid if

- (i) for $1 \leq i \leq p$ and $1 \leq j \leq q$, $P_i \cap Q_j$ is a path R_{ij} say
- (ii) for $1 \leq i \leq p$ the paths R_{i1}, \dots, R_{iq} are in order in P_i , and
- (iii) for $1 \leq j \leq q$ the paths R_{1j}, \dots, R_{pj} are in order in Q_j .

Our next main result is the following.

(4.5) *For every integer $p \geq 1$ there is an integer $p' \geq 1$ such that every digraph with (p', p') -mesh has a (p, p) -grid.*

To prove (4.5) we need the following lemma.

(4.6) *Let $r \geq 1$ be an integer, and let $p = \frac{1}{2}r(r+1)$. Let G have no directed circuit, and let P_1, \dots, P_p be disjoint paths in G . Let Q be a path in G meeting all of P_1, \dots, P_p . There is a path R in G so that $R \cap P_i$ is a path for at least r values of i ($1 \leq i \leq p$).*

Proof. Choose a path Q meeting all of P_1, \dots, P_p , so that $Q \cap (P_1 \cup \dots \cup P_p)$ has as few components as possible. Let the components of $Q \cap (P_1 \cup \dots \cup P_p)$ be C_1, \dots, C_n , numbered in order on Q . For $1 \leq i \leq n$, let $f(C_i) = j$ where $C_i \subseteq P_j$. Suppose that $Q \cap P_i$ is a path for $< r$ values of i . Hence for all other values of i , $Q \cap P_i$ has at least two components, and so

$$n > r + 2(p - r) = r^2.$$

Let us say C_i is *good* if $Q \cap P_j = C_i$, where $j = f(C_i)$. There are $< r$ good components, and so there exists i with $1 \leq i \leq n - r + 1$ so that $C_i, C_{i+1}, \dots, C_{i+r-1}$, are all not good. If $f(C_i), f(C_{i+1}), \dots, f(C_{i+r-1})$ are all distinct then we may take R to be the minimal subpath of Q containing $C_i, C_{i+1}, \dots, C_{i+r-1}$. Thus we may assume that there exist j, k with $i \leq j < k \leq i + r - 1$ so that $f(C_j) = f(C_k) = h$ say.

Choose such j, k with $k - j$ minimum. Let Q' be obtained from Q by replacing the path of Q between C_j and C_k by the path P_h between C_j and C_k (this exists and has no vertex in Q except its ends, since G has no directed circuits). Since none of C_{j+1}, \dots, C_{k-1} are good, and $f(C_{j+1}), \dots, f(C_{k-1})$ are all distinct by the minimality of $k - j$, it follows that Q' meets all of P_1, \dots, P_p , contradicting the minimality of n .

Thus, if $Q \cap P_i$ is a path for $< r$ values of i then the result holds. On the other hand, if it is a path for $\geq r$ values of i then we may take $R = Q$, and again the result holds. ■

Proof of (4.5).

Let $p' = p(\frac{1}{2}p(p+1))^p$. We claim that p' satisfies (4.5). For let G have a (p', p') -mesh $(P_1, \dots, P_{p'}, Q_1, \dots, Q_{p'})$. For $1 \leq i, j \leq p'$ let P_{ij} be the minimal subpath of P_i that contains $P_i \cap Q_j$. Then $P_{i1}, \dots, P_{ip'}$ are vertex-disjoint paths of P_i , in order in P_i .

Let $1 \leq j \leq p'$, and let $G_j = Q_j \cup P_{1j} \cup P_{2j} \cup \dots \cup P_{mj}$ where $m = \frac{1}{2}p(p+1)$. Then P_{1j}, \dots, P_{mj} are disjoint paths of G_j , and Q_j meets them all. By (4.6) there is a path R_j of G_j so that $R_j \cap P_{ij}$ is a path for $\geq p$ values of i ($1 \leq i \leq m$). Choose a sequence X_j of p distinct members of $\{1, \dots, m\}$, say i_1, \dots, i_p , so that for $1 \leq k \leq p$, $R_j \cap P_{i_k j}$ is a path and hence $R_j \cap P_{i_k}$ is a path, and moreover R_j meets P_{i_1}, \dots, P_{i_p} in order. Since there are $\leq m^p = (\frac{1}{2}p(p+1))^p = p'/p$ possibilities for X_j , there are at least p values of j for which X_j is the same; let these values be j_1, \dots, j_p say, where $j_1 < \dots < j_p$. Let the common value of X_j be $\{i_1, \dots, i_p\}$ say, where $i_1 < \dots < i_p$. Then

$$(P_{i_1}, \dots, P_{i_p}, R_{j_1}, \dots, R_{j_p})$$

is a (p, p) -grid. ■

Next, we need:

(4.7) *For all integers $p, q \geq 1$ there is an integer $p' \geq 1$ so that for every digraph G , if G contains a (p', p') -grid then it contains a (p, q) -fence.*

Proof. Let $p' = pq + 1$, and let $(P_1, \dots, P_{p'}, Q_1, \dots, Q_{p'})$ be a (p', p') -grid in G . For $1 \leq i \leq p'$ and $1 \leq j \leq p'$ let $T_{ij} = P_i \cap Q_j$, and let T_{ij} have first vertex $a(i, j)$ and last

vertex $b(i, j)$. For $1 \leq i \leq p$, let $i' = q(i - 1) + 1, j' = qi, j'' = j' + 1$, and let

$$\begin{aligned} R_{2i-1} &= P_{i'}[a(i', i'), b(i', j')] \\ R_{2i} &= Q_{j''}[a(i', j''), b(j', j'')]. \end{aligned}$$

For $j = 1, \dots, q$ let S_j be the union of all the paths

$$Q_j[a(1, j), b(q - j + 1, j)]$$

$$Q_{tq+j}[a(tq - j + 1, tq + j), b((t + 1)q - j + 1, tq + j)] \quad (1 \leq t \leq p - 1)$$

$$P_{tq-j+1}[a(tq - j + 1, (t - 1)q + j), b(tq - j + 1, tq + j)] \quad (1 \leq t \leq p - 1)$$

$$P_{pq-j+1}[a(pq - j + 1, (p - 1)q + j), b(pq - j + 1, pq + 1)].$$

It is routine to check that $(R_1, \dots, R_{2p}, S_1, \dots, S_q)$ is a (p, q) -fence. ■

Proof of (2.3).

Let $n \geq 1$ be an integer. Let $p_1, r_1 \geq 1$ satisfy (3.1) (with p, r replaced by p_1, r_1). Let $q_1 = 2r_1$. Let p_2 satisfy (4.7) (with p, q, p' replaced by p_1, q_1, p_2). Let p_3 satisfy (4.5) (with p, p' replaced by p_2, p_3). Let p_4 satisfy (4.1) (with p, p' replaced by p_3, p_4). We claim that $k = p_4$ satisfies (2.3). For the hypothesis of (2.3) implies that G has a (p_4, p_4) -web. By (4.1) it has a (p_3, p_3) -mesh. By (4.5) it has a (p_2, p_2) -grid. By (4.7) it has a (p_1, q_1) -fence. Let the top and bottom of the fence be A and B . There are $q_1 = 2r_1$ disjoint paths in G from A to B . Let (X, Y) be a separation of G with $B \subseteq X$ and $A \subseteq Y$. Let $C = B \cup (X - Y)$, $D = V(G) - C$. Thus $A \subseteq D$. Since there are $2r_1$ disjoint paths from A to B , there are at least $2r_1$ edges of G with tail in D and head in C . Since every vertex has indegree equal to outdegree, there are also $\geq 2r_1$ edges with head in D and tail in C . But for every such edge (with tail c and head d say) it is not the case that $c \in X - Y$ and $d \in Y - X$, since (X, Y) is a separation; and so either $c \in B \cap X \cap Y$, or $d \in X \cap Y - B$.

There are at most $2|B \cap X \cap Y|$ edges with tail in $B \cap X \cap Y$, since G is divalent, and similarly there are at most $2|X \cap Y - B|$ with head in $X \cap Y - B$. Hence

$$2r_1 \leq 2|B \cap X \cap Y| + 2|X \cap Y - B| = 2|X \cap Y|,$$

and so $|X \cap Y| \geq r_1$. From (1.4) it follows that there are r_1 disjoint paths in G from B to A . From (3.1), $\nu(G) \geq n$ as required. ■

5. AN ALGORITHM

In this section we use (1.3) to obtain a polynomial-time algorithm to test if a digraph G has $\nu(G) \geq n$, for fixed n . Indeed, we can solve a more general problem, as follows.

Let G, H be digraphs. An *embedding of H in G* is a function ϕ with domain $V(H) \cup E(H)$, such that

- (i) $\phi(v) \in V(G)$ for $v \in V(H)$, and $\phi(v_1) \neq \phi(v_2)$ for distinct $v_1, v_2 \in V(H)$
- (ii) for every edge e of H with tail u and head v , if $u \neq v$ then $\phi(e)$ is a directed path in G from $\phi(u)$ to $\phi(v)$, and if $u = v$ then $\phi(e)$ is a directed circuit with $\phi(u) \in V(\phi(e))$
- (iii) for all $e \in E(H)$ and $v \in V(H)$, if e is not incident with v then $\phi(v) \notin V(\phi(e))$
- (iv) for all distinct $e_1, e_2 \in E(H)$, $E(\phi(e_1) \cap \phi(e_2)) = \emptyset$, and for every vertex $w \in V(\phi(e_1) \cap \phi(e_2))$ there exists $v \in V(H)$ with $\phi(v) = w$.

In other words, there is an embedding of H in G if and only if there is a subdigraph of G that is isomorphic to a “directed subdivision” of H .

Let ψ be an injective function from a subset $dom(\psi)$ of $V(H)$ into $V(G)$. An embedding ϕ of H in G *extends* ψ if $\phi(v) = \psi(v)$ for all $v \in dom(\psi)$.

The main result of this section is the following.

(5.1) *Let n be a fixed integer, and H a fixed digraph. There is an algorithm, with running time bounded by a polynomial in $|V(G)| + |E(G)|$, which has*

(a) *input a digraph G and an injective function ψ as above, and*

(b) *output one of the following:*

(i) *ψ can be extended to an embedding of H in G*

(ii) *ψ cannot be so extended*

(iii) *$\nu(G) \geq n$.*

If we omit (iii) there is unlikely to be any such algorithm, because the problem was shown to be *NP*-complete by Fortune, Hopcroft and Wyllie [3], even when H has just two components, each a non-loop edge.

As a corollary we have

(5.2) *For every fixed integer n , there is a polynomial-time algorithm to decide whether $\nu(G) \geq n$ for a given input digraph G .*

Proof. We apply (5.1), taking H to be the digraph with n components each with one vertex and one (loop) edge, and taking $\text{dom}(\psi) = \emptyset$. ■

To prove (5.1) we need the following.

(5.3) *Let H be a fixed digraph. There is an algorithm, with running time polynomial in $|V(G)| + |E(G)|$, which has*

- (i) *input a digraph G and an injective function ψ from $V(H)$ into $V(G)$, so that every directed circuit of G meets $\{\psi(v) : v \in V(H)\}$*
- (ii) *output whether ψ can be extended to an embedding of H in G .*

Proof (sketch). “Split” each vertex of G in $\{\psi(v) : v \in V(H)\}$ into two vertices, one the head of all edges of G for which $\psi(v)$ was the head, and the other the tail of all edges of G for which $\psi(v)$ was the tail. Let the new digraph we obtain be G' . Similarly, split each vertex of H , forming H' , and define $\psi' : V(H') \rightarrow V(G')$ in the natural way. Now ψ extends to an embedding of H in G if and only if ψ' extends to an embedding of H' in G' ; and since G' has no directed circuits, the latter question can be answered by applying an algorithm of Fortune, Hopcroft and Wyllie [3].

Proof of (5.1) (sketch). First we find $X \subseteq V(G)$ meeting all directed circuits of G , with $|X| \leq t_n$. (If there is no such X then $\nu(G) \geq n$ by (1.3) and we are done.) If $v \in V(H) - \text{dom}(\psi)$, then there are only $|V(G)|$ possibilities for $\phi(v)$, and therefore we can enumerate them all and test them separately. Consequently, we can assume that $\text{dom}(\psi) = V(H)$. If $x \in X$ and $x \notin \{\psi(v) : v \in V(H)\}$, then either x is an internal vertex of exactly one path $\phi(e)(e \in E(H))$ or of none. We can examine the first case by subdividing e in H with a new vertex v say, and defining $\psi(v) = x$, and we can examine the second case by adding a new vertex v to H of degree 0, and defining $\phi(v) = x$. By repeating this for all such x , we may therefore assume that $X \subseteq \{\psi(v) : v \in V(H)\}$. But then the result follows from (5.3). ■

It is easy to modify the algorithm of (5.2) to *find* the n disjoint circuits if they exist. Roughly, we delete the vertices of G one at a time, and apply the algorithm of (5.2) at each step, until we have deleted so much that we find the circuits no longer exist. Replace the last vertex deleted. Now the circuits do exist, and also there is a bounded size set of

vertices meeting all directed circuits, so we can apply (5.3) as before (using a modification of (5.3) that finds the desired embedding of H).

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