SOME PROBLEMS IN DISCRETE GEOMETRY

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ABSTRACT OF THE DISSERTATION

Some Problems in Discrete Geometry

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Dissertation Director: Mario Szegedy and Vašek Chvátal

The Sylvester-Gallai theorem asserts that any non-collinear point set in the plane determines a line passing through exactly two points in the set. The problem was posed by Sylvester in 1893 and first solved by Gallai in 1930s. Many proof were found, including the surprisingly short proof of Kelly using Euclidean distances, and the one by Melchoir using Euler's formula. We survey the history of the theorem and related problems, including various proofs of the classical Sylvester-Gallai theorem, the lower bound on the number of Gallai lines, the deBruijn-Erdős theorem, the Scott's conjecture and Ungar's theorem, the Dirac conjecture, the magic configuration conjecture, the question on the number of Gallai points, and the colored version of the problem. We then present the recent work on the generalization of these problems in arbitrary metric space and hypergraphs. In particular, we present the Sylvester-Chvátal theorem and the problems related to the de Bruijn-Erdős theorem.

Another problem we study in this dissertation is the visibility of points and segments in the plane. We color the end points of each segments red and blue, and study, in particular, the visibility relations between the red points and the blue ones. We introduce the general frame work of the problem, and prove two main theorems about the visibility graph.

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Thus I listed a sampling of the (admirable masters $\cap$ pleasant people $\cap$ great friends) I got to know during the years.

Muito obrigado.

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Introduction

Let no one ignorant of Geometry enter

— people wrote that Plato wrote at the gate of Academy.

Geometry, the art of shapes and spatial relations, is one of the two mainstream fields of pre-modern mathematics. The recorded beginnings of Geometry are rooted in our earliest recorded civilizations — the ancient Egypt and the ancient Babylonia, initiated by the earthly art of construction, as well as the heavenly astronomy. Doubtlessly, the pre-history civilizations could not exist without solving some geometric problems. The ancient Egyptian and the ancient Babylonians were already capable of deriving some fine geometric principles. For instance, the area and volume of some simple shapes, the golden proportion, and some versions of the Pythagorean theorem — at least 1000 years before Pythagoras. Another culture, a little bit later in time, the ancient Indians, also recorded the study of many geometric problems. Inherited from the Egyptians, the ancient Greek culture and the Hellenistic period is our golden age of Geometry, with the super stars such as Thales, Plato, Euclid, and Archimedes. Euclid's *Elements* is the most influential work of science in the human history. The axiomatic method in this geometry textbook set the standard for all the rational reasonings in mathematics and other sciences. After the Hellenistic era, western mathematics entered the dark ages together with the western civilization. It is well known that the Islamic world saved and developed the sciences of ancient Greece and Rome. In the Renaissance, people witnessed important developments of Geometry. The projective geometry, initiated in the Hellenistic period, was developed by Desargues and Poncelet. Later people questioned the logical foundations of the Euclidean geometry and non-Euclidean ones were born. The analytic geometry was created by Descartes and Fermat. Equipped with the tools of Algebra, the analytic geometry should be considered as the first geometry that is not pure. After the 18th century, pure Geometry as the study of points and lines, of angles, lengths, and volumes, was no longer a main subject of mathematics. Thus Geometry died, and was reborn in many branches of mathematics.

The 20th century witnessed the renaissance of the geometry of points and lines as the new subject of *discrete geometry*, also known as *combinatorial geometry*. The background is the development of combinatorics as a new discipline; and the birth of modern computers introduced the need of algorithmic study of mathematics problems. During the new dark age of world wars and revolutions, the Europeans, in particular the Hungarians, contributed discrete mathematical problems that are simple looking and profound. Some of the problems form the main subject of discrete geometry.

One of the most celebrated problems in discrete geometry is the Sylvester problem proposed in 1893. It is probably the single problem in the area with the largest literature. The statement of the problem is accessible to people with minimum mathematical background. Yet for some reasons it was not solved (and almost has been forgotten) for 40 years. In the 1930s Erdős created the same problem independently, and soon the problem was solved by various solutions. Many of the solutions are accessible to people with minimum mathematical background. The first correct solution was due to Gallai, and the theorem is named as Sylvester-Gallai. Later, there were further studies around the Sylvester-Gallai theory; and some interesting questions are still open. The Sylvester-Gallai problem considers the incidence of points and lines. In nature, it is a problem in the real projective plane. Yet the most beautiful proofs of the theorem use the notion of Euclidean distance or Euler's formula of the surface. Recently, Vašek Chvátal initiated the work of generalizing the problems in arbitrary metric space, where the lines are derived from the distances. The main conjecture was that under an appropriate definition of the lines, the Sylvester-Gallai theorem holds in any metric space. This conjecture was proved as the Sylvester-Chvátal theorem by the author in 2003.

In this dissertation we first give a brief survey of the Sylvester-Gallai theory in the real projective plane, including the classical Sylvester-Gallai theorem, the lower bound on the number of Gallai lines, the deBruijn-Erdős theorem, Scott's conjecture and Ungar's theorem, the Dirac conjecture, the magic configuration conjecture, the question of the number of Gallai points, and the colored version of the problem. We briefly mention the generalized setting of the Sylvester-Gallai problems where the lines are replaced by pseudolines, and the closely related allowable sequences. The magic configuration conjecture leads to the question of lower bounds on the Gallai points. We present a conjecture in terms of allowable sequences, a positive answer to which will give the correct asymptotic order for the number of Gallai points as well as a complete solution to the magic configuration problem. Then we present the recent work on the effort of generalizing these problems in arbitrary metric spaces. We give Chvátal's definitions of lines in metric spaces. Then we present the proof of the Sylvester-Chvátal theorem and the work on problems analogous to the de Bruijn-Erdős in metric spaces and set systems.

Another problem we study in this dissertation is the visibility of points and segments in the plane. As in the usual setting of the visibility questions, the segments are considered as obstacles in the plane. The visibility questions are well studied. We introduce the complication, as in many classical geometry problems, by coloring the end points of each segments red and blue, and study, in particular, the visibility relations between the red points and the blue ones. We introduce the general framework of the problem, define the visibility graphs, pose the basic questions, and prove some basic facts in this setting. In particular, we prove two main theorems about the visibility graph.

Part I

The Sylvester-Gallai Theorem and Related Problems

Chapter 1

The Sylvester-Gallai Theorem

1.1 A history of the Sylvester-Gallai theorem

The following problem was brought to me, surely in Chinese, by Mr. Xiong Zhang, my middle school mentor and mathematics teacher, in those days when I was playing various mathematical competitions.

> *Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line.*

I was attracted by this problem very much; more so by the wonderful short solution. Some eight years later I read the history of this problem from Vašek Chvátal's paper [12], then from many other sources. The problem was first proposed by James Joseph Sylvester in a column of mathematical problems and solutions in the *Educational Times*, in 1893 [55]. Soon after an obviously incorrect proof by one of the readers (see [16]) appeared in the same column. Without a correct solution, the problem was kept silent for nearly 40 years, and seemed to have been forgotten. Erdős independently raised the same problem in the 1930s and since then many solutions were found. Erdős conjectured that Sylvester had a correct proof to his own problem (see [25]), but Coxeter assumed the contrary ([14], p. 69). The first open problem of this chapter is

Question 1.1. *Did Sylvester know a proof to his line problem in his* that *life?*

The first correct proof to Sylvester's problem is attributed to Gallai, therefore we have

Theorem 1.1. *(The Sylvester-Gallai theorem) Given P, a set of n points in the plane. Either all the points lie on the same line, or there is a line passing through exactly two points in P.*

Given a point set P, a line is called a *connecting line* if it is determined by two points in P. It is called an *ordinary line* if it contains exactly two points in P. We often use the synonym *Gallai line* instead of *ordinary line*. The Sylvester-Gallai theorem asserts that any non-collinear finite point set in the plane has a Gallai line. It is clear from the statement of the theorem that it does not matter whether we consider the plane to be Euclidean, affine, or, most generally, the real projective plane. Hence, it is equivalent to its dual

Theorem 1.2. *Given a set L of non-concurrent lines in the real projective plane, there exists an intersection point which is incident to exactly 2 lines of L.*

We first include L. M. Kelly's proof to the Sylvester-Gallai theorem. This is considered as the simplest proof to the theorem; also it happens to be the first proof I knew. Kelly never published the proof; One may find it in [13], [14], [1], and many other sources.

Proof. (of Theorem 1.1 by Kelly) Define the set

$$S = \{(a, \ell) : a \in P,\ \ell \text{ is a connecting line of } P \text{ and } a \notin \ell.\}$$

If the points in P do not lie on the same line, S is a non-empty finite set. We pick $(a, \ell) \in S$ such that the distance from a to ℓ is minimum. We claim that ℓ passes through exactly two points in P.

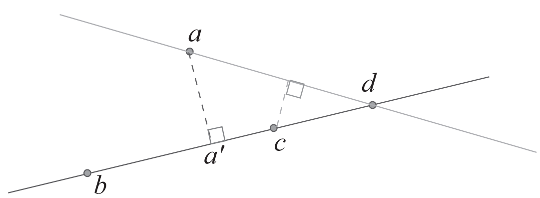


Figure 1.1: Kelly's Proof to the Sylvester-Gallai theorem

Assume the claim is not true, ℓ contains at least 3 points, b, c, and d in P. Let a' be the closest point to a on ℓ. By pigeonhole principle, there are two points in $\{b, c, d\}$ lie on the same side of a' on ℓ. We may assume a', c, and d appears in that order on ℓ, possibly $a' = c$. Let ℓ' be the line passing through d and a, the distance from c to ℓ' is smaller than the distance from a to ℓ, a contradiction. ∎

One may observe that Kelly's proof works not only for points in the plane, but also for any finite point set in $\mathbb{R}^n$.

As Erdős wrote in a reminiscence of Gallai ([23]), he reinvented the Sylvester problem in 1933, when he was reading the book *Anschauliche Geometrie* (in English, *Geometry and the Imagination*), "I expect it to be easy but to my great surprise and disappointment I could not find a proof. I told this problem to Gallai who very soon found an ingenious proof." Until 1943 Erdős posted the problem in the Monthly ([21]). In 1944 the Monthly published a proof by Steinberg, as well as Gallai's proof, which was submitted by Erdős with the problem ([22], [54]). It is stated in the editorial note of the solution that L. M. Kelly pointed out that the problem was proposed by Sylvester in 1893. The same editorial note states that correct solutions were also submitted by R. C. Buck and N. E. Steenrod, the former gives a proof similar to Gallai's proof; Buck said that his proof was not original, but he was uncertain about the origin. Also in the same note, "Erdős stated that, at Oslo, Karamata asked him about this problem which he had seen stated without proof in an old book about mechanics."

Here is Gallai's proof, which is the first known proof to Sylvester's problem.

Proof. (of Theorem 1.1 by Gallai) Let $a \in P$. If a is contained in a Gallai line then we are done. Otherwise, each connecting line passing through a contains at least two other points of P. Project a to a point at infinity so that those lines become a set of parallel horizontal lines, each containing (the images of) at least two points from P.

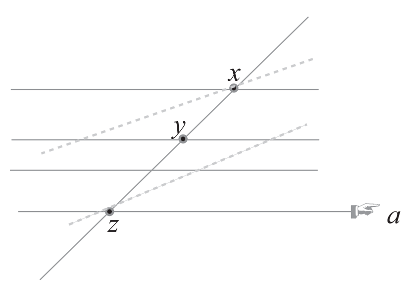


Figure 1.2: Gallai's Proof to the Sylvester-Gallai theorem

Consider all the connecting lines that are not incident to a, let ℓ be the one with the smallest slope. If ℓ is not a Gallai line, there are at least 3 points on ℓ. Let them be, from top down, x, y, and z. By our assumption, there is another point $y' \in P$ such that yy' is horizontal. If y' is to the left of y, then the connecting line xy' has a smaller slope than xyz. Otherwise the line zy' has a smaller slope than xyz. A contradiction. □

Parallel to the common history of the Sylvester-Gallai theorem, and seemingly unaware of the fact that the problem was raised by Sylvester, Melchior published a beautiful short proof to the dual version (Theorem 1.2) in a Nazi journal in 1940 ([42]. See also [16] and [1], p.62.). In fact, Melchior proved a stronger theorem that there are at least 3 Gallai lines if the point set is not collinear.

Proof. (of Theorem 1.2 by Melchior) Let S be a unit sphere under the plane, and project the plane onto S through the center of S. Now each point of L is projected to a pair of antipodal points on S; all the lines in L become great circles on S. Consider the graph on S with all the intersection points as vertices and segments of the projected great circles as edges. L is not concurrent implies that this is a simple planar graph. Therefore we have, through Euler's formula,

$$\sum_{v \in V} \deg v = 2|E| \leq 6|V| - 12. \tag{1.1}$$

An intersection point of L incident to t lines is projected to a pair of points with degree $2t$. From (1.1) it is clear there are at least 6 vertices with degree 4. i.e., there are at least 3 intersection points of L that are incident to exactly 2 lines. □

The proof above assumes minimal background of the real projective plane. In the rest of this dissertation, we usually think the lines in the real projective plane as the great circles on the sphere with antipodal points identified, as outlined in the proof above.

Unaware of Melchior's proof using Euler's formula, Coxeter in 1948 ([13]) commented on the proofs by Gallai and Kelly: "It seems to me that parallelism and distance are essentially foreign to this problem, which is concerned only with incidence and order." Furthermore, he commented in [14] (p. 181) that Kelly's proof "is like using a sledge hammer to crack an almond." According to Coxeter, the "nutcracker" proof only uses the axioms of incidence and order. The following projective proof is essentially due to Steinberg ([54], see also Coxeter [13] and [14], p.181).

Proof. (of Theorem 1.1 by Steinberg) Let $a \in P$ be a point, and a line ℓ through a but not incident to any other point in P. If P is not collinear, all the connecting lines of P intersect ℓ at at least one point other than a. Let p be such an intersection so that

there is no other intersection lies between a and p. (*)

Assume the connecting line incident to p is determined by b, c, d — if it is not a Gallai line. We may name them so that the segment bp contains c but not d. (Figure 1.3)

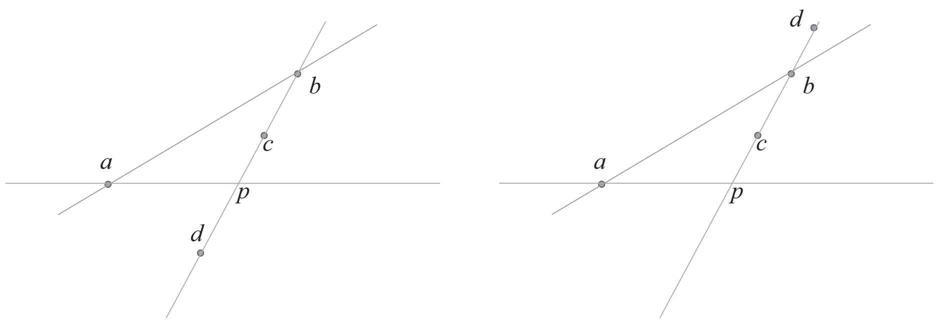


Figure 1.3: Steinberg's Proof to the Sylvester-Gallai theorem

If ab is not a Gallai line, there is a point $e \in P$ on the line ab. If e is between a and b, then ed intersects ℓ at a point between a and p. Otherwise ec intersects ℓ at a point between a and p. In either case we have a contradiction with (*). □

Definition 1.1. *Given a point set P and $a \in P$, suppose the connecting lines of $P \setminus a$ partition the real projective plane into regions. The region containing a is called the* residence *of a, and all those lines containing boundary edges of the residence of a are called the neighbors of a.*

In fact, Steinberg's proof gives the following result, upon which some later results are based. We give the projective translation of the same proof, since the proof is most elegant in the terms of projective geometry. We use the standard notation that $ab//cd$ denote the situation where the four points a, b, c, and d are collinear, and the pairs

(a, b) and (c, d) separate each other. In projective geometry, the betweenness relation of collinear points is not preserved by projectivity, but the separation relation is preserved.

Proposition 1.1. *Given a point set P and $a \in P$. If ℓ is a neighbor of a but not a Gallai line; and ℓ contains 3 points b, c, d from P. If $p \in \ell$ is a point on the boundary of the residence of a, $bp//cd$, then the line ab is a Gallai line.*

Proof. (As either picture in Figure 1.3.) Assume the contrary that there is another point $e \in P$ on the line ab. Denote $ed \cdot \ell$ and $ec \cdot \ell$ by d' and c', respectively. Then $bpcd$ is projected to $apd'c'$ through e. Therefore $ap//c'd'$, and p cannot be on the coundary of the residence of a. A contradiction. □

Proofs of the Sylvester-Gallai theorem were also given by A. Robinson (see [44]), G. D. W. Lang ([39]), V. C. Williams ([57]), and possibly others. Additional information on the Sylvester-Gallai theorem can be found in Borwein and Moser [4], Chvátal [12], Erdős and Purdy [25], and Pach and Agarwal [47].

One quick question is that whether the statement of the theorem still holds if the point set P is not necessary finite. A moment's thinking gives the negative answer to this question: Take P to be the set of integer lattice points in the plane, then every line passing through at least 2 points in P contains infinitely many points in P, but no line contains all the points. In fact, we may give a stronger example as follows.

Proposition 1.2. *There is an infinite set P of integer lattice points in the plane, not all lie on the same line, such that every connecting line of P contains exactly 3 points in P.*

Proof. We construct the points one by one. Let $a_1 = (0, 0)$, $a_2 = (0, 1)$, and $a_3 = (1, 0)$. Given $(a_1, a_2, \ldots, a_n)$, let $P_n = \{a_i : 1 \leq i \leq n\}$. Pick a_i and a_j such that the line passing through a_i and a_j does not contain any other point from P_n. By the Sylvester-Gallai theorem, these pairs exist. If there are more than one such pairs, we pick the one with $i + j$ minimized. If there is still a tie, pick arbitrary one, say, the one with smallest i. Now, the line $a_i a_j$ passes infinitely many integer lattice points. We may pick one of them to be a_{n+1} such that a_{n+1} is not co-linear with any three points from P.

Take $P = \cup_{n=3}^{\infty} P_n$. It is easy to see that P does not contain 4 co-linear points. And for any pair of points a_i and a_j in P, there is an $n \leq \binom{i+j}{2} + j + 1$ such that there is a point in P_n that is co-linear with a_i and a_j. $\square$

1.2 The Dirac conjecture on the number of Gallai lines

Given a non-collinear point set P, denote $t_i = t_i(P)$ the number of connecting lines that are incident to exactly i points of P. t_2 is the number of Gallai lines. In the dual setting, let L be a set of non-concurrent lines, denote $v_i = v_i(L)$ the number of intersection points that are incident to exactly i lines from L.

Definition 1.2. *Define $m(n)$ to be the minimum number of Gallai lines that are determined by any set of n non-collinear points. i.e.*

$$m(n) = \min_{|P|=n} t_2(P) = \min_{|L|=n} v_2(L).$$

In [18] de Bruijn and Erdős asked whether $m(n)$ tends to infinity with n, and noted that they could prove $m(n) \geq 3$. In fact, Melchior's simple proof by Euler's formula ([42]) establishes the same result. Dirac ([19]) also gave a lengthy proof of $m(n) \geq 3$ and made the following conjecture, which is still open today.

Conjecture 1.1. *(Dirac 1951 [19])*

$$m(n) \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

The conjecture is usually stated in a slightly stronger form as

$$m(n) \geq n/2 \quad \text{except for } n = 7 \text{ and } n = 13.$$

Motzkin first proved the fact that $m(n) \to \infty$.

Theorem 1.3. *(Motzkin 1951 [44])*

$$m(n) \in \Omega(\sqrt{n}).$$

Proof. (sketch) By Proposition 1.1, if we consider the regions of the planes that are partitioned by the Gallai lines, there is at most one non-Gallai point — a point that is not incident to any Gallai lines — in each region. The number of regions is at most $\binom{t_2}{2} + 1$, with the equality holds when the Gallai lines are in general position. On the other hand, the number of non-Gallai points is at least $n - 2t_2$. So

$$n - 2t_2 \leq \binom{t_2}{2} + 1$$

for any point set P. The theorem follows. □

In the real projective plane, the Euler formula states that for any planar simple graph

$$|V| - |E| + |F| = 1. \tag{1.2}$$

Consider a set of lines, view the intersection as vertices and the segments of the lines as edges. Suppose L is not concurrent, and denote f_i the number of faces with exactly i sides. Any intersection point lies on i lines has degree $2i$. We have

$$|V| = v_2 + v_3 + v_4 + \cdots \quad |F| = f_3 + f_4 + f_5 + \cdots \tag{1.3}$$

and

$$2|E| = \sum_{i \geq 3} i f_i = \sum_{i \geq 2} 2 i v_i. \tag{1.4}$$

By (1.2) and (1.4), we get

$$3 = \sum_{i \geq 2} (3 - i) v_i + \sum_{i \geq 3} (3 - i) f_i.$$

i.e., the equation due to Melchoir ([42])

$$v_2 = 3 + (v_4 + 2v_5 + 3v_6 + \cdots) + (f_4 + 2f_5 + 3f_6 + \cdots) \tag{1.5}$$

So,

$$v_2 \geq 3 + \sum_{i \geq 3} (i - 3) v_i = 3 + v_4 + 2v_5 + 3v_6 + \cdots \tag{1.6}$$

In the dual setting, for any set P of points

$$t_2 \geq 3 + t_4 + 2t_5 + 3t_6 + \cdots \tag{1.7}$$

This was Melchior's original proof of Sylvester-Gallai from Euler's formula. It is then clear that the existence of many lines with at least 4 points implies that there are many Gallai lines.

Using (1.7) Moser proved

Theorem 1.4. *(W. Moser 1957 [43])*

$$m(n) \geq (n+12)/6 \quad \textit{if } n \textit{ is even.}$$

Proof. If n is even, every point is incident to at least one line that has $k \neq 3$ points on it. So, counting the number of points by lines, and using (1.7),

$$n \leq 2t_2 + 4t_4 + 5t_5 + 6t_6 + \cdots \leq 2t_2 + 4(t_4 + 2t_5 + 3t_6 + \cdots) \leq 6t_2 - 12.$$

□

Observe Steinberg's proof of the Sylvester-Gallai theorem (and Proposition 1.1). For each point a we actually find a "local" Gallai line, that is, a Gallai line passing through a or close to a in some sense. Based on this, Kelly and Moser gave the first proof that $m(n) \in \Omega(n)$ for all n,

Theorem 1.5. *(Kelly and Moser 1958 [38])*

$$m(n) \geq \frac{3n}{7} \quad \textit{for all } n \geq 3.$$

Proof. (sketch) For each point a, define $f(a)$ to be the number of Gallai lines passing through a, and define $g(a)$ to be the neighbors of a which are Gallai lines. Apart from the trivial cases, we have

Claim 1. *For any point a where $f(a) \neq 2$, $f(a) + g(a) \geq 3$.*

Proof. (of the claim.) If $f(a) = 0$, by Proposition 1.1, all the neighbors of a are Gallai lines, so $g(a) \geq 3$ (except for the trivial cases).

If $f(a) = 1$, let ab be the only Gallai line passing through a. By Proposition 1.1, any neighbor that does not passing through b is a Gallai line. We are left with the case where a has 3 neighbors, two of them, ℓ and ℓ', passing through b and are not Gallai lines. Suppose ℓ contains b, c, and d from P. We may pick x on ℓ which is close to b and also on the boundary of the residence of a, so that $xd//bc$. By Proposition 1.1, ad is another Gallai line. □

It is not hard to show that, if ℓ is a neighbor of three points a, b, and c, then abc must be a triangle and the points from P on ℓ can only occur on the three lines of the triangle. If ℓ is a neighbor of 4 points a, b, c, and d, then it must pass two of the diagonal points of the complete quadrangle $abcd$, and must be a Gallai line. As a consequence, any connecting line can be the neighbor of at most 4 points.

We count the number of Gallai lines

$$t_2 \geq \frac{1}{6} \sum_{a \in P} (f(a) + g(a)), \tag{1.8}$$

since each Gallai line is counted in f twice, and counted in g at most 4 times.

Let k be the number of points such that $f(a) = 2$, we have

$$t_2 \geq k,$$

and, by Claim 1 and (1.8),

$$t_2 \geq \frac{3(n-k) + 2k}{6}.$$

The theorem follows. □

The proof in [38] has a minor error but can be easily fixed. (See [20].)

In his dissertation in 1981, Hansen ([33]) claimed a proof that $m(n) \geq n/2$ for $n \neq 7, 13$. But one of the main sub-theorem was found wrong by Csima and Sawyer ([16]), who corrected Hansen's proof and showed that

Theorem 1.6. *(Csima and Sawyer 1993 [16])*

$$m(n) \geq \frac{6n}{13} \quad \textit{for } n \neq 7.$$

They did their work in the dual setting. The essence of the proof is similar to that of Kelly and Moser — from each line one can associate at least one ordinary point on it or close to it in some sense, and each ordinary point is associated to at most a certain number of lines.

We now turn to the upper bound of $m(n)$. The Kelly-Moser example (see [38]) in Figure 1.4 shows that $m(7) \leq 3$. In the example, there are 7 points and 3 Gallai lines that are drawn as gray dashed lines.

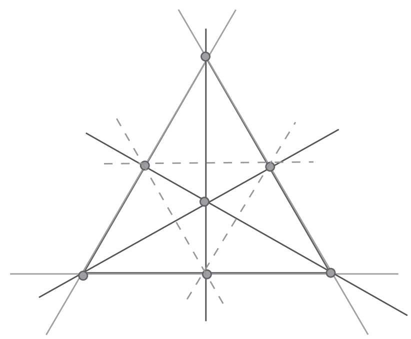


Figure 1.4: The Kelly-Moser configuration.

For a long time Dirac's conjecture was stated as $m(n) \geq n/2$ except for $n = 7$, until there came the McKee example (See Crowe and McKee [15]) showing that $m(13) \leq 6$. Figure 1.5 is the McKee configuration with 13 points in the real projective plane. $abcde$ and $abc'd'e'$ are two regular pentagons with the common edge ab in the Euclidean plane; and m is the middle point of ab. x, y, z, and w are 4 points at infinity as indicated in the figure. The 6 Gallai lines are shown as gray dashed lines.

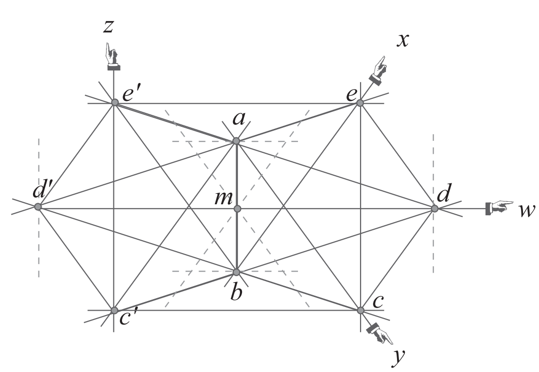


Figure 1.5: The McKee configuration.

For general n, the following configurations are due to Motzkin and Böröczky (see [15]).

Theorem 1.7. *For $n \geq 6$,*

$$m(n) \leq \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3(n-1)/4 & \text{if } n \equiv 1 \pmod 4, \\ 3(n-3)/4 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. The examples are best described in the real projective plane viewed as the Euclidean plane augmented with the line at infinity.

Case 1. $n = 2k$ for some $k \geq 3$. Construct a regular k-gon Q. All pairs of its vertices determine k directions. The point set P consists of the points at infinity on these k directions, as well as the k vertices of Q. There are k Gallai lines, each consists of a vertex of Q and a point at infinity.

Case 2. $n = 4t + 1$ for some $t \geq 2$. Construct the configuration for $4t$ points as in Case 1, and add a point o at the center of the $2t$-gon Q. In addition to the $2t$ Gallai lines as in Case 1, there are t Gallai lines passing through o.

Case 3. $n = 4t - 1$ for some $t \geq 2$. Construct the configuration for $4t$ points as in Case 1, and delete a point at infinity which is not a direction determined by any edge of Q. It is easy to see there are $2t - 2$ Gallai lines involve points at infinity, and $t - 1$ Gallai lines which are parallel diagonals of Q.

□

1.3 The magic configurations and the Gallai points

In the summer of 2006 I attended the conference in Montreal in celebration of Vašek Chvatál's 60th birthday. During a lunch I heard, among nice stories and jokes, the following problem from U. S. R. Murty. The origin of the problem is a paper by Murty 35 years ago [45].

Definition 1.3. *Let P be a finite set of points in the plane and $\sigma : P \to \mathbb{R}^+$ assigns each point a positive weight. (P, σ) is called a* magic configuration *if the sums of the weights on each connecting line are the same.*

Example 1.1. The following is a list of magic configurations.

1. P is a set of collinear points with arbitrary weights.

2. P is a set of points in general position with equal weights.

3. P consists of $n-1$ collinear points of weight x, and another point with weight $(n-2)x$.

4. P is a projective equivalent of the 7 points of the Kelly-Moser configuration (Figure 1.4). The weights are x on the vertices and the center of the triangle, and $2x$ on the remaining points.

$\diamond$

Problem 1.1. *Are there any magic configurations other than those listed in Example 1.1?*

Murty considered the integral valued functions. It is easy to see that the problem is unchanged when we consider rational or real valued functions. In [45], Murty also defined the *Sylvester graph.*

Definition 1.4. *Let P be a finite set of points. A point $x \in P$ is called a* Gallai point *if it is incident to any Gallai line. The* Sylvester graph of P *is defined as the graph with all the Gallai points as its vertices and $x \sim y$ if the line xy is a Gallai line.*

The following partial result towards Problem 1.1 is due to Murty ([45]).

Proposition 1.3. *Let (P, σ) be a magic configuration with at least 3 points. Then the Sylvester graph of P is a complete graph. And the set of Gallai points equals the set of points where σ is maximum.*

Proof. It is easy to check that all the configurations in Example 1.1 satisfy the proposition. We consider all the other configurations.

Assume the sum of each connecting line is k. We claim that all the Gallai points have weight $k/2$. Assume the contrary that there is a Gallai line xy where $\sigma(x) > k/2$ and $\sigma(y) < k/2$. Since the sum of the σ-value is k on every line, we have $\sigma(z) \leq$

$k - \sigma(x) < k/2$ for any $z \neq x$. It follows that every Gallai line passes through x. Let the Gallai lines be

$$xy_1, xy_2, \ldots, xy_t.$$

By Theorem 1.5, $t \geq 3n/7$. And, unless P is a pencil — one of the items in Example 1.1, there are at least $3(n-1)/7$ Gallai lines for the point set $P' = P \setminus \{x\}$. Each of the Gallai lines for P' passes x, since otherwise the sum on that line is less than k. So, we find, through the point x, at least $3n/7$ lines with one point other than x, and at least $3(n-1)/7$ lines with two points on each line other than x. The number of distinct points is at least

$$1 + \frac{3n}{7} + \frac{6(n-1)}{7} > n.$$

A contradiction.

Having proved the claim that each Gallai point has weight $k/2$, it is easy to see the Sylvester graph is complete and no other point has a weight greater than or equal to $k/2$. □

As Murty remarked in [45], the Sylvester graph of a magic configuration is a complete graph, yet it is not known whether there are any configurations (not necessarily magic) other than the Kelly-Moser configuration (or the trivial configurations in Example 1.1) for which the Sylvester graph is a complete graph. In the conversation with Murty, the author learnt that it is not even known whether there are configurations where the Sylvester graph is a K_4. In Section 2.4 we prove that there are no such configurations.

By the linear lower bound on the number of Gallai lines, we have an $\Omega(\sqrt{n})$ lower bound on the number of Gallai points. It is an interesting question whether this bound can be improved.

Conjecture 1.2. *For a set of n non-collinear points in the plane, there are at least $\Omega(n)$ Gallai points.*

We note that for the McKee configuration (Figure 1.5), the Sylvester graph is not complete. In the examples in Theorem 1.7, there are $\Theta(n)$ Gallai points — in fact almost all the points are Gallai points.

If the number of Gallai points is linear, and the constant is at least 1/2, we have a complete solution to the magic configuration problem. We will further discuss this in Section 2.4, where we give a stronger conjecture.

1.4 The number of connecting lines

The problem Erdős posed in [21] consists of two parts. The first part was the Sylvester's problem, and here is the second part:

> (2) Given n points which do not all lie on the same straight line, prove that if we join every two of them we obtain at least n distinct straight lines.

Erdős noted that "part (2) could be easily proved by induction using part (1)", i.e., the Sylvester-Gallai theorem. Correct solutions were provided by Steinberg and others. The theorem was published in 1948 in a de Bruijn and Erdős paper [18], where it gets its name, in a more general setting.

Theorem 1.8. *(de Bruijn - Erdos) Every non-collinear set of n points in the plane determines at least n distinct connecting lines.*

Proof. The base case $n \leq 2$ is trivial. For a set P of $n \geq 3$ non-collinear points, by the Sylvester-Gallai theorem, there is a line going through exactly two points a and b. Consider the set $P' = P \setminus \{a\}$. If P' consists of $n-1$ collinear points, then we have n lines: ax for each $x \in P'$ and the line containing all the points in P'. Otherwise, by induction there are $n-1$ lines determined by P', none of them is ab. $\square$

In the dual setting, the de Bruijn-Erdős theorem is

Theorem 1.9. *A set of n non-concurrent lines in the real projective plane determines at least n intersection points.*

Both theorems 1.8 and 1.9 are special cases of the following nonuniform Fisher's Inequality. There are many ways to look at a set system — we may look them as a hyper graph, or a bipartite graph with sets as one part and elements as another. We state the nonuniform Fisher's Inequality in the setting of a system of sets.

Theorem 1.10. *(Nonuniform Fisher's Inequality) If we have m sets on n elements, each with size bigger than λ and the intersection of any two of the sets has size λ, then $m \leq n$.*

Proof. Consider the incidence matrix M where

$$M_{ij} = \begin{cases} 1 & \text{if the i-th set contains the } j\text{-th element;} \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = M \cdot M^t$. All the off-diagonal elements of A are λ, and all the diagonal elements are bigger than λ. It is easy to show that A is positive definite, so it is nonsingular therefore has full rank. □

The original Fisher's Inequality [26] arises from the block designs. Our proof here could be found in D. Newman's nice little book *A Problem Seminar* ([46]), it is essentially due to R. C. Bose ([5]). The minor difference is that instead of showing A is positive definite, Bose directly computed the determinant of A to be non-zero. In [5], Bose gave the short proof to Fisher's Inequality, which is one of the first applications of linear algebra methods in combinatorics, without noticing that almost the same proof gives the nonuniform version of it. The nonuniform Fisher's Inequality was first discovered by Majumdar in 1953 ([41]) and later by Isbell in 1959 ([36]).

Both theorems 1.8 and 1.9 are special cases of the nonuniform Fisher's Inequality when $\lambda = 1$. For the original de Bruijn-Erdős theorem, we consider the lines as elements and consider each point as the set of lines that are incident to it. the only property we need about the lines and points is that each point is incident to at least 2 lines, and any two points determine a unique line. Thus, the de Bruijn-Erdős theorem can be understood in a setting that involves neither measurement of distances nor measurement of angles.

There are many generalizations with beautiful linear algebra proofs. For example, the theorems, including the uniform version, nonuniform version, and modular version, under the names Ray-Chaudhuri-Wilson and Frankl-Wilson. (See the standard but unpublished textbook by Babai and Frankl ([2]) for these generalizations, and the paper by Snevily ([52]) for a recent development.) However, all of these are generalizations of the nonuniform Fisher's Inequality in the setting of set systems with restricted intersection sizes, rather than extensions of the de Bruijn-Erdős theorem as a geometric problem.

Back to the direction of generalizing de Bruijn-Erdős theorem in the geometric setting, even the lower bound on the different directions of all the connecting lines is almost n. The following was conjectured by Scott in 1970 ([50]) and proved by Ungar in 1982 ([56]). It is easy to see that the bounds are sharp for the vertices of a regular n-gon if n is even, and a regular $(n-1)$-gon plus a point at the center when n is odd.

Theorem 1.11. *Let P be a set of n non-collinear points. The connecting lines of P show at least $2\lfloor n/2 \rfloor$ directions.*

Ungar's proof is a beautiful application of the *allowable sequences.* We will sketch the proof in Section 2.3.

1.5 Regions in the arrangement of the lines

For a set L of non-concurrent lines in the real projective plane, the quantities v_i and f_i are defined in Section 1.2 as the number of points that lie on i lines, and the number of faces in the arrangement with i sides. The Sylvester-Gallai theorem asserts that $v_2 \geq 1$, and in fact (Csima and Sawyer) $v_2 \geq 6n/13$ for $n \geq 13$. Analogously, a natural quantity to study is f_3, the number of triangles in the projective plane. (For a set of at least 3 non-concurrent lines, $f_2 = 0$.)

Using (1.2), (1.3), and (1.4), we have

$$-2 = (|E|-2|V|)+(|E|-2|F|) = (v_3+2v_4+3v_5+\cdots)-\frac{1}{2}f_3+\frac{1}{2}(f_5+2f_6+3f_7+\cdots).$$

Therefore, we have the equality (Grünbaum 1972 [32])

$$f_3 = 4 + (f_5 + 2f_6 + 3f_7 + \cdots) + 2(v_3 + 2v_4 + 3v_5 + \cdots). \tag{1.9}$$

Hence, $f_3 \geq 4$ in any arrangement. In fact, as the quantity t_2 (v_2), f_3 is at least linear. The following theorem is due to Levi.

Theorem 1.12. *(Levi 1926 [40]) In a set of non-concurrent lines in the real projective plane, every line is incident to at least 3 triangular regions.*

Immediately, we have

Theorem 1.13. *For a set of non-concurrent lines in the real projective plane,*

$$f_3 \geq n.$$

For a set of $n \geq 4$ lines in general position, it is easy to see that each edge is incident to at most one triangular face. So we have $f_3 \leq n(n-1)/3$. Brünbaum conjectured this is true for any arrangement of lines. It is proved by Roudneff.

Theorem 1.14. *(Roudneff 1996 [49]) For a set of $n \geq 9$ lines in the projective plane,*

$$f_3 \leq \frac{1}{3}n(n-1).$$

The equality holds in (1.6) if all the faces in the arrangement are triangles. Such arrangements are called *simplicial arrangements*. There are 3 infinite families of simplicial arrangements. The most obvious family is the *near pencil*, where all lines but one are concurrent. In the example for even $n = 2k$ in Theorem 1.7, we have

$$t_2 = k, t_3 = \binom{k}{2}, \text{ and } t_k = 1.$$

So,

$$t_2 = 3 + t_4 + 2t_5 + \cdots + (k-3)t_k + \cdots .$$

Thus, the dual of this example has equality hold in (1.6). This example gives another infinite family of simplicial arrangements, one for each even number n. The last infinite family is also from the examples in Theorem 1.7. For each $n = 4k + 1$, we have

$$t_2 = 2k, t_4 = k, \text{ and } t_k = 1.$$

It is easy to see that in the dual, the equality in (1.6) holds.

Besides the three infinite families introduced above, we know many sporadic simplicial arrangements, but the largest of the known examples is of size 37. For more information about the simplicial arrangements, see the manuscript [31].

1.6 The colored Sylvester-Gallai problems

The following question was asked by Graham and Newman:

> Given a finite set of points in the plane, each colored red or blue and not all collinear, must there be a monochromatic line?

The answer to the question is affirmative. The first published proof is due to Chakerian ([7]). (See also [1] for the proof.)

Theorem 1.15. *Given a set of non-concurrent lines in the real projective plane, each colored red or blue, there is an intersection point incident with only lines of one color.*

It is not necessary that we always have a monochromatic Gallai line, nor we always have a bi-chromatic Gallai line. Concerning the bi-chromatic lines, Pach and Pinchasi have the following result

Theorem 1.16. *(Pach and Pinchasi 2000 [48]) Given n red points and n blue points in the plane, not all collinear, then there exist more than $n/2$ bichromatic lines that pass through at most 2 red points and at most 2 blue points.*

Chapter 2

Computing the Gallai Numbers

Recall that $m(n)$ is defined as the minimum number of Gallai lines in any set of n non-collinear points (Definition 1.2). Similarly, we define $m^*(n)$ to be the minimum number of Gallai points.

Definition 2.1. *Define $m^*(n)$ to be the minimum number of Gallai points that are determined by any set of n non-collinear points.*

In this chapter we compute $m(n)$ and $m^*(n)$ for some small n.

2.1 The number of Gallai lines

The following is a table of the values $m(n)$ for some small n.

n	3	4	5	6	7	8	9	10	11	12	13	14
$m(n)$	3	3	4	3	3	4	6	5	6	6	6	7

n	15	16	17	18	19	20	21	22	23	24
$m(n)$	9	8		9		10		11		12

Table 2.1: The Gallai line number for small n's.

Some values of table 2.1 were established by Crowe and McKee ([15]), and later the list was extended by Brakke ([6]). Both of them used the upper bound of $m(n)$ from the Motzkin and Böröczky example as in Theorem 1.7, and the lower bound of $3n/7$ Gallai lines by Kelly and Moser (Theorem 1.5). Our (effortless) contributions to the table are $m(15)$, $m(20)$ and $m(24)$, which were not presented in Brakke's table. We get the first one because nowadays we have faster computers; we will discuss the computer aided search in Section 2.4. And the latter two are immediate consequences of the lower bound of $6n/13$ Gallai lines by Csima and Sawyer (Theorem 1.6), which was two decades later than Brakke's work.

Proposition 2.1. *For any set P of n points,*

(a)

$$\binom{n}{2} = \sum_{i\geq 2} \binom{i}{2} t_i. \tag{2.1}$$

(b)

$$\binom{n}{2} \equiv t_2 + t_5 + t_8 + \cdots \pmod 3 \tag{2.2}$$

Proof. The right side of (a) is couting all pairs of points by each connecting line. (b) is an easy consequence of (a). □

The main tool in computing the $m(n)$ values for small n is equation (1.7) and the Proposition 2.1. The work of Crowe and Mckee is complicated than necessary since they did not use (1.7). Brakke used some other observation in deciding the $m(n)$s for some even n, but the observations does not provide much advantage over the lower bound provided by Csima and Sawyer for those small n.

As an illustration, we prove the most non-trivial entry ($m(9)$) in Brakke's table.

Proposition 2.2. *$m(9) = 6$.*

Proof. By Theorem 1.6 and Theorem 1.7,

$$5 \leq m(9) \leq 6.$$

We prove that it is impossible for any set of 9 points to have $t_2 = 5$. By (1.7),

$$t_2 \geq 3 + t_4 + 2t_5 + 3t_6 + \cdots \tag{2.3}$$

If $t_2 = 5$, then $t_k = 0$ for any $k \geq 6$. And by (2.2), $5 + t_5 \equiv 0 \pmod 3$. Hence the only way to satisfy (2.3) and $t_2 = 5$ is $t_5 = 1$ and $t_4 = 0$. Now, let ℓ be the line with 5 points $y_i : i = 1, 2, 3, 4, 5$, and x_1, x_2, x_3, and x_4 be the 4 points off ℓ. We count the number of Gallai lines connecting one of the y_i's and one of the x_i's. x_1 is incident to at least 2 such Gallai lines, since, among the 5 lines $y_i x_1$, at most 3 can pass through one of the other x_i's. Similarly, each x_i is incident to at least 2 such Gallai lines, so $t_2 \geq 8$, a contradiction. □

2.2 The number of Gallai points

By Theorem 1.6, we have

Proposition 2.3. *For $n \neq 7$,*

$$\binom{m^*(n)}{2} \geq \frac{6n}{13}.$$

In the configuration of n points where one of the points is x and all the other points lie evenly on two lines passing x, x is not a Gallai point. We have

Proposition 2.4. *For $n \geq 5$, $m^*(n) \leq n - 1$.*

For $n \geq 9$, we have examples where the number of Gallai points is roughly $2n/3$.

Proposition 2.5. *(a) $m^*(3k) \leq 2k$ for all $k \geq 3$.*

(b) $m^*(3k+1) \leq 2k+2$ for all $k \geq 4$.

(c) $m^*(3k+2) \leq 2k+2$ for all $k \geq 3$.

(d) $m^*(6k+4) \leq 4k+2$ for all $k \geq 1$.

Proof. (a) Consider a regular $2k$-gon Q in the Euclidean plane. The edges of Q decide k directions, hence k points at infinity. Let P be the set of $2k$ vertices of Q and the k points at infinity. The set of Gallai points is the set of vertices of Q.

(b) As the construction in (a), consider the $(2k+2)$ vertices of a regular polygon Q, the $k+1$ points at infinity. Delete two of the points at infinity. The set of Gallai points is the vertex set of Q.

(c) Similar to (b), but just delete one point at infinity.

(d) As the construction in (a), let P be the $(4k+2)$ vertices of a regular polygon Q, the $2k+1$ points at infinity, and one point at the center of Q. The set of Gallai points is the vertex set of Q. □

Proposition 2.6. *Given a set P of n non-collinear points, if all the points in P lie on 2 lines, then there are at least $n-1$ Gallai points.*

Proof. Let ℓ_1 and ℓ_2 be the 2 lines containing all the points of P. Let x be the intersection of ℓ_1 and ℓ_2. For any $y \in \ell_1 \cap P$, $z \in \ell_2 \cap P$, $y \neq x$ and $z \neq x$, yz is a Gallai line. □

Since there is only one configuration on 3 points, trivially,

Proposition 2.7. $m^*(3) = 3$.

Proposition 2.8. $m^*(4) = 4$.

Proof. Since the 4 points are not collinear, each point is incident to at least 2 lines, one of these is a Gallai line. So all the points are Gallai points. □

Proposition 2.9. $m^*(5) = 4$.

Proof. Suppose there is a non-Gallai point x, there are at least 2 connecting lines passing x and each containing at least 2 points other than x. So all the points lie on 2 lines passing x. $m^*(5) = 4$ by Proposition 2.4 and Proposition 2.6. □

Similarly,

Proposition 2.10. $m^*(6) = 5$.

Proposition 2.11. $m^*(7) = 3$.

Proof. Since there are always at least 3 Gallai lines, so there are at least 3 Gallai points. The Kelly-Moser example (Figure 1.4) achieves as few as 3 Gallai points. □

Proposition 2.12. $m^*(8) = 6$.

Proof. We modify the Kelly-Moser example by adding one point on any non-Gallai connecting line, we get a configuration with 8 points, 6 of which are Gallai points.

Next we prove that there cannot be more than 2 non-Gallai points. Suppose x_0 is any non-Gallai point. By Proposition 2.6, there are at least 3 connecting lines passing through x_0, each with at least 2 points other than x_0. There are 8 points in total. We can classify the points on 3 lines. x_0, x_1, x_2, and x_3 on ℓ_x; x_0, y_1, and y_2 on ℓ_y; and x_0, z_1, and z_2 on ℓ_z. y_1 is a Gallai point, since at least one of y_1x_1, y_1x_2, and y_1x_3 is a Gallai line. Similarly y_2, z_1, and z_2 are Gallai points. We may assume x_0, x_1, and x_2 are 3 non-Gallai points. Now, since x_1 is not a Gallai point, x_1y_1 and x_1y_2 must intersect ℓ_z at z_1 and z_2. The same is true for x_2. So, x_0, x_1, and x_2 are the 3 diagonal points of the complete quadrangle $y_1y_2z_1z_2$. Contradicts the fact that the diagonal points of a complete quadrangle are never collinear. □

Proposition 2.13. $m^*(9) = 6$.

Proof. By Proposition 2.5, $m^*(9) \leq 6$.

Next we prove that there are always at least 6 Gallai points. Suppose x_0 is any non-Gallai point. By Proposition 2.6, there are at least 3 connecting lines passing through x_0, each with at least 2 points other than x_0. There are 9 points in total.

Suppose x_0 is incident to 3 lines and the number of points from P, excluding x, on these lines are 2, 2, and 4, respectively. Suppose x_0, y_1, and y_2 are on the first line; x_0, z_1, and z_2 are on the second line; and x_0, x_1, x_2, x_3, and x_4 are on the third line. y_1 must be a Gallai point, since one of the lines y_1x_1, y_1x_2, and y_1x_3 is a Gallai line. By the same reason, y_2, z_1, and z_2 are all Gallai points. If any x_i is a non-Gallai point, it must be a diagonal point of the complete quadrangle $y_1y_2z_1z_2$. Since the 3 diagonals are not collinear, there are at least 3 Gallai points among the x_i's. Thus there are at least 7 Gallai points.

Suppose x_0 is incident to 3 lines and the number of points from P, excluding x, on these lines are 2, 3, and 3, respectively. Suppose x_0, x_1, and x_2 are on the first line; x_0, y_1, y_2, and y_3 are on the second line; and x_0, z_1, z_2, z_3 are on the third line. y_1 must be a Gallai point, since one of the lines y_1z_1, y_1z_2, and y_1z_3 is a Gallai line. By the same reasoning, y_2, y_3, z_1, z_2, and z_3 are all Gallai points. So there are at least 6 Gallai points.

In the remaining case, each non-Gallai point is incident to 4 lines, with 3 points from P on each line. Let x_0 and x_1 be 2 of the non-Gallai point. There is a connecting line ℓ containing 3 points x_0, x_1, and x_2 from P. Project ℓ to infinity so that the other 3 lines incident to x_0 are parallel, as well as the other 3 lines incident to x_1. So, the 6 points other than the x_i's are distributed on the 3×3 lattice formed by the two sets of 3 parallel lines, with 2 points on each line. Consider the symmetry of the plane, we have two cases as in Figure 2.1.

Case 1. Figure 2.1 (a). We claim the six points $P' = \{a, b, c, d, e, f\}$ are all Gallai

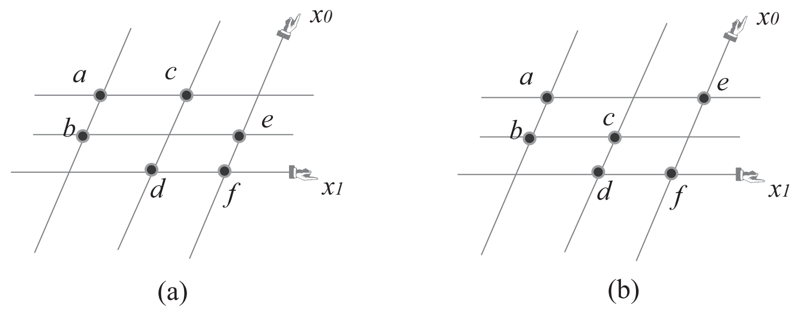


Figure 2.1: The proof of $m^*(9) = 6$.

points. To see a is a Gallai point, consider the lines ad, ae, and af, they do not pass any other point in P', nor do they pass x_0 nor x_1. One of them may pass x_2. So at least two of these lines are Gallai lines. The proofs for the other points in P' are similar.

Case 2. Figure 2.1 (b). Similar to the proof in Case 1, b, d, and e are Gallai points. The 3 lines ad, ce, bf are of different slopes, so at most one of them can pass the point x_2 at infinity. So there are at least 2 Gallai points in $\{a, c, f\}$. Furthermore, x_2 is a Gallai point, since it is impossible to find a direction in the figure which makes the 6 points into 3 parallel pairs other than the directions x_0 and x_1. □

It is possible to do a similar but more complicated proof for the fact that $m^*(10) = 6$. But we differ to Section 2.4 where we use a computer program to find $m^*(10)$.

2.3 Pseudoline arrangements and allowable sequences

A natural generalization of lines in the plane are the pseudolines. Pseudolines are not perfectly straight. It is first defined by Levi ([40]) where he proved the useful enlargement lemma. Many results in Chapter 1, including the Csima-Sawyer bound (Theorem 1.6), are still valid when we replace the word *lines* by *pseudolines*.

Definition 2.2. *A* pseudoline *in the real projective plane is a closed simple curve that does not disconnect the plane.*

Definition 2.3. *An* arrangement of pseudolines *is a set of pseudolines in the real projective plane such that any two of them meet at exactly one point, where they cross, and the intersection of all the pseudolines is empty.*

We define the counterpart of m and m^* in the setting of pseudoline arrangements.

Definition 2.4. *In an arrangement of pseudolines, an intersection point is ordinary if it is incident to exactly two pseudolines in the arrangement. Define $\widetilde{m}(n)$ to be the least possible number of ordinary points in any arrangement of n pseudolines. Define $\widetilde{m^*}(n)$ to be the least possible number of pseudolines that are incident to at least one ordinary point in any arrangement of n pseudolines.*

Any straight line is a pseudoline. Clearly we have

Proposition 2.14. *$m(n) \geq \widetilde{m}(n)$ and $m^*(n) \geq \widetilde{m^*}(n)$ for any n.*

The *allowable sequence* was first developed by Goodman and Pollack in the 1980s. It is a combinatorial object associated with point configuration, arrangements of lines or pseudolines.

For a point configuration, for each reference line of a certain direction that is not perpendicular to any connecting lines of the points, we get an ordering of all the points as they project on the reference line. We rotate the reference line, the ordering changes when we pass a direction that is perpendicular to some connecting line. We get the associated periodic sequence of the orderings for the point configuration.

For an arrangement of straight lines in the Euclidean plane, none of which is vertical, and no of the pairs parallel, we sweep a vertical line from the left to the right. At any moment whenever the vertical line is not incident to any intersection point of the arrangement, we have an ordering of the lines on the sweep line. The sequence we get is a half period of the periodic sequence for the arrangement. The ordering in the second half is gotten by reversing the corresponding ordering half period away.

For an arrangement of red lines in the real projective plane, viewed as the great circles on the sphere with antipodal points identified, and a point p not on any line, we sweep a yellow great circle passing through p. If the yellow circle does not pass through any intersection of two red circles, the orange points gives an ordering of all the lines in the arrangement. The ordering changes when the yellow circle passes any red intersection points. We keep sweeping the yellow circle, get a periodic sequence of the orderings associated to the arrangement.

There are some common properties to the sequence for the point configurations and arrangements. Goodman and Pollack defined the allowable sequences using these properties.

Definition 2.5. *An infinite periodic sequence of permutations of* $\{1, 2, \cdots, n\}$ *is an allowable* n*-sequence if it satisfies*

(1) Any permutation in the sequence is gotten by a move of reversing one or more disjoint blocks of consecutive positions in the previous permutation; and

(2) For any two pairs (i, j) *and* (i', j')*, the order between* i *and* j *is reversed exactly once between every two reversals of the order between* i' *and* j'*, unless they reverse in the same move, in which case they always reverse together.*

One consequence of the definition is that permutations of half a period away reverse each other.

For a point configuration, the associated sequence encodes many geometric information about the configuration. Any reversed block in a move corresponding to a connecting line of the point set. Such a block of size 2 corresponds to a Gallai line. Different blocks that are reversed in the same move corresponds to parallel lines.

There are allowable sequences that cannot be realized by points or straight line arrangements, but every allowable sequence is realized by an arrangement of pseudolines in the real projective plane. In fact, for any allowable sequence, there is an obvious *wiring diagram* of pseudoline arrangements which realizes it. Figure 2.2 shows such a wiring diagram for an allowable sequence which is not realizable by points or straight line arrangements.

The allowable sequences itself is a pure combinatorial object. From the sequence one can define analogue concepts, such as collinear points, extreme points, convex polygons, parallel lines, and so on. Along this line, we give the obvious analogous definitions for the Sylvester-Gallai theory, and redefine the values $\widetilde{m}$ and $\widetilde{m^*}$.

Definition 2.6. *Let* Γ *be an allowable sequence of permutations on a set* P*. Any set* $L \subseteq P$ *is a* line *if the elements of* P *forms a reversed block in some move of* Γ*. A line* L *is a* Gallai line *if* $|L| = 2$*. An element* $x \in P$ *is a* Gallai point *if it belongs to some*

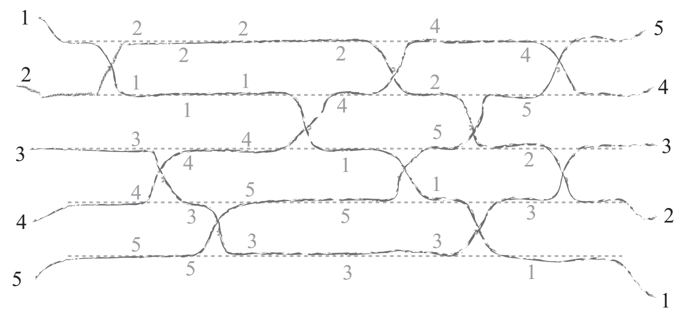


Figure 2.2: A wiring diagram of an allowable sequence which is not realizable by point configurations.

Gallai line.

Definition 2.7. *For any $n \geq 3$,*

$$\widetilde{m}(n) := \min_{\Gamma}(\textit{the number of Gallai lines in } \Gamma)$$

and

$$\widetilde{m^*}(n) := \min_{\Gamma}(\textit{the number of Gallai points in } \Gamma),$$

where the sequences Γ are taken over all the allowable sequences on n-element set with half period length at least 2.

In [56], Ungar proved Scott's conjecture. The proof is a beautiful application of allowable sequences. We briefly sketch the proof.

Theorem 2.1. *(Restating Theorem 1.11) Let P be a set of n non-collinear points. The connecting lines of P show at least $2\lfloor n/2 \rfloor$ directions.*

Proof. (Sketch) It is enough to prove the theorem for even numbers $n = 2k$. We want to prove that any allowable sequence on n elements must have half period length n, unless it is the trivial sequence with half period length 1.

Consider the allowable sequence associated to the point set. Each element is a permutation on n numbers. We focus on the "central barrier" between the k-th element and the $(k+1)$-st. Call a move *crossing* if some block crossing the barrier is reversed in the move, otherwise call it *ordinary*. Let the crossing moves be m_1, m_2, *cdots*, m_t in the half period. Suppose the reversed block crossing the barrier has l_i positions on the left side and r_i positions on the right side. Let $d_i = \min(l_i, r_i)$, so the number of elements that moving across the barrier in m_i is d_i. In the half period, the total number of elements moved across the barrier is at least n, so

(1) $n \leq 2d_1 + 2d_2 + \cdots + 2d_t$.

Next, we prove that,

(2) The number of ordinary moves between m_i and m_{i+1} is at least $d_i + d_{i+1} - 1$; and

(3) The number of ordinary moves before m_1 and after m_t is at least $d_t + d_1 - 1$.

The facts (1), (2), and (3) conclude that the total number of moves in the half period is at least n.

To prove (2), notice that the two elements near the center after m_i cannot be the same pair of elements near the center before the move m_{i+1}. We assume the right neighbor is changed. (The other case is similar.) Consider the d_i elements to the right of the barrier after m_i, they are

$$a_1 > a_2 > \ldots > a_{d_i};$$

and the d_{i+1} elements to the right of the barrier before m_{i+1} are

$$b_1 < b_2 < \ldots < b_{d_{i+1}}.$$

Denote M the period after m_i and before m_{i+1}. The order among the a's are not changed in M, so is the order among the b's. Notice that $a_1 \neq b_1$. The two sets of a's and b's do not have a common element, and all the pairs a_p and b_q are crossed once during M. It is easy to see that the following pairs are crossed in different moves, from the earliest to the latest:

$$a_{d_i}b_1, a_{d_i-1}b_1, \ldots, a_1b_1, a_1b_2, \ldots, a_1b_{d_{i+1}}.$$

Thus (2) is proved. (3) is actually the same as (2), since we can freely choose the starting position of the half period in the periodic sequence, and rename the elements so that the first permutation is $(12\cdots n)$.

□

As another application of the allowable sequences, we give a new proof to the Sylvester-Gallai theorem.

Proof. (of Theorem 1.1) Any permutation of n elements has $n-1$ adjacent pairs. We call the position between each pair a *wall.* Denote w_i the wall between position i and $i+1$.

In the Sylvester-Gallai problem, we may assume that in each move only one block is reversed, and the first move is to reverse a block $(12\cdots t)$ for some t. (See next section for more details.) For an allowable sequence, we are going to define the walls, after each move, into two classes — the crossable walls and the non-crossable ones. We want to define them so that

$$\text{The walls marked as non-crossable can not be crossed in the next move.} \tag{2.4}$$

On the other hand, it is possible that a crossable wall is between a pair of descending elements.

We define the sequence of walls associated to the half period of an allowable sequence as

(1) In the beginning, all the walls are crossable.

(2) If a move reverses a block of positions i, $i+1$, $\cdots$, j, then the marks are changed from the previous moment as the following: w_i, w_{i+1}, $\cdots$, w_{j-1} become non-crossable; w_{i-1} and w_j (if exist) become crossable.

It is easy to see that the condition (2.4) is satisfied. Certainly, a move where the reversed block has size 2 is called a *Gallai move.*

A configuration of marked walls is called *pending* if it ends with two non-crossable walls, and there are no consecutive crossable walls.

Claim 2. *For any moment of the walls $w_1, w_2, \cdots, w_k, \cdots, w_{n-1}$ associated to an allowable sequence. If the first k walls form a pending configuration, and the wall w_{k-1} will become crossable before the end of the half period, then there is a Gallai move before w_{k-1} becomes crossable.*

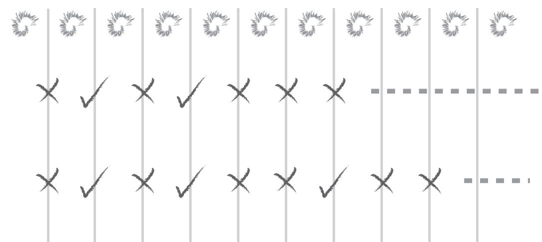


Figure 2.3: A proof of Sylvester-Gallai theorem by allowable sequences.

The proof to the claim is an easy induction on $n-k$. The base case is trivial. For the inductive case, before w_{k-1} becomes crossable, either there is a Gallai move before the position k, or there is a moment w_1, w_2, $\cdots$, w_{k-1} are not changed, and w_k becomes crossable because a reverse of a block of positions $(k+1, k+2, \ldots, k+d)$. If $d=2$, it is a Gallai move. Otherwise, the walls after this move has the first $k+d-1$ walls forming a pending configuration. After this moment, before w_{k-1} becomes crossable, either there is a Gallai move before the position $k+d$, or w_{k+d-2} will become crossable. In the later case we use inductive hypothesis.

If the first move, which reversing the elements on the first t positions, is not a Gallai line, the first $t>2$ walls after the first move form a pending configuration. Since $t<n$, x_{t-2} will be crossed by the element n before the end of the half period, so x_{t-2} will become crossable. We conclude there is a Gallai line. $\square$

For more information about the pseudoline arrangements and allowable sequences, we refer to the surveys Goodman 1997 ([28]), Goodman and Pollack 1981 and 1993 ([29] and [30]).

2.4 Computer aided enumerations

In this section we provide computer programs to decide the values $\widetilde{m}(n)$ and $\widetilde{m^*}(n)$ for the small n's.

Because of the relationship between pseudoline arrangements and the allowable sequences, we are allowed to write simple computer programs to check all the n-allowable sequences when n is small. The helpful picture we keep in mind is the wiring diagram of the pseudolines from the left to the right, and we sweep a vertical line to get the orderings of the pseudolines. The number of all n-allowable sequences grows rapidly with n, we trim the enumeration by observations when they do not affect the $\widetilde{m}$ or $\widetilde{m^*}$ values.

We generate the half period of the sequence. We may rename the pseudolines so that the first element in the sequence is the identity permutation $(12\cdots n)$.

Property (2) of the allowable sequence states that in the half period, the order of every pair of numbers is reversed exactly once. If we start the half period with $(12\cdots n)$ and end with $(n\cdots 21)$, the property translates to a simple rule that we can reverse a block if and only if the numbers in the block are in ascending order.

In a move from one permutation to the next one, by property (1) of the allowable sequence, we need to pick one or more blocks to reverse. However, if we slightly disturb the wiring diagram so that no two intersection points form a vertical line, the set of ordinary points are not changed. (For the point configuration, this is the fact that parallelism is not essential to the Sylvester-Gallai problems.) So, when several disjoint blocks are to be reversed, we may pick any one of them first. In our program, we just need to pick one block in any move.

We further utilize this observation. If B_1, B_2, $\ldots$, B_t are t disjoint blocks, from left to right, that are reversed in a sequence of consecutive moves. We may reorder the moves so that the blocks are reversed in left to right order. In terms of the pictures, if we have a sequence of consecutive, from left to right, intersection points of the pseudolines in the wiring diagram, where the intersection points involve disjoint set of pseudolines, then we may stretch the picture so that the higher points occur earlier. To use this observation, we simply add a rule in the program: If the last move reverses the block $[i..j]$, and the current move reverses the block $[i'..j']$, then either they are not disjoint, or the clock in the current move is to the right of the previous block. In other words, $j' \geq i$.

We do a small optimization by assuming the second permutation in the sequence is always $(21345\cdots n)$. The reason is that there is always an ordinary point by the Sylvester-Gallai theory, and we can alway choose, in the sphere modal of the projective plane, the reference point and the start moment so that the first event is the swap of 1 and 2. Similarly, if we decide there is a move with a block of size k, we may, by choosing a reference point and a start direction, reverse the first k elements in the permutation in our first move.

We use the obvious rule that whenever the search is sure to produce an $\widetilde{m}$ ($\widetilde{m^*}$) value that is no less than the current known upper bound, we stop the branch and backtrack. In the beginning, the upper bound is the trivial ones or the known result such as those from Theorem 1.6. The bound changes when we find an example with a better value. We further utilize this rule as the following in some of our programs. In the beginning the user specify some estimate m' for the upper bound of the value. If the program terminate with a value $m < m'$, we are sure m is the correct result. Otherwise, the program terminates with m', and we get the partial answer that the correct value is at least m'.

To decide whether we reach the end of a half period in the recursive search, we do not use the slow way of checking if the permutation is $(n \cdots 21)$. Instead we use the identity on the number of pairs (Proposition 2.1). The total number of pairs to be reversed is $\binom{n}{2}$. Whenever we reverse a block of size t, we count $\binom{t}{2}$ pairs.

In the program computing $\widetilde{m}$, we use the equation (1.7) and the Proposition 2.1 to generate the possible vectors $(t_2, t_3, \ldots)$, where t_i is the number of connecting lines that are incident to exactly i points in the point configuration. In our setting, t_i is the number of intersection points that are incident to exactly i pseudolines. So, blocks of size i are reversed t_i times in the half period.

We present the results of the computer search in table 2.4, and provide the programs with comments and some output in the appendix.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\widetilde{m}(n)$	3	3	4	3	3	4	6	5	6	6	6	7	9	8
$\widetilde{m^*}(n)$	3	4	4	5	3	6	6	6	6	6	7	8	9	8

Table 2.2: The Gallai numbers for pseudoline arrangements by computer enumeration.

From the computer enumeration results we complete some new entries that were not proved before.

Proposition 2.15. $m^*(10) = 6$.

Proof. $m^*(10) \leq 6$ by the example in Proposition 2.5 (d), and $m^*(10) \geq \widetilde{m^*}(10) = 6$ as in table 2.4. □

Proposition 2.16. $m(15) = 9$.

Proof. $m(15) \leq 9$ by the example of Motzkin and Böröczky (Theorem 1.7). And $m(15) \geq \widetilde{m}(15) = 9$ as in table 2.4. □

As suggested by Jacob Goodman, the magic configuration conjecture may be true in the general setting of pseudolines. Based on the $\widetilde{m^*}(n)$ values for small n's in table 2.4, we have the following conjecture, which is stronger than Conjecture 1.2.

Conjecture 2.1. *For $n \neq 7$, $\widetilde{m^*}(n) \geq n/2$.*

We note that a positive answer to Conjecture 2.1 implies a complete solution to the magic configuration conjecture in the general setting of pseudoline arrangements.

The magic configuration with the Kelly-Moser example has K_3 as the Sylvester graph. As we learnt from Murty that it is not known if there is any configuration with K_4 as the Sylvester graph, i.e., a configuration with 4 Gallai points and 6 Gallai lines. We prove that the answer is negative.

Proposition 2.17. *There is no configuration on $n \geq 5$ points with K_4 as the Gallai graph.*

Proof. Let x be any non-Gallai point. Since any connecting line of Gallai points is a Gallai line, x is not collinear with any two Gallai points. So each of the 4 lines connecting x with the Gallai points contains at least one other non-Gallai point. It is clear there are at least 9 points.

When $n > 13$, the number of Gallai lines is at least 7 by Theorem 1.6. When $9 \leq n \leq 13$, the number of Gallai points is at least $m^*(n) \geq \widetilde{m^*}(n) \geq 6$ as in table 2.4. □

Chapter 3

Generalization of Points and Lines

There is only one natural way to generalize the notion of points into any metric space and set systems — in fact the metric spaces are defined on the set of elements which we call points. There is more than one way to generalize the notion of *lines*. We give some of the definitions that will derive interesting results. However, as we will show in this dissertation, none of the definitions are perfectly satisfactory. In the seminars and conferences during the past years, one of the frequently asked questions is that why we define the lines in our way. The confusion should be eased when we consider the word *lines* as just a name we borrowed for the animals we defined; and we shall be happy for the similarity between them and the geometrical lines, not their difference in some aspects.

3.1 Definitions of lines in hypergraphs and matric spaces

In this section we give two types of definitions of lines in hypergraphs and metric spaces. Given a metric space (X, ρ), it is quite natural to call 3 points x, y, and z "collinear" if they satisfy

$$\rho(x, z) = \rho(x, y) + \rho(y, z).$$

We denote this by $[xyz]$.

A straightforward definition of a line similar to the one in Euclidean space is that "the line of x and y consists of all those points that are collinear with x and y". i.e.

$$\overline{xy} = \{x, y\} \cup \{p : [pxy]\} \cup \{p : [xpy]\} \cup \{p : [xyp]\}. \tag{3.1}$$

The other definition we will consider is called the closure-line. This is the line V.

Chvátal defined in [12] which gives a successful generalization of the Sylvester-Gallai theorem.

Both definitions rely on the ternary relation $[\cdot]$. We can similarly define the lines in hypergraphs.

We first give a definition of a line in an arbitrary hypergraph.

Definition 3.1. *(Lines in a hypergraph.) Given $(X, \mathcal{H})$, where X is a point set and $\mathcal{H}$ is a hypergraph on X, the* line $\overline{uv}$ *in $(X, \mathcal{H})$ is defined as*

$$\overline{uv} = \{u, v\} \cup \{p : \{u, v, p\} \subseteq H \textit{ for some } H \in \mathcal{H}\}$$

for any two distinct vertices u and v.

We will mostly work on 3-uniform hypergraphs. In fact, for any hypergraph $\mathcal{H}$, we define

$$\mathcal{H}' = \{\{a, b, c\} : \{a, b, c\} \subseteq H \text{ for some } H \in \mathcal{H}\}.$$

The lines defined in $(X, \mathcal{H})$ and $(X, \mathcal{H}')$ are the same.

Definition 3.2. *Let (X, ρ) be a metric space, the associated hypergraph is*

$$\mathcal{H}(\rho) = \{\{a, b, c\} : [abc]\}.$$

All the hypergraphs $\mathcal{H}(\rho)$ are 3-uniform, but some 3-uniform hypergraphs do not arise from any metric space (X, ρ) as $\mathcal{H}(\rho)$. For example, it has been shown in [12] and [9] that the Fano plane does not arise from any metric space.

Definition 3.3. *(Lines in a metric space.) Given a metric space (X, ρ), for any two distinct points u and v, the* line $\overline{uv}$ *in (X, ρ) is defined to be the line $\overline{uv}$ in $(X, \mathcal{H}(\rho))$.*

We note that this definition of the line in a metric space is the same as in (3.1).

Definition 3.4. *(Closure-lines in hypergraphs) Given $(X, \mathcal{H})$, where X is a point set and $\mathcal{H}$ is a hypergraph on X, a set $T \subseteq X$ is called* affinely closed *if and only if every edge that shares at least 2 vertices with T is fully contained in T. For any $S \subseteq X$, the* affine closure *of S, denote by* $\mathrm{aff}(S)$*, is defined to be*

$$\mathrm{aff}(S) = \bigcap \{T : S \subseteq T, T \textit{ is affinely closed}\}.$$

For two distinct points u and v, the closure-line $\widetilde{uv}$ *is defined to be* $\text{aff}(\{u, v\})$.

We remark that $\text{aff}(S)$ is well defined since X is affinely closed, and $\text{aff}(S)$ is the unique minimal affinely closed set that contains S.

Definition 3.5. *(Closure-lines in a metric space) Given a metric space (X, ρ), for any two distinct points u and v, the* closure-line $\widetilde{uv}$ *in (X, ρ) is defined to be the closure-line $\widetilde{uv}$ in $(X, \mathcal{H}(\rho))$.*

3.2 Example Gallery

When X is a subset of a Euclidean space and ρ is the Euclidean metric, both $\overline{uv}$ and $\widetilde{uv}$ coincide with the Euclidean line.

Example 3.1. (V. Chvátal)

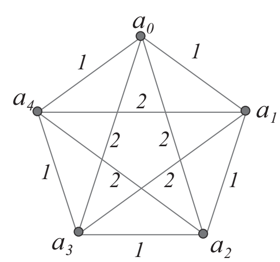


Figure 3.1: Example 3.1 of lines and closure lines.

Let $X = \{a_i : 0 \le i < 5\}$,

$$\rho(a_0a_1) = \rho(a_1a_2) = \rho(a_2a_3) = \rho(a_3a_4) = \rho(a_4a_0) = 1,$$

and

$$\rho(a_0a_2) = \rho(a_1a_3) = \rho(a_2a_4) = \rho(a_3a_0) = \rho(a_4a_1) = 2.$$

Then every line contains 3 or 4 points in X, and X is the only closure line. (Figure 3.1)

$\diamond$

In general, two points in X may be contained in more than one lines; the intersection of two lines may have more than one points; and it is possible that one line is a proper subset of another. The same is true for the closure lines.

Example 3.2. If X is a subset of $\mathbb{R}^2$ and ρ is the ℓ_1 distance. $[abc]$ is equivalent to the condition that b lies in the (possibly degenerate) closed axis-parallel rectangle determined by a and c.

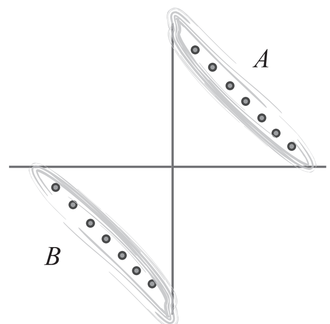


Figure 3.2: Example 3.2 of lines and closure lines.

Let $A = \{(i, n+1-i) : i \in \mathbb{Z}, 1 \leq i \leq n\}$ and $B = \{(-i, i-n-1) : i \in \mathbb{Z}, 1 \leq i \leq n\}$. Then

$$\overline{uv} = \widetilde{uv} = \begin{cases} \{u, v\} & \text{if } |\{u, v\} \cap A| = 1, \\ A & \text{if } \{u, v\} \subseteq A, \\ B & \text{if } \{u, v\} \subseteq B. \end{cases}$$

$\diamond$

Example 3.3. Consider the metric space (X, ρ), where $X = \{x_k : 1 \leq k \leq n\}$ is a subset of $\mathbb{R}^2$ with

$$x_1 = (1, 3), \quad x_2 = (2, 4), \quad x_3 = (3, 1), \quad x_4 = (4, 2),$$
$$x_k = (k, n + 5 - k) \text{ whenever } 5 \leq k \leq n,$$

and ρ is the ℓ_1 metric.

Since $\mathcal{H}(\rho)$ consists of all $\{x_1, x_2, x_k\}$ with $5 \leq k \leq n$, all $\{x_3, x_4, x_k\}$ with $5 \leq k \leq n$,

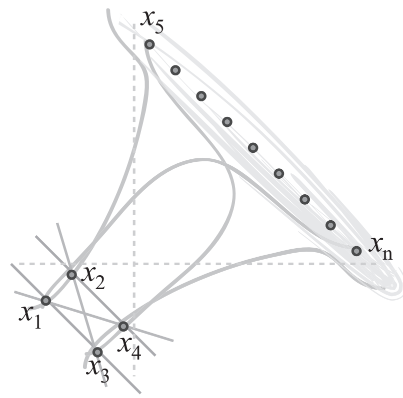


Figure 3.3: Example 3.3 of lines and closure lines, with the 7 closure lines indicated.

and all $\{x_i, x_j, x_k\}$ with $5 \le i < j < k \le n$, we have

$$\begin{aligned}
\widetilde{x_1x_2} &= X - \{x_3, x_4\}, \\
\widetilde{x_3x_4} &= X - \{x_1, x_2\}, \\
\widetilde{x_ix_j} &= X - \{x_1, x_2, x_3, x_4\} \text{ whenever } 5 \le i < j \le n, \\
\widetilde{x_ix_j} &= \{x_i, x_j\} \text{ whenever } 1 \le i \le 2 \text{ and } 3 \le j \le 4, \\
\widetilde{x_ix_j} &= X - \{x_3, x_4\} \text{ whenever } 1 \le i \le 2 \text{ and } 5 \le j \le n, \\
\widetilde{x_ix_j} &= X - \{x_1, x_2\} \text{ whenever } 3 \le i \le 4 \text{ and } 5 \le j \le n.
\end{aligned}$$

These are the 7 closure-lines in (X, ρ). The number of lines is much bigger. For example, We have $\overline{x_1x_i} = \{x_1, x_2, x_i\}$ for any $i \ge 5$. $\diamond$

Chapter 4

The Sylvester-Chvátal Theorem

4.1 A history of the Sylvester-Chvátal theorem

In 1998 Vašek Chvátal considered the possibility of generalizing the Sylvester-Gallai theorem in arbitrary metric spaces (see [12]). He observed the fact that the similar statement of the Sylvester-Gallai theorem does not hold if one uses the most straightforward definition of a *line*. He gave the simple example (3.1) where $|\overline{uv}|$ is either 3 or 4 for any pair of points u and v. However, if we take the definition of the closure lines (Definition 3.5) things seem different. The main conjecture of [12] is that the Sylvester-Gallai theorem generalizes to any finite metric space if we consider the closure lines. Here is Conjecture 3.2 of [12]

> If (V, ρ) is a metric space such that $1 < |V| < \infty$, then V includes distinct points a, b such that $\widetilde{ab}$ is $\{a, b\}$ or V.

Chvátal [12] proved the single digit version of this conjecture, i.e., for all metric spaces with at most nine points. Chvátal also gave the affirmative answer to the conjecture in the case where the metric spaces is induced by connected graphs with unit weights. In 2001 we [8] proved the conjecture for all (finite) subspaces of ℓ_1^2, the two-dimensional space with the ℓ_1-metric. Later we developed techniques that allowed us to give a much simpler proof for the ℓ_1^2 case, as well as proofs for metric spaces induced by graphs with some special weights. In 2003 (see [9]) we completely proved the conjecture and made it to the

Theorem 4.1. *(Sylvester-Chvátal theorem) If (V, ρ) is a finite metric space, then V contains two distinct points a and b such that the $\widetilde{ab} = \{a, b\}$ or $\widetilde{ab} = V$.*

The original Sylvester-Gallai theorem is a special case of Theorem 4.1, where (V, ρ) is a finite subspace of the Euclidean plane.

In the next sections we give the proofs for the ℓ_1^2 case and then give the proof to the whole theorem.

4.2 The Manhattan Sylvester-Chvátal theorem

When Vašk first talked to me about this problem, he told me the question in the ℓ_1^2 setting. This is an interesting special case of the conjecture. From the question and the proof one may get a better understanding of Vašek's definition of lines in metric spaces. In fact, the problem and the first proof I gave has a very different taste than the original Sylvester problem.

Here was the question in the special setting:

> Given a finite set of points P in the plane. Three points are called *friendly* if one is in the axis-parallel rectangle determined by the other two. Start from any two points u and v in P, we grow a set L_{uv}. At any moment, if there is a point in P that is not in L_{uv} and is friendly with 2 points in L_{uv}, then we include that point to L_{uv}. Prove that we can always find two points u and v such that the final L_{uv} contains all the points in P, or is just $\{u, v\}$.

The following theorem, appeared in [8] gives the positive answer to the question.

Theorem 4.2. *Let V be a finite set of points in $\mathbb{R}^2$, and ρ is induced by l_1 metric, then there are two points a and b such that the closure line $\widetilde{ab}$ is either $\{a, b\}$ or V.*

As usual, the way we write down the proof is the opposite way as we find the proof. One might get a better understanding if we first concentrate on the cases where there are no two points that lie on a horizontal line or a vertical line. We also note that it can be easily seen from the proof that there is a linear time algorithm to find a pair ab such that $\widetilde{ab}$ is $\{a, b\}$ or V.

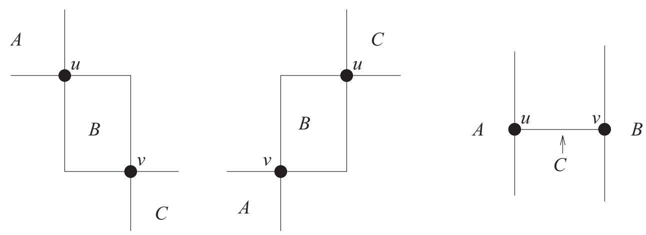
Proof. Let V be a set of n points. We say $\{a, b, c\}$ is collinear if $[abc]$. As in Definition 3.4, for any subset W of V, $\mathrm{aff}(W)$ is the closure of W with respect to the collinear relation. Call $\{u, v\}$ *nilpotent* if the closure line $\widetilde{uv}$ is $\{u, v\}$, call it *omnipotent* if the closure line $\widetilde{uv}$ is V. We want to prove that, if there are no nilpotent pairs, then there is an omnipotent pair.

Let us look at some easy observations.

Fact 1. Easy to see that if we have three points u, v, and w that lie on the same horizontal line (or vertical line), then any two of them are omnipotent.

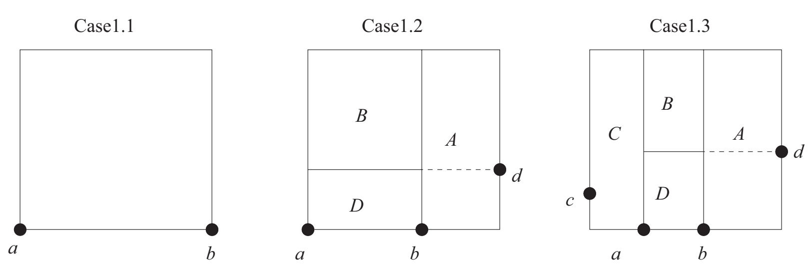
Fact 2. If we have u, v, w such that uv is horizontal and wv is vertical, then any two of them are omnipotent.

Fact 3. As in the picture below, if (u, v) is nilpotent, then there is no any other point lies in the regions A, B, or C, including the boundaries.



Now, consider the bounding box of all the points. In the proof below, we use the first capital letters A, B, C, $\ldots$ to denote rectangular regions with boundaries included. The notation $(u, v) \to Y$ means any point in the set Y is collinear with u and v.

Case 1. There are two points a and b lie on the same edge of bounding box. Without loss of generality, they are on the bottom edge. There are three cases as in the picture below.

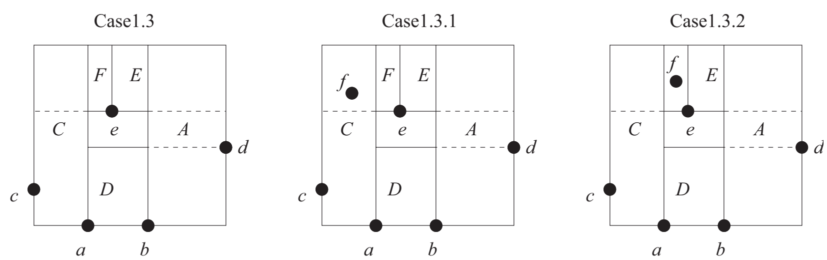


Case 1.1. a is also a leftmost point, and b a rightmost point. Then either we have a case as in Fact2, or (a, b) is nilpotent.

Case 1.2. a is a leftmost point, and b is not rightmost. The proof is essentially the same as in Case 1.3.

Case 1.3. a is not a leftmost point, and b is not rightmost. Pick a leftmost point c and a rightmost one d. Since $(a, b) \to \{A, C\}$, and $(a, d) \to D$, so (a, b) is omnipotent if there is no point in B. Otherwise, pick the lowest point e in B, if there are more than one, pick the rightmost one. As the picture below, if we have the set $\{a, b, e\}$, then $(a, b) \to \{A, C\}$, $(a, e) \to E$, $(b, e) \to F$, by the choice of e, we get all the points in $\mathcal{C}(a, b, e)$.

Since (e, d) is not nilpotent, there is point f collinear with d and e. By our choice of e, we have two cases.



Case 1.3.1. f is in the region A or C, then $(a, b) \to \{d, f\}$, $(d, f) \to e$. We get $\{a, b, e\}$ from (a, b), so (a, b) is omnipotent.

Case 1.3.2. Otherwise, f is in F. In this case, $(f, e) \to \{b, d\}$, $(b, d) \to \{a\}$, so (f, e) is omnipotent.

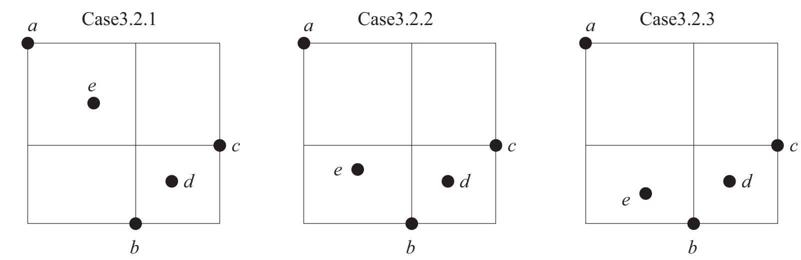
Case 2. There are two points a and b on the bounding box and have the same x-coordinates. Then, either (a, b) is nilpotent, or we have the same situation as in Fact1 or Fact2.

Case 3. If there is one point located on one corner of bounding box, but we do not have Case 1 or 2. Assume a is on the upper-left corner.

Case 3.1. If there is another point b on the lower-right corner, then (a, b) is omnipotent.

Case 3.2. Otherwise, we can pick up the lowest point b and the rightmost one c. As the picture below, it is easy to check $\text{aff}(a, b, c) = V$.

Since (b, c) is not nilpotent, there is a point d in the rectangle decided by b and c. Also, there is a point e lies in the rectangle decided by (a, b). We discuss the possible position of E.



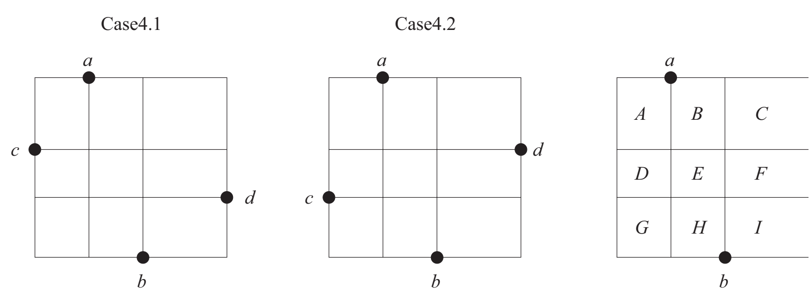
Case 3.2.1. e is higher than c, then $(a, e) \to \{b, c\}$, so (a, e) is omnipotent.

Case 3.2.2. e is not higher than c, but higher than d, then $(a, e) \to \{b, d\}$ and $(b, d) \to \{c\}$. Thus (a, e) is omnipotent.

Case 3.2.3. e is not higher than d, then $(c, d) \to \{e, b\}$ and $(b, e) \to \{a\}$, therefore (c, d) is omnipotent.

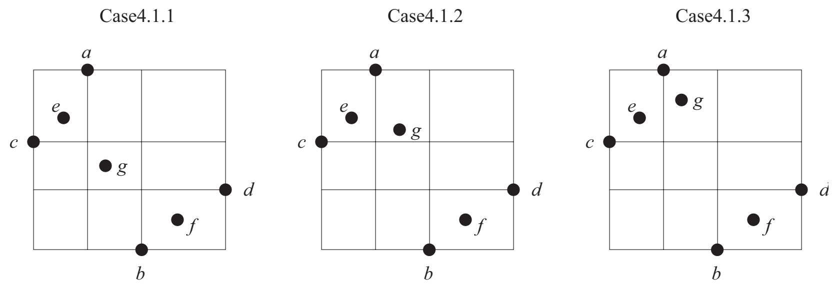
Otherwise, we come to the general case.

Case 4. As pictured below, a, b, c and d are the top, bottom, leftmost, rightmost points, respectively. And the line ab is not vertical, cd is not horizontal. Without loss of generality, the x-coordinate of a is smaller than that of b. We label the nine regions as A, B, ..., I.



We have two cases. Easy to check that, in both case, $\text{aff}(a, b, c, d) = V$. So, if we can prove $\{a, b, c, d\} \subseteq \text{aff}(u, v)$, then (u, v) is omnipotent.

Case 4.1. c is higher than d. Since (a, c), (b, d) are not nilpotent, there is a point e in A and a point f in I. Because (a, b) is not nilpotent, there is a point g in one of the regions B, E, or H. Without loss of generality, assume it's above d, we have three cases.



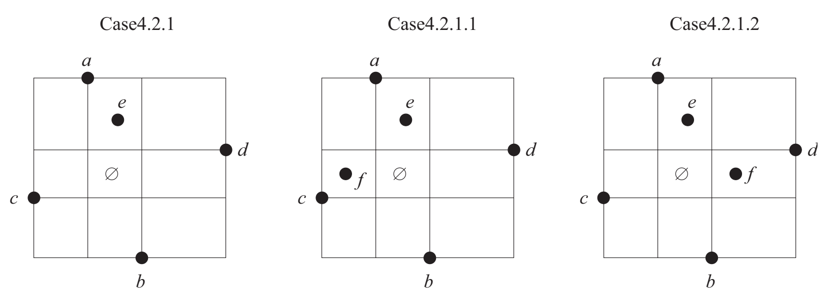
Case 4.1.1. g is in E, then $(a,g) \to \{b,d\}$, $(b,g) \to \{c\}$. So (a,g) is omnipotent.

Case 4.1.2. g is in B and below e, then $(a,g) \to \{b,d\}$, $(b,g) \to \{e\}$, and $(a,e) \to \{c\}$. Again, (a,g) is omnipotent.

Case 4.1.3. g is in B and not lower than e. In this case (c,e) is omnipotent, because $(c,e) \to \{a,g\}$, and $(a,g) \to \{b,d\}$.

Case 4.2. c is lower than d.

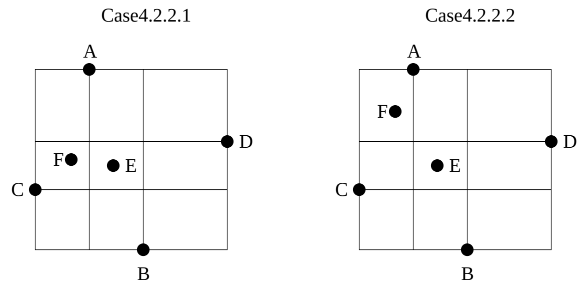
Case 4.2.1. There is no point in the region E. Since (a,b) is not nilpotent, there is a point e lies either in B or in H. Without loss of generality, e is in B. Similarly, there is a point f lies in either D or F.



Case 4.2.1.1. f is in D, then (c,f) is omnipotent, since $(c,f) \to \{a,e\}$ and $(a,e) \to \{b,d\}$.

Case 4.2.1.2. f is in F. (a,e) is omnipotent. Clearly in the picture above this is the Case 4.2.1.1 rotated by 90 degree:)

Case 4.2.2. There is a point e in the region E. Since (a,c) is not nilpotent, there is a point f lies either in A or in D.



Case 4.2.2.1. f is in D, then (c, d) is omnipotent, since $(c, d) \to \{e, f\}$, $(c, f) \to \{a\}$, and $(a, e) \to \{b\}$.

Case 4.2.2.2. f is in A, then (b, e) is omnipotent, since $(b, e) \to \{a, f\}$, $(a, f) \to \{c\}$, then $(c, e) \to \{d\}$. □

Another special case of the Sylvester-Chvátal theorem is when the metric space is induced by a graph with small weights. It turned out that working on the graphs with small integer weights reveals some important aspects of our final proof. As a by-product, one week before we discovered the final proof, we found the following short proof for the ℓ_1^2 case (Theorem 4.2). Both proofs involves case discussions. The first proof focuses on the (usually 4) points on the bounding box, while the new proof focuses on the maximal line. In the latter the number of cases is smaller. In particular, the situation where we have two points on a horizontal line or a vertical line is no longer a headache.

Proof. (of Theorem 4.2) In the two-dimensional space with ℓ_1 norm, $[abc]$ is the same as saying that the point c is inside the rectangle decided by a and b.

Consider a maximal closure line L. Assume $|L| > 2$, there are three points $a, b, c \in L$ such that $[acb]$ and $L = \widetilde{ab} = \widetilde{bc} = \widetilde{ca}$. For any point v, we write its x-coordinate and y-coordinate as $x(v)$ and $y(v)$, respectively. Without loss of generality, we assume $x(a) \leq x(c) \leq x(b)$, and $y(a) \leq y(c) \leq y(b)$. Aiming a contradiction, we assume $L \neq V$, pick a point z such that $z \in V \setminus L$. It is enough to show there is a line contains $\{a, c, z\}$. By the symmetry of the plane, we may assume that the point z is above the straight line passing through a and b, and $x(z) \leq x(a)$.

In this case $y(z) > y(c)$, otherwise $[zcb]$ implies $z \in L$. Since the line zc contains at least one other point, there is a point v such that $[vza]$ or $[zva]$ or $[zav]$.

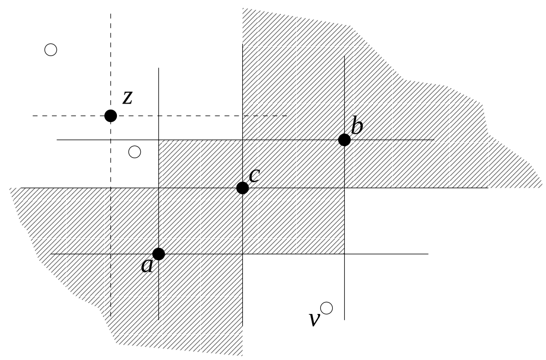


Figure 4.1: The proof to Theorem 4.2. Points in the shaded region will be included in the line ab in the first step. Unfilled dots are the possible places for v, if v is not in the shaded region.

Case 1. $[vza]$, i.e. $x(v) \leq x(z)$ and $y(v) \geq y(z)$. Now, we have $[vza]$ and $[vzc]$, so $\{a, c, z\} \subseteq \widetilde{vz}$.

Case 2. $[zva]$, i.e. $x(z) \leq x(v) \leq x(a)$ and $y(a) \leq y(v) \leq y(z)$.

Case 2.1. If $y(v) \geq y(c)$, we have $[vza]$ and $[vzc]$, so $\{a, c, z\} \subseteq \widetilde{vz}$.

Case 2.2. If $y(v) < y(c)$, we have $[bcv]$ and $[avz]$, so $\{a, c, z\} \subseteq \widetilde{ab}$.

Case 3. $[zav]$, i.e. $x(v) \geq x(a)$ and $y(v) \leq y(a)$.

Case 3.1. If $y(v) \leq y(c)$, we have $[bcv]$ and $[vaz]$, so $\{a, c, z\} \subseteq \widetilde{ab}$.

Case 3.2. If $y(v) > y(c)$, we have $[zcv]$ and $[zav]$, so $\{a, c, z\} \subseteq \widetilde{vz}$. □

It is hopeful to apply the similar proof to $\mathbb{R}^3$ with the ℓ_1 metric, probably with the aid of computers. However, the same idea will seemingly produce too complicated approach in proving the same thing in higher dimensions.

4.3 Proof of the Sylvester-Chvátal theorem

For a finite, un-directed, connected graph $G = (V, E, w)$ with positive weights, there is naturally an induced metric space (V, ρ), where $\rho(x, y)$ is defined to be the distance — the length of a shortest path — between x and y in G. Note that every finite metric space is induced by such a graph, so it suffices to prove the conjecture for finite metric spaces induced by graphs. Although the proof does not need the notations in graphs, this was the setting where we worked on the conjecture.

As mentioned in the previous section, the key idea that leads us to the final proof of the Sylvester-Chvátal theorem was found when we were focus on the graphs with small weights. With some observations, we found short proof to some special cases of Theorem 4.1, including the following.

1. Triangle-free graphs.

2. Graphs with weights coming from the set $\{w_1, w_2, \cdots, w_k, w_{k+1}, w_{k+2}\}$, where $k \geq 1$, $w_i > 0$ for all i, w_i/w_{i-1} is an integer for all $1 < i \leq k+1$, and w_{k+2}/w_k is an integer.

3. Graphs with weights coming from the set $\{\alpha, \beta\}$, where $0 < \alpha < \beta$ are any two positive reals.

4. Graphs with weights coming from the set $\{w_i : i \geq 1\}$, where $w_{3k+1} = 6^k$, $w_{3k+2} = 2 \cdot 6^k$, and $w_{3k+3} = 3 \cdot 6^k$, i.e. (w_i) is the sequence $1, 2, 3, 6, 12, 18, 36, \cdots$.

And finally we have the proof to Theorem 4.1. We provide the proof in this section.

We reserve the letter V for the ground-set of a finite metric space with at least two points and we reserve the letter ρ for the metric of this space. Our proof splits into two parts.

Proposition 4.1. *If every three points of V are contained in some closure line, then some closure line consists of all points of V.*

Proposition 4.2. *If some three points of V are contained in no closure line, then some closure line consists of precisely two points.*

Proof. (of Proposition 4.1) Consider a closure line, $L = \widetilde{ab}$, which is maximal with respect to set-inclusion; we claim that $L = V$. To justify this claim, assume the contrary: some point, c, of V lies outside L. By assumption, a, b, c are contained in some closure line L'; then L' contains $L \cup \{c\}$, contradicting the maximality of L. $\square$

Proof. (of Proposition 4.2). The following proof is published in [9]. We use the notation there, write *lines* instead of *closure lines* in this proof for simplicity.

By a *simple edge*, we mean any edge — ab — of the complete graph with vertex-set V such that no point x of V satisfies $[axb]$. By a *simple triangle*, we mean any three points — a, b, c — of V such that all of ab, bc, ca are simple edges. Now consider the following statements:

(i) some three points of V are contained in no line,

(ii) some simple triangle is contained in no line,

(iii) some line consists of precisely two points.

Proof of (i)⇒(ii). By (i), there are three points — a, b, c — of V such that

$$\text{no line contains } \{a, b, c\}; \tag{4.1}$$

among all such triples, choose one that minimizes $\rho(a, b) + \rho(b, c) + \rho(a, c)$; we claim that a, b, c is a simple triangle.

To justify this claim, assume the contrary: without loss of generality, there is a point d such that $[adb]$. Note first that $d \neq c$ (else (4.1) is contradicted by $[acb]$) and then that $\rho(d, c) < \rho(d, b) + \rho(b, c)$ (else (4.1) is contradicted by $[dbc]$ and $[adb]$). It follows that

$$\begin{aligned} & \rho(a, d) + \rho(d, c) + \rho(a, c) \\ < \; & \rho(a, d) + \rho(d, b) + \rho(b, c) + \rho(a, c) \\ = \; & \rho(a, b) + \rho(b, c) + \rho(a, c); \end{aligned}$$

now minimality of a, b, c implies that some line contains $\{a, d, c\}$; by $[adb]$, the same line contains $\{a, b, c\}$, contradicting (4.1).

Proof of (ii)⇒(iii). For each ordered triple u, v, w of points of V, let us write

$$\Delta(u, v, w) = \rho(u, v) + \rho(v, w) - \rho(u, w).$$

By (ii), some simple triangle — a, b, c — satisfies (4.1); among all such simple triangles, choose one that minimizes $\Delta(a, b, c)$; we claim that line $\widetilde{ac}$ consists of precisely two points.

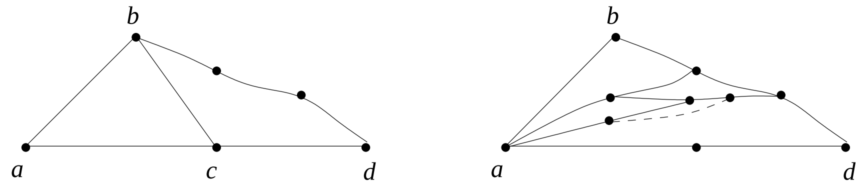


Figure 4.2: The proof of Theorem 4.1. If $\widetilde{ac}$ consists of at least three points, the point d will cause infinite trouble.

To justify this claim, assume the contrary: line $L(a,c)$ consists of at least three points. This means that some point d satisfies $[dac]$ or $[adc]$ or $[acd]$; since ac is a simple edge, $[adc]$ is excluded; now symmetry allows us to assume $[acd]$. Among all such points d, we choose one that minimizes $\rho(c,d)$; this property of d guarantees that cd is a simple edge. (Figure 4.2.)

Let us show that

$$bd \text{ is not a simple edge.} \tag{4.2}$$

If (4.2) is false, then (b,c,d) is a simple triangle; $[acd]$ and (4.1) guarantee that this simple triangle is contained in no line. It follows that $\Delta(b,c,d) \geq \Delta(a,b,c)$, which means $\rho(c,d) - \rho(b,d) \geq \rho(a,b) - \rho(a,c)$; since $[acd]$, we conclude $[abd]$; but then (4.1) is contradicted.

In addition, let us observe that

$$\rho(a,b) + \rho(b,d) < \rho(a,d) + \Delta(a,b,c) : \tag{4.3}$$

if (4.3) is false, then $[acd]$ guarantees $\rho(b,d) \geq \rho(b,c) + \rho(c,d)$, and so $[bcd]$; but then (4.1) is contradicted.

By a *path*, we mean any sequence — $(a_1, a_2, \ldots, a_k)$ — of points of V; we define its *length* as

$$\textstyle\sum_{i=1}^{k-1} \rho(a_i, a_{i+1});$$

if the path is denoted P, then we denote its length $\ell(P)$. By a *special path*, we mean a path $(a_1, a_2, \ldots, a_k)$ such that $a_1 = a$, $k \geq 3$, $a_k = d$,

(α) no line contains $\{a_1, a_2, a_3\}$, and

(β) at least one of a_1a_2 and a_2a_3 is not a simple edge.

Note that (a, b, d) is a special path: here, (α) follows from $[acd]$ with (4.1) and (β) follows from (4.2). Now we can choose a shortest special path, $(a_1, a_2, \ldots, a_k)$; let us denote it P. Since (a, b, d) is a special path, (4.3) guarantees that

$$\ell(P) < \rho(a, d) + \Delta(a, b, c). \tag{4.4}$$

To complete the proof of (ii)⇒(iii), we shall distinguish between three cases.

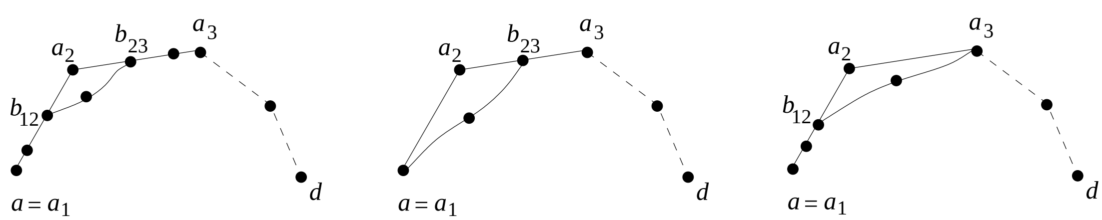


Figure 4.3: Three cases of the special path P in the proof of Theorem 4.1.

CASE 1: *Neither a_1a_2 nor a_2a_3 is a simple edge.*

By assumption of this case, there is a point b_{12} such that $[a_1b_{12}a_2]$. Among all such points b_{12}, we choose one that minimizes $\rho(b_{12}, a_2)$; this property of b_{12} guarantees that $b_{12}a_2$ is a simple edge. Similarly, there is a point b_{23} such that $[a_2b_{23}a_3]$ and such that a_2b_{23} is a simple edge. Note that

$$\text{no line contains } \{b_{12}, a_2, b_{23}\}: \tag{4.5}$$

else $[a_1b_{12}a_2]$, $[a_2b_{23}a_3]$ would guarantee that the same line contains $\{a_1, a_2, a_3\}$, contradicting property (α) of P. Let P' denote the path $(a_1, b_{12}, b_{23}, a_3, \ldots, a_k)$. From $[a_1b_{12}a_2]$ and $[a_2b_{23}a_3]$, we have

$$\ell(P) - \ell(P') = \Delta(b_{12}, a_2, b_{23});$$

(4.5) guarantees that $\Delta(b_{12}, a_2, b_{23}) > 0$, and so P' is shorter than P; now minimality of P implies that P' is not special. Since no line contains $\{a_1, b_{12}, b_{23}\}$ (else $[a_1 b_{12} a_2]$ would guarantee that the same line contains $\{b_{12}, a_2, b_{23}\}$, contradicting (4.5)) and yet P' is not special, both $a_1 b_{12}$ and $b_{12} b_{23}$ are simple edges. Since $b_{12} b_{23}$ is a simple edge, b_{12}, a_2, b_{23} is a simple triangle, and so (4.5) implies $\Delta(b_{12}, a_2, b_{23}) \geq \Delta(a, b, c)$. But then

$$\ell(P) = \ell(P') + \Delta(b_{12}, a_2, b_{23}) \geq \rho(a, d) + \Delta(a, b, c),$$

contradicting (4.4).

Case 2: *$a_1 a_2$ is a simple edge and $a_2 a_3$ is not.*

As in Case 1, there is a point b_{23} such that $[a_2 b_{23} a_3]$ and such that $a_2 b_{23}$ is a simple edge. Note that

$$\text{no line contains } \{a_1, a_2, b_{23}\}: \tag{4.6}$$

else $[a_2 b_{23} a_3]$ would guarantee that the same line contains $\{a_1, a_2, a_3\}$, contradicting property (α) of P. Let P' denote the path $(a_1, b_{23}, a_3, \ldots, a_k)$. From $[a_2 b_{23} a_3]$, we have

$$\ell(P) - \ell(P') = \Delta(a_1, a_2, b_{23});$$

(4.6) guarantees that $\Delta(a_1, a_2, b_{23}) > 0$, and so P' is shorter than P; now minimality of P implies that P' is not special. Since no line contains $\{a_1, b_{23}, a_3\}$ (else $[a_2 b_{23} a_3]$ would guarantee that the same line contains $\{a_1, a_2, b_{23}\}$, contradicting (4.6)) and yet P' is not special, both $a_1 b_{23}$ and $b_{23} a_3$ are simple edges. Since $a_1 b_{23}$ is a simple edge, a_1, a_2, b_{23} is a simple triangle, and so (4.6) implies $\Delta(a_1, a_2, b_{23}) \geq \Delta(a, b, c)$. But then

$$\ell(P) = \ell(P') + \Delta(a_1, a_2, b_{23}) \geq \rho(a, d) + \Delta(a, b, c),$$

contradicting (4.4).

Case 3: *$a_2 a_3$ is a simple edge and $a_1 a_2$ is not.*

As in Case 1, there is a point b_{12} such that $[a_1 b_{12} a_2]$ and such that $b_{12} a_2$ is a simple edge. Note that

$$\text{no line contains } \{b_{12}, a_2, a_3\}: \tag{4.7}$$

else $[a_1b_{12}a_2]$ would guarantee that the same line contains $\{a_1, a_2, a_3\}$, contradicting property (α) of P. Let P' denote the path $(a_1, b_{12}, a_3, \ldots, a_k)$. From $[a_1b_{12}a_2]$, we have

$$\ell(P) - \ell(P') = \Delta(b_{12}, a_2, a_3);$$

(4.7) guarantees that $\Delta(b_{12}, a_2, a_3) > 0$, and so P' is shorter than P; now minimality of P implies that P' is not special. Since no line contains $\{a_1, b_{12}, a_3\}$ (else $[a_1b_{12}a_2]$ would guarantee that the same line contains $\{b_{12}, a_2, a_3\}$, contradicting (4.7)) and yet P' is not special, both a_1b_{12} and $b_{12}a_3$ are simple edges. Since $b_{12}a_3$ is a simple edge, b_{12}, a_2, a_3 is a simple triangle, and so (4.7) implies $\Delta(b_{12}, a_2, a_3) \geq \Delta(a, b, c)$. But then

$$\ell(P) = \ell(P') + \Delta(b_{12}, a_2, a_3) \geq \rho(a, d) + \Delta(a, b, c),$$

contradicting (4.4). This completes the proof of Proposition 4.2, and also the proof of Theorem 4.1. □

In retrospect, there is some similarity between our proof and Kelly's proof to the original Sylvester problem, although the latter one was not what we kept in mind when we were working on this problem. The star in Kelly's proof is the Euclidean distance in the plane and he picks a pair (a, ℓ) such that point a is outside the line ℓ and a is as close to ℓ as possible. In other words, the pair is as *flat* as possible. In a general metric space, after the lines (closure lines) are defined, it is not clear how to define the distance from a point to a line. In our proof, we defined the simple triangles, and picked a pair of point a and edge bc such that (a, b, c) is a simple triangle and we minimize the quantity

$$\rho(a, b) + \rho(a, c) - \rho(b, c).$$

In some sense, this captures the notion of the flatness.

We also remark that a straightforward translation of our proof into the ℓ_2^n space gives yet another proof to the original Sylvester-Gallai theorem.

4.4 An application in block designs

A (v, k, λ) *design* is a hypergraph on a set V of v points with the property that any pair of two points is contained in exactly λ edges with k points in each edge. We say a block design on V is *realizable as a metric space* if there is a metric space (V, ρ) such that for any three points $a, b, c \in V$ we have

$$\{a, b, c\} \in \mathcal{H}(\mathcal{B}(\rho)) \text{ if and only if } \{a, b, c\} \text{ is contained in some edge of the design.}$$

Recall that a finite projective plane of order n is an $(n^2+n+1, n+1, 1)$ design, a finite affine plane of order n is an $(n^2, n, 1)$ design, and a Steiner triple system is a $(v, 3, 1)$ design. If a $(v, k, 1)$ design is realizable as a metric space, then every line in the metric space contains exactly k points. Therefore, immediately following Theorem 4.1 we have

Corollary 4.1. *No $(v, k, 1)$ design with $k \geq 3$ and $v > k$ is realizable as a metric space. In particular, no projective plane of order higher than* 1*, nor any affine plane of order higher than* 2*, nor any Steiner triple system with more than* 3 *points is realizable as a metric space.*

Mario Szegedy, in a discussion on this subject, first asked the question whether there is a short proof for the fact that no projective plane of order higher than 1 is realizable as a metric space. To the best of the author's knowledge and ability, no simple proof is available; even the question is new to the literature. The special case of the Fano plane, i.e., the $(7, 3, 1)$ design, was solved by Chvátal in [12]; the proof was not simple. We do not know a simple straightforward proof to Corollary 4.1.

4.5 Related questions

The closure lines give a successful generalization of the Sylvester-Gallai theorem in any finite metric space. However, we briefly note in this section that, for some of problems related to the classical Sylvester-Gallai theorem in Chapter 1, the analogue questions are not very interesting.

In the classical setting, de Bruijn and Erdős asked whether the number of Gallai lines tends to infinity with the number of points. The affirmative answer was provided and the linear lower bound was obtained (Section 1.2). In metric spaces, Example 3.1 shows that when there is no line contains every point, it is possible that every line is of size bigger than 2. The Sylvester-Chvátal theorem asserts that there exists a closure line of size 2. However, Example 3.3 shows that

Proposition 4.3. *For any $n \geq 6$, there is a metric space (V, ρ) on n points where V is not a closure line, and there are only 7 closure lines in total. 4 of the closure lines are of size 2.*

So, the number of Gallai lines does not tend to infinity. This also denies a generalization of de Bruijn-Erdős theorem (1.8). In the next chapter we present further efforts related to this theorem.

Again, observe Example 3.3. We color x_1 and x_2 red, x_3 and x_4 blue, and half of $\{x_5, \cdots, x_n\}$ red and the other half blue, then we do not have any monochromatic closure lines. If we color x_1, $\cdots$, $x_{[n/2]}$ red and the rest blue, then any bi-chromatic closure line contains $\Theta(n)$ red points as well as $\Theta(n)$ blue points. Thus, the results of Chakerian (Theorem 1.15) and Pach and Pinchasi (Theorem 1.16) are not generalized to the metric spaces and closure lines.

Chapter 5

Problems Related to a de Bruijn - Erdős Theorem

In this chapter we study the analogous problems of the de Bruijn-Erdős theorem (Theorem 1.8) in the setting of lines and closure-lines in hypergraphs and metric spaces. This is a joint work with V. Chvátal. Presently the content of this chapter appears as [10].

5.1 Lines in metric spaces

There are many generalizations of the de Bruijn-Erdős theorem on the number of connecting lines (Theorem 1.8) as we discussed in Section 1.4. In this chapter, we study the generalized problem in metric spaces and, more generally, hypergraphs. As we see in Proposition 4.3 (Example 3.3), when there are no closure lines containing all the points, the number of closure lines $\widetilde{uv}$ can be very small. On the other hand, we do not have examples where the number of lines $\overline{uv}$ is small. We have the following conjecture, which, if it is true, would be a generalization of Theorem 1.8.

Conjecture 5.1. *In any finite metric space (X, ρ), either there is a pair of points x and y such that $\overline{xy} = X$, or there are at least $|X|$ lines.*

5.2 Lines in hypergraphs

Let $m(n, k)$ denote the smallest number of lines in a hypergraph on n vertices where every line consists of at most k vertices. Showing that $m(n, n-1) \geq n$ would give a generalization of de Bruijn-Erdős theorem. However, as we are going to prove, $m(n, n-1)$ grows slower than any power of n.

Lemma 5.1. *If n, ℓ, a are positive integers such that $2 \leq n - \ell \leq a^{\ell}$, then*

$$m(n, n-1) \leq 2^{\ell} + \ell a.$$

Proof. Write $P = \{1, 2, \ldots, \ell\}$ and let A be a set of size a. By assumption, there is a set S of strings of length ℓ over alphabet A such that $|S| = n - \ell$ and such that, for each i in P, some two strings in S differ in their i-th position. For each choice of i in P and x in A, set

$$E_{ix} = \{i\} \cup \{x_1 x_2 \ldots x_\ell \in S : x_i = x\}.$$

Now consider all the lines $\overline{uv}$ in the hypergraph

$$(P \cup S,\ \{P, S\} \cup \{E_{ix} : i \in P, x \in A\}).$$

If $u, v \in P$, then $\overline{uv} = P$. If $u \in P$ and $v \in S$, then $\overline{uv} = E_{ux}$ with x the u-th character in v. If $u, v \in S$, then $\overline{uv} = S \cup P'$ with P' the set of positions in which u and v agree; P' is a proper (and possibly empty) subset of P. So the hypergraph has n vertices, none of its lines consists of all n vertices, and there are at most $1 + \ell a + (2^{\ell} - 1)$ lines. □

Theorem 5.1. *There are positive constants n_0 and c such that*

$$n \geq n_0 \ \Rightarrow \ m(n, n-1) \leq c^{\sqrt{\ln n}}. \tag{5.1}$$

for all n.

Proof. Let $\alpha, \beta, \gamma, \delta$ be arbitrary constants such that

$$0 < \alpha < 1 < \beta < \gamma < 2 < \delta.$$

There is a positive integer ℓ_0 such that

$$\ell \geq \ell_0 \ \Rightarrow \ \alpha\ell < \ell - 1,\ \beta^{\ell} < \gamma^{\ell} - 1,\ \ell\gamma^{\ell} < 2^{\ell},\ 2^{\ell+1} < \delta^{\ell}.$$

We claim that (5.1) holds as long as

$$n \geq n_0 \ \Rightarrow \ n - \left\lceil \sqrt{\frac{\ln n}{\ln \beta}} \right\rceil \geq 2$$

and

$$\ln n_0 \geq \ell_0^2 \ln \beta, \quad \ln c \geq \frac{\ln \delta}{\alpha \sqrt{\ln \beta}}.$$

To justify this claim, consider an arbitrary n such that $n \geq n_0$ and set

$$\ell = \left\lceil \sqrt{\frac{\ln n}{\ln \beta}} \right\rceil, \quad a = \lfloor \gamma^{\ell} \rfloor.$$

Now $\ell \geq \ell_0$, $a > \beta^{\ell}$, and so $\ell \ln a > \ell^2 \ln \beta \geq \ln n$. Lemma 5.1 guarantees that

$$m(n, n-1) \leq 2^{\ell} + \ell a;$$

since

$$\ell < \frac{\ell - 1}{\alpha} < \frac{1}{\alpha}\sqrt{\frac{\ln n}{\ln \beta}}$$

we have

$$2^{\ell} + \ell a < 2^{\ell+1} < \delta^{\ell} < c^{\sqrt{\ln n}}.$$

□

We do not know the order of growth of $m(n, n-1)$; our best lower bound is only logarithmic in n. (We follow the convention of letting lg stand for the logarithm to base 2.)

Theorem 5.2. *$m(n, n-1) \geq \lg n$.*

Proof. Consider an arbitrary hypergraph with n vertices and m lines where no line consists of all n vertices. Let us observe that

$$\begin{gathered}\text{for every two distinct vertices } u \text{ and } v,\\ \text{there is a line which includes } u \text{ and does not include } v:\end{gathered} \tag{5.2}$$

by assumption, some vertex w is not included in line $\overline{uv}$, and so no edge includes all three vertices u, v, w, and so line $\overline{uw}$ includes u and does not include v. For each vertex x, let S_x denote the set of all lines that include x. Property (5.2) guarantees that these n sets are all distinct, and so $n \leq 2^m$. □

Actually, property (5.2) guarantees that the n sets S_x form an *antichain* in the sense that none of them is a subset of another. This observation allows a negligible improvement of the bound in Theorem 5.2: first, the classic result of Sperner ([53]) asserts that an antichain on a ground set of size m has at most

$$\binom{m}{\lfloor m/2 \rfloor}$$

sets; next, by Stirling's formula,

$$\binom{m}{\lfloor m/2 \rfloor} \sim \frac{2^m}{\sqrt{\pi m/2}};$$

finally, if $m = \lg n + \frac{1}{2} \lg \lg n + c$, then

$$\frac{2^m}{\sqrt{\pi m/2}} = (2/\pi)^{1/2} e^c n.$$

It follows that for every positive ε there is an n_0 such that

$$n \geq n_0 \;\Rightarrow\; m(n, n-1) > \lg n + \frac{1}{2} \lg \lg n + \frac{1}{2} \ln \frac{\pi}{2} - \varepsilon.$$

Since $m(n, k)$ is a nonincreasing function of k, Theorem 5.2 guarantees that $m(n, k) \geq \lg n$ whenever $2 \leq k < n$. For small values of k, this bound can be much improved.

Theorem 5.3.

$$m(n, k) \geq \frac{n(n-1)}{k(k-1)}$$

whenever $n \geq k \geq 2$.

Proof. Consider an arbitrary hypergraph with n vertices and m lines where every line consists of at most k vertices. Trivially,

for every two distinct vertices u and v,

there is a line which includes both u and v. (5.3)

Let P denote the set of all pairs $(L, \{u, v\})$ such that L is a line and u, v are two distinct vertices in L. On the one hand, every line includes at most k points, and so

$$|P| \leq m \binom{k}{2}.$$

On the other hand, property (5.3) guarantees that

$$|P| \geq \binom{n}{2}.$$

The lower bound on m follows by comparing the two bounds on $|P|$. □

When the value of k is fixed, the lower bound of Theorem 5.3 is asymptotically optimal:

Theorem 5.4.

$$\lim_{n\to\infty} m(n,k)\cdot\frac{k(k-1)}{n(n-1)} = 1$$

whenever $k \geq 2$*.*

Proof. Theorem 5.3 guarantees that

$$\liminf_{n\to\infty} m(n,k)\cdot\frac{k(k-1)}{n(n-1)} \geq 1\,.$$

In every k-uniform hypergraph (X,H) such that

$$\text{every two edges share at most one vertex,} \tag{5.4}$$

each line is either an edge or a set of two vertices that is not a subset of any edge, and so there are

$$|H| + \left(\binom{|X|}{2} - |H|\binom{k}{2}\right)$$

lines altogether. In particular, with $f(n,k)$ standing for the largest number of edges in a k-uniform hypergraph with n vertices and with property (5.4), we have

$$m(n,k) \leq \binom{n}{2} - f(n,k)\left(\binom{k}{2} - 1\right);$$

Erdős and Hanani [24] proved that

$$\lim_{n\to\infty} f(n,k)\cdot\frac{k(k-1)}{n(n-1)} = 1\,;$$

it follows that

$$\limsup_{n\to\infty} m(n,k)\cdot\frac{k(k-1)}{n(n-1)} \leq 1\,.$$

□

5.3 Closure-lines in metric spaces and hypergraphs

In the last chapter, the closure lines provides a successful generalization of Sylvester-Gallai theorem in metric spaces. However, as we noted in Proposition 4.3, it falls far short of providing a translation of the de Bruijn-Erdős theorem to the metric spaces.

Let $\overline{m}(n,k)$ denote the smallest number of closure-lines in a hypergraph on n vertices where every closure-line consists of at most k vertices. Our proof of Theorem 5.3 with "lines" replaced by "closure-lines" shows that

$$\overline{m}(n,k) \geq \frac{n(n-1)}{k(k-1)} \tag{5.5}$$

whenever $n \geq k \geq 2$; in turn, our proof of Theorem 5.4 with "lines" replaced by "closure-lines" yields the following conclusion.

Theorem 5.5.

$$\lim_{n\to\infty} \overline{m}(n,k) \cdot \frac{k(k-1)}{n(n-1)} = 1$$

whenever $k \geq 2$.

The order of growth of $\overline{m}(n,k)$ is given by its lower bound (5.5):

Theorem 5.6. *There is a positive constant* c *such that*

$$\frac{n(n-1)}{k(k-1)} \;\leq\; \overline{m}(n,k) \;\leq\; c \cdot \frac{n(n-1)}{k(k-1)}$$

whenever $n \geq k \geq 2$.

Proof. For every integer k greater than 1, Theorem 5.5 guarantees the existence of a constant c_k such that

$$\overline{m}(n,k) \leq c_k \cdot \frac{n(n-1)}{k(k-1)} \text{ whenever } n \geq k. \tag{5.6}$$

With c any constant such that

$$c \geq 12 \quad \text{and} \quad c \geq c_k \text{ whenever } 2 \leq k < 12,$$

we propose to show that, for every integer k greater than 1,

$$\overline{m}(n,k) \;\leq\; c \cdot \frac{n(n-1)}{k(k-1)} \text{ whenever } n > k. \tag{5.7}$$

(Trivially, $\overline{m}(n,k) = 1$ whenever $2 \leq n \leq k$.) For this purpose, consider an arbitrary but fixed integer k greater than 1. If $k < 12$, then (5.7) follows from (5.6); if $k \geq 12$, then we will use induction on n to prove that $\overline{m}(n,k) \leq cn^2/k^2$ whenever $n > k$.

Set

$$p = 2\left\lceil \frac{n+1}{k} \right\rceil$$

and note for a future reference that

$$4 \le p < 2\left(\frac{n+1}{k} + 1\right) \le \frac{4n}{k}\,.$$

Take a set X such that $|X| = n$, take a subset X_0 of X such that $|X_0| = p - 1$, and partition $X - X_0$ into pairwise disjoint sets V_i ($1 \le i \le p$) whose sizes are as nearly equal as possible. Since

$$\frac{k}{4} - 1 < \frac{n - (p-1)}{p} \le \frac{k}{2} - 1\,,$$

we have

$$2 \le \min|V_i| \le \max|V_i| \le \frac{k-1}{2}\,.$$

In some hypergraph $(X_0, \mathcal{H}_0)$, every closure-line consists of at most k vertices and there are precisely $\overline{m}(p-1,k)$ distinct closure-lines altogether. A theorem of Behzad, Chartrand, and Cooper, Jr. [3] (the chromatic index of the complete graph K_{2s} is $2s-1$) guarantees the existence of a mapping

$$\phi : \{S :\ S \subset \{1,2,\ldots,p\},\ |S| = 2\} \to X_0$$

with the following property:

for every i in $\{1,2,\ldots,p\}$ and for every w in X_0
there is precisely one j in $\{1,2,\ldots,p\}$ such that $\phi(\{i,j\}) = w$.

Set

$$\begin{aligned} H_1 &= \{\{u,v,w\} :\ \text{there are } i \text{ and } j \text{ with } u \in V_i,\ v \in V_j,\ \phi(\{i,j\}) = w\},\\ H_2 &= \{S :\ |S| = 3 \text{ and there is an } i \text{ with } S \subseteq V_i\}, \end{aligned}$$

and $H = H_0 \cup H_1 \cup H_2$. Since closure-lines in hypergraph (X, H) are

- all the closure-lines in hypergraph (X_0, H_0),

- all the sets $V_i \cup V_j \cup \{\phi(\{i,j\})\}$ such that $1 \le i < j \le p$, and

- all the sets V_i such that $1 \le i \le p$,

we have

$$\overline{m}(n,k) \le \overline{m}(p-1,k) + \binom{p}{2} + p\,.$$

If $p-1 > k$, then (as $p-1 < n/3$) the induction hypothesis guarantees that

$$\overline{m}(p-1,k) \le c\left(\frac{p-1}{k}\right)^2 < \frac{c}{9}\left(\frac{n}{k}\right)^2;$$

if $p-1 \le k$, then

$$\overline{m}(p-1,k) = 1 < \frac{c}{9}\left(\frac{n}{k}\right)^2;$$

finally,

$$\binom{p}{2} + p = \binom{p+1}{2} < 10\left(\frac{n}{k}\right)^2.$$

We conclude that

$$\overline{m}(n,k) \le \frac{c}{9}\left(\frac{n}{k}\right)^2 + 10\left(\frac{n}{k}\right)^2 \le c \cdot \frac{n^2}{k^2}\,.$$

□

Part II

Visibility with Colors

Chapter 6

Visibility with Colors

Our study of visibility questions starts with the following mathematical puzzle.

> We have two armies, A and B, of silkworms in the plane in general position. Prove that there always exists one silkworm x who sees the tail of another from the other army.
> Furthermore, if army A is heading one direction and army B is heading another, then we can always find such x in both army.

In another setting, we have creatures in the form of segments in the flatland. Each creature has its one end as the *head* and on the other end it has a *gun*. We will be interested in the visibility question of which guns can shoot which heads without penetrating the body of any other creature.

The key ingredient in our picture is the colors on the end points of the segments — red and blue, head and tail, or head and gun.

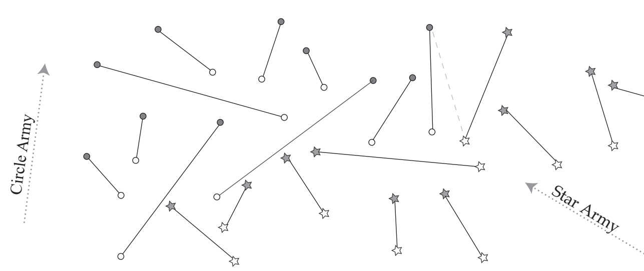


Figure 6.1: Two directed armies of silkworms. No army can hide their tails from the enemy's heads.

6.1 A concise history of the problem

The visibility problem in the plane is among one of the oldest genre in discrete geometry. In this kind of problems, a set of segments in the plane are considered as obstacles. Two points are visible to each other if the straight line joining them does not pass through any of the obstacles. The visible region from a single point is a (possibly unbounded) star-shaped polygon.

In a conversation that happened in 1973, V. Klee posed to V. Chvátal the Art Gallery Problem. The problem was quickly solved by Chvátal ([11]) and became the famous Art Gallery theorem. Since then there has been extensive research on the problems related to the Art Gallery theorem. (See the survey by Shermer [51].) We note that a short proof to the theorem was discovered by Fisk ([27]).

Traditional art galleries are posed as a simple polygon. The variation on an arrangement of disjoint segments is also studied, with a focus on guarding the whole plane or guarding the entire set of the segments (See [51]).

Our investigation of visibility questions with colors starts when the author received the invitation for a submission to the *Akiyama-Chvátal Festschrift*, a special edition of *Graphs and Combinatorics*. In search of a good topic, the author "invented" the problem of visibility questions with colors, and, in particular, the puzzle stated in the beginning of this chapter. We believed that the topic is new. The work resulted in a manuscript which is the content of this chapter. In particular, we give positive answers to both parts of the puzzle. Part of the origin of the problem is also stimulated from some work related to the Ramsey type results by G. Károlyi, J. Pach, G. Tóth, and P. Valtr ([37]).

Later we discovered that the colored visibility is a studied problem. In particular, our main result, which gives the positive answer to the first part of the puzzle in the beginning of this chapter, is proved in 2004 by two groups of researchers. Their work, which proves our Theorem 6.8, appeared as [35].

Since the conference publication of [35], there are several new results. The strongest result is

Theorem 6.1. *([34]) Given a vertex colored (disconnected) planar straight line graph, with no singleton components, one can add edges such that we get a vertex colored connected planar straight line graph and every vertex is incident to at most two new* *edges.*

In this chapter we present our independent work on the topic. Although the proof of the main theorem appeared as a conference paper, we provide our detailed solution. Our interest is slightly different from those in the literature. Some focuses of our study are the graph theoretical aspects of the problem.

6.2 Definitions, Questions, Examples

6.2.1 Basic settings

Let $\mathcal{L}$ be a set of segments in the plane. We say that $\mathcal{L}$ is in *general position* if no 3 end points of the segments in it are collinear. Two distinct points u and v in the plane *see* each other if the interior of the segment uv does not intersect any segment in $\mathcal{L}$. We will be interested in the visibility problems, especially the visibility between end points of the segments.

A remark on the definition is in order. We have the visibility between u and v if no segments in $\mathcal{L}$ intersects the interior of the segment uv. Thus, if u (or v) is an end point of one segment in $\mathcal{L}$, u itself, being on that segment, is not considered as an obstacle for u to see other points. On the other hand, if uv is a segment in $\mathcal{L}$, then they do not see each other.

We call a set of segments 2-colored if each segment has one end point colored red and the other end point blue. In most situations, we study the visibility relation between points of different colors.

For a set of segments $\mathcal{L}$, $P(\mathcal{L})$ is the set of end points of the segments in $\mathcal{L}$. Unless otherwise specified, $n = |\mathcal{L}| > 1$, $\mathcal{L} = \{a_1, \ldots, a_n\}$, and we assume that the $2n$ points in $P(\mathcal{L})$ are distinct and in general position.

As we will discuss, we can not say much about the visibility in a set of segments that are not disjoint. In most situations we focus on a set of 2-colored disjoint segments in the plane in general position. For convenience, we call such $\mathcal{L}$ a *set of needles*. For a set of needles, we usually write a_i as R_iB_i, where R_i is the red end and B_i is the blue end.

We will call a set $\mathcal{L}$ of needles directed along $\overrightarrow{x}$ if for each i, R_i appears before B_i on the direction $\overrightarrow{x}$. i.e., the inner product of the vectors $\overrightarrow{R_iB_i}$ and $\overrightarrow{x}$ are positive.

We may simply create many visibility questions.

Question 6.1. *(a) Is it true that every point in the plane sees some colored point?*

(b) Is it true that every point in the plane sees points of both color?

(c) Is it true that every colored point sees some other colored points?

(d) Is it true that every colored point sees some point of a different color?

(e) Is it true that at least one colored point sees some point of a different color?

It is not hard to see, as we will show in the beginning of the next subsection, that the answer could be *no* to all these questions if the segments in $\mathcal{L}$ are not disjoint. On the contrast, if $\mathcal{L}$ is a set of disjoint segments, we have much better lower bound on the amount of visibility. For a set of needles, the answers to (a), (c), and (e) are *yes*.

We construct a directed graph G_L, which is called the *visibility graph*. It is a graph on $[n] = \{1, 2, \ldots, n\}$ where there is a directed edge $i \to j$ iff R_i sees B_j. We also construct an undirected graph $G_L^{(u)}$ which is a simple graph gotten from G_L by omitting the directions. We may create more questions, some of which are just the same question as we asked above, reformulated in graph theoretic terms.

Question 6.2. *Does the visibility graph has isolated points?*

Question 6.3. *What is the lower bound on the number of edges in the visibility graph?*

Question 6.4. *Determine the upper bound and lower bound on the length of the longest path in the visibility graph, in terms of functions of n.*

6.2.2 A gallery of examples

In Figure 6.2(a) we show an example where there is an end point does not see any other end point at all. This gives the negative answers to questions 6.1 (a) to (d) in the case where L is not necessary disjoint, even when all segments in $\mathcal{L}$ meet a common line.

Figure 6.2(b) shows an example where the visibility graph G_L has only two edges and $G_L^{(u)}$ has only one edge. It is easy to see that, when the convex hull of the red points and the convex hull of the blue points are disjoint, G_L has at least 2 edges and $G_L^{(u)}$ has at least 1 edge.

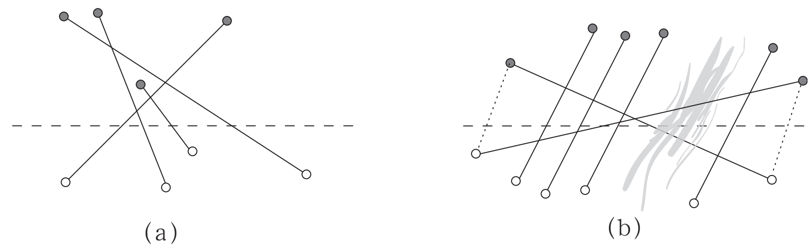


Figure 6.2: (a) There is an end point does not see any other end point. (b) The visibility graph of a set of two colored segments has only two edges, as dotted.

In the general case, if the segments are allowed to intersect and they are not restricted to be intersecting a common line, the visibility graph may be empty, as shown in Figure 6.3. During a conversation with Mario Szegedy we got this graph. This gives the negative answer to Question 6.3 (e).

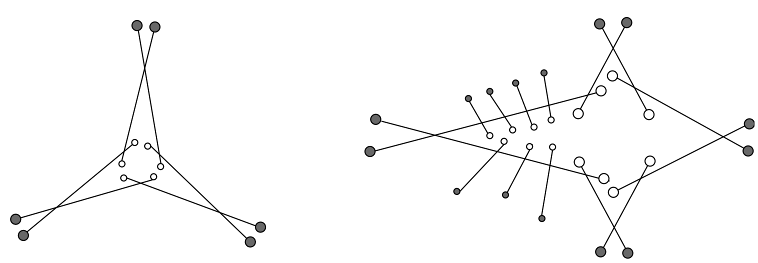


Figure 6.3: Example(s) where the visibility graph for a set of two colored segments is empty.

For disjoint segments, it is easy to show a linear lower bound on the number of edges of the visibility graph. Figure 6.4 shows an example where there are n edges in G_L. We will prove that this is the exact lower bound. (Proposition 6.3).

For disjoint segments, we will prove that every point has either positive in-degree or positive out-degree in G_L. But there may exist points with in-degree 0 or out-degree 0, as shown in Figure 6.5. In particular, G_L is not necessarily strongly connected.

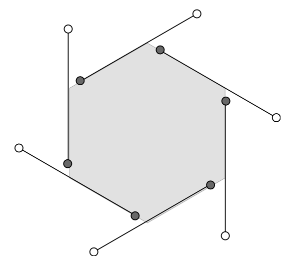


Figure 6.4: A set of disjoint segments where the visibility graph is a cycle.

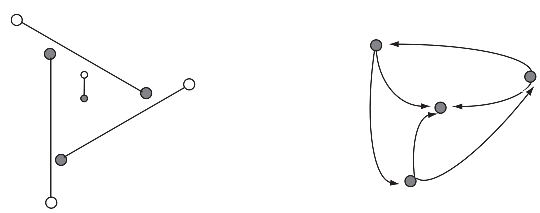


Figure 6.5: A visibility graph that is not strongly connected.

6.2.3 More definitions on visibility graphs

There are may ways to define a relation of visibility between segments. We have

Definition 6.1. *Let $\mathcal{L}$ be a set of n 2-colored segments. We define the following types of visibility graphs on $[n]$.*

- $G_L^{(g)}$ *(g for general) is the undirected graph where $i \sim j$ iff some point on a_i sees some point on a_j.*
- $G_L^{(m)}$ *(m for monochromatic) is the undirected graph where $i \sim j$ iff some end point of a_i sees some end point of a_j.*
- $G_L^{(u)}$ *(u for undirected) is the undirected graph where $i \sim j$ iff some end point of a_i sees some end point of a_j of a different color. i.e., R_i sees B_j or B_i sees R_j.*
- $G_L^{(rb)}$ *is the directed graph where $i \to j$ iff R_i sees B_j. $G_L^{(rr)}$, $G_L^{(br)}$, and $G_L^{(bb)}$ are defined similarly. Our most interested visibility graph is $G_L = G_L^{(rb)}$.*

It is clear that for any $\mathcal{L}$, $G_L^{(u)}$ is a subgraph of $G_L^{(m)}$, which is a subgraph of $G_L^{(g)}$. $G_L^{(br)}$ is the graph gotten by reverse the edges in G_L. $G_L^{(u)}$ is the union of G_L and $G_L^{(br)}$, it is the simple undirected graph gotten by omitting the directions on edges of G_L. Both $G_L^{(rr)}$ and $G_L^{(bb)}$ are undirected, and

$$G_L^{(m)} = G_L^{(rr)} \cup G_L^{(bb)} \cup G_L^{(u)} = G_L^{(rr)} \cup G_L^{(bb)} \cup G_L^{(rb)} \cup G_L^{(br)}.$$

In the rest of this chapter, we will mostly focus on the study of G_L and $G_L^{(u)}$. Here we briefly discuss the connectivity of the other two types.

Proposition 6.1. *$G_L^{(g)}$ is connected for any $\mathcal{L}$.*

Proof. Let A and B be a partition of $\mathcal{L}$. Consider the closest pair of points between A and B. □

$G_L^{(m)}$ is not necessary connected. Indeed, in Figure 6.2 (a) we have a gadget of 3 intersecting segments that enclose a triangular region. Any segment we put inside the region cannot see any end point from the outside world. We prove $G_L^{(m)}$ is connected when L is a set of disjoint segments.

Proposition 6.2. *$G_L^{(m)}$ is connected if L is a set of disjoint segments.*

Proof. In the beginning, let R_i be very close to B_i so that $G_L^{(m)}$ is the complete graph. Now start grow R_i towards its destination. In the moving, the graph changes only when 3 end points become collinear. If a, b and c are collinear, and a and c are blocked by the edge with end point b afterwards (so the edge ac is removed), we still have at least 2 edges, ab and bc, after the graph chages. So a and c are still connected after the event. □

We note that the last proposition hold even when ℓ is not in general position. The proofs are the same, except in the proof of Proposition 6.2, we do not have the complete graph in the beginning, but a connected graph.

Also notice that the condition that the segments do not intersect is important. Otherwise we have one more type of event that can change the visibility graph, namely, the point R_i penetrates some segment. As we know, this kind of event may disconnect the graph $G_L^{(m)}$.

6.3 Visibility in a set of needles

From now on we study the visibility questions for a set of needles. In this section we consider the general case where the viewpoint is any point in the plane. For convenience, usually we have some general position assumption, i.e., if the view point is not one of the end points in $\mathcal{L}$, then it is not collinear with any 2 points in $P(\mathcal{L})$.

For a set of n disjoint segments, the visibility graph can be the complete directed graph with $n(n-1)$ edges, as seen in the example where we have n red points in general position, and each blue point is very close to its corresponding red point.

For the lower bound, Figure 6.4 shows an example where G_L has n edges. We prove that this is the exact lower bound. The star in the picture is the triangulation of a point set with some noncrossing segments given.

Proposition 6.3. *Let $\mathcal{L}$ be a set of $n > 1$ needles. Then each segment has at least one end point that sees another point of different color. i.e., $G_L^{(u)}$ has no isolated vertices. Moreover, G_L has at least n edges; the lower bound is tight for all $n > 1$.*

Proof. It is well known that (See [17]), given any point set S with m points, not all collinear, any set of noncrossing edges connecting points in S can be completed to a triangulation of S. Any triangulation of a point set with m points with k points on its convex hull has $2m - 2 - k$ triangles and $3m - 3 - k$ edges.

In our setting, $\mathcal{L}$ is a set of n disjoint segments. Let $S = P(\mathcal{L})$. We have triangulations of S that complete $\mathcal{L}$. For any such triangulation, each edge $R_iB_i \in \mathcal{L}$ is contained in at least one triangle, without loss of generality, $R_iB_iR_j$ for some $j \neq i$. Thus, either R_i sees some blue point, or B_i sees some red point.

Now we prove that G_L has at least n edges by induction on n. It is easy to see that when $n = 2$, the visibility graph is always the 2-cycle.

For $n \geq 2$. If there is a segment a_{i_0} which is also an edge of the convex hull of $P(\mathcal{L})$, i.e., all the other points are on the same side of the line a_{i_0}, then the presence of this edge does not affect any visibility relation in the other segments. By induction, the subgraph of G_L induced by $\mathcal{L} \setminus \{a_{i_0}\}$ has at least $n-1$ edges. And since i_0 in the visibility graph also has some positive in-degree or out-degree, so there are at least n edges.

If no such a_{i_0} exists, every edge R_iB_i is contained in at least two triangles in any triangulation. Thus the summation of all the in-degrees and out-degrees is at least $2n$, and there are at least n edges. □

It is easy to see, from the picture of triangulations, that any point in the plane sees at least 2 colored points in a set of needles. As to the question of whether every point in the plane sees points of both colors, the answer is negative even for disjoint segments as we have seen in Figure 6.4 and 6.5. Notice that, in both examples, the point cannot see a certain color because it is enclosed by segments around on every direction. Lemma 6.1 and its corollaries show that this is basically the only situation when a point does not see end points of a certain color.

Let $\mathcal{L}$ be a set of disjoint segments in the plane. For each point x in the plane and any direction, we define the *view* of x at that direction. Formally,

Definition 6.2. *Let $\mathcal{L}$ be a set of disjoint segments in the plane and x is a point in the plane. Define*

$$\mathcal{L}_x = \begin{cases} \mathcal{L} & \text{if } x \text{ does not belong any segment in } \mathcal{L}, \\ \mathcal{L} \setminus a & \text{if } x \in a \in \mathcal{L}. \end{cases}$$

We further assume that x is not collinear with any two end points in $\mathcal{L}_x$. Let S^1 be the unit circle centered at the origin. We identify each point v on S^1 as the (vector) direction from the origin to v. For each point $x \in \mathbb{R}^2$ and each direction $v \in S^1$, denote $R_{x,v}$ the ray starting from x with direction v. The view of x *is a function $V : S^1 \to \mathcal{L}_x \cup \{\infty\}$, where*

$$V_x(d) = \begin{cases} a & \text{if the first intersection of } R_{x,v} \text{ and } \mathcal{L}_x \text{ is a point on } a, \\ \infty & \text{if the ray } R_{x,v} \text{ does not intersect any segment in } \mathcal{L}_x. \end{cases}$$

Note that the view function is well defined, since the segments are disjoint. It is easy to see that, S^1 is partitioned by V_x into several intervals. Since the segments in $\mathcal{L}_x$ are disjoint, at the end points of these intervals on S^1, the ray $R_{x,v}$ meets the end point of some segments in $\mathcal{L}_x$.

Lemma 6.1. *Let $\mathcal{L}$ be a set of needles in the plane and x is a point in the plane which is not collinear with any two end points in $\mathcal{L}_x$. If x does not see any red point, then S^1 is partitioned by V_x into finitely many half closed intervals, where the points in the same interval are mapped to the same element in $\mathcal{L}_x$.*

Proof. Suppose x only sees blue points. Pick any direction v where x sees a blue point B_i. Without loss of generality, assume the direction $\overrightarrow{xR_i}$ is clockwise to the direction $\overrightarrow{xB_i}$. We start from v and rotate the view direction clockwise. The event where the V_x value changes is when we see the other end of the segment a_i, or we start to see another segment $a_{i'}$. Since x does not see any blue point. The only possible event is that x we see another blue point $B_{i'}$ before we leave a_i. And since we came from the clockwise direction and first meet $B_{i'}$, we know that the direction $\overrightarrow{xR_{i'}}$ is clockwise to the direction $\overrightarrow{xB_{i'}}$.

We continue this procedure. Since there are finitely many possible events, finally we get back to the starting direction and see B_i again. □

The following corollary is immediate.

Corollary 6.1. *$\mathcal{L}$ be a set of needles in the plane and x is a point in the plane which is not collinear with any two end points in $\mathcal{L}_x$. If there is a ray from x that does not intersect any segment in $\mathcal{L}_x$, then x sees both red and blue end points in $\mathcal{L}_x$.*

Corollary 6.1 immediately gives the fact that any point (not collinear with any 2 end points in $\mathcal{L}$) outside the convex hull of $\mathcal{L}$ can see end points with both colors. We are going to prove something more general.

Lemma 6.2. *Let $\mathcal{L}$ be a set of needles. Let x be a point that is not collinear with any 2 end points in $\mathcal{L}$. Assume there is a line ℓ passing through x, dividing the plane into two open half spaces ℓ^+ and ℓ^-, such that there is at least one blue point in ℓ^+, and* *any segment in $\mathcal{L}$ that intersects ℓ has its blue end in ℓ^+, then x sees at least one blue point in ℓ^+.*

Proof. We may throw away all the segments that are completely contained in ℓ^-, since they do not affect the visibility from x to points in ℓ^+.

We may also assume that all the red points are in ℓ^+. To see this, let R_iB_i be an edge with $R_i \in \ell^- \cup \ell$. We connect x with all the blue points in ℓ^+. Some of the connecting segments intersect R_iB_i, but all the intersections are in ℓ^+. We may change R_i to a point on the same segment in ℓ^+ but very close to ℓ. By doing this we do not change the visibility of x to any blue point in ℓ^+. Now the statement follows from Corollary 6.1. □

Corollary 6.2. *Let $\mathcal{L}$ be a set of needles. Let x be a point outside the convex hull of all the blue points and x is not collinear with any 2 end points in $\mathcal{L}$, then x sees at least one blue point. The same is true for the red points.*

While studying the Ramsey type results of geometric graphs ([37]), we formed the following puzzle. This was the starting point of our research on this topic.

> If all the segments meet a line, is it true that any end point sees at least one end point on the opposite side of the line?

As it turns out, the important condition is not that the segments meet the same line, but the fact that every segment has its red point higher than its blue point. The following is a corollary of Lemma 6.2. We note that it is also a special case of Theorem 6.2.

Corollary 6.3. *Let $\mathcal{L}$ be a set of $n > 1$ needles where each R_i is higher than B_i. Then every end point sees some other end point of a different color. In terms of the visibility graph, for this kind of $\mathcal{L}$, each vertex of G_L has non-zero in-degree and out-degree.*

Proof. By symmetry, we just need to prove the statement for R_n. If R_n is below all the other blue points, it is outside the convex hull of $\mathcal{L} \setminus \{a_i\}$ and we apply Corollary 6.2. Otherwise we apply Lemma 6.2. □

6.4 The timberland

In this section we give a positive answer to the second half of the puzzle we posed in the beginning of this chapter. We will go through some conceptual changes, see the nice picture of timberlands, and grow the woods continuously with the invariant of visibility.

Theorem 6.2. *Let $\mathcal{L} = \{a_i : 1 \leq i \leq n\}$ be a set of needles. Let A and B be a partition of $[n]$ such that $\{a_i : i \in A\}$ is directed along some $\overrightarrow{x}$ and $\{a_i : i \in B\}$ is directed along some $\overrightarrow{y}$. Then there exist a blue point in A and a blue point in B that see each other.*

By reversing $\overrightarrow{x}$, $\overrightarrow{y}$, or both, we have red-blue, blue-red, and red-red visibility relations as well. In terms of the visibility graph, the following theorem is immediate form Theorem 6.2.

Theorem 6.3. *Let $\mathcal{L}$ be a set of needles directed along some direction $\overrightarrow{x}$, then G_L, $G_L^{(br)}$, $G_L^{(rr)}$, $G_L^{(bb)}$ are all strongly connected.*

Our approach is based on the following obvious observation.

Lemma 6.3. *In general, we have a set $\mathcal{L}$ of segments in the plane. Let Q be a subset of $P(\mathcal{L})$. Suppose uv is a segment in $\mathcal{L}$ such that $v \notin Q$. Then, if we extend uv by moving v further away from u, the visibility between the points in Q is not increased.*

We define somehow the *worse case* of a segment set when we consider the visibility between points in a set Q.

Definition 6.3. *Let $\mathcal{L} = \{a_1, \ldots, a_n\}$ be a set of segments in the plane. Let $Q \subseteq P(\mathcal{L})$. We call $\mathcal{T}$* a timberland *of $\mathcal{L}$ with respect to Q if $\mathcal{T} = \{C, b_1, \ldots, b_n\}$, where*

(1) C is a convex polygon that contains the convex hull of P in its interior;

(2) Given $\{b_1, \ldots, b_{i-1}\}$, b_i is defined as follows. For any end point $v \notin Q$ of a_i, we extend a_i by moving v further away from the other end until we hit one of $\{C, b_1, \ldots, b_{i-1}, a_{i+1}, \ldots, a_n\}$.

Note that there might be many timberlands for $\mathcal{L}$ with respect to Q, since the order in which we list the segments in $\mathcal{L}$ matters, as well as the choice of C. The purpose of using the polygon C is merely to bound our objects to be compact. In Most cases we pick C as a rectangular bounding box of the whole picture. The following statement follows Lemma 6.3 immediately.

Lemma 6.4. *Let $\mathcal{T}$ be a timberland of $\mathcal{L}$ with respect to Q, and $u, v \in Q$. If u sees v in $\mathcal{T}$, then they see each other in $\mathcal{L}$.*

We often study the case where $\mathcal{L}$ is a set of disjoint segments in general position and Q meets any segment in $\mathcal{L}$ exactly once. In this situation, a timberland is the bounding polygon C and a set of segments with no interior intersection. any segment $uv \in \mathcal{L}$ where $v \in Q$ is extended to a segment $u'v \in \mathcal{T}$ where v does not lie on any other elements in $\mathcal{T}$, and u' lies on the interior of another element in $\mathcal{L}$.

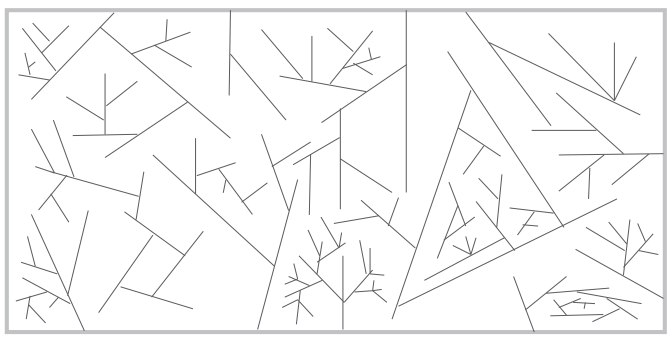


Figure 6.6: A well formed timberland.

Definition 6.4. *We call a timberland $\mathcal{T}$ of $\mathcal{L}$ with respect to Q* well formed *if $\mathcal{L}$ is a set of disjoint segments in general position, and each segment in $\mathcal{L}$ has exactly one end point in Q. In this case, each segment s in $\mathcal{T}$ has one end point $v \in Q$ which does not lie on any other element in $\mathcal{L}$, and the other end point u which lies in the interior of another element $s' \in \mathcal{T}$. We call v the* tip *of s, u the* root *of s, and we say s is rooted on s'.*

Definition 6.5. *Let $\mathcal{T}$ be a well formed timberland and uv be a segment in $\mathcal{T}$ with root u. Let $\overrightarrow{x}$ be a direction. If u is before v on the direction $\overrightarrow{x}$, we call uv $\overrightarrow{x}$-ward, and we call v an $\overrightarrow{x}$-ward tip.*

Now let us come back to the setting of Theorem 6.2. Here we set

$$Q = \{B_i : i \in A\} \cup \{B_j : j \in B\}.$$

Let $\mathcal{T}$ be a well formed timberland of $\mathcal{L}$ with respect to Q. Notice that for $i \in A$, B_i is an $\overrightarrow{x}$-ward tip in $\mathcal{L}$; and for $j \in B$, B_j is a $\overrightarrow{y}$-ward tip. Thus, Theorem 6.2 is implied by the following

Theorem 6.4. *Let $\mathcal{T}$ be a well formed timberland. Let $\overrightarrow{x}$ and $\overrightarrow{y}$ be 2 (not necessary distinct) directions. Let A and B be a partition of the tips in $\mathcal{T}$. If every tip in A is $\overrightarrow{x}$-ward and every tip in B is $\overrightarrow{y}$-ward, then there exist $a \in A$ and $b \in B$ such that a sees b in $\mathcal{T}$.*

Proof. We use an induction on the number of segments in $\mathcal{T}$, i.e., $n = |A| + |B|$. We construct a directed graph G on $[n]$ such that $i \to j$ if the i-th segment is rooted on the j-th segment. If the i-th segment is rooted on the bounding polygon C, then the out-degree of i is 0, otherwise it is 1.

Base Case 1. $n = 2$. In this case $|A| = |B| = 1$, and no segment can block the visibility of the two tips.

Base Case 2. Each point in G has positive in-degree. In this case, every vertex in G has in-degree 1 and out-degree 1. G is a collection of cycles. Consider the convex hull of all the segments, i.e., the convex hull of all the end points of the segments. All the extreme points of the convex hull must be a tip, since any root lies on the interior of another segment and no root is on the bounding box. In particular, along the direction $\overrightarrow{x}$ the first point on the convex hull (if there are 2, pick any one) is not $\overrightarrow{x}$-wards, and the first point on the convex hull along $\overrightarrow{y}$ direction is not $\overrightarrow{y}$-wards.

Therefore, we can find adjacent extreme points on the convex hull, say, v_1 and v_2, such that v_1 is $\overrightarrow{x}$-wards and v_2 is $\overrightarrow{y}$-wards. Since the original segments are in general position, v_1 sees v_2 in the timberland.

Inductive Step. Let $\mathcal{D}$ be the set of segments that does not contain any root of other segments. If $|\mathcal{D}| \geq 2$, note that $n \geq 3$, we can pick $s \in \mathcal{D}$ such that the tip of s is not the single $\overrightarrow{x}$-ward or single $\overrightarrow{y}$-ward tip in $\mathcal{T}$. If $|\mathcal{D}| = 1$, we just pick s to be the single element in $\mathcal{D}$. If we delete s from $\mathcal{L}$, it is still a well formed timberland. Now we grow s back from the root u, and moving the tip, say, v' towards v. Without loss of generality, we assume that v is an $\overrightarrow{x}$-ward tip.

We claim that there is an $\overrightarrow{x}$-ward tip sees a $\overrightarrow{y}$-ward tip in the beginning. By our pick of s, if $|\mathcal{D}| \geq 2$, then the claim holds by induction, since v' is close enough to u so it dos not block the visibility.

In the other case s is the only $\overrightarrow{x}$-ward tip and $|\mathcal{D}| = 1$. After we delete s, the picture $\mathcal{T} \setminus \{s\}$ is a well formed timberland with only $\overrightarrow{y}$-ward tips. There must be a segment s' that is not rooted by any other in $\mathcal{T} \setminus \{s\}$. Otherwise, similar to the base case 2, along $\overrightarrow{y}$, the first point on the convex hull of $\mathcal{L} \setminus \{s\}$ is not $\overrightarrow{y}$-ward. Now, since $s' \notin \mathcal{D}$, s must have its root u on s'. And no other root is on s'. In the beginning, v' is very close to u so that v' sees the tip of s'.

Consider the moments where a pair of visible $\overrightarrow{x}$-ward and $\overrightarrow{y}$-ward tips a and b becomes not visible. We have finitely many such moments, namely, when v' is collinear with two end points in $\mathcal{L}$. Note that our segment uv' will not go cross any other segment in $\mathcal{T}$. And any moment will be eventually crossed when we finally arrive at v, since $\mathcal{L}$ is in general position.

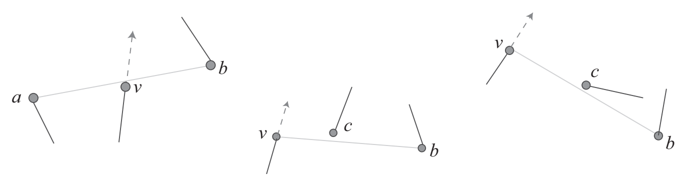


Figure 6.7: The inductive step in the proof of Theorem 6.4.

If $a \neq v'$, it must be the moment where v' crosses the segment connecting a and b. Before the moment, there is a narrow polygon W containing the segment ab such that the interior of W does not touch any segment in $\mathcal{T}$. So, after v' crosses the segment ab, v' sees b.

If $a = v'$, there is another end point c comes in between v' and b at this moment. c must be a tip, otherwise the edge where c is rooted blocks the visibility of v' and b even before the moment. Similar to the previous case, we have 2 visible pairs $\{v', c\}$ and $\{c, b\}$. One of these is an $\overrightarrow{x}$-ward and $\overrightarrow{y}$-ward pair. $\square$

6.5 The connectivity of the visibility

This section is dedicated to the proof of the following theorem.

Theorem 6.5. *Let $\mathcal{L}$ be a set of needles. Let A and B be a partition of $[n]$. Then either there is a blue point in A sees a red point in B, or there is a red point in A sees a blue point in B.*

Equivalently, the graph $G_L^{(u)}$ is connected for any set of needles $\mathcal{L}$.

To simplify the discussion, we use the symbols $\bullet$, $\circ$, $\blacktriangle$, and $\triangle$ to denote a red point in A, a blue point in A, a red point in B, and a blue point in B, respectively. Our goal is to find a visible $\bullet\ \triangle$ pair or a visible $\blacktriangle\ \circ$ pair. We extend $\mathcal{L}$ to a triangulation of $P(\mathcal{L})$, there is an end point in A sees and end point in B. If they are of different colors then we are done.

In the rest of the discussion, we assume the theorem is not true. Without loss of generality, we assume there is a visible $\bullet\ \blacktriangle$ pair, say, R_iR_j, where $i \in A$ and $j \in B$, and we may assume that, from viewpoint of R_i, the direction $\overrightarrow{R_iB_j}$ lies clockwise to the direction $\overrightarrow{R_iR_j}$. This is our starting point. We start from R_i, look at R_j, and rotate our view clockwise, i.e., we observe the function V_{R_i}, start from the direction $\overrightarrow{R_iR_j}$. We keep seeing the segment a_i until the event happens when we see another end point x. We extend R_ix meet the segment a_j at y (where it is possible $x = y$). We know that the interior of the triangle R_iR_jy is disjoint from the segments in $\mathcal{L}$. So, both R_i and R_j see x. Thus, being an end point, x can only be $\blacktriangle$ or $\bullet$. If x is $\blacktriangle$, we keep rotating clockwise and find the next event. We can not always get x being $\blacktriangle$ and turn around back to the starting direction, since when we pass the direction R_iB_i, the last $\blacktriangle$ will see the $\circ$ point B_i.

Thus, let $r_0 := R_i$ and r_1 be the new $\bullet$ point. We may conclude that r_1 is the first $\bullet$ point r_0 sees in clockwise order from the direction R_iR_j. We are going to build a polygonal line, called the *gray line*, with r_0r_1 being its first segment. The following general step is very similar to what we did above.

Given $r_{i-1}r_i$, we switch our view point to r_i, start from the direction $\overrightarrow{r_{i-1}r_i}$ and keep rotating clockwise. Let R_j be the last $\blacktriangle$ point, i.e., $V_{r_i}(\overrightarrow{r_{i-1}r_i}) = j$ and $\overrightarrow{r_iB_j}$ is

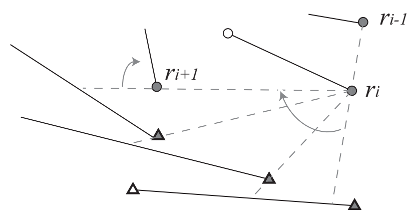


Figure 6.8: A general step in constructing the polygonal line in the proof of Theorem 6.5. The interior of any triangle in this picture is disjoint from all the segments in $\mathcal{L}$.

clockwise to $\overrightarrow{r_{i-1}r_i}$. The function value of V_{r_i} changes when there is a ▲ or • enters the picture. If it is a ▲ , we keep rotating clockwise. There are finitely many events before we reach the direction r_ib_i, where b_i is the other end point of the segment that contains r_i. Since there is no visible ○ ▲ pair, we must see a • before that moment. We set this point as r_{i+1}, and add r_ir_{i+1} to the gray line.

Fact 6.1. *r_{i+1} is completely determined by r_{i-1} and r_i. The direction from $\overrightarrow{r_{i-1}r_i}$ to $\overrightarrow{r_ir_{i+1}}$ is always a right turn. From r_i, starting from the direction $\overrightarrow{r_{i-1}r_i}$ and rotating clockwise, r_{i+1} is the first • point we see. Furthermore, in the process r_i does not see any blue point.*

The following lemma will be used several times in our proof.

Lemma 6.5. *Let c_1, …, c_k be a part of the gray line, such that their x-coordinates are in increasing order, so the chain $c_1 \ldots c_k$ is an upper convex chain. Let c_0 be the previous point in the gray line such that its x-coordinate is no bigger than that of c_k. Let l be a vertical line between the x-coordinates of c_{k-1} and c_k, and two points u and v on l such that u is above the segment $c_{k-1}c_k$ and v is on or below the segment $c_{k-1}c_k$.*

By our choice of the polygonal line, there exists a point c_t where $1 \le t \le k-1$ such that uc_t is the tangent from u to the upper convex chain.

Assume c_t does not see u, then u does not see v.

Proof. (See Figure 6.9.) Accompanied with the view function V_x, we also define a distance function D_x. $D_x(d)$ is the distance from x to the first intersection of the ray $R_{x,d}$ and $\mathcal{L}_x$. If on an interval V_x is never ∞, then D_x is well defined. And if we further

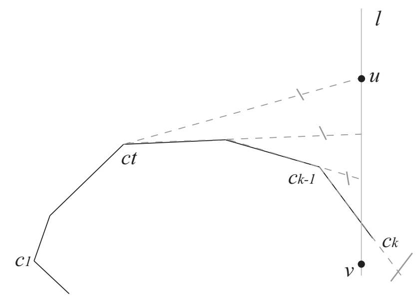


Figure 6.9: The proof of Lemma 6.5

assume each time the V_x changes when a new segment enters the picture rather than we leave an old segment, then D_x is continuous, except at finitely many points where it jumps to a smaller value.

When we start the rotation from c_t, in the beginning the direction is $\overrightarrow{c_{t-1}c_t}$ and at the end the direction is $\overrightarrow{c_tc_{t+1}}$. The direction of $\overrightarrow{c_tu}$ is in the middle.

For the interval of directions from $\overrightarrow{c_tu}$ to c_tc_{t+1}, we define $D'(d)$ to be the distance from c_t to the intersection of l and the ray $R_{c_t,d}$. $D'(d)$ is a continuous function. Since c_t does not see u, we have $D_{c_t} < D'$ in the beginning of the interval. By the properties of D' and D, if D_{c_t} ever exceeds D', then there is a direction d in the interval where $D_{c_t}(d) = D'(d)$, this means there is a point in $\mathcal{L}$ lies on the line l between u and v, and we are done. If $D_{c_t} < D'$ at the end of the interval, we repeat this argument starting from c_{t+1} and from the direction $\overrightarrow{c_tc_{t+1}}$. At the end, since $c_{k-1}c_k$ crosses the line l, which implies that from c_{k-1}, just before the direction $c_{k-1}c_k$, the value of $D_{c_{k-1}}$ exceeds the corresponding D', so u does not see v.

□

Claim 3. *No two polygonal segments on the gray line intersect in the interior.*

Proof. Assume the contrary, let the first of these intersection be the intersection of r_ir_{i+1} with r_jr_{j+1}, where $i < j$. We may take r_{j+1} to be c_k and depending on cases either $(u, v) = (r_{i+1}, r_i)$ or $(u, v) = (r_i, r_{i+1})$. In either cases we apply Lemma 6.5 and conclude that r_i does not see r_{i+1}, contradicts the basic property of the gray line. □

Since there are finitely many • points and r_{i+1} is determined by r_{i-1} and r_i, the points on the gray line eventually enters a loop. In fact we can say something more.

Claim 4. *The sequence of points on the gray line eventually form a loop. The points on the loop form a convex polygon $\mathcal{P}$ in clockwise order. Furthermore, (i) any segment in $\mathcal{L}$ that intersects this $\mathcal{P}$ are completely contained in $\mathcal{P}$; and (ii) all the segments on the gray line are completely contained in $\mathcal{P}$.*

Proof. We walk along the polygonal loop, each time make a right turn, it is enough to show that this loop does not intersect itself. By Claim 3 and the general position assumption, we only need to show that the loop cannot cross itself at a point on the loop. i.e., there are points r_{i-1}, r_i, r_{i+1}, and r_{j-1}, r_j, r_{j+1} such that $r_i = r_j = r$.

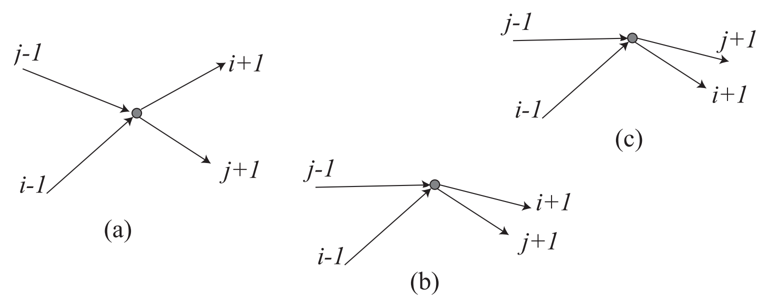


Figure 6.10: Three cases where the gray line is self intersecting.

We may assume $\overrightarrow{r_{j-1}r}$ is clockwise to $\overrightarrow{r_{i-1}r}$. We discuss 3 cases (Figure 6.10). (a) From $\overrightarrow{r_{j-1}r}$ to $\overrightarrow{rr_{i+1}}$ is a left turn. We take $r_j = c_k$, view rr_{i+1} as a vertical line, set $(u, v) = (r_{i+1}, r)$, and apply Lemma 6.5. We get r_i does not see r_{i+1}. (b) From $\overrightarrow{r_{j-1}r}$ to $\overrightarrow{rr_{i+1}}$ is a right turn and r_{i+1} comes before r_{j+1}, this contradicts the definition of r_{j+1} from r_{j-1} and r_j, since r_{j+1} should be the first • point we see from r_j, and clockwise starting from $\overrightarrow{r_{j-1}r_j}$. (c) From $\overrightarrow{r_{j-1}r}$ to $\overrightarrow{rr_{i+1}}$ is a right turn and r_{i+1} comes after r_{j+1}, this contradicts the definition of r_{i+1} from r_{i-1} and r_i.

(i). Since any adjacent vertex on $\mathcal{P}$ are visible, no segment in $\mathcal{L}$ can cross the boundary of $\mathcal{P}$. We only need to prove B_i is in $\mathcal{P}$ if R_i is a vertex of $\mathcal{P}$. We may assume $R_i = r_1$. If B_i is outside $\mathcal{P}$, we have two cases. (a) $r_0r_1B_i$ is a left turn. We pick $u = B_i$, v to be a point in $\mathcal{P}$ very close to r_1 such that uv intersects the segment r_0r_1, and $c_k = r_1$. Apply Lemma 6.5 we get that u does not see v, a contradiction. (b) $r_0r_1B_i$ is a right turn, but the ray R_{r_1,B_i} goes into $\mathcal{P}$ and comes out from another edge. Similar to the previous case, we can apply Lemma 6.5 and get a contradiction. (c) $r_0r_1B_i$ is a right turn and B_i comes before r_2 in clockwise order. This contradicts the definition of r_2.

(ii). We shall rule out the possibility that the initial segments of the gray line, before we enter the loop, lies outside $\mathcal{P}$. By Claim 3, we may assume there is an edge r_ir_{i+1} of $\mathcal{P}$ and a point $r_{i'}$ such that $r_{i'+1} = r_{i+1}$. The proof is almost the same as the proof for (i). □

Claim 5. *Let v be a point in $\mathcal{P}$ and u be a point outside $\mathcal{P}$. Assume u sees v and u is an end point in ℓ, then u is ▲ .*

Proof. View uv as the vertical line with u on the top. If v is not the leftmost point of $\mathcal{P}$, let r_t be the left tangent from u to $\mathcal{P}$. r_t does not see u if u is a blue point or a • outside $\mathcal{P}$. By Lemma 6.5, u does not see v. For the special case when v is the leftmost point in $\mathcal{P}$, by the procedure of how we get r_{t+1} from r_{t-1} and r_t, u must be a ▲ . □

Now we are ready to finish the proof of the main theorem.

Proof. (of Theorem 6.5) We know there are both segments inside $\mathcal{P}$ and outside $\mathcal{P}$. Consider all the segments in $\mathcal{P}$. All the segments containing extreme points of $\mathcal{P}$ are from the set A. If there is at least one segment from B. By induction there is a visible pair of • △ or ▲ △ points. And any segment outside $\mathcal{P}$ will not harm this visibility.

The remaining case is when all the segments in $\mathcal{P}$ are from A. Intuitively, in this situation all the ○ points in $\mathcal{P}$ are facing a firing squad of ▲ all around. Formally, we pick a pair of visible points (u, w) such that u is ▲ , w is any point in $\mathcal{P}$, such that the distance between u and w is minimized. (Obviously, w is on the boundary of $\mathcal{P}$.) By Corollary 6.2, if we only consider the segments inside $\mathcal{P}$, u sees some blue point v inside $\mathcal{P}$. All we need to prove is that this visibility is not obscured by any other segment outside $\mathcal{P}$. We offer our final picture.

Suppose there are segments block the visibility uv, these are segments outside $\mathcal{P}$, so they are disjoint from $\mathcal{P}$. Also notice that uw is by definition a visible pair, so none of these segments intersects uw. Thus, these segments enters and triangle uvw from the

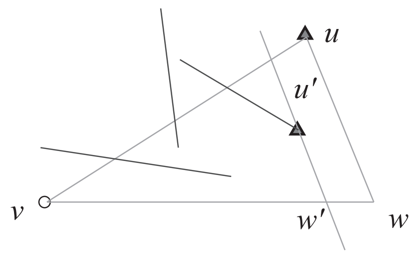


Figure 6.11: In Proof of Theorem 6.5: The proof of the fact that u sees v.

edge uv and never come out. In particular, we have an end point u' such that (*) u' is the closest to the line uw among all the end points. By Claim 5 u' must be ▲ since it is seen by w. Draw the line through u' and parallel to uw, intersect vw at w'. By convexity, w' is a point in $\mathcal{P}$ and the length $|u'w'|$ is smaller than $u'w'$ is smaller than uw. By (*), $u'w'$ is not blocked by any other segments outside $\mathcal{P}$, if it is blocked by some segment in $\mathcal{P}$, we find another $w'' \in \mathcal{P}$ with $|u'w''|$ even smaller. In any case, we have a contradiction with the minimality of $|uw|$. □

Thus, the answer to the puzzles in the beginning of this chapter is complete. In both cases we have the positive answer.

6.6 Planar geometric connection of two colors

Given a set of needles $\mathcal{L}$, let us (temporarily) call a segment in the plane *legal* if it connects one red point and one blue point. Theorem 6.5 states that in a set of needles, no matter how we partition the segments into two parts, we can always draw a legal edge connecting the two parts such that the edge does not cross any segment in $\mathcal{L}$. The natural generalization would be, given a set of needles, find a set of legal edges connecting all the segments in $\mathcal{L}$ such that these connections along with segments in $\mathcal{L}$ are non-crossing. The question was first asked by Mario Szegedy. In this section we give the positive answer. We will show in fact the seemingly stronger theorem follows easily from Theorem 6.5.

Theorem 6.6. *Given a set of 2-colored disjoint segments $\mathcal{L}$ in general position. We* *view $\mathcal{L}$ as a partial triangulation of the end points in $\mathcal{L}$. Then there is another triangulation $\mathcal{T}$ such that (a) $\mathcal{T}$ extends $\mathcal{L}$; (b) All the segments in $\mathcal{T}$ connects points of different colors; and (c) All the end points in $\mathcal{L}$ are connected in $\mathcal{T}$.*

In fact, we can find such a $\mathcal{T}$ in an arbitrary manner. As long as the whole point set can be partitioned into two parts that are not connected yet, we can find a red-blue visible pair between them and extend the partial triangulation by one more edge. Theorem 6.6 is implied by the following.

Theorem 6.7. *Let P be a set of points in the plane in general position and each point colored either red or blue. Let $\mathcal{G}$ be a partial triangulation of P where all the edges are between points of different colors. We view $\mathcal{G}$ as a planar graph on P. Assume there are no isolated points in $\mathcal{G}$, and there is a partition A and B of P such that points in A are not connected to points in B in $\mathcal{G}$, then we can either find a red point in A that sees a blue point in B, or a blue point in A that sees a red point in B.*

Follow the proof of Theorem 6.5 carefully, one might find that we just need to change some words to get a proof of Theorem 6.7. However, what we will do here is to show that Theorem 6.7 is easily implied by Theorem 6.5.

Proof. Given the situation $\mathcal{G}$, we perturb it a little bit to get a set of disjoint segments. Without loss of generality, we may assume there is a $\bullet$ v with degree $d > 1$. We split it to d new $\bullet$ points as follows. When $d > 2$, pick any edge uv and shrink it a little bit at the v end, then rotate it a little bit with center u, so that v is in general position with all the old points. When $d = 2$, say, there are two edges uv and wv. We extend uv a little bit and shrink wv at the v end. We can do this to keep the general position assumption, and keep the property that if there is a $\bullet$ $\triangle$ (or $\blacktriangle$ $\circ$) pair visible in the new arrangement, then their corresponding original points are also visible in $\mathcal{G}$. Now we apply Theorem 6.5 to the new arrangement. $\square$

Recall a *geometric graph* is a drawing of a graph in the plane such that all edges are straight line segments. The following fact is well known.

Proposition 6.4. *Given n red points and n blue points in the plane, there is a geometric matching on these $2n$ points where the edges are non-crossing and each edge connects a red point and a blue point.*

Proof. Consider all the $n!$ matchings, pick one σ with the smallest total length. We claim the edges determined by the matching are non-crossing. Suppose $u\sigma_u$ and $v\sigma_v$ cross, then we may switch σ_u and σ_v to get a new matching with smaller total length. □

The following statement can be proved by almost the identical proof. We also note that it follows from the proposition easily. Assume there are d more red points than blue points, we may pick one blue point, copy it $d+1$ times, and perturb them a little. We have

Corollary 6.4. *Given $m > 0$ red points and $n > 0$ blue points in the plane, there is a geometric graph on these $m + n$ points where the edges are non-crossing and there are no isolated points and each edge connects a red point and a blue point.*

Now, given a finite set of points in the plane, and take any partition of the points into two colors, we can find a geometric non-crossing graph with no isolated points. From that point we apply Theorem 6.7, we get a geometric non-crossing graph connecting all the points. We have

Theorem 6.8. *Given $m > 0$ red points and $n > 0$ blue points in the plane, one can connect all the points by a set of non-crossing segments between red and blue points.*

Appendix A

Programs for the Gallai numbers

We provide the commented programs for the Gallai points and Gallai lines. The program `gpoints.cpp` computes the values $\widetilde{m^*}(n)$ and `glines.cpp` computes $\widetilde{m}(n)$. Both programs are in standard `C++`. We compiled the programs using Microsoft Visual Studio .NET 2003 in the release mode. We ran the programs on our personal computer with 2.80GHz Intel Pentium 4 CPU and 1GB memory.

A.1 `gpoints.cpp`

```
#include <iostream>
#include <fstream>

using namespace std;

int C[30]; // C[n] stores n choose 2
int n, mn;
// mn: the current upper bound
int s[30]; // the permutation
int cnt[30]; // cnt[i] counts the number of Gallai lines passing i
int deg; // the longest block can be reversed

void input()
{
    cout<<"n = "; cin>>n;
    cout<<"Guess or known upper bound: "; cin>>mn;
```

```
}

void init()
{
    for(int i=0;i<30;i++) C[i]=i*(i-1)/2;
}

void play(int a, int b, int lastBlock, int dep=0)
// recursive function to generate the next move
// a: number of reversed pairs; b: number of Gallai points so far
// lastBlock: the left boundary of the block in the last move
{
    if(b>=mn) return; //not possible to beat the current value

    // for user to watch
    if(dep<5)
    {
        for(int i=0;i<n;i++) cout<<s[i]<<" ";
        cout<<": "<<dep<<" "<<mn<<endl;
    }

    if(a>=C[n]) // the end of a half period
    {
        mn=b;
        return;
    }

    int i,j,len, ii,jj, G,tmp;

    //try to reverse the block [i..j], with length len
```

```
for(i=0;i<n;i++)
{
    j=i+1; len=2;
    while(j<n && s[j]>s[j-1] && len<=deg)
    {
        if(j>=lastBlock)
        {
            G=0; // to compute G as the number of new Gallai points
            if(len==2)
            {
                // a new Gallai point x is found if
                // cnt[x] changes from 0 to 1
                cnt[s[i]]++; if(cnt[s[i]]==1) G++;
                cnt[s[j]]++; if(cnt[s[j]]==1) G++;
            }

            // reverse s[i..j]
            for(ii=i,jj=j; ii<jj; ii++,jj--)
            { tmp=s[ii]; s[ii]=s[jj]; s[jj]=tmp; }
            // recursively call the next move
            play(a+C[len], b+G, i, dep+1);
            // restore the state
            for(ii=i,jj=j; ii<jj; ii++,jj--)
            { tmp=s[ii]; s[ii]=s[jj]; s[jj]=tmp; }
            if(len==2)
            {
                cnt[s[i]]--;
                cnt[s[j]]--;
            }
        }
```

```
            j++; len++;
        }
    }
}

int main()
{
    input();
    init();
    for(deg=n-1; deg>=3; deg--)
    {
        // set the initial permutation
        for(int i=0; i<n; i++) s[i]=i;
        // and reverse the first deg elements
        for(int i=0; i<deg; i++) s[i]=deg-1-i;
        for(int i=0; i<n; i++) cnt[i]=0;
        if(deg==2) {cnt[0]=cnt[1]=1; play(1,2,0);}
        else play(C[deg],0,0);
    }
    cout<<"m*("<<n<<") = "<<mn<<endl;
}
```

We found the values $\widetilde{m}^*(n)$ for $n \leq 16$ using this program. We note that a good guess of the upper bound in the beginning makes the search much faster. For both $n = 15$ and $n = 16$ the program runs in several days (for the best guesses $\widetilde{m}^*(n) + 1$). The running time for $n = 15$ is much slower than that for $n = 16$, since $\widetilde{m}^*(15) > \widetilde{m}^*(16)$.

A.2 glines.cpp

```
#include <iostream>
```

```
#include <fstream>
#include <algorithm>

using namespace std;

int C[30]; // C[n] stores n choose 2
int n, mn;
int s[30];
int t[30], tUsed[30];
// t[..] stores the t vector
// tUsed[i] stores the number of reversed i-blocks so far
int fg;  // a flag in the search indicating whether suceeded
int deg; // the highest i such that t[i] is nonzero


void input()
{
    cout<<"n = "; cin>>n;
}

void init()
{
    for(int i=0;i<30;i++) C[i]=i*(i-1)/2;
    // initialize the permutation to be (2 1 ... n)
    mn=C[n];
    // the upper bound by Motzkin and Boroczky
    if(n>7)
    {
        if((n%2)==0) mn=n/2;
        if((n%4)==1) mn=3*(n-1)/4;
```

```
        if((n%4)==3) mn=3*(n-3)/4;
    }
    mn=min(mn, n-1);
}

void play(int a, int lastBlock)
// a: number of reversed pairs so far
// lastBlock: the left boundary of the block in the last move
{
    if(fg) return;
    if(a>=C[n]) // the end of a half period
    {
        fg=1;
        mn=min(t[2], mn);
        return;
    }
    int i,j,len, ii,jj,tmp;
    // try the current block to be [i..j], with length len
    for(i=0;i<n;i++)
    {
        j=i+1; len=2;
        while(j<n && s[j]>s[j-1] && len<=deg)
        {
            if(j>=lastBlock && tUsed[len]<t[len])
            {
                // reverse s[i..j]
                for(ii=i,jj=j; ii<jj; ii++,jj--)
                { tmp=s[ii]; s[ii]=s[jj]; s[jj]=tmp; }
                tUsed[len]++;
                play(a+C[len], i);
```

```
                // restore the state
                tUsed[len]--;
                for(ii=i,jj=j; ii<jj; ii++,jj--)
                { tmp=s[ii]; s[ii]=s[jj]; s[jj]=tmp; }
            }
            j++; len++;
        }
    }
}

void launch(int a, int b, int lb)
// generate t[a]
// b=\sum_{a \leq i < n} \binom{i}{2} t[i]
// lb: lower bound from 3+t4+2t5+...
{
    if(a>2 && lb>t[2]) return;
    if(a>=n)
    {
        // check if \sum \binom{t[i]}{2} = \binom{n}{2}
        if(b!=0) return;
        // output the t-vector
        for(int i=0; i<n; i++) if(t[i])
            cout<<"t["<<i<<"]="<<t[i]<<" ";
        // find the highest i such that t[i]!=0
        for(int i=0;i<n;i++) if(t[i]) deg=i;
        // set the permutation to be
        // (t, t-1, ..., 1, t+1, t+2, ..., n)
        for(int i=0;i<n;i++) s[i]=i+1;
        for(int i=0;i<deg;i++) s[i]=deg-i;
        int firstBlock=deg;
```

```
        t[firstBlock]--;
        for(int i=0;i<n;i++) tUsed[i]=0;
        // find the highest t[i]!=0 after the first block reversed
        for(int i=0;i<n;i++) if(t[i]) deg=i;
        fg=0;
        play(C[firstBlock], 0);
        t[firstBlock]++;
        if(fg) cout<<"found"<<endl; else cout<<"nop"<<endl;
        return;
    }
    if(a==2)
    {
        int CsimaSawyer=(6*n+12)/13;
        if(n==7) CsimaSawyer=3;
        for(t[2]=CsimaSawyer;t[2]<mn;t[2]++)
            launch(3, b-t[2], 3);
        return;
    }
    for(t[a]=0; C[a]*t[a]<=b; t[a]++)
        launch(a+1, b-C[a]*t[a], lb+(a-3)*t[a]);
}

int main()
{
    input();
    init();
    launch(2,C[n],3);
    cout<<"m("<<n<<") = "<<mn<<endl;
}
```

Below is the output for the result $\widetilde{m}(15) = 9$, including all the possible t-vectors checked. The program runs in this case for about 2 to 3 hours.

```
n = 15
t[2]=7 t[3]=26 t[5]=2 nop
t[2]=8 t[3]=23 t[8]=1 nop
t[2]=8 t[3]=23 t[4]=3 t[5]=1 nop
t[2]=8 t[3]=24 t[5]=1 t[6]=1 nop
t[2]=8 t[3]=25 t[4]=2 t[5]=1 nop
t[2]=8 t[3]=27 t[4]=1 t[5]=1 nop
t[2]=8 t[3]=29 t[5]=1 nop
m(15) = 9
```

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