TAMKANG JOURNAL OF MATHEMATICS
Volume 38, Number 4, 277-289, Winter 2007

ON CONFORMAL TRANSFORMATION OF CERTAIN FINSLER SPACES

S. C. RASTOGI AND A. K. DWIVEDI

0. Introduction

M. S. Knebelman [4] first defined the conformal theory of Finsler metrics, such that two metric functions L and $\underline{L}$ are conformal if the length of an arbitrary vector in the space with the metric L is proportional to the length in the space with metric $\underline{L}$, that is, if $\underline{g}_{ij} = \phi g_{ij}$, where g_{ij} and $\underline{g}_{ij}$ are the metric tensors corresponding to metric functions L and $\underline{L}$ respectively and ϕ is a function of coordinates. Conformal transformations in Finsler spaces have further been studied by various authors namely Hashiguchi [1], Izumi [2, 3], Matsumoto [8] and others. The purpose of the present paper is to sutdy conformal transformation of $L(\alpha, \beta)$-metric (Matsumoto [5, 7]) and its special case related to Randers' space [9]. Throughout the present paper we shall follow the notations used in Matsumoto's monograph [6].

1. Preliminaries

Let (M^n, L) be an n-dimensional Finsler space equipped with the fundamental function $L(x, y)$ on a differentiable manifold M^n. Let $(M^n, {}^*L)$ be another Finsler space equipped with the fundamental function ${}^*L(x, y)$ such that Matsumoto [5]:

$$ {}^*L(x, y) = L(x, y) + \beta(x, y). \tag{1.1}$$

where $\beta(x, y) = b_i(x)dx^i$.

We also have for $l_i = \partial L/\partial y^i$, ${}^*l_i = \partial^* L/\partial y^i$ and $b_i = \partial\beta/\partial y^i$,

$$ {}^*l_i = l_i + b_i. \tag{1.2}$$

If $h_{ij} = g_{ij} - l_i l_j = LL_{ij}$ we have ${}^*h_{ij}/{}^*L = h_{ij}/L$ or $L_{ij} = {}^*L_{ij}$ such that [5]

$$ {}^*g_{ij} = \tau(g_{ij} - l_i l_j) + {}^*l_i {}^*l_j, {}^*g^{ij} = \tau^{-1} g^{ij} + \mu l^i l^j - \tau^{-2}(l^i b^j + l^j b^i), \tag{1.3}$$

where $\mu = (Lb^2 + \beta)/({}^*L\tau^2)$, $b^2 = b_i b^i$, $b^i = g^{ij} b_j$, $\tau = {}^*L/L$.

Received October 21, 2004.
2000 *Mathematics Subject Classification.* 53B40.
Key words and phrases. Conformal transformation, Finsler space, Randers' space.

Since ${}^*l^i = \tau^{-1}l^i$, therefore for $m_i = b_i - (\beta/L)l_i$, we can obtain

$$ {}^*C_{ijk} = \tau C_{ijk} + (h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)/2L. \tag{1.4}$$

The h- and v-covariant derivatives of a covariant vector field X_i are defined as

$$X_{i|j} = \partial_j X_i - N^r_j(\Delta_r X_i) - X_r F^r_{ij} \tag{1.5}$$

and

$$X_{i\big|j} = \Delta_j X_i - X_r C^r_{ij}, \tag{1.6}$$

where the symbol ∂_j and Δ_j stands for $\partial/\partial x^j$ and $\partial/\partial y^j$ respectively, $(F_j{}^i{}_k, N^i_j, C^i_{jk})$ are connection parameters of F^n such that $N^i_k = F^i_{0k} = y^r F^i_{rk}$ and $C^r_{jk} = g^{ir}C_{ijk}$.

If ${}^*F^i_{jk}$ denotes the Cartan's connection of ${}^*F^n$, then it is given by Matsumoto [5]

$$ {}^*F^i_{jk} = F^i_{jk} + D^i_{jk}, \tag{1.7}$$

where D^i_{jk} is a tensor of type (1,2) such that it satisfies [5]

$$L_{ijr}D^r_{ok} + L_{rj}D^r_{ik} + L_{ir}D^r_{jk} = 0, L_{ri}D^r_{oj} + (l_r + b_r)D^r_{ij} = b_{i|j}, \tag{1.8}$$

$$D^i_{00} = 2LF^i_0 + \tau^{-1}(E_{00} - 2LF_{r0}b^r)l^i, \tag{1.9}$$

$$D^i_{0j} = LG^i_j + \tau^{-1}l^i(G_j - LG_{mj}b^m) \tag{1.10}$$

and

$$D^i_{jk} = LH_{rjk}(g^{ir} - l^i b^r \tau^{-1}) + l^i\tau^{-1}H_{jk}, \tag{1.11}$$

where

$$2F_{jk} = b_{j|k} - b_{k|j}, \quad 2E_{jk} = b_{j|k} + b_{k|j}. \tag{1.12}$$

$$G_{ij} = F_{ij} - L_{ijr}D^r_{00}/2, \quad G_j = E_{j0} - L_{jr}D^r_{00}/2, \quad G^i_j = g^{ir}G_{rj} \tag{1.13}$$

$$H_{ijk} = (L_{jkr}D^r_{0i} - L_{kir}D^r_{0j} - L_{ijr}D^r_{0k})/2 \tag{1.14}$$

and

$$H_{jk} = E_{jk} - (L_{jr}D^r_{0k} + L_{kr}D^r_{0j})/2. \tag{1.15}$$

The T-tensor in a Finsler space is defined by [6]:

$$T_{hijk} = LC_{hij\big|k} + C_{hij}l_k + C_{hik}l_j + C_{hjk}l_i + C_{ijk}l_h, \tag{1.16}$$

which in a space with generalized (α, β) metric can be expressed as [5]

$$ {}^*T_{hijk} = {}^*L{}^*C_{hij\big|k} + {}^*C_{hij}{}^*l_k + {}^*C_{hik}{}^*l_j + {}^*C_{hjk}{}^*l_i + {}^*C_{ijk}{}^*l_h. \tag{1.17}$$

2. Conformal transformation

Let us assume that there exists a conformal transformation of Finsler spaces which transform in such away that $\underline{L} = Le^{\sigma}$, $(\sigma = \sigma(x))$, $^*\underline{L} = {}^*Le^{\sigma}$. From equation (1.1), (1.2), (1.3) and (1.4) we can obtain for $\underline{\tau} = \tau$, $\underline{\mu} = \mu$,

$$\underline{l}_i = l_i e^{\sigma},\ \underline{l}^i = l^i e^{-\sigma},\ {}^*\underline{l}_i = {}^*l_i e^{\sigma},\ {}^*\underline{l}^i = {}^*l^i e^{-\sigma},\ \underline{b}_i = b_i e^{\sigma},\ \underline{b}^i = b^i e^{-\sigma},$$
$$\underline{\beta} = \beta e^{\sigma},\ \underline{y}_i = y_i e^{2\sigma},\ \underline{y}^i = y^i,\ {}^*\underline{y}_i = e^{2\sigma}{}^*y_i,\ {}^*\underline{y}^i = {}^*y^i, \tag{2.1}$$

$$\underline{g}_{ij} = g_{ij}e^{2\sigma},\ {}^*\underline{g}_{ij} = {}^*g_{ij}e^{2\sigma},\ {}^*\underline{h}_{ij} = \tau e^{2\sigma}h_{ij},\ {}^*\underline{L}_{ij} = L_{ij}e^{\sigma}, \tag{2.2}$$

$$\underline{g}^{ij} = g^{ij}e^{-2\sigma},\ {}^*\underline{g}^{ij} = {}^*g^{ij}e^{-2\sigma},\ {}^*\underline{h}^{ij} = {}^*h^{ij}e^{-2\sigma}, \tag{2.3}$$

and

$$\underline{C}_{ijk} = C_{ijk}e^{2\sigma},\ {}^*\underline{C}_{ijk} = {}^*C_{ijk}e^{2\sigma},\ {}^*\underline{L}_{ijk} = L_{ijk}e^{\sigma} \text{ and } (\underline{\Delta}_k\underline{h}_{ij}) = e^{2\sigma}(\Delta_k h_{ij}). \tag{2.4}$$

From equations (2.1), (2.2), (2.3) and (2.4) we can obtain

Theorem 2.1. *Under the given conformal transformation following entities are conformally invariant* $^*l_i{}^*L^{-1}$; $^*L^*l^i$; $L^{-1}b_i$; Lb^i; $L^{-1}\beta$; $^*g_{ij}L^{-2}$; $^*g^{ij}{}^*L^2$; $^*h_{ij}{}^*L^{-2}$; $^*h^{ij}{}^*L^2$; $^*C_{ijk}{}^*L^{-2}$.

We know that Izumi [2, 3]

$$\underline{G}^i = G^i + B^{ih}\sigma_h, \quad \underline{G}^i_j = G^i_j + b^i_j, \quad \underline{G}^i_{jk} = G^i_{jk} + b^i_{jk}, \tag{2.5}$$

where

$$B^{ih} = y^i y^h - L^2 g^{ih}/2, \quad b^i_j = (\Delta_j B^{ih})\sigma_h, \quad b^i_{jk} = (\Delta_k(\Delta_j B^{ih}))\sigma_h. \tag{2.6}$$

From equation (1.12) and $F^i_0 = F_{j0}g^{ij}$ we can obtain

$$\underline{N}^i_j = N^i_j + b^i_j, \tag{2.7}$$
$$2\underline{F}_{jk} = e^{\sigma}[2F_{jk} + b_j\sigma_k - b_k\sigma_j], \tag{2.8}$$
$$2\underline{E}_{jk} = e^{\sigma}[2E_{jk} + b_j\sigma_k + b_k\sigma_j] \tag{2.9}$$

and

$$\underline{F}^i_0 = e^{-\sigma}[F^i_0 + (b^i\sigma_0 - \beta g^{ij}\sigma_j)/2]. \tag{2.10}$$

From $\underline{L} = Le^{\sigma}$, we can write Log $\underline{L}$ = Log $L + \sigma$, which gives

$$\sigma_k = \underline{L}^{-1}(\partial_k\underline{L}) - L^{-1}(\partial_k L), \quad \sigma_0 = \{\underline{L}^{-1}(\partial_k\underline{L}) - L^{-1}(\partial_k L)\}y^k. \tag{2.11}$$

Hence from equations (2.8), (2.9) and (2.10) we have:

Theorem 2.2. *Under the given conformal transformation following entities are conformally invariant:*

(a) $(b_j\sigma_k - b_k\sigma_j)/L$;
(b) $(b_j\sigma_k + b_k\sigma_j)/L$;
(c) $L(b^i\sigma_0 - \beta g^{ij}\sigma_j)$;
(d) $L^{-1}[2F_{jk} - L^{-1}(b_j\partial_k L - b_k\partial_j L)]$;
(e) $L^{-1}[2E_{jk} - L^{-1}(b_j\partial_k L + b_k\partial_j L)]$;
(f) $L[F^i{}_0 - (1/2)(\partial_j L)(b^i y^i - \beta g^{ij})]$.

From equation (1.9) we can obtain

$$\underline{D}^i_{00} = D^i_{00} + B^i_{00} \tag{2.12}$$

where

$$B^i_{00} =: \{L(\sigma_0 b^i - \beta\sigma_p g^{pi}) - y^i\tau^{-1}(b^2\sigma_0 - \beta\sigma_p b^p - L^{-1}\beta\sigma_0)\}/2. \tag{2.13}$$

Equation (2.12) with the help of (2.13) gives on simplification

Theorem 2.3. *Under the given conformal transformation tensor* ${D^*}^r_{00}$ *defined by*

$${D^*}^r_{00} =: D^r_{00} - (\partial_k L)[b^r y^k - \beta g^{kr} - l^r\tau^{-1}\{(b^2 - \beta L^{-1})y^k - \beta b^k\}]/2 \tag{2.14}$$

is conformally invariant.

From equation (1.13) we get

$$\underline{G}_{ij} = e^\sigma[G_{ij} + (b_i\sigma_j - b_j\sigma_i)/2 - L_{ijr}B^r_{00}], \tag{2.15}$$

and

$$\underline{G}_j = e^\sigma[G_j + (b_j\sigma_0 + \beta\sigma_j)/2 - L_{jr}B^r_{00}]. \tag{2.16a}$$

Since $G_j = E_{j0} - F_{j0}$, we can also obtain

$$\underline{G}_j = e^\sigma(G_j + \beta\sigma_j). \tag{2.16b}$$

Comparing equations (2.16a) and (2.16b), we get

$$L_{jr}B^r_{00} = (b_j\sigma_0 - \beta\sigma_j)/2. \tag{2.17}$$

From equations (2.15) and (2.17), we get

$$\underline{G}_{ij} = e^\sigma[G_{ij} - L_{ir}(\Delta_j B^r_{00})]. \tag{2.18}$$

From equations (2.15) and (2.16), we can obtain

Theorem 2.4. *Under the given conformal transformation following entities are conformally invariant:*
(a) $L^{-1}\{G_k - \beta L^{-1}(\partial_k L)\}$,
(b) $L^{-1}[G_{ij} - L^{-1}\{b_i(\partial_j L) - b_j(\partial_i L)\}] - L^{-1}L_{ijr}(\partial_k L)[b^r y^k - \beta g^{kr} - l^r\tau^{-1}\{(b^2 - \beta L^{-1})y^k - \beta b^k\}]/2$.

With the help of equations (1.10), (2.17) and (2.18) we can obtain

$$\underline{D}^i_{0j} = D^i_{0j} + B^i_{0j}. \tag{2.19}$$

where

$$B^i_{0j} =: \tau^{-1} l^i \beta \sigma_j - (\Delta_j B^r_{00})(h^i_r - \tau^{-1} l^i m_r). \tag{2.20}$$

From equations (2.19) and (2.20) with the help of (2.12), we can obtain

Theorem 2.5. *Under the given conformal transformation tensor* ${D^*}^i_{0j}$ *defined by*

$${D^*}^i_{0j} =: D^i_{0j} - [\tau^{-1} l^i \beta L^{-1}(\partial_j L) - (\Delta_j D^r_{00})({h^i}_r - \tau^{-1} l^i m_r)], \tag{2.21}$$

is conformally invariant.

Multiplying (2.19) by y^j comparing the resulting equation with (2.12) and using (2.13) we obtain on simplification

$$L[\beta \sigma_p (g^{pi} - l^i b^p \tau^{-1}) - \sigma_0 (b^i - l^i b^2 \tau^{-1})] = (\Delta_j P^r)(h^i_r - \tau^{-1} l^i m_r) y^j, \tag{2.22}$$

which implies

Theorem 2.6. *Under the given conformal transformation, there exists a scalar $\sigma(x)$, which satisfies equation* (2.22).

Since from equation (1.13) we can obtain

$$G_{kj} = \Delta_k G_j - (l_r + b_r)(\Delta_k D^r_{0j}), \tag{2.23}$$

therefore by virtue of equations (2.14) and (2.23) we can obtain on simplification

$$(l_r + b_r)\Delta_k B^r_{0j} - L_{jkr} B^r_{00} = E_{jk} \tag{2.24a}$$

and

$$\Delta_k \{(\Delta_j B^t_{00}) A^r_t\} = (l_r + b_r)\Delta_k(\tau^{-1} l^r \beta \sigma_j) - L_{kr}(\Delta_j B^r_{00}) - b_k \sigma_j, \tag{2.24b}$$

where $A^i_r = (h^i_r - \tau^{-1} l^i L^{-1} m_r)$.

Hence we have:

Theorem 2.7. *Under the given conformal transformation, there exists a scalar $\sigma(x)$, for which the tensors B^r_{00} and A^i_r satisfy* (2.24).

From equations (1.14) and (1.15), we can obtain

$$\underline{H}_{ijk} = e^\sigma [H_{ijk} + (1/2)\{L_{jkr} B^r_{0i} - L_{kir} B^r_{0j} - L_{ijr} B^r_{0k}\}], \tag{2.25}$$

and

$$\underline{H}_{jk} = e^\sigma [H_{jk} + \{b_j \sigma_k + b_k \sigma_j + L_{jr}(\Delta_k B^r_{00}) + L_{kr}(\Delta_j B^r_{00})\}]. \tag{2.26}$$

From equations (2.25) and (2.26) on simplification we can obtain

Theorem 2.8. *Under the given conformal transformation, following entities are conformally invariant*

$$L^{-1}[\tau^{-1}\beta(\partial_i L)L^{-1} + (\Delta_i D^r_{00})(l_r + \tau^{-1}m_r)], \tag{2.27a}$$
$$L^{-1}[H_{jk} - (L_{jr}\Delta_k D^r_{00} + L_{kr}\Delta_j D^r_{00}) - L^{-1}(b_k\partial_j L + b_j\partial_k L)], \tag{2.27b}$$
$$L^{-1}[H_{ijk} - (1/2)\{L_{jkr}D^r_{0i} - L_{kir}D^r_{0j} - L_{ijr}D^r_{0k}\}]. \tag{2.27c}$$

From equation (1.11), we can obtain

$$\underline{D}^i_{jk} = D^i_{jk} + B^i_{jk}, \tag{2.28}$$

where

$$B^i{}_{jk} =: (1/2)L\{L_{jkt}B^t_{0r} - L_{krt}B^t_{0j} - L_{jrt}B^t_{0k}\}(g^{ir} - l^i b^r\tau^{-1}) \\ +l^i\tau^{-1}\{b_j\sigma_k + b_k\sigma_j + L_{jr}(\Delta_k B^r_{00}) + L_{kr}(\Delta_j B^r_{00})\}. \tag{2.29}$$

With the help of equations (2.28) and (2.29), we can obtain

Theorem 2.9. *Under the given conformal transformation, the tensor defined by*

$$D^{*i}_{\ \ jk} =: D^i_{jk} - (1/2)L\{L_{jkt}D^t_{0r} - L_{krt}D^t_{0j} - L_{rjt}D^t_{0k}\}(g^{ir} - l^i b^r\tau^{-1}) \\ -l^i\tau^{-1}\{L^{-1}(b_j\partial_k L + b_k\partial_j L) + L_{jr}\Delta_k D^r_{00}) + L_{kr}\Delta_j D^r_{00}\}, \tag{2.30}$$

is conformally invariant.

From equations (1.16) and (1.17), we can easily obtain ${}^*\underline{T}_{hijk} = e^{3\sigma}{}^*T_{hijk}$, which implies

Theorem 2.10. *Under the given conformal transformation, the tensor* ${}^*L^{-3}{}^*T_{hijk}$ *is conformally invariant.*

3. Conformal transformation of connection parameters

From equation (1.7) with the help of equation (2.28), (2.29) and Hashiguchi [1]

$$\underline{F}^i_{jk} = F^i_{jk} + U^i_{jk}, \tag{3.1}$$

where

$$U^i_{jk} = \delta^i_j\sigma_k + \delta^i_k\sigma_j + C^i_{jm}B^m_k + C^i_{km}B^m_j - g^{in}C_{jkm}B^m_n - g_{jk}\sigma^i, \tag{3.2}$$

we can obtain

$${}^*\underline{F}^i_{jk} = {}^*F^i_{jk} + {}^*U^i_{jk} \tag{3.3}$$

where

$$^*U^i_{jk} =: [\delta^i_j\sigma_k + \delta^i_k\sigma_j + C^i_{jm}B^m_k + C^i_{km}B^m_j - g^{in}C_{jkm}B^m_n - g_{jk}\sigma^i$$
$$+(1/2)L(L_{jkt}B^t_{0r} - L_{krt}B^t_{0j} - L_{jrt}B^t_{0k})(g^{ir} - l^ib^r\tau^{-1})$$
$$+l^i\tau^{-1}\{b_j\sigma_k + b_k\sigma_j + L_{jr}(\Delta_k B^r_{00}) + L_{kr}(\Delta_j B^r_{00})\}]. \tag{3.4}$$

From equation (3.3) on multiplication by y^j we can obtain by virtue of ${}^*U^i{}_{jk}y^j =: {}^*b^i{}_k$

$$^*\underline{N}^i_k = {}^*N^i_k + {}^*b^i_k, \tag{3.5}$$

where

$$^*b^i_k =: [y^i\sigma_k + \delta^i_k\sigma_0 + L^2\sigma^m C^i_{km} - y_k\sigma^i - (1/2)L(L_{kt}B^t_{0r} + L_{krt}B^t_{00}$$
$$-L_{rt}B^t_{0k})(g^{ir} - l^ib^r\tau^{-1}) + l^i\tau^{-1}\{\beta\sigma_k + b_k\sigma_0 + L_{kr}(\Delta_j B^r_{00})y^j\}]. \tag{3.6}$$

Hence we have:

Theorem 3.1. *Under the given conformal transformation in a space with generalized (α,β)-metric the entities ${}^*U^i_{jk}$ and ${}^*b^i_k$ given by* (3.4) *and* (3.6) *respectively are conformally invariant.*

From equations (3.3), (3.4), (3.5) and (3.6) we can obtain

$$N^{*i}_{\ k} =: {}^*N^i_k + {}^*M^i_k \tag{3.7}$$

and

$$F^{*i}_{\ jk} =: {}^*F^i_{jk} + {}^*M^i_{jk}, \tag{3.8}$$

where

$$^*M^i_k =: (1/2)L(L_{kt}D^t_{0r} + L_{krt}D^t_{00} - L_{rt}D^t_{0k})(g^{ir} - l^ib^r\tau^{-1})$$
$$-(\partial_k L)L^{-1}(y^i + L^2C^i_{rm}g^{rm} + l^i\tau^{-1}\beta)$$
$$-(\partial_r L)(L^{-1}y^r\delta^i_k - y_kg^{ir} + l^il^r\tau^{-1}b_k) - l^i\tau^{-1}L_{kr}(\Delta_j D^r_{00})y^j \tag{3.9}$$

and

$$^*M^i_{jk} =: -L^{-1}(\delta^i_k\partial_j L + \delta^i_j\partial_k L) - L^{-1}(\partial_r L)\{C^i_{km}(y_jg^{rm} - \delta^m_j y^r - L^2C^{mr}_j)$$
$$+C^i_{jm}(y_kg^{rm} - \delta^m_k y^r - L^2C^{mr}_k) + g^{in}C_{jkm}(y_ng^{rm} - \delta^m_n y^r - L^2C^{mr}_n)$$
$$+g_{jk}g^{ri}\} - (1/2)L\{(L_{jkt}D^t_{0r} - L_{krt}D^t_{0j} - L_{jrt}D^t_{0k})(g^{ir} - l^ib^r\tau^{-1})\}$$
$$-l^i\tau^{-1}\{L(b_j\partial_k L + b_k\partial_j L) + L_{jr}(\Delta_k D^r_{00}) + L_{kr}(\Delta_j D^r_{00})\}. \tag{3.10}$$

Theorem 3.2. *Under the given conformal transformation in a space with generalized (α,β)-metric the entities defined by $N^{*i}_{\ k}$ and $F^{*i}_{\ jk}$ are conformally invariant.*

4. Conformal transformation of torsion and curvature tensors

The h-torsion tensor R^i_{jk} is expressed as [6]:

$$R^i_{jk} = \c{C}_{(j,k)}\{\partial_k N^i{}_j - N^r_k \Delta_r N^i_j\}, \tag{4.1}$$

therefore by virtue of ${}^*N^i_j = N^i_j + D^i_{0j}$, we can easily obtain

$${}^*R^i_{jk} = R^i_{jk} + I^i_{jk}, \tag{4.2}$$

where

$$I^i{}_{jk} =: \c{C}_{(j,k)}[D^i_{0j|k} + D^m_{0j}\{(\Delta_m {}^*F^i_{sk})y^s + D^i_{mk}\}] \tag{4.3}$$

and $\c{C}_{(j,k)}$ means interchange of j and k and subtraction.

From equation (4.2) it is easy to get

$${}^*\underline{R}^i_{jk} = {}^*R^i_{jk} + J^i_{jk}, \tag{4.4}$$

where

$$J^i_{jk} =: \c{C}_{(j,k)}[{}^*b^i_{j\|k} + {}^*b^m_j(\Delta_m {}^*N^i_k - {}^*F^i_{mk})] \tag{4.5}$$

and symbol $\|k$, means covariant derivative corresponding to ${}^*F^i_{jk}$.

The hv-torsion tensor P^i_{jk} is expressed as [6]:

$$P^i_{jk} = \Delta_k N^i_j - F^i_{jk}, \tag{4.6}$$

therefore we can obtain

$${}^*P^i_{jk} = P^i_{jk} + \Delta_k D^i_{0j} - D^i_{jk}, \tag{4.7}$$

which on conformal transformation gives

$${}^*\underline{P}^i_{jk} = {}^*P^i_{jk} + \Delta_k {}^*b^i_j + {}^*U^i_{jk}. \tag{4.8}$$

Hence we have:

Theorem 4.1. *The torsion tensors of a space with generalized (α, β)-metric, when conformally transformed, satisfy equations* (4.4) *and* (4.8) *such that entities J^i_{jk} and $(\Delta_k {}^*b^i_j + {}^*U^i_{jk})$ are conformally invariant.*

Further with the help of equations (3.7), (3.8), (3.9), (3.10), (4.4) and (4.5), we can define

$$R^{*i}{}_{jk} =: {}^*R^i_{jk} + \c{C}_{(j,k)}({}^*M^i_{j\|k} + 2{}^*M^m_j {}^*F^i_{mk}) \tag{4.9a}$$

and

$$P^{*i}{}_{jk} =: {}^*P^i_{jk} + \Delta_k {}^*M^i_j + {}^*M^i_{jk}, \tag{4.9b}$$

which give

Theorem 4.2. *Under the given conformal transformation in a space with generalized (α,β)-metric the entities $R^{*i}_{\ jk}$ and $P^{*i}_{\ jk}$ defined by (4.9a, b) are conformally invariant.*

We know that the h-curvature tensor R^i_{hjk} is given as [6]:

$$R^i_{hjk} = \c{C}_{(j,k)}\{\partial_k F^i_{hj} - N^m_k(\Delta_m F^i_{hj}) + F^m_{hj}F^i_{mk}\} + C^i_{hm}R^m_{jk}, \tag{4.10}$$

implying

$$\begin{aligned} {}^*R^i_{hjk} = R^i_{hjk} + C^i_{hm}I^m_{jk} + M^i_{hm}{}^*R^m_{jk} + \c{C}_{(j,k)}\{D^i_{hj|k} + D^m_{0j}(\Delta_m {}^*F^i_{hk}) \\ + D^m_{hj}D^i_{mk}\} \end{aligned} \tag{4.11}$$

and

$${}^*\underline{R}^i_{hjk} = {}^*R^i_{hjk} + {}^*C^i_{hm}J^m_{jk} - {}^*N^i_{hjk}, \tag{4.12}$$

where

$${}^*N^i_{hjk} =: \c{C}_{(j,k)}\{{}^*U^i_{hj\|k} + {}^*b^m_k(\Delta_m {}^*\underline{F}^i_{hj}) + {}^*U^i_{mk}{}^*\underline{F}^m_{hj} - {}^*U^m_{hj}{}^*U^i_{mk}\}. \tag{4.13}$$

From equations (4.5), (4.12) and (4.13) on simplification, we can obtain

$$\underline{R}^{*i}_{\ hjk} = R^{*i}_{\ hjk} + {}^*U^i_{hjk}, \tag{4.14}$$

where

$$\begin{aligned} R^{*i}_{\ hjk} =: {}^*R^i_{hjk} + \c{C}_{(j,k)}\{{}^*F^i_{hj\|k} + {}^*N^m_k(\Delta_m {}^*F^i_{hj}) \\ - {}^*C^i_{hm}({}^*N^m_{j\|k} + {}^*N^r_j\Delta_r {}^*N^m_k)\} \end{aligned} \tag{4.15}$$

and

$${}^*U^i_{hjk} =: \c{C}_{(j,k)}\{{}^*C^i_{hm}({}^*\underline{N}^r_k\Delta_r {}^*b^m_j + {}^*F^m_{rj}{}^*b^r_k) + {}^*N^r_k\Delta_r {}^*U^i_{hj} + {}^*U^i_{rj}{}^*F^r_{hk}\}, \tag{4.16}$$

which leads to

Theorem 4.3. *Under the given conformal transformation in a space with generalized (α,β)-metric the entity $R^{*i}_{\ hjk}$ is conformally invariant if and only if ${}^*U^i_{hjk}$ vanishes.*

We know that the hv-curvature tensor P^i_{hjk} is given by [6]:

$$P^i_{hjk} = \Delta_k F^i_{hj} - C^i_{hk|j} + C^i_{hm}P^m_{jk}, \tag{4.17}$$

therefore we can obtain

$$\begin{aligned} {}^*P^i_{hjk} = P^i_{hjk} + M^i_{hk\|j} + \Delta_k D^i_{hj} + {}^*b^m_j\Delta_k C^i_{hm} + {}^*C^i_{hm}\Delta_k D^m_{0j} + M^i_{hm}P^m_{jk} \\ + C^i_{mk}D^m_{hj} - M^i_{hm}D^m_{jk} - C^m_{hk}D^i_{mj}, \end{aligned} \tag{4.18}$$

where

$$M^r_{jk} = (2L\tau)^{-1}[(h^r_j - \tau^{-1}l^r m_j)m_k + (h^r_k - \tau^{-1}l^r m_k)m_j + h_{jk}\{m^r - l^r\tau^{-1}(b^2 - \beta^2/L^2)\}]. \tag{4.19}$$

The curvature tensor ${}^*P^i_{hjk}$ on conformal transformation leads to

$${}^*\underline{P}^i_{hjk} = {}^*P^i_{hjk} - \Delta_k {}^*U^i_{hj} + {}^*C^i_{hm}(\Delta_k {}^*b^m_j + {}^*U^m_{jk}). \tag{4.20}$$

From equation (4.20), we can easily obtain $\underline{P}^{*i}{}_{hjk} = P^{*i}{}_{hjk}$, where

$$P^{*i}{}_{hjk} =: {}^*P^i_{hjk} + \Delta_k {}^*F^i_{hj} - {}^*C^i_{hm}(\Delta_k {}^*N^m_j + {}^*F^m_{jk}). \tag{4.21}$$

Theorem 4.4. *Under the given conformal transformation in a space with generalized* (α, β)*-metric the entity* $P^{*i}{}_{hjk}$ *defined by* (4.21) *is conformally invariant.*

We know that the v-curvature tensor S^i_{hjk} is given by [6]

$$S^i_{hjk} = \c{C}_{(j,k)}\{\Delta_k C^i_{hj} + C^m_{hj} C^i_{mk}\}, \tag{4.22}$$

therefore by virtue of

$${}^*C^r_{jk} = C^r_{jk} + M^r_{jk}, \tag{4.23}$$

and (2.4), the conformal transformation of generalized v-curvature tensor, satisfies the invariant property ${}^*\underline{S}^i_{hjk} = {}^*S^i_{hjk}$. Hence we have:

Theorem 4.5. *The curvature tensors of a space with generlized* (α, β)*-metric under a conformal transformation satisfy equations* (4.12) *and* (4.20) *such that entities* $({}^*C^i_{hm} J^m_{jk} - {}^*N^i_{hjk})$ *and* $\{\Delta_k {}^*U^i_{hj} - {}^*C^i_{hm}(\Delta_k {}^*b^m_j + {}^*U^m_{jk})\}$ *are conformally invariant.*

Multiplying equation (4.12) by ${}^*y^j$ and comparing the resulting equation with (4.4), on simplification, we obtain equation

$$\c{C}_{(j,k)}\{{}^*U^i_{mj} {}^*N^m_k + {}^*b^m_j \Delta m {}^*b^i_k\} = 0. \tag{4.24}$$

which implies:

Theorem 4.6. *Under the given conformal transformation in a space with generalized* (α, β)*-metric the tensors* ${}^*U^i_{mj}$, ${}^*N^m_k$ *and* ${}^*b^m_j$ *satisfy equation* (4.24).

5. Some special cases

Case I. Randers' space: The v-curvature tensor in a Randers' space is expressed in the following form [5]

$${}^*L^{2*}S_{hijk} = \c{C}_{(j,k)}(h_{hk}m_{ij} + h_{ij}m_{hk}), \tag{5.1}$$

where the v-Ricci tensor is given by

$$ {}^*L^{2*}S_{ik} = -\{(n-1)m^2/4\tau\}{}^*h_{ik} - \{(n-3)/4\}m_i m_k, \tag{5.2}$$

where $m_{ij} = (\tau/4)\{(m^2/2)h_{ij} + m_i m_j\}$.

From equations (5.1) and (5.2), we can easily obtain

$$ {}^*\underline{S}_{hijk} = e^{2\sigma *}S_{hijk}, \quad {}^*\underline{S}_{ik} = {}^*S_{ik}. \tag{5.3}$$

Hence we have:

Theorem 5.1. *In a Randers' space* ${}^*L^{-2*}S_{hijk}$ *and* ${}^*S_{ik}$ *are conformally invariant.*

In a Randers' space the $(v)hv$-torsion tensor is given by [5]

$$ {}^*P_{hjk} = h_{hj}P_k + h_{jk}P_h + h_{kh}P_j, \tag{5.4}$$

where

$$2P_j = ({}^*L/L^2)F_{j0} + E_{j0}/L - F_{\beta j} - Pl_j - Gm_j \tag{5.5a}$$

and

$$G = (E_{00} - 2LF_{\beta 0})/(2L^*L), \quad P = \tau(2G + F_{\beta 0}/{}^*L). \tag{5.5b}$$

From equation (5.5b), we can easily obtain

$$\underline{G} = e^{-\sigma}[G + \{\sigma_0(\beta - Lb^2) + L\beta\sigma_a b^a\}/(2L^*L)] \tag{5.6a}$$

and

$$\underline{P} = e^{-\sigma}[P + \beta\sigma_a b^a/(2L) + \sigma_0(2\beta - Lb^2)/(2L^2)]. \tag{5.6b}$$

With the help of equations (5.5a) and (5.6a,b), we can obtain on simplification

$$\begin{aligned}2\underline{P}_j = 2P_j + \sigma_0[\tau b_j + m_j + (b^2 - \beta/L)(l_j + \tau^{-1}m_j)]/(2L) - \sigma_j\{\beta(\tau+1) - b^2L\}/(2L) \\ + (1/2)\sigma_a b^a m_j(1 - \beta/{}^*L),\end{aligned} \tag{5.7}$$

which implies for $P_j y^j = P_0$,

$$\underline{P}_0 = P_0 + (1/2)(b^2 - \beta/L)\sigma_0. \tag{5.8}$$

From equation (2.11), we can obtain

$$L(\sigma_a b^a) = (\partial_a \underline{L})\underline{b}^a - (\partial_a L)b^a. \tag{5.9}$$

From equations (5.6a,b) with the help of equations (2.11), (5.7) and (5.8) together with $T = {}^*LG + 2P_0L$, we can obtain

$$\sigma_a b^a = 2\beta^{-1}(\underline{T} - T). \tag{5.10}$$

From equation (5.7) on simplification we can obtain $\underline{Q}_j = Q_j$, where

$$Q_j =: 4P_j - 2L^{-1}P_0\{(\tau b_j + m_j)(b^2 - \beta/L)^{-1} - (l_j + \tau^{-1}m_j)\} + 2Tm_j(\tau\beta)^{-1} + L^{-2}\{\beta(\tau+1) - b^2L\}(\partial_j L). \tag{5.11}$$

Hence we have:

Theorem 5.2. *In a Randers' space entities $L\sigma_a b^a$, $\beta\sigma_a b^a$ and Q_j are conformally invariant.*

From equations (5.4) and (5.7) with the help of equation (5.11) on simplification we can obtain $\underline{P}^*{}_{hjk} = P^*{}_{hjk}$, where

$$P^*{}_{hjk} =: {}^*L^{-1}[{}^*P_{hjk} + (1/4)\{h_{hj}Q_k + h_{jk}Q_h + h_{kh}Q_j\}]. \tag{5.12}$$

Hence we have:

Theorem 5.3. *In a Randers' space the entity $P^*{}_{hjk}$ defined by* (5.12) *is conformally invariant.*

Cast II. Landsberg space: If Randers' space reduces to a Landsberg space, we can write [5]

$${}^*R^i_{hjk} = R^i_{hjk} + {}^*C^i_{hr}R^r_{0jk}, \tag{5.13}$$

where R^i_{hjk} is well known Riemannian curvature tensor.

Taking conformal transformation of (5.13), we can obtain

$${}^*\underline{R}^i_{hjk} = {}^*R^i_{hjk} + {}^*C^i_{hr}X^r_{0jk} + X^i_{hjk}, \tag{5.14}$$

where

$$X^i_{hjk} = \underline{R}^i_{hjk} - R^i_{hjk}. \tag{5.15}$$

From equation (5.14), we can obtain $\underline{R}^{*i}{}_{hjk} = R^{*i}{}_{hjk}$, where

$$R^{*i}{}_{hjk} =: {}^*R^i_{hjk} - \text{Ç}_{(j,k)}[\delta^i_k\{R_{hj} - Rg_{hj}/2(n-1)\} + g^{il}g_{hj}\{R_{lk} - Rg_{lk}/2(n-1)\} + {}^*C^i_{hr}[\{R_{0j} - Ry_j/2(n-1)\} - y_j g^{rl}\{R_{lk} - Rg_{lk}/2(n-1)\}]]/(n-2). \tag{5.16}$$

Hence we have:

Theorem 5.4. *In a Landsberg space the entity $R^{*i}{}_{hjk}$ defined by* (5.16) *is conformally invariant.*

References

[1] M. Hashiguchi, *On conformal transformations of Finsler metrics*, J. Math. Kyoto Univ. **16**(1976), 25–50.

[2] H. Izumi, *Conformal transformation of Finsler space I*, Tensor, N. S. **31**(1977), 33–41.

[3] H. Izumi, *Conformal transformation of Finsler space II*, Tensor, N. S. **34**(1980), 337–359.

[4] M. S. Knebelman, *Conformal geometry of generalized metric space*, Proc. Nat. Acad. Sci. U. S. A., **15**(1929), 376–379.

[5] M. Matsumoto, *On Finsler spaces with Randers' metric and special forms of important tensors*, J. Math. Kyoto Univ. **14**(1974), 477–498.

[6] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press. Saikawa, Otsu, 520, Japan, 1986.

[7] M. Matsumoto, *Theory of Finsler spaces with* (α,β)*-metric*, Rep. Math. Phys. **31**(1992), 43–83.

[8] M. Matsumoto, *Conformal changes of Finsler spaces with* 1*-form metric*, Analele Stiitifice Ale Universitatii"A. L. I. Cuza", Iasi, **XL**(1994), 1–6.

[9] G. Randers, *On an asymmetric metric in the four space of general relativity*, Phys. Rev.(2) **59**(1941), 195–199.

Department of Mathematics, Shri Ramswaroop Memorial, College of Engineering and Management, Tewarigenj, Faizabad Road, Lucknow-227105, India.

E-mail: sureshrastogi @rediffmail.com

Department of Mathematics, Northern India Engineering College, Faizabad Road, Lucknow-227105, India.

E-mail: dwivediashwini@yahoo.co.in