Introduction to Numerical Algebraic Geometry[*]

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Abstract. In a 1996 paper, Andrew Sommese and Charles Wampler began developing a new area, "Numerical Algebraic Geometry", which would bear the same relation to "Algebraic Geometry" that "Numerical Linear Algebra" bears to "Linear Algebra".

To approximate all isolated solutions of polynomial systems, numerical path following techniques have been proven reliable and efficient during the past two decades. In the nineties, homotopy methods were developed to exploit special structures of the polynomial system, in particular its sparsity. For sparse systems, the roots are counted by the mixed volume of the Newton polytopes and computed by means of polyhedral homotopies.

In Numerical Algebraic Geometry we apply and integrate homotopy continuation methods to describe solution components of polynomial systems. One special, but important problem in Symbolic Computation concerns the approximate factorization of multivariate polynomials with approximate complex coefficients. Our algorithms to decompose positive dimensional solution sets of polynomial systems into irreducible components can be considered as symbolic-numeric, or perhaps rather as numeric-symbolic, since numerical interpolation methods are applied to produce symbolic results in the form of equations describing the irreducible components.

Applications from mechanical engineering motivated the development of Numerical Algebraic Geometry. The performance of our software on several test problems illustrate the effectiveness of the new methods.

1 Introduction

The goal of this paper is to provide an overview of the main ideas developed so far in our research program to implement numerical algebraic geometry, initiated in [86].

We are concerned with numerically solving polynomial systems. While the homotopy continuation methods of the past were limited to approximating only the isolated roots, we developed tools to describe all positive dimensional irreducible components of the solution set of a polynomial system. In particular, our algorithms produce for every irreducible component a *witness*

set, whose cardinality equals the degree of the component, as this set is obtained by intersecting the component with a general linear space of complementary dimension. A point of a witness set corresponds to what is known in algebraic geometry as a generic point. Our main results [77, 78, 79, 80, 81, 82, 83, 84, 85] can be summarized in four items:

1. In [77] we presented a cascade of homotopies to find candidate witness points for every component of the solution set. Separating the junk from the candidate witness points was done in [78], where factorization methods based on interpolation implemented a numerical irreducible decomposition. The use of central projections and a homotopy membership test to filter junk were the improvements of [79].

2. An open problem in symbolic-numerical computing is the factorization of multivariate polynomials with approximate coefficients [38]. While we first struggled with the same challenge, the discovery of monodromy [80] followed by the validation by the linear trace [81] enabled us to deal with very high degree components, using only machine floating point numbers. In [82] we defined these methods for multiple components and applied it to the factorization of multivariate polynomials in [85].

3. Our new homotopy algorithms have been implemented and tested using the path trackers in the software package PHCpack [93]. In [84] we outlined the new tools in PHCpack and describe a simple interface to Maple. Our software found the degrees of all irreducible components of the cyclic 8 and 9 roots problems, which previously could only be done via Gröbner bases (and only by the very best implementation [18]).

4. Polynomial systems with positive dimensional components occur naturally when designing mechanical devices which permit motion. We investigated a special case of a moving platform, discovering through a numerical irreducible decomposition [81] a component missed by experts [36]. This and other applications of our tools to systems coming from mechanical design are described in [83].

In this paper we will introduce these results, after first describing homotopy continuation methods in the next section.

Recent and exciting new developments in fields related to numerical algebraic geometry will not be explained, but we cannot refrain from mentioning numerical Schubert calculus ([32], [34], [55], [91], [102]) and numerical jet geometry [70].

2   Homotopy Continuation Methods

Homotopy continuation methods operate in two stages. Firstly, homotopy methods exploit the structure of the system $f(\mathbf{x}) = 0$ to find a root count and to construct a start system $g(\mathbf{x}) = 0$ that has exactly as many regular solutions as the root count. This start system is embedded in the *homotopy*

$$h(\mathbf{x}, t) = \gamma(1 - t)g(\mathbf{x}) + tf(\mathbf{x}) = 0, \quad t \in [0, 1], \tag{1}$$

with $\gamma \in \mathbb{C}$ a random number. Secondly, as t moves from 0 to 1, numerical continuation methods trace the paths that originate at the solutions of the start system towards the solutions of the target system. The good properties we expect from a homotopy are (borrowed from [45]):

1. (*triviality*) The solutions for $t = 0$ are trivial to find.
2. (*smoothness*) No singularities along the solution paths occur (because of γ).
3. (*accessibility*) An isolated solution of multiplicity m is reached by exactly m paths.

Continuation or path-following methods are standard numerical techniques ([1, 2, 3], [57], [107, 108]) to trace the solution paths defined by the homotopy using *predictor-corrector* methods. The smoothness property of complex polynomial homotopies implies that paths never turn back, so that during correction the parameter t stays fixed, which simplifies the set up of path trackers. The adaptive step size control determines the step length while enforcing quadratic convergence in Newton's method to avoid path crossing. At the end of the path, end games ([33], [62, 63, 64], [87]) deal with diverging paths and paths leading to singular roots.

Following [32], we say that a homotopy is *optimal* if every path leads to one solution. The classification in Table 1 (from [94]) contains key words for three classes of polynomial systems for which optimal homotopies are available in PHCpack [93]. These homotopies have no diverging paths for generic instances of polynomial systems in their class.

system	model	theory		space
dense	highest degrees	Bézout	$\mathbb{P}^n$	projective
sparse	Newton polytopes	Bernshteĭn	$(\mathbb{C}^*)^n$	toric
determinantal	localization posets	Schubert	G_{mr}	Grassmannian

Table 1: Key words of the three classes of polynomial systems.

The earliest applications of homotopies for solving polynomial systems ([8], [15], [24], [25], [43], [47] [56], [113], [114]) belong to the dense class, where the number of paths equals the product of the degrees in the system. Multi-homogeneous homotopies were introduced in [58, 59] and applied in [103, 104], see also [105]. Similar are the random product homotopies [48, 49], see also [44] and [51]. Methods to construct linear-product start systems were introduced in [96], and extended in [97, 98], [54] and [112]. A general approach to exploit product structures was developed in [65].

Almost all systems have fewer terms than allowed by their degrees. Implementing constructive proofs of Bernshteĭn's theorems [5], polyhedral homotopies were introduced in [30] and [101] to solve sparse systems more efficiently. These methods provided ways to start cheater's homotopies ([50], [52]) and special instances of coeffient-parameter polynomial continuation ([60, 61]). The root count requires the calculation of the mixed volume, for which a lift-and-prune approach was presented in [16]. Exploitation of symmetry was studied in [99] and the dynamic lifting of [100] led to incremental polyhedral continuation. See [95] for a Toric Newton. Extensions to count all affine roots (also those with zero components) were proposed in in [17], [23], [31], [53], [71, 72], [73]. Very efficient calculations of mixed volumes are described in [13], [21, 22], [40], [46], and [92].

Determinantal systems (with equations like $\det(A|X) = 0$) arise in problems of enumerative geometry. The homotopies in numerical Schubert calculus first appeared explicitly in [32], originating from questions in real enumerative geometry [88, 89]. While real enumerative geometry [90] is interesting on its own, these homotopies solve the pole placement problem ([6], [68, 69], [74], [75]) in control theory. Recent improvements and applications can be found in [34], [55], [91], and [102].

We end this section noting that homotopies have a wider application range than "just" solving polynomial systems, see for instance [109] for a survey, [110], and [111] for a description of HOMPACK. The speedup of continuation methods on multi-processor machines has been addressed in [4, 7, 29].

3   A Dictionary

Kempf writes in [39] that "Algebraic geometry studies the delicate balance between the geometrically plausible and the algebraically possible". With our numerical tools, we feel closer to the geometrical than to the algebraic side, because we are not calculating with polynomials in the algebraic sense. In [84] we outlined the structure of a dictionary, presented as Table 2.

Numerical Algebraic Geometry Dictionary		
Algebraic Geometry	example in 3-space	Numerical Analysis
variety	collection of points, algebraic curves, and algebraic surfaces	polynomial system + union of witness sets, see below for the definition of a witness point
irreducible variety	a single point, or a single curve, or a single surface	polynomial system + witness set + probability-one membership test
generic point on an irreducible variety	random point on an algebraic curve or surface	point in a witness set; a witness point is a solution of polynomial system on the variety and on a random slice whose codimension is the dimension of the variety
pure dimensional variety	one or more points, or one or more curves, or one or more surfaces	polynomial system + set of witness sets of same dimension + probability-one membership tests
irreducible decomposition of a variety	several pieces of different dimensions	polynomial system + array of sets of witness sets and probability-one membership tests

Table 2: Dictionary to translate algebraic geometry into numerical analysis.

The probability-one membership test determines whether a given point $\mathbf{p}$ lies on a pure dimensional solution set. Suppose we have witness points defined by a polynomial system $f(\mathbf{x}) = \mathbf{0}$ and hyperplanes $L(\mathbf{x}) = \mathbf{0}$. A homotopy method implements the probability-one membership test:

1. Define $K(\mathbf{x}) = L(\mathbf{x}) - L(\mathbf{p})$. As $K(\mathbf{p}) = \mathbf{0}$, the hyperplanes K pass through $\mathbf{p}$.

2. Consider the homotopy

$$h(\mathbf{x}, t) = \begin{pmatrix} f(\mathbf{x}) \\ K(\mathbf{x}) \end{pmatrix} (1 - t) + \begin{pmatrix} f(\mathbf{x}) \\ L(\mathbf{x}) \end{pmatrix} t = \mathbf{0}. \qquad (2)$$

At $t = 1$ we start tracking paths at the witness set and find their end points at $t = 0$.

3. If $\mathbf{p}$ belongs to the solution set of $h(\mathbf{x}, 0) = \mathbf{0}$, then it is also a witness point of the pure dimensional solution set.

Notice that this test does not move the point $\mathbf{p}$, which may be a highly singular point. This observation is important for the numerical stability of this test.

4 Witness Sets and a Cascade of Homotopies

A witness set is the basic concept of numerical algebraic geometry as it allows us to apply numerical methods for isolated solutions to positive dimensional solution components.

Every irreducible component of a solution set is presented by a witness set whose cardinality equals the degree of the irreducible component. To reduce a solution set of dimension k to a set of isolated points, we cut the k degrees of freedom by adding k random hyperplanes $L(\mathbf{x}) = \mathbf{0}$ to the system $f(\mathbf{x}) = \mathbf{0}$ which defines the entire solution set.

One obstacle is that we have to deal with systems whose number of equations in not necessarily the same as the number of unknowns. If there are fewer equations than unknowns, we simply add enough random hyperplanes to make up for the difference, so underdetermined systems are easy to handle.

Let us consider overdetermined systems, say f consists of 5 equations in 3 variables. To turn f into a system of N equations in N variables where N is either 3 or 5, we can respectively apply the following techniques:

randomization: Choosing random complex numbers a_{ij}, we add random combinations of the last two polynomial to the first three polynomials:

$$\begin{cases} f_1(\mathbf{x}) + a_{11}f_4(\mathbf{x}) + a_{12}f_5(\mathbf{x}) = 0 \\ f_2(\mathbf{x}) + a_{21}f_4(\mathbf{x}) + a_{22}f_5(\mathbf{x}) = 0 \\ f_3(\mathbf{x}) + a_{31}f_4(\mathbf{x}) + a_{32}f_5(\mathbf{x}) = 0 \end{cases} \tag{3}$$

slack variables: We introduce two new variables z_1 and z_2 (so-called slack variables) and add random multiples of these variables to every equation:

$$\begin{cases} f_1(\mathbf{x}) + a_{11}z_1 + a_{12}z_2 = 0 \\ f_2(\mathbf{x}) + a_{21}z_1 + a_{22}z_2 = 0 \\ f_3(\mathbf{x}) + a_{31}z_1 + a_{32}z_2 = 0 \\ f_4(\mathbf{x}) + a_{41}z_1 + a_{42}z_2 = 0 \\ f_5(\mathbf{x}) + a_{51}z_1 + a_{52}z_2 = 0 \end{cases} \tag{4}$$

While the randomization technique might seem at first more attractive because we are left with fewer equations, working with slack variables provides a cascade of homotopies to compute candidate witness points on all positive dimensional components.

In particular, considering f_4 and f_5 as hyperplanes L_1 and L_2 to cut the solution set of the first three equation in f, we have the following systems in the cascade:

$$\begin{cases} f_1(\mathbf{x}) + a_{11}z_1 + a_{12}z_2 = 0 \\ f_2(\mathbf{x}) + a_{21}z_1 + a_{22}z_2 = 0 \\ f_3(\mathbf{x}) + a_{31}z_1 + a_{32}z_2 = 0 \\ L_1(\mathbf{x}) + z_1 = 0 \\ L_2(\mathbf{x}) + z_2 = 0 \end{cases} \quad \begin{cases} f_1(\mathbf{x}) + a_{11}z_1 + a_{12}z_2 = 0 \\ f_2(\mathbf{x}) + a_{21}z_1 + a_{22}z_2 = 0 \\ f_3(\mathbf{x}) + a_{31}z_1 + a_{32}z_2 = 0 \\ L_1(\mathbf{x}) + z_1 = 0 \\ z_2 = 0 \end{cases} \quad \begin{cases} f_1(\mathbf{x}) + a_{11}z_1 = 0 \\ f_2(\mathbf{x}) + a_{21}z_1 = 0 \\ f_3(\mathbf{x}) + a_{31}z_1 = 0 \\ z_1 = 0 \\ z_2 = 0 \end{cases} \tag{5}$$

We start with the largest system at the left. Solutions with $z_1 = 0$ and $z_2 = 0$ define witness points on the two dimensional solution components. Solutions with $z_1 \neq 0$ and $z_2 \neq 0$ provide start points in the homotopy which removes L_2 from the system. The paths defined by this move end at witness points on the one dimensional components, picked out by $z_1 = 0$. Solutions with $z_1 \neq 0$ are used in the homotopy which removes L_1 to lead to the isolated solutions of the systems. We next give a specific example of this cascade.

5 A Numerical Irreducible Decomposition

Consider the following example:

$$f(\mathbf{x}) = \begin{cases} (x_1 - 1)(x_2 - x_1^2) = 0 \\ (x_1 - 1)(x_3 - x_1^3) = 0 \\ (x_1^2 - 1)(x_2 - x_1^2) = 0 \end{cases} \tag{6}$$

From its factored form we see that $f(\mathbf{x}) = \mathbf{0}$ has two solution components: the two dimensional plane $x_1 = 1$ and the twisted cubic $\{ (x_1, x_2, x_3) \mid x_2 - x_1^2 = 0, \ x_3 - x_1^3 = 0 \}$.

To describe the solution set of this system, we use a cascade of homotopies, the chart in Figure 1 illustrates the flow of data for this example.

Because the top dimensional component is of dimension two, we add two random hyperplanes to the system and make it square again by adding two slack variables z_1 and z_2:

$$e(\mathbf{x}, z_1, z_2) = \begin{cases} (x_1 - 1)(x_2 - x_1^2) + a_{11}z_1 + a_{12}z_2 = 0 \\ (x_1 - 1)(x_3 - x_1^3) + a_{21}z_1 + a_{22}z_2 = 0 \\ (x_1^2 - 1)(x_2 - x_1^2) + a_{31}z_1 + a_{32}z_2 = 0 \\ c_{10} + c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + z_1 = 0 \\ c_{20} + c_{21}x_1 + c_{22}x_2 + c_{23}x_3 + z_2 = 0 \end{cases} \tag{7}$$

where all constants a_{ij}, $i = 1, 2, 3$, $j = 1, 2$, and c_{kl}, $k = 1, 2$, $l = 0, 1, 2, 3$ are randomly chosen complex numbers. Observe that when $z_1 = 0$ and $z_2 = 0$ the solutions to $e(\mathbf{x}, z_1, z_2) = \mathbf{0}$ satisfy $f(\mathbf{x}) = \mathbf{0}$. So if we solve $e(\mathbf{x}, z_1, z_2) = \mathbf{0}$ we will find a single witness point on the two dimensional solution component $x_1 = 1$ as a solution with $z_1 = 0$ and $z_2 = 0$. Using polyhedral homotopies, this requires the tracing of six solutions paths.

The embedding was proposed in [77] to find generic points on all positive dimensional solution components with a cascade of homotopies. In [77] it was proven that solutions with slack variables $z_i \neq 0$ are regular and, moreover, that those solutions can be used as start solutions in a homotopy to find witness points on lower dimensional solution components. At each stage of the algorithm, we call solutions with nonzero slack variables *nonsolutions*.

In the solution of $e(\mathbf{x}, z_1, z_2) = \mathbf{0}$, one path ended with $z_1 = 0 = z_2$, the five other paths ended in regular solutions with $z_1 \neq 0$ and $z_2 \neq 0$. These five "nonsolutions" are start solutions for the next stage, which uses the homotopy

$$h_2(\mathbf{x}, z_1, z_2, t)$$
$$= \begin{cases} (x_1 - 1)(x_2 - x_1^2) + a_{11}z_1 + a_{12}z_2 = 0 \\ (x_1 - 1)(x_3 - x_1^3) + a_{21}z_1 + a_{22}z_2 = 0 \\ (x_1^2 - 1)(x_2 - x_1^2) + a_{31}z_1 + a_{32}z_2 = 0 \\ c_{10} + c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + z_1 = 0 \\ z_2(1 - t) + (c_{20} + c_{21}x_1 + c_{22}x_2 + c_{23}x_3 + z_2)t = 0 \end{cases} \tag{8}$$

where t goes from one to zero, replacing the last hyperplane with $z_2 = 0$. Of the five paths, four of them converge to solutions with $z_1 = 0$. Of those four solutions, one of them is found to lie on the two dimensional solution component $x_1 = 1$, the other three are generic points on the twisted cubic. As there is one solution with $z_1 \neq 0$, we have one candidate left to use as a start point in the final stage, which searches for isolated solutions of $f(\mathbf{x}) = \mathbf{0}$. The homotopy for this stage is

$$h_1(\mathbf{x}, z_1, t) = \begin{cases} (x_1 - 1)(x_2 - x_1^2) + a_{11}z_1 = 0 \\ (x_1 - 1)(x_3 - x_1^3) + a_{21}z_1 = 0 \\ (x_1^2 - 1)(x_2 - x_1^2) + a_{31}z_1 = 0 \\ z_1(1 - t) + (c_{10} + c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + z_1)t = 0 \end{cases} \tag{9}$$

which as t goes from 1 to 0, replaces the last hyperplane $z_1 = 0$. At $t = 0$, the solution is found to lie on the twisted cubic, so there are no isolated solutions.

The calculations are summarized in Figure 1. The breakup into irreducibles will be explained in the next section.
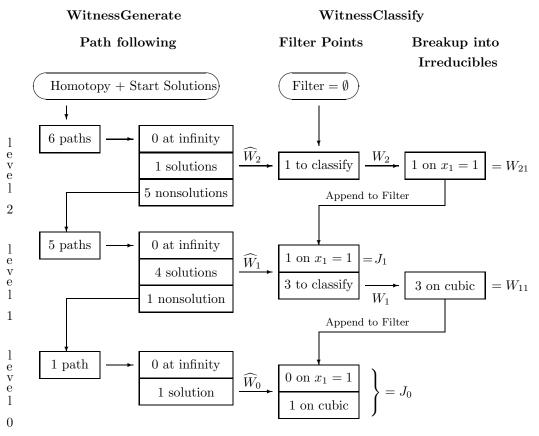

Figure 1: Numerical Irreducible Decomposition of a system whose solutions are the 2-dimensional plane $x_1 = 1$, the twisted cubic, and one isolated point. At level i, for $i = 2, 1, 0$, we filter candidate witness sets $\widehat{W}_i$ into junk sets J_i and witness sets W_i. The sets W_i are partitioned into witness sets W_{ij} for the irreducible components.

6   Factorization Methods

A recent trend in computer algebra is the adaption of symbolic methods to deal with approximate input data, which leads to the use of hybrid methods [12]. One such problem is the factor of multivariate polynomials, listed as a challenge in [38]. Recent papers on this problem are [9, 10], [19, 20], [37], and [76].

6.1 Monodromy to Partition Witness Point Sets

We can see whether a curve factors or not by looking at its plot in complex space, i.e.: we consider the curve as a Riemann surface. Figure 2 was made with Maple (see [11] for instructions).
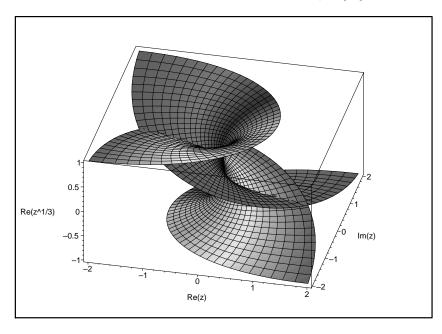

Figure 2: The Riemann surface of $z^3 - w = 0$. The height of the surface is the real part of $w = z^{1/3}$, while the gray scale corresponds to the imaginary part of $w = z^{1/3}$. Observe that a loop around the origin permutes the order of points.

Looking at Figure 2, imagine a line which intersects the surface in three points. Taking one complete turn of the line around the vertical axis $z = 0$ will cause the points to permute. For example, the point which was lowest will have moved up, while another point will have come down. Such a permutation can only happen if the corresponding algebraic curve is irreducible.

Based on this observation, we can decompose any pure dimensional set into irreducible components. Our monodromy algorithm returns a partition of the witness set for a pure dimensional component: points in the same subset of the partition belong to the same irreducible component. Recall that witness points are defined by a system $f(\mathbf{x}) = \mathbf{0}$ and a set of hyperplanes $L(\mathbf{x}) = \mathbf{0}$. With the homotopy

$$h_{KL}(\mathbf{x}, t) = \lambda \left(\begin{array}{c} f(\mathbf{x}) \\ K(\mathbf{x}) \end{array} \right) (1-t) + \left(\begin{array}{c} f(\mathbf{x}) \\ L(\mathbf{x}) \end{array} \right) t = \mathbf{0}, \quad \lambda \in \mathbb{C}, \tag{10}$$

we find new witness points on the hyperplanes $K(\mathbf{x}) = \mathbf{0}$, starting at those witness points satisfying $L(\mathbf{x}) = \mathbf{0}$, letting t move from one to zero. Choosing another random constant $\mu \neq \lambda$, we move back from K to L, using the homotopy

$$h_{LK}(\mathbf{x}, t) = \mu \left(\begin{array}{c} f \\ L \end{array} \right) (1-t) + \left(\begin{array}{c} f \\ K \end{array} \right) t = \mathbf{0}, \quad \mu \in \mathbb{C}. \tag{11}$$

The homotopies $h_{KL}(\mathbf{x}, t) = \mathbf{0}$ and $h_{LK}(\mathbf{x}, t) = \mathbf{0}$ implement one loop in the monodromy algorithm, moving witness points from L to K and then back from K to L. At the end of the loop we have the same witness set as the set we started with, except possibly permuted. Permuted points belong the same irreducible component.

Notice that the monodromy algorithm does not know the locations of the singularities. See [14] for the algorithms to compute the monodromy group of an algebraic curve in Maple (package algcurves).

6.2 Linear Traces to Validate the Partition

When we run the monodromy algorithm, we may not have made enough loops to group as many witness points as the degree of each factor, i.e.: the partition predicted by the monodromy might be too fine. For a k-dimensional solution component, it suffices to consider a curve on the component cut out by $k - 1$ random hyperplanes. The factorization of the curve tells the decomposition of the solution component. Therefore, we restrict our explanation of using the linear trace to the case of a curve in the plane.

Suppose we have three points in the plane obtained as (projections of) witness points from some polynomial system. If the monodromy found loops between those points, then we know that these points lie on an irreducible factor of degree at least three. Whence our question: is this irreducible factor on which the given three points lie of degree three?

To answer this question we represent the factor by a cubic polynomial f in the form

$$
\begin{aligned}
f(x, y(x)) &= (y - y_1(x))(y - y_2(x))(y - y_3(x)) \\
&= y^3 - t_1(x)y^2 + t_2(x)y - t_3(x)
\end{aligned}
\tag{12}
$$

Since $\deg(f) = 3$, $\deg(t_1) = 1$, so t_1 is the *linear trace*: $t_1(x) = c_1 x + c_0$.

We now proceed as follows. Via interpolation we find the coefficients c_0 and c_1. We first sample the cubic at $x = x_0$ and $x = x_1$. The samples are $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$. To find c_0 and c_1 we then solve the linear system

$$
\begin{cases}
y_{00} + y_{01} + y_{02} = c_1 x_0 + c_0 \\
y_{10} + y_{11} + y_{12} = c_1 x_1 + c_0
\end{cases}
\tag{13}
$$

With t_1 we can predict the sum of the y's for a fixed choice of x. For example, samples at $x = x_2$ are $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$. So our test consists in computing $t_1(x_2)$ in two ways:

$$
c_1 x_2 + c_0 = y_{20} + y_{21} + y_{22}.
\tag{14}
$$

If the equality holds, then the answer to our question is yes.

6.3 Efficiency and Numerical Stability

The validation with the linear trace is fast. Therefore, our implementation does this validation each time a new loop with the monodromy algorithm is found. Even as we do not know the locations of the singularities, practical experiences on many systems all lead to a rapid finding of permutations. While this approach is suitable for irreducible factors of very large degree (e.g., one thousand), strategies purely based on traces perform often better for smaller degrees.

Related to the efficiency is the good numerical stability: if we can compute witness points with standard machine arithmetic, then we can also factor using standard machine arithmetic. This feature is very important when the accuracy of coefficients of the polynomial system is limited.

7 Software

We agree with the statement: "It can be argued that the "mission" of numerical analysis is to provide the scientific community with effective software tools." (taken from the preface to [26]). Aside from our missionary intentions, software has helped us refining our algorithms, along the lines of the quote (from [41]): "Another reason that programming is harder than the writing of books and research papers is that programming demands a significant higher standard of accuracy."

The software package PHCpack [93] is currently undergoing the transition from being a toolbox/blackbox for various homotopy continuation methods to approximate all isolated solutions to a complete solving environment with capabilities to handle positive dimensional solution components efficiently, both in terms of computer operations and user manipulations. By the latter we hint at the search to find the right user interface, identifying the right data flow and trying to balance the toolbox with the blackbox approach.

While PHCpack offered the first reliable implementation of polyhedral homotopies, its efficiency is currently surpassed by the implementations described in [21, 22, 46] and [13, 28, 40, 92]. To interact better with other codes, we are currently developing an interface from the Ada routines in PHCpack to routines written in C. Another (but related) interface concerns the interaction with computer algebra software. In [84] we describe a very simple interface to Maple.

8 Applications

A benchmark suite for systems with positive dimensional solution components is gradually taking shape. Rather than listing summaries of a benchmark, we choose to treat two very typical applications: the cyclic n-roots problem from computer algebra and a special Stewart-Gough platform from mechanical design.

The cyclic n-roots problem. This problem is already interesting by its compact formulation and widespread fame in the computer algebra community, but by known theoretical results concerning the number of isolated roots when n is prime.

For $n = 8$, there are 16 one dimensional irreducible components: eight quadrics and eight curves of degree 16. While approximations to all 1,152 isolated cyclic 8-roots were found already in the first release of PHCpack, monodromy was needed to factor the curve of degree 144 into irreducibles. To compute all witness points for the cyclic 9-roots problem, the software of [46] was essential. While the factorization of a two dimensional component of degree 18 into six cubics posed no difficulty, the homotopy membership test was required to certify that among the 6,642 isolated ones 162 cyclic 9-roots occured with multiplicity four. In addition, multi-precision arithmetic was used to confirm this result.

The isolated cyclic n-roots (up to $n = 13$, for which 2,704,156 paths were traced) can be found on `http://www.is.titech.ac.jp/~kojima/polynomials/cyclic13`. These roots have been computed with PHoM [28].

A special Stewart-Gough platform. The Stewart-Gough platform is a parallel robot which attracted lots of interest from computational kinematicians and researchers in computer algebra. That the platform has forty isolated solutions was first established computationally by continuation [67] and elimination methods [42, 66], later proved analytically in [35] and [106].

A six-legged platform (similar to the general Stewart-Gough platform) which permits motion was presented by [27] and first analyzed in [36]. Instead of forty isolated solutions we now

consider a curve. In our formulation of the two cases we studied, twelve lines corresponded to degenerate cases deemed uninteresting from a mechanisms point of view. In the first case we were then left with one irreducible component of degree 28, while in the second case we found five components, four of degree six (one sextic was missed by the analysis of [36]), and one component of degree four, see Figure 3.
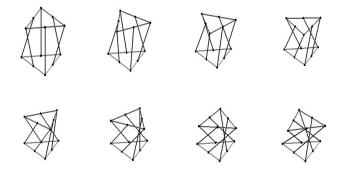

Figure 3: One component of the Griffis-Duffy platform. Starting at the configuration at the left above, we see the clockwise rotation of the end platform.

It is interesting to note that the running times for the factorization with the monodromy-traces method does not seem to depend on the particular geometry of the system, i.e.: the execution times are about the same in both cases, when we deal with one irreducible factor of high degree or with several factors of smaller degrees.

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