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On a Combinatorial Problem in Group Theory*

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Abstract. Let n be a positive integer or infinity (denote ∞). We denote by $W^*(n)$ the class of groups G such that, for every subset X of G of cardinality $n+1$, there exist a positive integer k, and a subset $X_0 \subseteq X$, with $2 \le |X_0| \le n+1$ and a function $f : \{0,1,2,\ldots,k\} \longrightarrow X_0$, with $f(0) \ne f(1)$ and non-zero integers $t_0, t_1, \ldots, t_k$ such that $[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = 1$, where $x_i := f(i)$, $i = 0, \ldots, k$, and $x_j \in H$ whenever $x_j^{t_j} \in H$, for some subgroup $H \ne \left\langle x_j^{t_j} \right\rangle$ of G. If the integer k is fixed for every subset X we obtain the class $W_k^*(n)$. Here we prove that

(1) Let $G \in W^*(n)$, n a positive integer, be a finite group, $p > n$ a prime divisor of the order of G, P a Sylow p-subgroup of G. Then there exists a normal subgroup K of G such that $G = P \times K$.
(2) A finitely generated soluble group has the property $W^*(\infty)$ if and only if it is finite-by-nilpotent.
(3) Let $G \in W_k^*(\infty)$ be a finitely generated soluble group, then G is finite-by-(nilpotent of k-bounded class).

Keywords: combinatorial conditions, finitely generated soluble groups

1. Introduction and Results

B. H. Neumann has proved [19] that a group is center-by-finite if and only if every infinite subset contains a commuting pair of distinct elements. This result was an affirmative answer to a question of P. Erdös. Other problems of this type have been the object of several articles, for example [1]-[12], [15]-[17], [19], [23]-[25].

Our notation and terminology are standard and can be found in [20]. In particular for a group G and elements $x, y, x_1, x_2, \ldots, x_k \in G$ we write

$$[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2 = x_1^{-1} x_1^{x_2}, \quad [x_1, \ldots, x_k] = [[x_1, \ldots, x_{k-1}], x_k]$$

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$$[x,\ _0y] = x, \quad [x,\ _ky] = [[x,\ _{k-1}y], y].$$

A group is said to be a k-Engel group (respectively, Engel group) if for all $x, y \in G$ $[x,\ _ky] = 1$ (respectively, there exists a positive integer t depending on x and y such that $[x,\ _ty] = 1$). The class of k-Engel (respectively, Engel) groups will be denoted by $\mathcal{E}_k$ (respectively, $\mathcal{E}$).

Let k be a positive integer, n be a positive integer or infinity (denote ∞). We denote by $\mathcal{E}_k(n)$ (respectively, $\mathcal{E}(n)$) the class of all groups G such that, for every subset X of cardinality $n+1$, there exist distinct elements $x, y \in X$ such that $[x,\ _ky] = 1$ (respectively, $[x,\ _ty] = 1$ for some positive integer t depending on x, y). Longobardi and Maj [16] (see also [9]) noticed that a finitely generated soluble group G has the property $\mathcal{E}(\infty)$ if and only if G is finite-by-nilpotent. Abdollahi [2] has showed that finite $\mathcal{E}(2)$-groups (respectively, $\mathcal{E}(15)$-groups) are nilpotent (respectively, soluble) and a finitely generated residually finite $\mathcal{E}_k(n)$-group is finite-by-nilpotent.

In order to generalize the classes of groups mentioned above, we define a new class of groups as follows. Let n be a positive integer or infinity. We denote by $W(n)$ the class of groups G such that, for every subset X of G of cardinality $n+1$, there exist a subset $X_0 \subseteq X$, with $2 \leq |X_0| \leq n+1$, such that the following condition holds

There exists a positive integer k and a function $f : \{0, 1, 2, \ldots, k\} \longrightarrow X_0$, with $f(0) \neq f(1)$ and non-zero integers $t_0, t_1, \ldots, t_k$ such that $[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = 1$, where $x_i := f(i)$, $i = 0, \ldots, k$.

The class $W^*(n)$ is defined exactly as $W(n)$, with additional conditions "$x_j \in H$ whenever $x_j^{t_j} \in H$, for some subgroup $H \neq \left\langle x_j^{t_j} \right\rangle$ of G". If the integer k is the same for any subset X of G, we say that G is in the class $W_k(n)$. Similarly we define $W_k^*(n)$. Clearly the classes $W(n)$ and $W^*(n)$ are subgroup and quotient closed.

If the subset X_0 in the definition of $W(n)$ has always 2 elements we obtained the class $\Omega(n)$ [24]. If in addition $t_i \in \{\pm 1\}$ one obtains the class $E^{\#}(n)$ [7]. If $X_0 = \{x, y\}$ and the function f is always of the form $f(0) = x$ and $f(i) = y$ and $t_0 = t_1 = \cdots = t_k = 1$ one can obtain the class $\mathcal{E}(n)$. Note that

$$\mathcal{E}(n) \subseteq E^{\#}(n) \subseteq W(n) \subseteq W(n+1) \quad \text{and} \quad W^*(n) \subseteq W(n) \subseteq \Omega(n).$$

Trabelsi [25] has recently proved that a finitely generated soluble group G is nilpotent-by-finite if and only if for every pair X, Y of infinite subsets of G there exist x in X, y in Y and two positive integers $m = m(x, y), n = n(x, y)$ satisfying $[x,\ _ny^m] = 1$. The author [24] (see also [7]) has generalized the result of Trabelsi and showed that a finitely generated soluble group G has the property $\Omega(\infty)$ if and only if G is nilpotent-by-finite. In [7] it is shown that every finitely generated soluble $E^{\#}(\infty)$-group is finite-by-nilpotent.

Our first result is about finite groups in $W^*(n)$:

Theorem 1.1. *Let $G \in W^*(n)$, n a positive integer, be a finite group, $p > n$ a prime divisor of the order of G, and P a Sylow p-subgroup of G. Then there exist a normal subgroup K of G such that $G = P \times K$.*

Generalizing some results mentioned above, also we prove that

Theorem 1.2. *Let G be a finitely generated soluble group. Then $G \in W^*(\infty)$ if and only if G finite-by-nilpotent.*

Also we prove that

Theorem 1.3. *Let $G \in W_k^*(\infty)$ be a finitely generated soluble group. Then G is finite-by-(nilpotent of k-bounded class).*

2. Proof of Theorem 1.1

First note that if A is an Abelian normal subgroup of a group G, $a_0, a_1, \ldots, a_k \in A$, $t_0, t_1, \ldots, t_k \in \mathbf{Z}$, and $g \in G$ then

$$\begin{aligned}[(g^{a_0})^{t_0}, (g^{a_1})^{t_1}, (g^{a_2})^{t_2}, \ldots, (g^{a_k})^{t_k}] &= [(g^{a_0})^{t_0}, (g^{a_1})^{t_1}, g^{t_2}, \ldots, g^{t_k}] \\ &= [g^{t_0}, a_0 a_1^{-1}, g^{t_1}, g^{t_2}, \ldots, g^{t_k}]\end{aligned}$$

Now as in Lemma 2 of [24] we can easily prove the following Lemma

Lemma 2.1. *Let G be an infinite group in $W^*(\infty)$, and A be a normal Abelian subgroup of G. If there exist a torsion free element g of G such that $C_A(g^m) = 1$, the centralizer of g^m in A, for all integers m, then A is finite.*

Proof. Suppose that A in infinite. Then the set $g^A = \{g^a | a \in A\}$ is infinite, as $C_A(g) = 1$. Now, since $G \in W^*(\infty)$, there exist elements $a_i \in A$ and integers t_i, $i = 0, 1, \ldots, k$, such that $[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = 1$, where $x_i \in \{g^{a_0}, g^{a_1}, \ldots, g^{a_k}\}$, $i = 0, 1, 2, \ldots k$, and $x_0 \neq x_1$. Thus, by the above observation, $[g^{t_0}, a_0 a_1^{-1}, g^{t_1}, g^{t_2}, \ldots, g^{t_k}] = 1$. Now, since A is normal Abelian, $u = [g^{t_0}, a_0 a_1^{-1}, g^{t_1}, g^{t_2}, \ldots, g^{t_{k-1}}] \in C_A(g^{t_k}) = 1$. So $u = 1$. Continuing in this way we find that $[g^{t_0}, a_0 a_1^{-1}] \in C_A(g^{t_1}) = 1$. So $a_0 a_1^{-1} \in C_A(g^{t_0}) = 1$, a contradiction. ∎

The following Lemma is proved similarly

Lemma 2.2. *Let G be group in $W^*(n)$, A be a normal Abelian subgroup of G. If there exist $g \in G$, such that $C_A(g^m) = 1$, for all integers m, with $g^m \neq 1$, then $|A| \leq n$.*

The proof of the Theorem 1.1 is based on the following Proposition

Proposition 2.3. *Let G be a finite group, p a prime divisor of $|G|$. If every proper subgroup and every proper quotient group of G can be expressed as a direct product of a Sylow p-subgroup and a p'-subgroup, but G itself does not, then G is splitting extension of a cyclic group of order q by a cyclic group of order p in which p divides $q-1$.*

Proof. First note that, by hypothesis, G is not a p-group. Let P be a p-subgroup of G, then $N_G(P)$, the normalizer of P in G, is a proper subgroup of G and so we have $N_G(P) = P \times Q$, where Q is a p'-subgroup of G. Thus $Q \leq C_G(P)$ and it follows that $N_G(P)/C_G(P)$ is a p-group. So by a Theorem of Frobenius (see [21, Theorem 10.47, p. 262]) G is p-nilpotent. Now let P be a Sylow p-subgroup of G and H be the normal complement of P in G. Thus $G = PH$, $P \cap H = 1$. Suppose that q be any prime divisor of the order of H, and Q be a Sylow q-subgroup of G. Then, since $Q \leq H \leq G$, $G = HN_G(Q)$ and so $P \leq N_G(Q)$. Now if PQ is a proper subgroup of G we have $PQ = P \times Q$, and thus $[P, Q] = 1$. If this is true for all prime divisor q of $|H|$ then, since p'-elements generate H, $[P, H] = 1$, which is contradiction. Therefore there exist a prime divisor q of $|H|$, such that $G = PQ$. Thus $H = Q$, and G is a $\{p, q\}$-group.

Now if $Q' \neq 1$, the derived subgroup of Q then, by hypothesis, G/Q' is the direct product of its Sylow p-subgroup and Sylow q-subgroup and so that it is nilpotent. Hence by a result of P. Hall (see [20, Theorem 5.2.10, p. 129]) G is nilpotent, a contradiction. Thus $Q' = 1$ and Q is Abelian.

Let x be any non-trivial element of P. If $\langle x \rangle Q$ is a proper subgroup of G then, by hypothesis, $\langle x \rangle Q = \langle x \rangle \times Q$ and $[x, Q] = 1$. Since $[P, Q] \neq 1$, there exists $1 \neq x \in P$ such that $\langle x \rangle Q = PQ$ and so $P = \langle x \rangle$.

Note that if $Z(G) \neq 1$, then $G/Z(G)$ is proper quotient of G and , by hypothesis, is nilpotent and so G is nilpotent. Hence $Z(G)$=1. Suppose, if possible, that $1 \neq \langle x^m \rangle$ is proper subgroup of P, then by hypothesis, $x^m \in Z(G) = 1$. Thus $|P| = p$. Now Q is Abelian and, as above we can see that $Q = \langle a \rangle$ is cyclic of order q. Since $P = \langle x \rangle$ acts faithfully on $Q = \langle a \rangle$ it follows that p divides $q - 1$. This completes the proof. ■

Proof of Theorem 1.1 Suppose the assertion of the Theorem is false and choose a counter-example G of smallest order. Then, by Proposition 2.3, G is the splitting extension of a cyclic group $Q = \langle a \rangle$ of order r by a cyclic group $P = \langle x \rangle$ of order s in which s divides $r - 1$, where r, s are primes. Now, since $C_Q(x) \leq Z(G) = 1$, we have $r = |Q| \leq n$, by Lemma 2.2. If $p = s$, then $n < p = s < r \leq n$, and if $p = r$, then $n < p = r \leq n$, a contradiction. ■

Corollary 2.4. *If $G \in W^*(n)$, n a positive integer, is a finite group, then $G = P_1 \times P_2 \times \cdots \times P_r \times H$, where P_i is a Sylow p_i-subgroup, with $p_i > n$, and H is a Hall π-subgroup, where $\pi = \{q \mid q \leq n\}$.*

In particular, in Corollary , if $n = 2$, then $\pi = \{2\}$, and so H is nilpotent. Also, if $n = 4$, then $\pi = \{2, 3\}$, and so H is soluble. Thus we have

Corollary 2.5. *Let $G \in W^*(2)$ be finite group, then G is nilpotent.*

Corollary 2.6. *Let $G \in W^*(4)$ be finite group, then G is soluble.*

3. Proof of Theorem 1.2

As in the proof of the [24, Lemma 4] using Lemma 1.1, one can show that the restricted wreath product of a cyclic group A of prime order with the infinite cyclic group has not the property $W(\infty)$. Therefore, exactly as in the proof of [24, Theorem 2] (see also [7, Theorem 1.1]), we have

Proposition 3.1. *Let G be a finitely generated soluble group. The $G \in W(\infty)$ if and only if G is nilpotent-by-finite.*

Since the dihedral group is nilpotent-by-finite it satisfies the condition $W(\infty)$. The following Lemma shows that the class $W^*(\infty)$ is properly contained in $W(\infty)$.

Lemma 3.2. *The infinite dihedral group has not the property $W^*(\infty)$.*

Proof. Let $G = \langle a, x \mid x^2 = 1, a^x = a^{-1} \rangle$ be the infinite dihedral group. Then $\{ax, a^2x, a^3x, \ldots\}$ is an infinite set of elements of order 2. Since

$$\begin{aligned}[][a^{i_0}x, a^{i_1}x, a^{i_2}x, \ldots, a^{i_k}x] &= [a^{i_0}x, a^{i_1}x,_{k-1} x] \\ &= [a^{2(i_1-i_0)},_{k-1} x] \neq 1,\end{aligned}$$

G does not satisfies the condition $W^*(\infty)$. ∎

Let H be group and A a H-module. Recall that A is rationally irreducible, if A is torsion free Abelian group of finite rank and $V = A \otimes_{\mathbf{Z}} \mathbf{Q}$ is an irreducible $\mathbf{Q}H$-module. If A is torsion free Abelian group of finite rank, then A is rationally irreducible if and only if every non-zero H-submodule of A has finite index in A (see [22, p. 23]). To prove the Theorem 1.2 we need the following key Lemma

Lemma 3.3. *Let $G = \langle A, x \rangle$, where A is a normal Abelian torsion free of finite rank on which $\langle x \rangle$ acts rationally irreducibly. If $G \in W(\infty)$ then for some positive integer d, $\langle x^d \rangle$ acts trivially on A.*

Proof. Since G is a finitely generated meteabelian group, there exists a positive integer d, depending only on G, such that $C_G(g^m) \leq C_G(g^d)$, for all $g \in G$ (see

[14, Theorem B]).
Let a be a non-identity element of A. If $x^{a^i} = x^{a^j}$, for some positive integers i, j, then $[a^{i-j}, x] = 1$ and so $[a, x] = 1$, as A is Abelian normal and torsion free. So we may assume that the set $\{x^a, x^{a^2}, x^{a^3}, \ldots\}$ is infinite. Since $G \in W(\infty)$, there exists a positive integer k, positive integers $i_0, i_1, \ldots, i_k$, and non-zero integers $t_0, t_1, \ldots, t_k$, such that $[(x^{a^{i_0}})^{t_0}, (x^{a^{i_1}})^{t_1}, \ldots, (x^{a^{i_k}})^{t_k}] = 1$ and so $[(x^{a^{i_0}})^{t_0}, (x^{a^{i_1}})^{t_1}, x^{t_2}, \ldots, x^{t_k}] = 1$. Since $C_G(x^{t_k}) \leq C_G(x^d)$, we have $[(x^{a^{i_0}})^{t_0}, (x^{a^{i_1}})^{t_1}, x^{t_2}, \ldots, x^d] = 1$. Now, as G is metabelian, for any $u \in G'$ and $v, w, x_1, x_2, \ldots, x_k \in G$, we have $[u, v, w] = [u, w, v]$ and $[x_1, x_2, x_3, \ldots, x_k]^{-1} = [x_2, x_1, x_3, \ldots, x_k]$. It follows that

$$\begin{aligned} 1 &= [(x^{a^{i_0}})^{t_0}, (x^{a^{i_1}})^{t_1},_{k-1} x^d] \\ &= [x^{t_0}, a^{i_0 - i_1}, x^{t_1},_{k-1} x^d] \\ &= [x^{t_0}, a^{i_0 - i_1},_k x^d] \\ &= [a^{i_0 - i_1}, x^{t_0},_k x^d]^{-1}, \end{aligned}$$

and so $[a^{i_0-i_1},_{k+1} x^d] = 1$. Since A is Abelian normal and torsion free, we have $[a,_{k+1} x^d] = 1$.
Let $V = A \otimes_{\mathbf{Z}} \mathbf{Q}$. Then V is an irreducible $\mathbf{Q}\langle x \rangle$-module and, by Schur's Lemma, $D = \mathrm{End}_{\mathbf{Q}\langle x \rangle} V$ is a division ring of finite dimension over $\mathbf{Q}$. Now the image of $\langle x \rangle$ in $\mathrm{End}_{\mathbf{Q}} V$ lies in D and generates D. Hence D is an algebraic number field. As D-space, V is one dimensional. Let α be the image of x in D. Then we can identify V with $\mathbf{Q}(\alpha)$ under addition and the action of x on V being that multiplication by α. If β corresponds to a in the isomorphism of V and $\mathbf{Q}(\alpha)$, then the equality $[a,_{k+1} x^d] = 1$ translates into

$$\beta(1 - \alpha^d)^{k+1} = 0.$$

Therefore $\alpha^d = 1$ and α is a root of unity. This means that x^d acts trivially on A and the proof is complete. ∎

Now we are ready to prove the Theorem 1.2

Proof of Theorem 1.2 Let $G \in W^*(\infty)$ and suppose, for a contradiction, that G is not finite-by-nilpotent. By Proposition 3.1, G is nilpotent-by-finite and so G satisfies the maximal condition on normal subgroups. Hence there exists among the normal subgroups of G one, say N, which is maximal subject to G/N is not finite-by-nilpotent. Replacing G by G/N, we may assume that every proper quotient of G is finite-by-nilpotent but G itself is not. Let T be the maximal finite normal subgroup of G. Again replacing G by G/T, we may assume that G has no non-trivial finite normal subgroup.

Let $K := \mathrm{Fitt}(G)$, the Fitting subgroup of G. If $K \neq G$, then let any $x \in G \backslash K$ with x^p, where p is a prime. Suppose we have shown that $\langle K, x \rangle$, where K is torsion free, is nilpotent. Since G/K is finite and soluble, there is a subnormal series

$$K = K_0 < K_1 < \cdots < K_m = G$$

with prime order factors. Then $K_i = K_{i-1}\langle x_i \rangle$, where $x_i^{p_i} \in K_i$. Note that $\tau(K)$, the torsion subgroup of K, is a finite normal subgroup of G. So $\tau(K) = 1$ and $K = K_0$ is torsion free. Now $K_1 = \langle K_0, x_1 \rangle$ is nilpotent, and $\tau(K_1)$ is a normal finite subgroup of $K_2 = \langle K_1, x_2 \rangle$. Using bars (temporarily) to denote the factor groups modulus $\tau(K_1)$, we have $\overline{K_2} = \langle \overline{K_1}, \overline{x_2} \rangle$. Now $\overline{K_1}$ is torsion free and so $\overline{K_2}$ is nilpotent. Continuing in this way we get that G is finite-by-nilpotent, which is contradiction.

Thus we may assume that $G = \langle K, x \rangle$, $x^p \in K$, and K is torsion free to show that G is nilpotent.

If $Z^*(G) \not\leq K$, where $Z^*(G)$ is the hypercenter of G, then $G = Z^*(G)K$ and $G/Z^*(G) \simeq K/Z^*(G) \cap K$ and so G is nilpotent. For let $ax^i \in Z^*(G) \backslash K$, where $a \in K$ and i is an integer which is prime to p, then $x^i = a^{-1}(ax^i) \in KZ^*(G)$. There exist integers c_1, c_2 with $c_1 i + c_2 p = 1$ so $x = x^{c_1 i} x^{c_2 p} \in KZ^*(G)K = KZ^*(G)$, and so $G = Z^*(G)K$ as required. So assume that $Z^*(G) \leq K$.

Now since $Z(G) \leq Z(K)$ and $K/Z(K)$ is torsion free, so is $K/Z(G)$. Also since $Z_j(G) \leq Z_j(K)$ and $Z_j(K)/Z_{j-1}(K)$ is torsion free, for all $j \geq 1$, we conclude that $K/Z^*(G)$ is torsion free. Therefore $G/Z^*(G) = \langle K/Z^*(G), \langle x \rangle Z^*(G) /Z^*(G) \rangle$. If $G/Z^*(G)$ is nilpotent then G is nilpotent. So, in order to obtain a contradiction, we may assume that $Z^*(G) = 1$. In particular $Z(G) = 1$.

Let A be a non-trivial normal subgroup of G, lying in the center of $\mathrm{Fitt}(G)$ of least Hirsch length. Note that $K \leq \mathrm{Fitt}(G)$ and $A \leq Z(\mathrm{Fitt}(G)) \leq C_G(K)$. A is torsion free, since G has no non-trivial finite normal subgroup. Now $\langle x \rangle$ acts rationally irreducibly on A and so, by Lemma 3.3, $\langle x^d \rangle$, for some positive integer d acts trivially on A. Thus $[A, x^d] = 1$, that is $x^d \in C_G(A)$. Since $A \leq Z(\mathrm{Fitt}(G))$ is a normal subgroup of G, it is a non-trivial rationally irreducible G-module. Now $\overline{G} := G/C_G(A) = \langle xC_G(A) \rangle$, since $A \leq C_G(K)$. Note that $\overline{G}$ is an irreducible linear group over $\mathbf{Q}$. Since $x^d \in C_G(A)$, $\overline{G}$ is a finite cyclic irreducible group and so it is necessarily of order 2. Therefore $x^2 \in C_G(A)$.

Now we use the notations of the Lemma 3.3. Let $a \in A$, then $1 = [a, x^2] = a^{-1+x^2}$ and so in $\mathbf{Q}(\alpha)$ we have $\beta(1 - \alpha^2) = 0$, where β corresponds to a in the isomorphism of V and $\mathbf{Q}(\alpha)$. Thus $\alpha^2 = 1$ and so $\alpha = \pm 1$. Now A is a finitely generated group which is isomorphic to a subgroup of $\mathbf{Q}(\alpha) = \mathbf{Q}$ and therefore $A = \langle a \rangle$ is cyclic.

If $\alpha = 1$ then $\beta(1 - \alpha) = 0$ and so $[a, x] = 1$, that is $\langle x \rangle$ acts trivially on A. Hence $A \leq Z(G) = 1$, a contradiction.

If $\alpha = -1$ then $\beta(1 + \alpha) = 0$ and so $a^{1+x} = 1$, that is $a^x = a^{-1}$. Let $N = A\langle x \rangle = \langle a, x \rangle$, then $a^x = a^{-1}$ and $x^2 \in Z(N)$. Therefore $N/Z(N)$ is an infinite dihedral group which is not belong to $W^*(\infty)$, by Lemma 3.2, a contradiction. ■

If we argue as in [7, Lemma 2.1], we can prove that a finitely generated metabelian torsion nilpotent group G has the property $W_k^*(\infty)$ if and only if

$G = Z_k(G)$. We give sketch of proof: By induction on the nilpotency class of G we assume that $G = Z_{k+1}(G)$, and prove that G is k-Engel group (see [20, 12.3.3 (i)]). If G is not k-Engel then there exist $x, y \in G$ such that $[x,_k y] \neq 1$. Now since $H = \langle x, y\rangle$ is a residually finite 2-group there exist a normal subgroup N such that $[x,_k y] \notin N$ and $|H/N| = 2^r$. Considering the set $\{x^{2^{r+m}} y \mid m \in \mathbf{N}\}$ we obtain that $[(x^{c_0}y)^{t_0}, (x^{c_1}y)^{t_1}, \ldots, (x^{c_k}y)^{t_k}] = 1$, where $c_i = 2^{n+m_i}$, and $c_0 \neq c_1$. Since $G = Z_{k+1}(G)$, we have $[x^{c_0}y, x^{c_1}y, \ldots, x^{c_k}y]^{t_0t_1\cdots t_k} = 1$, and so $[x^{c_0}y, x^{c_1}y, \ldots, x^{c_k}y] = 1$, as G is torsion free. Now

$$\begin{aligned}[x^{c_0}y, x^{c_1}y, \ldots, x^{c_k}y] &= [[x^{c_0}, y]^y[y, x^{c_1}]^y, x^{c_2}y, \ldots, x^{c_k}y]\\ &= [x^{c_0}, y, x_2^c y, \ldots, x^{c_k}y]^y[y, x^{c_1}, x^{c_2}y, \ldots, x^{c_k}y]^y\\ &= [x, y, x^{c_2}y, \ldots, x^{c_k}y]^{(c_0-c_1)y}.\end{aligned}$$

Thus $[x, y, x^{c_2}y \ldots, x^{c_k}y] = 1$, since G is torsion free and $c_0 \neq c_1$. Now 2^r divides $c_j \in N$ and so $[x,_k y]N = N$, which is contradiction.

Therefore we can prove that

Corollary 3.4. *Let G be a finitely generated metabelian group. Then $G \in W_k^*(\infty)$ if and only if $G/Z_k(G)$ is finite.*

Proof. By Theorem 1.2, G is finite-by-nilpotent and so there exist a finite normal subgroup H of G such that G/H is nilpotent. Let T be the torsion subgroup of H, then T/H is a finitely generated nilpotent group and so it is finite. Since H is finite so is T. Also G/T is torsion free nilpotent. By the above remark $\Gamma_{k+1}(G/T) = 1$, where $\Gamma_{k+1}(G)$ denotes the kth term of the lower central series of G. Thus $\Gamma_{k+1}(G) \leq T$ is finite, and so $G/Z_k(G)$ is finite, as G is finitely generated. ■

4. Proof of Theorem 1.3

In order to prove the Theorem 1.3, firstly we prove that the nilpotency class of a torsion free nilpotent $W_k(\infty)$-group is bounded by a function of k. Observe that if $a_0, a_1, \ldots, a_k \in A$, where A is Abelian normal subgroup of a group G and $x \in G$, and i_j, t_j are positive integers, then

$$\begin{aligned}[(a_0^{i_0}x)^{t_0}, (a_1^{i_1}x)^{t_1}, \ldots, (a_k^{i_k}x)^{t_k}] &= [(a_0^{i_0}x)^{t_0}, (a_1^{i_1}x)^{t_1}, x^{t_2} \ldots, x^{t_k}]\\ &= [x^{t_0}b, x^{t_1}c, x^{t_2}, \ldots, x^{t_k}],\end{aligned}$$

where $b = a_0^{i_0(x^{t_0}+x^{t_0-1}+\cdots+x)}$ and $c = a_1^{i_1(x^{t_1}+x^{t_1-1}+\cdots+x)}$.

Lemma 4.1. *Let $G \in W_k(\infty)$, be torsion free nilpotent. Then the nilpotency class of G is bounded by a function of k.*

Proof. Let G be nilpotent of class c. Then $\Gamma_{[c/2]}(G)$ is Abelian, where $[c/2]$ equals $(c+2)/2$ if c is even and $(c+1)/2$ if c is odd ($\Gamma_{s+1}(G)$ is the sth term of lower central series of G). Let $A = \{x \in G \mid x^m \in \Gamma_{[c/2]}(G),\quad \text{for some non-zero integer } m\}$ denote the isolator of $\Gamma_{[c/2]}(G)$. Then A is Abelian, since G is torsion free. Now let $a \in A$ and $g \in G$. Considering the elements $ag, a^2g, a^3g \ldots$, we find positive integers $i_0, i_1, \ldots, i_k$, $i_0 \neq i_1$, such that

$$1 = [(a^{i_0}g)^{t_0}, (a^{i_1}g)^{t_1}, \ldots, (a^{i_k}g)^{t_k}] = [(a^{i_0}g)^{t_0}, (a^{i_1}g)^{t_1}, g^{t_2}, \ldots, g^{t_k}].$$

Now, since G is metabelian, $[a, u, v] = [a, v, u]$, for all $a \in G'$ and $u, v \in G$. It follows that in the above equation we may replace t_j by $|t_j|$, for $j \geq 2$. Now suppose that $t_0 < 0$, then since A is Abelian normal, we have

$$\begin{aligned}
[(a^{i_0}g)^{-t_0}, (a^{i_1}g)^{t_1}] &= [(a^{i_0}g)^{t_0}, (a^{i_1}g)^{t_1}]^{-(a^{i_0}g)^{-t_0}} \\
&= [(a^{i_1}g)^{t_1}, (a^{i_0}g)^{t_0}]^{g^{-t_0}a^{i_0(g^{-t_0}+g^{-t_0-1}+\cdots+g)}} \\
&= [(a^{i_1}g)^{t_1}, (a^{i_0}g)^{t_0}]^{g^{-t_0}}.
\end{aligned}$$

Therefore, since the identity $[x_1, x_2, x_3, \ldots, x_k]^{-1} = [x_2, x_1, x_3, \ldots, x_k]$ holds in a metabelian group, we have

$$\begin{aligned}
[(a^{i_0}g)^{-t_0}, (a^{i_1}g)^{t_1}, g^{t_2}, \ldots, g^{t_k}] &= [(a^{i_1}g)^{t_1}, (a^{i_0}g)^{t_0}, g^{t_2}, \ldots, g^{t_k}]^{g^{-t_0}} \\
&= [(a^{i_0}g)^{t_0}, (a^{i_1}g)^{t_1}, g^{t_2}, \ldots, g^{t_k}]^{-g^{-t_0}} \\
&= 1^{-g^{-t_0}} = 1
\end{aligned}$$

Thus we may replace t_0 by $|t_0|$. Similarly we may replace t_1 by $|t_1|$. Hence we may assume that $t_i > 0$, for all $i = 0, 1, \ldots, k$. We treat A as a $\mathbf{Z}\langle g\rangle$-module and show that $A(g-1)^N = 0$, where N is a function of k. If $g \in A$, then $A(g-1) = 0$ and we are done. So suppose that $g \notin A$. Now, since $[g^{t_0}a^{i_0(g^{t_0}+g^{t_0-1}+\cdots+g)}, g^{t_1}a^{i_1(g^{t_1}+g^{t_1-1}+\cdots+g)}, g^{t_2}, \ldots, g^{t_k}] = 1$, we have $af_1(g) = 0$, where

$$\begin{aligned}
f_1(x) = \big(&i_1(x^{t_1} + x^{t_1-1} + \cdots + x)(x^{t_0} - 1) + \\
&i_0(x^{t_0} + x^{t_0-1} + \cdots + x)(x^{t_1} - 1)\big)(1 - x^{t_2}) \cdots (1 - x^{t_k}).
\end{aligned}$$

Put $f(x) := (x-1)f_1(x) = (i_1 + i_0)x(x^{t_0} - 1)(x^{t_1} - 1)(1 - x^{t_2}) \cdots (1 - x^{t_k})$. Then $af(g) = 0$ and $f(x) = \sum_{i=1}^{t} c_i x^{m_i}$, where $c_i \neq 0$, $t \leq N$, and N is a function of k. Let $A_1 = A \otimes_{\mathbf{Z}} \mathbf{Q}$. We consider g as an operator on A_1, and obtain that $af(g) = 0$. Since $\langle A_1, g\rangle$ is also nilpotent of class at most c, $(g-1)^c$ annihilates a, as $f(g)$ does. Now if $(x-1)^e$ divides $f(x)$, then $f(1) = f'(1) = \cdots = f^{(e-1)}(1) = 0$. Thus

$$\begin{cases}
c_1 + \cdots + c_t = 0 \\
m_1c_1 + \cdots + m_tc_t = 0 \\
\vdots \quad \vdots \quad \vdots \\
m_1^{e-1}c_1 + \cdots + m_t^{e-1}c_t = 0.
\end{cases}$$

Note that $0 = f''(1) = m_1(m_1-1)c_1 + \cdots + m_t(m_t-1)c_t$ implies that $m_1^2c_1 + \cdots + m_t^2c_t = 0$, since $m_1c_1 + \cdots + m_tc_t = 0$, and so on. If $e \geq N$, then $e \geq t$ and since the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ m_1 & m_2 & \cdots & m_t \\ \vdots & \vdots & \ddots & \vdots \\ m_1^{t-1} & m_2^{t-1} & \cdots & m_t^{t-1} \end{bmatrix}$$

is invertible, the only solution of the system is $c_i = 0$, for all $i = 1, 2, \ldots, t$. This means that $f(x) = 0$, a contradiction. Therefore $e < N$. Thus $a(g-1)^N = 0$, for all $a \in A$ and $g \in G$. In multiplicative notation of the group G, we have

$$[A, \underbrace{g, \ldots, g}_{N}] = 1.$$

Since G is torsion free, a result of Zelmanov (see [26, p. 166]) implies that, A lies in $Z_{\mu(N)}(G)$, the $\mu(N)$th center of G, where $\mu(N)$ is a function of N and independent of the number of generators of G. Thus the nilpotency class of G is at most $[c/2] + \mu(N)$ and hence $c \leq 2\mu(N)$. ■

Now we are ready to prove the Theorem 1.3

Proof of Theorem 1.3 By Theorem 1.2, there is a finite normal subgroup N of G such that G/N is nilpotent. Let T/N be the torsion subgroup of G/N. Then G/T is torsion free, and so by Lemma 4.1, its nilpotency class is bounded by a function of k. Now T/N is finite, since it is a finitely generated nilpotent torsion group, and so T is finite, as N is finite. This completes the proof. ■

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