On the Griffith-Irwin Fracture Theory

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The purpose of the present paper is to give a new mathematical formulation of the criterion for crack extension according to the Griffith-Irwin theory in terms of a path-independent line integral. The derivation will be made for the case of a crack in a plate where the plane-stress theory applies, but it may be generalized easily to other cases (three-dimensional elasticity, shells, etc.). An account of the Griffith-Irwin theory as such may be found in [1][2] and in the papers referred to in that paper.

Analysis

Suppose we have a plate with arbitrary boundaries on which we prescribe arbitrary boundary conditions (possibly time-dependent). The plate is supposed to contain a crack of length $L(\alpha)$ where α is some parameter which increases when L increases. The edges of the crack itself are assumed stress-free and all inertia effects are neglected. We will assume that the (quasistatic) solution of this plane-stress problem is known, that is, σ_{ij}, ϵ_{ij}, and u_i are known functions of x, y, t, and α. Let C be some simple closed curve surrounding the crack. According to the Griffith-Irwin theory, the following energy balance must hold during a (virtual) extension of the crack:

> The rate at which work is being done by forces acting across C equals the rate of increases of strain energy stored in the material inside C plus the rate at which energy is dissipated by the growing crack.

The last term in this equality is assumed to be proportional to

$$\frac{dL}{dt} = \frac{dL}{d\alpha}\frac{d\alpha}{dt} = L'\dot{\alpha}$$

In symbols the equality reads:

$$\int_C T_i \frac{du_i}{dt}\, ds = \frac{1}{2}\frac{d}{dt}\int_C T_i u_i\, ds + GL'\dot{\alpha} \tag{1}$$

where T_i are the boundary tractions, and G is a constant which depends on the power required to extend the crack at a given rate. Equation (1) may be transformed into

$$\frac{1}{2}\dot{\alpha}\int_C \left(T_i \frac{\partial u_i}{\partial \alpha} - u_i \frac{\partial T_i}{\partial \alpha}\right) ds + \frac{1}{2}\int_C (T_i\dot{u}_i - u_i\dot{T}_i) ds = GL'\dot{\alpha} \tag{2}$$

where

$$\frac{du_i}{dt} = \dot{u}_i + \dot{\alpha}\frac{\partial u_i}{\partial \alpha}$$

and similarly for dT_1/dt. Now the second integral on the left is easily shown to vanish, and so we have the result:

$$\left[\frac{1}{2}\int_C \left(T_i \frac{\partial u_i}{\partial \alpha} - u_i \frac{\partial T_i}{\partial \alpha}\right) ds - GL'\right]\dot{\alpha} = 0 \tag{3}$$

Ordinarily equation (3) has no solution other than $\dot{\alpha} = 0$ and we have here a kind of eigenvalue problem. For certain values of the crack length (or load level) which may be called critical the following equation holds:

$$I = \frac{1}{2}\int_C \left(T_i \frac{\partial u_i}{\partial \alpha} - u_i \frac{\partial T_i}{\partial \alpha}\right) ds = GL' \tag{4}$$

In the usual case a crack length longer than that determined by (4) would extend and lead to failure of the structure. However, in other cases the solution of equation (4) may determine the conditions under which a running crack would be arrested (neglecting inertia effects) and so the proper physical interpretation of the solution to (4) must be left to depend on the case at hand. The most useful feature of the present form of the criterion (4) is that the integral I is independent of the path C (by the manner of its derivation). One obvious advantage is that there will be no difficulties connected with calculating the strain energy in infinite regions. A simple modification of (4) is that C need not necessarily be a closed path but may begin and end on a free boundary in such a way as to enclose the tip of the crack.

Since the integral I is independent of the path C, one might expect it to take a simple form if the solution of the plane-stress problem is known in terms of the two analytic functions ϕ and ψ of the Muskhelishvili formulation of plane-stress theory [2]. This is indeed the case. According to Muskhelishvili,

$$E(u - iv) = (3 - \nu)\bar{\phi} - (1 + \nu)\bar{z}\phi' - (1 + \nu)\psi \tag{5}$$

where $u_1 = u$, $u_2 = v$, and a bar over a quantity denotes the complex conjugate. Let $T_1 ds = dX$, $T_2 ds = dY$, and let a subscript α denote differentiation with respect to α. The integral I may be written

$$\begin{aligned} I &= \frac{1}{2}\,\mathrm{Re}\left\{\int_C [(u_\alpha + iv_\alpha)d(X - iY) - (u + iv)d(X_\alpha - iY_\alpha)]\right\} \\ &= \frac{1}{2E}\,\mathrm{Im}\left\{\int_C [(1 + \nu)z\bar{\phi}_\alpha' - (3 - \nu)\phi_\alpha + (1 + \nu)\psi_\alpha]d(\bar{z}\phi' + \bar{\phi} + \psi) \right. \\ &\quad \left. - [(1 + \nu)z\bar{\phi}' - (3 - \nu)\phi + (1 + \nu)\psi]d(\bar{z}\phi_\alpha' + \bar{\phi}_\alpha + \psi_\alpha)\right\} \end{aligned} \tag{6}$$

where Re or Im means the real or imaginary part of, and where $d(X - iY) = id(\bar{z}\phi' + \bar{\phi} + \psi)$. After some manipulation (6) becomes:

$$I = -\frac{2}{E}\,\mathrm{Im}\left\{\int_C (\psi_\alpha\phi' + \psi'\phi_\alpha)dz\right\} \tag{7}$$

If the path C begins at point A and ends at point B on a free boundary where the boundary condition is $\bar{z}\phi' + \bar{\phi} + \psi = 0$, then equation (7) must be modified to read:

$$I = -\frac{2}{E}\,\mathrm{Im}\left\{[\bar{z}\phi'\phi_\alpha]_A{}^B + \int_A^B (\psi_\alpha\phi' + \psi'\phi_\alpha)dz\right\} \tag{8}$$

Applications

The following two examples are not new results but are presented to illustrate the application of the criterion (4).

Consider the problem of a cracked rectangular plate loaded on two opposite edges which are constrained to remain straight and free on the other two edges. On the first loaded edge let $u_1 = u_2 = 0$. Let C be the boundary of the plate; then the only contribution to I comes from the second loaded edge. We have:

$$I = \frac{1}{2}\frac{\partial U}{\partial \alpha}\int T_2\, ds - \frac{1}{2}U\frac{\partial}{\partial \alpha}\int T_2 f ds$$

or

$$= \frac{1}{2}\frac{\partial U}{\partial \alpha}F - \frac{1}{2}U\frac{\partial F}{\partial \alpha} \tag{9}$$

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[2] Numbers in brackets indicate References at end of Note.

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where $F = \int T_2 ds$, $u_1 = 0$, and $u_2 = U$ on the second loaded edge. For the present purpose, all we need to know about the solution of the plane-stress problem is that, for a fixed value of α, there exists a linear relation between the resultant load F and the displacement U, namely

$$F = k(\alpha)U \tag{10}$$

From the foregoing we can show that the critical crack length for a fixed grip test ($\partial U/\partial\alpha = 0$) is the same as for a dead load test ($\partial F/\partial\alpha = 0$). In the first case we have from (9) and (10)

$$I = -\frac{1}{2}U\frac{\partial F}{\partial\alpha} = -\frac{1}{2}U^2k' \tag{11}$$

In the second case from (9) and (10)

$$I = \frac{1}{2}F\frac{\partial U}{\partial\alpha} = -\frac{1}{2}\left(\frac{F}{k}\right)^2 k' = -\frac{1}{2}U^2k' \tag{12}$$

The value of I is the same in both cases and hence so is the critical crack length. This result has been noted before by several authors.

For another example, consider the problem of an infinite plate with a periodic array of cracks of length $2a$ along the x-axis with a distance l between the centers of the cracks. The plate is subjected to a uniform stress $\sigma_y = \sigma$ at infinity. The problem of determining the distribution in this case was solved by Westergaard [3]. In terms of the Muskhelishvili functions the solution is:

$$\phi = -\frac{\sigma}{4}\left[z + \frac{2il}{\pi}\cosh^{-1}\left(\frac{\cos\frac{\pi z}{l}}{\cos\frac{\pi a}{l}}\right)\right] \tag{13}$$

$$\psi = -z\phi' + \phi + \frac{1}{2}\sigma z \tag{14}$$

Let C enclose the crack which extends from $z = -a$ to $z = a$. By using (14) we have

$$\begin{aligned} I &= -\frac{2}{E}\,\mathrm{Im}\left\{\int_C(\psi_a\phi' + \psi'\phi_a)dz\right\} \\ &= -\frac{2}{E}\,\mathrm{Im}\left\{\int_C\left(2\phi' + \frac{1}{2}\sigma\right)\phi_a dz\right\} \\ &= -\frac{\sigma^2}{E}\tan\frac{\pi a}{l}\,\mathrm{Im}\left\{\int_C\frac{\sin\frac{\pi z}{l}\cos\frac{\pi z}{l}}{\cos^2\frac{\pi z}{l} - \cos^2\frac{\pi a}{l}}dz\right\} \end{aligned} \tag{15}$$

The path C encloses the two simple poles at $z = a$ and $z = -a$ so the integral may be evaluated easily. After the calculations have been performed the criterion reads

$$\frac{2\sigma^2 l}{E}\tan\frac{\pi a}{l} = 2G \tag{16}$$

and hence the critical stress for a crack length $L = 2a$ is

$$\sigma = \left[\frac{GE}{l}\cot\frac{\pi a}{l}\right]^{1/2} \tag{17}$$

Conclusions

The Griffith-Irwin criterion for crack extension has been expressed in the form of a contour integral. The path independence of the integral makes it clear that the amount of strain energy available in the cracked structure is irrelevant. The quantity that does matter is the strength of the square-root singularity at the tip of the crack. This strongly suggests that the Griffith-Irwin approach to fracture mechanics via energy concepts is equivalent to an approach via stress-concentration factors (which reinforces a conclusion reached in Ref. [4]).

References

1 G. R. Irwin, "Analysis of Stresses and Strains Near the End of a Crack Traversing a Plate," JOURNAL OF APPLIED MECHANICS, vol. 24, TRANS. ASME, vol. 79, 1957, pp. 361–364.

2 N. I. Muskhelishvili, "Some Basic Problems of the Mathematical Theory of Elasticity," P. Noordhoff Ltd., Groningen, 1953.

3 H. M. Westergaard, "Bearing Pressures and Cracks," JOURNAL OF APPLIED MECHANICS, vol. 6, TRANS. ASME, vol. 61, 1939, pp. A-49–A-53.

4 G. R. Irwin, J. A. Kies, and H. L. Smith, "Fracture Strengths Relative to Onset and Arrest of Crack Propagation," Proceedings of the ASTM, vol. 58, 1958, pp. 640–660.

On the Flexure of Plastic Plates[1]

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IN THEIR pioneering work on bending of rigid, perfectly plastic plates, Hopkins and Prager[4] used the yield condition of Tresca and the associated flow rule. Hopkins and Wang[5] later extended the analysis to arbitrary yield conditions and the associated flow rules. On account of the mathematical difficulties presented by other shapes, work in this area has been almost exclusively concerned with circular plates under rotationally symmetric loading. Attempts at experimental verification of the theory have likewise been restricted to circular plates under rotationally symmetric loading.[6] Such tests can at best provide indirect verification of the theory because they deal with the over-all deformation of the circular plate under a nonuniform distribution of radial and circumferential bending moments. The purpose of the present note is to draw attention to a simple test that provides a direct check on the fundamental assumptions of the theory, when the two principal bending moments have opposite signs.

Analysis

Fig. 1 illustrates the proposed test: A horizontal rhomboid plate of the uniform thickness h and semidiagonals of the lengths a and b is subjected to four vertical loads of the same intensity P, two of which act downward at the end points of one diagonal, while the remaining two act upward at the end points of the other diagonal. As was first pointed out by Kelvin and Tait, this type of loading is compatible with a distribution of bending and twist-

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[4] H. G. Hopkins and W. Prager, "The Load-Carrying Capacity of Circular Plates," *Journal of the Mechanics and Physics of Solids*, vol. 2, 1952, pp. 1–13.

[5] H. G. Hopkins and A. J. Wang, "Load-Carrying Capacities for Circular Plates of Perfectly Plastic Material With Arbitrary Yield Conditions," *Journal of the Mechanics and Physics of Solids*, vol. 3, 1954, pp. 117–129.

[6] R. M. Haythornthwaite and E. T. Onat, "The Load-Carrying Capacity of Initially Flat Circular Steel Plates Under Reversed Loading," *Journal of the Aeronautical Sciences*, vol. 22, 1955, pp. 867–869.

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