Central extensions of the Ptolemy-Thompson group arising in quantized Teichmüller theory*

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Abstract

The central extension of the Thompson group T that arises in the quantized Teichmüller theory is 12 times the Euler class. This extension is obtained by taking a (partial) abelianization of the so-called braided Ptolemy-Thompson group introduced and studied in [17]. We describe then the cyclic central extensions of T by means of explicit presentations.

1 Introduction and statements

Fock and Goncharov ([14, 15]) improved on previous work of Faddeev, Kashaev ([12, 23]), Chekhov and Fock ([9]) and defined new families of projective unitary representations of modular groupoids associated to cluster algebras. In the particular case of $\mathrm{SL}(2,\mathbb{R})$ one obtained (projective) representations of the Ptolemy modular groupoids associated to triangulations of surfaces arising in the quantification of the Teichmüller space. The main ingredient is Faddeev's dilogarithm which permits to deform the natural action of the modular groupoid on the Teichmüller space. Thereby they will be called dilogarithmic representations. These are actually projective unitary representations, or equivalently, representations of suitable central extensions.

These representations – depending on a deformation parameter – are infinite dimensional. The general belief is that they collapse at roots of unity to finite dimensional representations which can be identified with the mapping class group representations arising from the quantum group $U_q(\mathrm{SL}(2,\mathbb{R}))$. Moreover, they should also coincide with the representations coming from the quantum hyperbolic invariants introduced by Baseilhac and Benedetti (see [1]).

There exists an universal setting for these constructions where the surface is the hyperbolic plane $\mathbb{H}^2$ (endowed with a specific triangulation, namely the Farey triangulation) and the universal Teichmüller space is the one constructed by Penner in [26]. The associated modular groupoid is the Ptolemy groupoid of flips on the triangulation. As it is well-known (see [26]) there is a group structure underlying the groupoid structure that identifies the Ptolemy group of flips on the Farey triangulation to the Thompson group T of piecewise-$\mathrm{PSL}(2,\mathbb{Z})$ homeomorphisms of the circle (see [8]). Our aim is to identify the central extension $\widehat{T}$ of T arising in the dilogarithm representations constructed in ([15], section 10, [16], section 3). We refer to $\widehat{T}$ as the dilogarithmic central extension of T.

Following [21] the cohomology ring $H^*(T)$ is generated by two classes $\alpha, \chi \in H^2(T)$, which are called the discrete Godbillon-Vey class and respectively, the Euler class. One can obtain χ as the Euler class of the action of T on the circle.

Moreover, central extensions of T are classified up to isomorphism by their extension classes in $H^2(T)$. For instance the Euler class χ is the extension class of the central extension

$$1 \to \mathbb{Z} \to \widetilde{T} \to T \to 1$$

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where $\widetilde{T} \subset \mathrm{Homeo}^+(\mathbb{R})$ is the group of lifts of piecewise-PSL$(2,\mathbb{Z})$ homeomorphisms of the circle to homeomorphisms of the real line $\mathbb{R}$.

Our first goal is the identification of the extension class of $\widehat{T}$. Specifically our main result is the following:

Theorem 1.1. *The class $c_{\widehat{T}}$ of the dilogarithmic central extension $\widehat{T}$ is $c_{\widehat{T}} = 12\chi \in H^2(T)$.*

Remark 1.1. It is worthy to notice that the central extensions arising in the quantization of representations spaces into other Lie groups coincide with $\widehat{T}$. It is only the central charge that depends on the particular Lie group.

Our approach consists of first relating the dilogarithmic central extension $\widehat{T}$ to (one of) the braided Ptolemy-Thompson group T^* introduced and studied in [17, 18]. This will provide a description of $\widehat{T}$ by means of an explicit group presentation by generators and relations. There is a general setup for studying central extensions of a finitely presented group in which all relations are given arbitrary central lifts in the extension. In the case of the group T this provides a series of group presentations depending on five integer parameters $T_{n,p,q,r,s}$. The dilogarithmic extension $\widehat{T}$ appears in this series as $T_{1,0,0,0,0}$.

Our second goal is to understand all central extensions of T in terms of group presentations depending on these integers. Although specific computations with $T_{1,0,0,0,0}$ would suffice to prove the theorem above we found more instructive to unravel the general case. This amounts to understand the cohomology 2-classes of T by means of their associated extensions.

There exists a simple proof (using however results from [17]) which shows that the dilogarithmic extension class is a multiple of the Euler class because the extension splits over the smaller Thompson group $F \subset T$. However, in general we face new complications due to the presence of the discrete Godbillon-Vey class.

The extension classes behave linearly on the parameters and, in order to find their coefficients, one has to make use of several explicit (central) extensions. There are only a few such extensions in the litterature, namely those coming from mapping class groups of planar surfaces with infinitely many punctures. The Ptolemy-Thompson group T^* is one such mapping class group, but it is useless in computing the coefficients of the Godbillon-Vey class. The other extension of the same kind is the Greenberg-Sergiescu acyclic extension (see [22]) and hence the core of the proof consists of computations in the Greenberg-Sergiescu extension. This lead us to a formula for $c_{T_{n,p,q,r,0}}$ in theorem 3.1 from which theorem 1.1 will follow immediately.

All over this paper the Ptolemy-Thompson group T appears in every one of its instances, as a group of homeomorphisms of the circle, as the group of flips, as the group $PPSL(2,\mathbb{Z})$ and as a mapping class group. The main difficulty is to pass from one description to another one.

The plan of the paper is at follows. We introduce in the next section the braided Ptolemy-Thompson T^* and relate it to the dilogarithmic extension $\widehat{T}$, by using results of Fock and Goncharov. In the third section we describe the central extensions of T by means of presentations and compute their cohomology classes. It appears that there exists a method for computing such invariants, similar to Milnor's algorithm which calculates the Euler class of a surface group representation. The last section contains some speculations on what should be the geometric extensions of T on one hand, and the central extensions of the mapping class group of a finite (punctured) surface arising in the quantized Teichmüller theory.

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2 The central extension induced from the braided Ptolemy-Thompson group T^*

2.1 The Thompson groups F and T

The Thompson group F is the group of dyadic piecewise affine homeomorphism of $[0,1]$. We describe every element γ of F as follows. There exist two partitions of $[0,1]$ into consecutive intervals $I_1, I_2, \ldots, I_k$ on one side and $J_1, J_2, \ldots, J_k$ whose end points are dyadic numbers and γ sends affinely each interval I_j into its counterpart J_j, for all $j \in \{1,2,\ldots,k\}$. Thus the restriction of γ to any interval I_j is given by $\gamma(x) = a_j x + b_j$, where $a_j = 2^{n_j}, n_j \in \mathbb{Z}$ and b_j belong to the set of dyadic numbers i.e. $b_j = \frac{p_j}{2^{m_j}}$, $p_j, m_j \in \mathbb{Z}$.

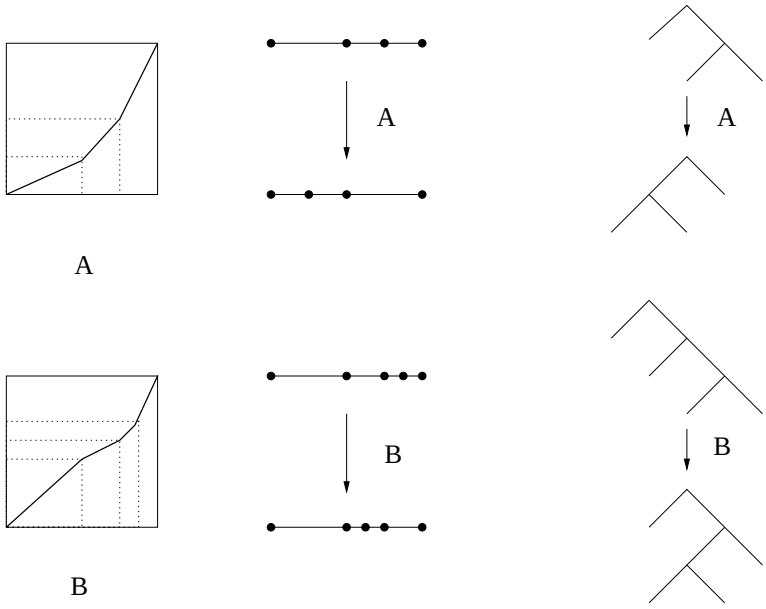
The group F is generated by the two elements A and B below:

$$A(x) = \begin{cases} \frac{x}{2}, & \text{if} \quad x \in [0, \frac{1}{2}] \\ x - \frac{1}{4}, & \text{if} \quad x \in [\frac{1}{2}, \frac{3}{4}] \\ 2x - 1, & \text{if} \quad x \in [\frac{3}{4}, 1] \end{cases}, \quad B(x) = \begin{cases} x, & \text{if} \quad x \in [0, \frac{1}{2}] \\ \frac{x}{2} + \frac{1}{4}, & \text{if} \quad x \in [\frac{1}{2}, \frac{3}{4}] \\ x - \frac{1}{8}, & \text{if} \quad x \in [\frac{3}{4}, \frac{7}{8}] \\ 2x - 1, & \text{if} \quad x \in [\frac{7}{8}, 1] \end{cases}$$

Moreover, the group F has the presentation

$$F = \langle A, B; [AB^{-1}, A^{-1}BA] = 1,\ [AB^{-1}, A^{-2}BA^2] = 1 \rangle$$

There is a geometric encoding of elements of F as pairs of (stable) finite rooted binary trees. Each finite rooted binary tree encodes a subdivision of $[0,1]$ into dyadic intervals. Adding two descending edges to a vertex amounts to subdivide the respective interval into two equal halves. Given the two subdivisions the element of F sending one into the other is uniquely determined. The pair of trees is determined up to stabilization, namely adding extra couples of descending edges to corresponding vertices in both trees.



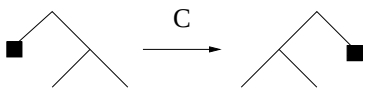
The Thompson group T is the group of dyadic piecewise affine homeomorphisms of $S^1 = [0,1]/0 \sim 1$. It is not hard to see that A, B and C generate T, where

$$C(x) = \begin{cases} \frac{x}{2} + \frac{3}{4}, & \text{if} \quad x \in [0, \frac{1}{2}] \\ 2x - 1, & \text{if} \quad x \in [\frac{1}{2}, \frac{3}{4}] \\ x - \frac{1}{4}, & \text{if} \quad x \in [\frac{3}{4}, 1] \end{cases}$$

The presentation of T in terms of the generators A, B, C consists of the two relations above with four more relations to be added:

$$T = \left\langle \begin{array}{ll} A, B, C; & CA = (A^{-1}CB)^2, \ (A^{-1}CB)(A^{-1}BA) = B(A^{-2}CB^2), \\ & C^3 = 1, \ C = BA^{-1}CB, \ [AB^{-1}, A^{-1}BA] = 1, \ \ [AB^{-1}, A^{-2}BA^2] = 1 \end{array} \right\rangle$$

We can associate a pair of (stable) trees to encode an element of T, as well, but we have to specify additionally, where the origin is sent to. We denote the image of the origin in the trees by marking the leftmost leaf of its domain. Usually the origin is fixed to be the leftmost leaf in the first tree. For instance, the element C has the following description:



Remark 2.1. The group T has also another presentation with generators α and β and relations

$$\alpha^4 = \beta^3 = 1$$

$$[\beta\alpha\beta, \alpha^2\beta\alpha\beta\alpha^2] = 1$$

$$[\beta\alpha\beta, \alpha^2\beta^2\alpha^2\beta\alpha\beta\alpha^2\beta\alpha^2] = 1$$

$$(\beta\alpha)^5 = 1$$

If we set $A = \beta\alpha^2$, $B = \beta^2\alpha$ and $C = \beta^2$ then we obtain the generators A, B, C of the group T, considered above. Then the two commutativity relations above are equivalent to

$$[AB^{-1}, A^{-1}BA] = 1, \ \ [AB^{-1}, A^{-2}BA^2] = 1$$

2.2 Mapping class groups of infinite surfaces

Definition 2.1. *A* rigid structure *d on the surface Σ is a decomposition of Σ into 2-disks with disjoint interiors, called elementary pieces. We suppose that the closures of the elementary pieces are still 2-disks.*

We assume that we are given a family F, called the family of admissible subsurfaces of Σ, of compact subsurfaces of Σ such that each member of F is a finite union of elementary pieces.

Given the data (Σ, d, F) we can associate the asymptotic mapping class group $\mathcal{M}(\Sigma, d, F)$ as follows. We restrict first to those homeomorphisms that act in the simplest possible way at infinity.

Definition 2.2. *A homeomorphism φ between two surfaces endowed with rigid structures is* rigid *if it sends the rigid structure of one surface onto the rigid structure of the other.*

The homeomorphism $\varphi : \Sigma \to \Sigma$ is said to be asymptotically rigid *if there exists some admissible subsurface $C \subset \Sigma$, called a support for φ, such that $\varphi(C) \subset \Sigma$ is also an admissible subsurface of Σ and the restriction $\varphi|_{\Sigma - C} : \Sigma - C \to \Sigma - \varphi(C)$ is rigid.*

As it is customary when studying mapping class groups we consider now isotopy classes of such homeomorphisms.

Definition 2.3. *The group $\mathcal{M}(\Sigma, d, F)$ of isotopy classes of asymptotically rigid homeomorphisms is called the* asymptotic mapping class group *of Σ corresponding to the rigid structure d and family of admissible subsurfaces F.*

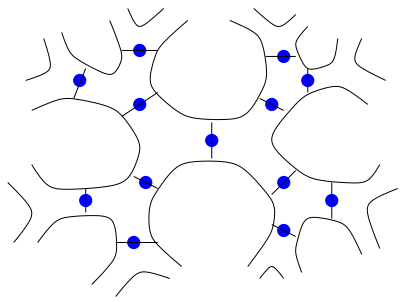
Remark 2.2. Two asymptotically rigid homeomorphisms that are isotopic should be isotopic among asymptotically rigid homeomorphisms.

2.3 T and T^* as mapping class groups

The surfaces below will be oriented and all homeomorphisms considered in the sequel will be orientation-preserving, unless the opposite is explicitly stated. Actions in the sequel are left actions and the composition of maps is the usual one, namely we start composing from right to the left.

Definition 2.4. *The ribbon tree D is the planar surface obtained by thickening in the plane the infinite binary tree. We denote by D^* the ribbon tree with infinitely many punctures, one puncture for each edge of the tree. A homeomorphism of D^* is a homeomorphism of D which permutes the punctures D^*.*

Definition 2.5. *A* rigid structure *on D or D^* is a decomposition into hexagons by means of a family of arcs whose endpoints are on the boundary of D. Each hexagon contains no puncture within its interior but each arc passes through a unique puncture in the case of D^*. It is assumed that these arcs are pairwise non-homotopic in D, by homotopies keeping the boundary points of the arcs on the boundary of D. The choice of a rigid structure of reference is called the* canonical rigid structure. *The canonical rigid structure of the ribbon tree D is such that each arc of this rigid structure crosses once and transversely a unique edge of the tree. The canonical rigid structure on D^* is assumed to coincide with the canonical rigid structure of D when forgetting the punctures. See the Figure below:*



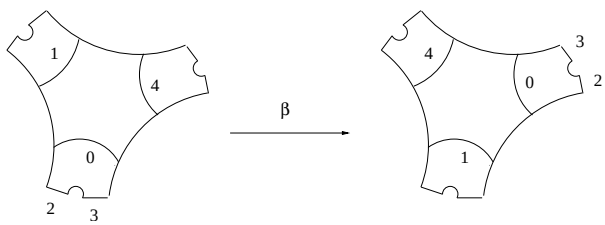
A planar subsurface of D or D^* is admissible *if it is a connected finite union of hexagons coming from the canonical rigid structure.*

One denotes by T and T^* the group of isotopy classes of asymptotically rigid homeomorphisms of D and D^*, respectively.

Remark 2.3. There exists a cyclic order on the frontier arcs of an admissible subsurface induced by the planarity. An asymptotically rigid homeomorphism necessarily preserves the cyclic order of the frontier for any admissible subsurface.

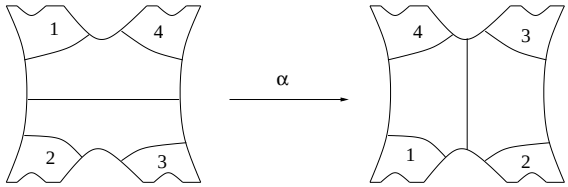
The mapping class group T is isomorphic to the Thompson group which is commonly denoted T. This fact has been widely developed in [24] and [17]. We consider the following elements of T, defined as mapping classes of asymptotically rigid homeomorphisms:

- The support of the element β is the central hexagon on the figure below. Further, β acts as the counterclockwise rotation of order three which permutes the three branches of the ribbon tree issued from the hexagon.



  In fact, β is globally rigid.

- The support of α is the union of two adjacent hexagons, one of them being the support of β. Then α rotates counterclockwise the support of angle $\frac{\pi}{2}$, by permuting the four branches of the ribbon tree issued from the support.



Note that α is not globally rigid, but α^2 is.

2.4 The relative abelianization of the braided Ptolemy-Thompson group T^*

Recall from [17, 18] that there exists a natural surjection homomorphism $T^* \to T$ between the two mapping class groups, which is obtained by *forgetting the punctures*. Its kernel is the infinite braid group B_∞ consisting of those braids in the punctures of D^* that move non-trivially only finitely many punctures. In other words B_∞ is the direct limit of an ascending sequence of braid groups associated to an exhaustion of D^* by punctured disks. This yields the following exact sequence description of T^*:

$$1 \to B_\infty \to T^* \to T \to 1$$

Observe that $H_1(B_\infty) = \mathbb{Z}$. Thus, the abelianization homomorphism $B_\infty \to H_1(B_\infty) = \mathbb{Z}$ induces a central extension T^*_{ab} of T, where one replaces B_∞ by its abelianization $H_1(B_\infty)$, as in the diagram below:

$$\begin{array}{ccccccc} 1 \to & B_\infty & \to & T^* & \to & T & \to 1 \\ & \downarrow & & \downarrow & & \| & \\ 1 \to & \mathbb{Z} & \to & T^*_{ab} & \to & T & \to 1 \end{array}$$

Then T^*_{ab} is the relative abelianization of T^* over T. We are not only able to make computations in the mapping class group T^* and thus in T^*_{ab}, but also to interpret the algebraic relations in T^*_{ab} in geometric terms.

Proposition 2.1. *The group T^*_{ab} has the presentation with three generators α^*_{ab}, β^*_{ab} and z and the relations*

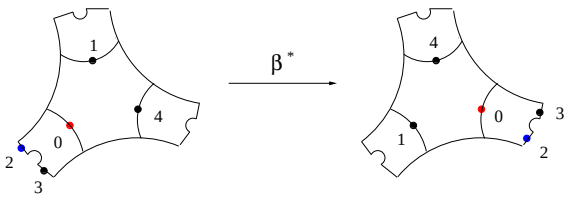
$$\alpha^{*\,4}_{ab} = \beta^{*\,3}_{ab} = 1, (\beta^*_{ab}\alpha^*_{ab})^5 = z, [\alpha^*_{ab}, z] = 1, [\beta^*_{ab}, z] = 1$$

$$[\beta^*_{ab}\alpha^*_{ab}\beta^*_{ab}\,,\, \alpha^{*\,2}_{ab}\beta^*_{ab}\alpha^*_{ab}\beta^*_{ab}\alpha^{*\,2}_{ab}] = [\beta^*_{ab}\alpha^*_{ab}\beta^*_{ab}\,,\, \alpha^{*\,2}_{ab}\beta^{*\,2}_{ab}\alpha^{*\,2}_{ab}\beta^*_{ab}\alpha^*_{ab}\beta^*_{ab}\alpha^{*\,2}_{ab}\beta^*_{ab}\alpha^{*\,2}_{ab}] = 1$$

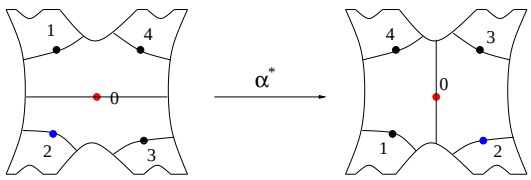
Moreover the projection map $T^*_{ab} \to T$ sends α^*_{ab} to α, β^*_{ab} to β and z to identity.

Proof. Recall from [17] that T^* is generated by two elements α^* and β^* below.

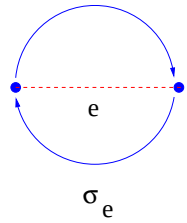
- The support of the element β^* of T^* is the central hexagon. Further β^* acts as the counterclockwise rotation of order three which permutes cyclically the punctures. One has $\beta^{*3} = 1$.



- The support of the element α^* of T^* is the union of two adjacent hexagons, one of them being the support of β^*. Then α^* rotates counterclockwise the support of angle $\frac{\pi}{2}$, by keeping fixed the central puncture. One has $\alpha^{*4} = 1$.



Let now e be a simple arc in D^* which connects two punctures. We associate a braiding $\sigma_e \in B_\infty$ to e by considering the homeomorphism that moves clockwise the punctures at the endpoints of the edge e in a small neighborhood of the edge, in order to interchange their positions. This means that, if γ is an arc transverse to e, then the braiding σ_e moves γ on the left when it approaches e. Such a braiding will be called *positive*, while σ_e^{-1} is *negative*.

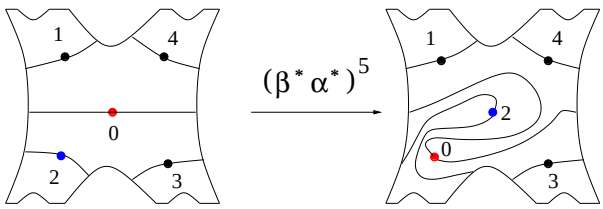


It is known that B_∞ is generated by the braids σ_e where e runs over the edges of the binary tree with vertices at punctures. Let $\iota : B_\infty \to T^*$ be the inclusion. It is proved in [17] that

The braid generator $\sigma_{[02]}$ associated to the edge joining the punctures numbered 0 and 2 has the image

$$\iota(\sigma_{[02]}) = (\beta^*\alpha^*)^5$$

because we have



Recall next that all braid generators σ_e are conjugate and call z their image in T^*_{ab}. It follows that T^*_{ab} is an extension of T by $\mathbb{Z}$. Moreover, it is simple to check that $\alpha^*\sigma_{[02]}\alpha^{-1}$ is also a braid generator, namely $\sigma_{[\alpha^*(0)\alpha^*(2)]}$. The same holds true for $\beta^*\sigma_{[02]}\beta^{*-1} = \sigma_{[\beta^*(0)\beta^*(2)]}$. This implies that the extension T^*_{ab} is central.

In particular, a presentation of T^*_{ab} can be obtained by looking at the lifts of relations in T, together with those coming from the fact that z is central.

The first set relations above are obviously satisfied by T^*_{ab}. Finally recall from [17] that T^* splits over the smaller Thompson group F and thus the following relations hold true in T^*:

$$[\beta^*\alpha^*\beta^*\,,\,\alpha^{*2}\beta^*\alpha^*\beta^*\alpha^{*2}] = [\beta^*\alpha^*\beta^*\,,\,\alpha^{*2}\beta^{*2}\alpha^{*2}\beta^*\alpha^*\beta^*\alpha^{*2}\beta^*\alpha^{*2}] = 1$$

Thus the second set of relations are automatically verified in T^*_{ab}. Since these relations form a complete set of lifts of relations presenting T and z is central, then they represent a complete system of relations in T^*_{ab}. This ends the proof. □

2.5 The Ptolemy-Thompson group and the Ptolemy groupoid

We will use the terms triangulation of $\mathbb{H}^2$ for ideal *locally finite* triangulations of the hyperbolic space $\mathbb{H}^2$ i.e. a countable locally finite collection of geodesics whose complementary regions are triangles. The *vertices* of the triangulation are the asymptotes of the constitutive geodesics, viewed as points of the circle at infinity S^1_∞.

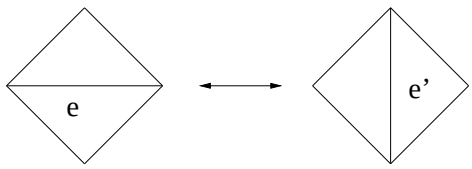
Consider the basic ideal triangle having vertices at $1, -1, \sqrt{-1} \in S^1_\infty$ in the unit disk model $\mathbb{D}$ of $\mathbb{H}^2$ and G be the group of isometries of $\mathbb{H}^2$ generated by the hyperbolic reflections in the sides of the triangle. The orbits of the sides of this triangle by the group G form the Farey triangulation τ_0.

A triangulation τ is *marked* if one fixes a distinguished oriented edge (abbreviated d.o.e.) $\vec{a}$ of it. The standard marking of the Farey triangulation τ_0 is the oriented edge $\vec{a_0}$ joining -1 to 1.

We define next a *marked tesselation* of $\mathbb{H}^2$ to be an equivalence class of marked triangulations of $\mathbb{H}^2$ with respect to the action of $PSL(2,\mathbb{R})$. Since the action of $PSL(2,\mathbb{R})$ is 3-transitive each tesselation can be uniquely represented by its *canonical marked triangulation* containing the basic ideal triangle and whose d.o.e. is $\vec{a_0}$. The marked tesselation is of Farey-type if its canonical marked triangulation has the same vertices as the Farey triangulation. Unless explicitly stated otherwise all tesselations considered in the sequel will be Farey-type tesselations. In particular, the ideal triangulations have the same vertices as τ_0 and coincide with τ_0 for all but finitely many ideal triangles.

Definition 2.6. *The objects of the* (universal) Ptolemy groupoid *Pt are marked tesselations. The morphisms between two objects $(\tau_1, \vec{a_1})$ and $(\tau_2, \vec{a_2})$ are* eventually trivial *permutations maps $\phi : \tau_1 \to \tau_2$ such that $\phi(\vec{a_1}) = \vec{a_2}$. When marked tesselations are represented by their canonical triangulations, the later coincide for all but finitely many triangles. Then ϕ is eventually trivial if the induced correspondence at the level of canonical triangulations is the identity for all but finitely many edges.*

We define now particular elements of Pt called flips. Let e be an edge of the marked tesselation represented by the canonical marked triangulation $(\tau, \vec{a})$. The result of the flip F_e on τ is the triangulation $F_e(\tau)$ obtained from τ by changing only the two neigbouring triangles containing the edge e, according to the picture below:



This means that we remove e from τ and further add the new edge e' in ordre to get $F_e(\tau)$. In particular there is a natural correspondence $\phi : \tau \to F_e(\tau)$ sending e to e' and identity for all other edges.

If e is not the d.o.e. of τ then $F_e(\vec{a}) = \vec{a}$. If e is the d.o.e. of τ then $F_e(\vec{a}) = \vec{e'}$, where the orientation of $\vec{e'}$ is chosen so that the frame $(\vec{e}, \vec{e'})$ is positively oriented.

We define now the flipped tesselation $F_e((\tau, \vec{a}))$ to be the tesselation $(F_e(\tau), F_e(\vec{a}))$. It is proved in [26] that flips generate the Ptolemy groupoid i.e. any element of Pt is a composition of flips.

For any marked tesselation $(\tau, \vec{a})$ there is defined a characteristic map $Q_\tau : \mathbb{Q} - \{-1, 1\} \to \tau$. Assume that τ is the canonical triangulation repersenting this tesselation. We label first by $\mathbb{Q} \cup \infty$ the vertices of τ, by induction:

1. -1 is labeled by $0/1$, 1 is labeled by $\infty = 1/0$ and $\sqrt{-1}$ is labeled by $-1/1$.
2. If we have a triangle in τ having two vertices already labeled by a/b and c/d then its third vertex is labeled $(a + c)/(b + d)$. Notice that vertices in the upper half-plane are labeled by negative rationals and those from the lower half-plane by positive rationals.

As it is well-known this labeling produces a bijection between the set of vertices of τ and $\mathbb{Q} \cup \infty$.

Let now e be an edge of τ, which is different from $\vec{a}$. Let $v(e)$ be the vertex opposite to e of the triangle Δ of τ containing e in its frontier andlying in the component of $\mathbb{D} - e$ which does not contain $\vec{a}$. We associate then to e the label of $v(e)$. This assignment establishes a bijection $Q_\tau : \mathbb{Q} - \{-1, 1\} \to \tau$.

Remark that if $(\tau_1, \vec{a_1})$ and $(\tau_2, \vec{a_2})$ are marked tesselations then there exists a unique map f between their vertices sending triangles to triangles and marking on the marking. Then $f \circ Q_{\tau_1} = Q_{\tau_2}$.

The role played by Q_τ is to allow flips to be indexed by the rationals and not on edges of τ.

Definition 2.7. *Let $\mathcal{T}$ be the set of marked tesselations (of Farey-type). Define the action of the free monoid generated by $\mathbb{Q} - \{-1, 1\}$ on $\mathcal{T}$ as follows*

$$q \cdot (\tau, \vec{a}) = Q_\tau(q)(\tau, \vec{a}), \quad \textit{for } q \in \mathbb{Q} - \{-1, 1\}, (\tau, \vec{a}) \in \mathcal{T}$$

The kernel of this action is the submonoid acting trivially on $\mathcal{T}$. The Ptolemy group T is the quotient of the free monoid by the kernel.

In particular it makes sense to speak of flips in the present case and thus flips generate the Ptolemy group.

The notation T for the Ptolemy group is not misleading because this group is isomorphic to the Thompson group T and for this reason, we preferred to call it the Ptolemy-Thompson group.

Given two marked tesselations $(\tau_1, \vec{a_1})$ and $(\tau_2, \vec{a_2})$ the combinatorial isomorphism $f : \tau_1 \to \tau_2$ from above provides a map between the vertices of the triangulations, which are identified with $P^1(\mathbb{Q}) \subset S^1_\infty$. This map extends continuously to a homeomorphism of S^1_∞, which is piecewise-$PSL(2, \mathbb{Z})$. This establishes an isomorphism between the Ptolemy group and the group of piecewise-$PSL(2, \mathbb{Z})$ homeomorphisms of the circle.

An explicit isomorphism with the group T in the form introduced above was provided by Lochak and Schneps (see [25]). Send α to the flip F_a of $(\tau_0, \vec{a_0})$ and β to the element $(\tau_0, \vec{a_0}), (\tau_0, \vec{a_1})$ of the Ptolemy group, where $\vec{a_1}$ is the oriented edge in the basic triangle of the Farey triangulation τ_0 next to $\vec{a_0}$.

2.6 Chekhov-Fock algebras and the quantization of the Teichmüller space

The dilogarithm representation is obtained by using a quantification of some family of algebras A_Γ associated to ribbon graphs Γ. In our case Γ is the planar embedding of the binary tree, as the graph dual to a Farey-type marked ideal triangulation of $\mathbb{H}^2$.

When Γ is planar one defines the *Chekhov-Fock algebra* A_Γ following ([2, 9, 15]) as the non-commutative algebra generated by the generators Z_e, where e are edges of the ideal triangulation (equivalently of Γ), with the relations

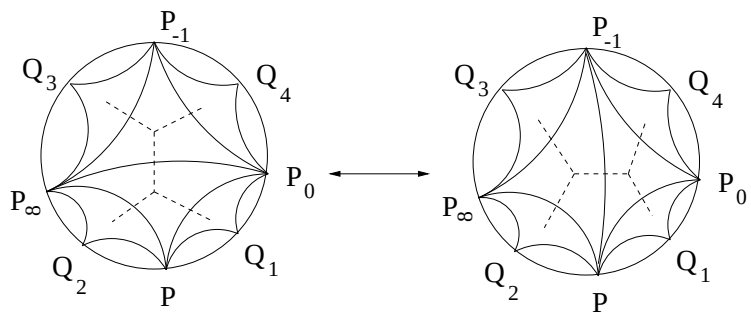
$$[Z_e, Z_f] = 2\pi\sqrt{-1}h$$

where h is a deformation parameter, whenever e and f are incident at some vertex and obtained one from the other by a positive rotation around their common vertex.

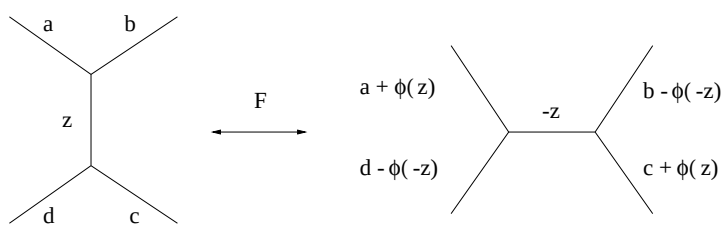
Notice that if Γ were not a planar graph (as in the case of surfaces of positive genus) the relations are slightly different (see [2, 9, 15]).

In [15] the authors constructed a projective representation of the Ptolemy groupoid into the groupoid of endomorphisms of the algebras A_Γ. This induces then a projective representation of the Ptolemy-Thompson group T.

Remark that a flip F_e in the edge e replaces the edge e by another edge e' of a different embedding of the binary tree which we denote Γ'.



The flip F associated to the transformation of Γ into Γ' yields a natural isomorphism between the algebras A_Γ and $A_{\Gamma'}$. Specifically, if one identifies each coordinate Z_e with a label on the edge e then the action of the isomorphism F is given by the formula



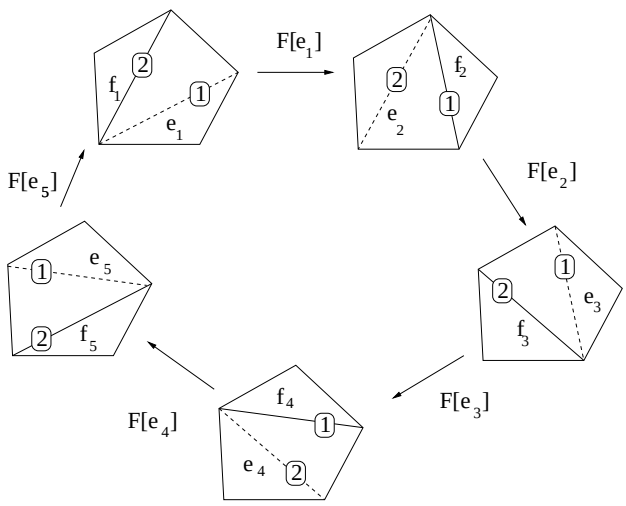
Here ϕ is the function

$$\phi(z) = -\frac{\pi h}{2}\int_{-\infty}^{\infty}\frac{\exp(itz)}{\operatorname{sh}(\pi t)\operatorname{sh}(\pi h t}dt$$

We need to complete the algebras A_Γ in such a way that $\phi(z)$ makes sense by functional calculus.

Flips act transitively on the embeddings of the binary tree in the plane that are asymptotically congruent. Thus, given two embeddings Γ and Γ' we can use a sequence of flips transforming one graph into the other and thus an isomorphism $K(\Gamma,\Gamma')$ between the algebras A_Γ and $A_{\Gamma'}$. In order to get a well-defined isomorphism $K(\Gamma,\Gamma')$ independent on the choice of the sequence of flips we chose the flips should verify some compatibility conditions.

1. The first condition is that flips on disjoint edges should commute.
2. The second condition is subtler and it has no analog yet for general cluster modular groupoids. It amounts to understand what happens when we have the pentagon relation in the groupoid of flips:



Projective representations are equivalent to representations of central extensions. We can summarize the results of [15] (see also the equivalent construction at the level of Chekhov-Fock algebras in [2]) as follows:

Proposition 2.2. *The dilogarithmic representation of the dilogarithmic central extension $\widehat{T}$ has the following properties:*

1. *Images of disjoint flips commute;*
2. *The composition of the images of the five flips from the pentagon relation above is a scalar operator.*

Proof. This is a consequence of the main result of [15] that the flips defined by the dilogarithm above verify the two compatibility conditions above in a weaker sense, namely the pentagon condition is satisfied only up to a non-zero scalar. When the scalar is not the unit then we have a central extension associated to the modular groupoid.

This also follows from the description of the deformation of the flip action on the Teichmüller space. In fact the action on the algebra A_Γ is given by algebraic functions (see [15, 2, 3]). □

2.7 Dilogarithmic representations of T

Notice that we did not make use of the expression of operators in the dilogarithmic representation. However, for the sake of the completeness we will give a sketchy construction of the operators.

The algebras A_Γ look very much like the Heisenberg algebra, having slightly more general but still constant equations. An analog of the Stone von Neumann theorem holds true then for these algebras. Specifically,

there exists an unique unitary irreducible Hilbert space representation of A_Γ that is integrable i.e. which can be integrated to the corresponding Lie group. Notice that there exist in general also non-integrable unitary representations. Further there exists a simple way to construct one such irreducible representation. It suffices to select a Lagrangian subspace in the generators of A_Γ and further to mimick the Heisenberg algebra representation. Thus we obtain a space $L^2(\mathbb{R}^N)$ where N is the set of vertices of Γ and the generators Z_e act either as linear combinations of the operators p_e, q_e, where

$$p_e(f) = z_e f, \text{for} f \in L^2(R^N)$$

and z_e are the coordinates in $\mathbb{R}^N$, and

$$q_e(f) = 2\pi i h \frac{\partial f}{\partial z_e},$$

for $f \in L^2(R^N)$.

It is clear that for the smallest possible N (which is identified with the set of vertices of Γ) the representations obtained this way is irreducible, unitary with respect to the $*$-algebra structure $Z_e^* = Z_e$ and integrable; the Stone von Neumann theorem alluded above implies that the representations of A_Γ and $A_{\Gamma'}$ are isomorphic and the conjugation isomorphism $J(\Gamma, \Gamma')$ yields a (projective) unitary representation of the modular groupoid. The exact form of $J(\Gamma, \Gamma')$ will not matter in the sequel.

2.8 Identifying the two central extensions of T

The main result of this section is the following:

Proposition 2.3. *The dilogarithmic extension $\widehat{T}$ is identified to T^*_{ab}.*

Proof. Since $\widehat{T}$ is a central extension of T then it is generated by the lifts $\widehat{\alpha}, \widehat{\beta}$ of α and β together with the generator z of the center. Let us see what are the relations arising in the group $\widehat{T}$. According to proposition 2.2 lifts of disjoint flips should commute. By a simple computation we can show that the elements $\beta\alpha\beta$, $\alpha^2\beta\alpha\beta\alpha^2$ and $\alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2$ act as disjoint flips on the Farey triangulation. In particular we have the relations

$$[\widehat{\beta}\widehat{\alpha}\widehat{\beta}, \widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2] = [\widehat{\beta}\widehat{\alpha}\widehat{\beta}, \widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}^2\widehat{\alpha}^2] = 1$$

satisfied in $\widehat{T}$. Moreover, by construction we have also

$$\widehat{\beta}^3 = \widehat{\alpha}^4 = 1$$

meaning that the $\widehat{\alpha}$ is still periodic while $\widehat{\beta}$ is not deformed.

Eventually the only non-trivial lift of relations comes from the pentagon relation $(\widehat{\beta}\widehat{\alpha})^5$. The pentagon equation is not anymore satisfied and the keypoint of the proof in [15] is to prove that under the dilogarithmic representation the operator associated to $(\widehat{\beta}\widehat{\alpha})^5$ is the multiplication by a scalar λ. Since z is the generator of the kernel $\mathbb{Z}$ of $\widehat{T} \to T$ it follows that the the lift of the pentagon equation from T to $\widehat{T}$ is given by

$$(\widehat{\beta}\widehat{\alpha})^5 = z$$

According to proposition 2.1 all relations presenting T^*_{ab} are satisfied in $\widehat{T}$. Since $\widehat{T}$ is a nontrivial central extension of T by $\mathbb{Z}$ it follows that the groups are isomorphic. □

3 Classification of central extensions of the group T

3.1 Statement of the main result

Our main concern here is to identify the extension class of the central extension $\widehat{T}$ in $H^2(T)$. Before doing that we consider a series of central extensions $T_{n,p,q,r,s}$ of T by $\mathbb{Z}$, having similar properties to $\widehat{T}$.

Definition 3.1. *The group $T_{n,p,q,r,s}$, is presented by the generators $\overline{\alpha}, \overline{\beta}, z$ and the relations:*

$$(\overline{\beta}\overline{\alpha})^5 = z^n$$

$$\overline{\alpha}^4 = z^p$$
$$\overline{\beta}^3 = z^q$$
$$[\overline{\beta}\overline{\alpha}\overline{\beta}, \overline{\alpha}^2\overline{\beta}\overline{\alpha}\overline{\beta}\overline{\alpha}^2] = z^r$$
$$[\overline{\beta}\overline{\alpha}\overline{\beta}, \overline{\alpha}^2\overline{\beta}\overline{\alpha}^2\overline{\beta}\overline{\alpha}\overline{\beta}\overline{\alpha}^2\overline{\beta}^2\overline{\alpha}^2] = z^s$$
$$[\overline{\alpha}, z] = [\overline{\beta}, z] = 1$$

Let us denote $T_{n,p,q,r} = T_{n,p,q,r,0}$ *and* $T_{n,p,q} = T_{n,p,q,0,0}$.

According to [17] we can identify $\widehat{T}$ with $T_{1,0,0}$. In fact the group T^* is split over the smaller Thompson group $F \subset T$ and thus $\widehat{T}$ is split over F. Further F is generated by the elements $\beta^2\alpha$ and $\beta\alpha^2$ and thus relations of F are precisely given by the commutation relations above. Thus the last two relations hold true, while z is central and thus $\widehat{T}$ is given by the presentation above.

Remark 3.1. We considered in [17] the twin group $T^\sharp$ and gave a presentation of it. Then, using a similar procedure there is a group obtained from $T^\sharp$ by abelianizing the kernel B_∞, which is identified actually to $T_{3,1,0}$.

Proposition 3.1. *Central extensions of* T *by* $\mathbb{Z}$ *are exhausted by the set of extensions* $T_{n,p,q,r}$.

Proof. Consider the relations in $T_{n,p,q,r,s}$ other than

$$[\overline{\beta}\overline{\alpha}\overline{\beta}, \overline{\alpha}^2\overline{\beta}\overline{\alpha}^2\overline{\beta}\overline{\alpha}\overline{\beta}\overline{\alpha}^2\overline{\beta}^2\overline{\alpha}^2] = z^s$$

It suffices to see that these relations already force

$$[\overline{\beta}\overline{\alpha}\overline{\beta}, \overline{\alpha}^2\overline{\beta}\overline{\alpha}^2\overline{\beta}\overline{\alpha}\overline{\beta}\overline{\alpha}^2\overline{\beta}^2\overline{\alpha}^2] = 1$$

The commutator $[\beta\alpha\beta, \alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2]$ is the trivial element of T and thus it leads by means of Hopf theorem to a 2-cycle on T, given by the formula:

$$\delta = (\beta\alpha\beta, \alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2) - (\alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2, \beta\alpha\beta)$$

However the commutator above can be written as a commutator in the subgroup F as we already remarked that

$$[\beta\alpha\beta, \alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2] = [AB^{-1}, A^{-2}BA^2]$$

where $A = \beta\alpha^2$ and $B = \beta^2\alpha$ are the generators of F. Moreover the last commutator defines the 2-cycle

$$\epsilon = (AB^{-1}, A^{-2}BA^2) - (A^{-2}BA^2, AB^{-1})$$

in $H_2(F)$ and the inclusion $i : F \subset T$ sends the class of $[\epsilon]$ into $i_*([\epsilon]) = [\delta]$. vanishes in $H_2(T)$. However, it is known from [21] that $[\epsilon]$ is in the kernel of $i_* : H_2(F) \to H_2(T)$ and thus $[\delta] = 0$. This shows that $\langle c_{T_{n,p,q,r,s}}|_F, [\epsilon]\rangle = 0$, where $\langle , \rangle : H^2(T) \times H_2(T) \to \mathbb{Z}$ is the obvious pairing. Thus the restriction of $T_{n,p,q,r,s}$ to F splits over the subgroup generated by the element $[AB^{-1}, A^{-2}BA^2]$. As the later subgroup is contained in F, this amounts to the splitting of the extension $T_{n,p,q,r,s}$ over the subgroup generated by $[\beta\alpha\beta, \alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2]$. Thus up to changing each one of the lifts $\overline{\alpha}, \overline{\beta} \in T_{n,p,q,r,s}$ of α, β by a central factor we have

$$[\overline{\beta}\overline{\alpha}\overline{\beta}, \overline{\alpha}^2\overline{\beta}\overline{\alpha}^2\overline{\beta}\overline{\alpha}\overline{\beta}\overline{\alpha}^2\overline{\beta}^2\overline{\alpha}^2] = 1$$

Hovewer central factor will cancel each other in the commutator above and thus the identity above holds for any choice of the lifts. In particular we have $s = 0$ always. □

The aim of this section is to prove the following:

Theorem 3.1. *The class* $c_{T_{n,p,q,r}} \in H^2(T)$ *of the extension* $T_{n,p,q,r}$ *is given by*

$$c_{T_{n,p,q,r}} = (12n - 15p - 20q - 60r)\chi + r\alpha$$

We denote by $\chi(n, p, q, r)$ the coefficient of χ and $\alpha(n, p, q, r)$ the coefficient of α in $c_{T_{n,p,q,r}}$.

3.2 Computing $\chi(n,p,q,0)$

Proposition 3.2. *We have $\chi(n,p,q,0) = 12n - 15p - 20q$.*

Proof. We observe first that

Lemma 3.1. *The class $c_{T_{n,p,q}}$ is a multiple of the Euler class χ.*

Proof. Because of the last two commutation relations the extension $T_{n,p,q}$ splits over the Thompson group F. Thus the restriction of $c_{T_{n,p,q}}$ to F vanishes and a fortiori the restriction to the commutator subgroup $F' \subset F$. According to [21] we have $H^2(F') = \mathbb{Z}\alpha$ where α is the discrete Godbillon-Vey class. Thus the map $H^2(T) \to H^2(F')$ is the projection $\mathbb{Z}\alpha \oplus \mathbb{Z}\chi \to \mathbb{Z}\alpha$. Since $c_{\widehat{T}}$ belongs to the kernel of $H^2(T) \to H^2(F')$. This proves that $c_{T_{n,p,q}} \in \mathbb{Z}\chi$. □

Lemma 3.2. *The function $\chi(n,p,q,r) : \mathbb{Z}^4 \to \mathbb{Z}$ is linear.*

Proof. The group T contains all finite cyclic subgroups. Let us fix some positive integer k and set $y_k \in T$ for an element of order k. Denote by $G_k \subset T$ the cyclic subgroup generated by y_k.

We want to compute the restriction $c_{T_{n,p,q,r}}|_{G_k}$, namely the image of $c_{T_{n,p,q,r}}$ under the obvious morphism $H^2(T) \to H^2(G_k)$. Since G_k is cyclic $H^2(G_k)$ is cyclic $\mathbb{Z}/k\mathbb{Z}$ of order k, generated by the restriction of the Euler class $\chi|_{G_k}$. In particular,

$$c_{T_{n,p,q,r}}|_{G_k} = \chi(n,p,q,r)\chi|_{G_k} \in \mathbb{Z}/k\mathbb{Z} = H^2(G_k)$$

On the other hand we can compute the Euler class of a central extension of a cyclic subgroup of $T \subset \mathrm{Homeo}^+(S^1)$ by means of the following Milnor-Wood algorithm. Assume that $y_k = w_k(\alpha,\beta)$ is given by a word in the generators α, β of T. Consider next the word $\overline{y_k} = w_k(\overline{\alpha},\overline{\beta}) \in T_{n,p,q}$. Then $\overline{y_k}^k = z^e$ for some e and the Euler class is given by the value of e modulo k.

However $\overline{y_k}^k$ is an element of the center of $T_{n,p,q,r}$ and thus it is a product of conjugates of $(\overline{\beta}\overline{\alpha})^5$, $\overline{\alpha}^4$, $\overline{\beta}^3$ and the almost-commutation relations not involving z. If $\varphi_{n,p,q,r}(w)$ is the additive map that associates to any subword of w from the subset $\{(\overline{\beta}\overline{\alpha})^5, \overline{\alpha}^4, \overline{\beta}^3, [\overline{\beta}\overline{\alpha}\overline{\beta}, \overline{\alpha}^2\overline{\beta}\overline{\alpha}^2\overline{\beta}\overline{\alpha}\overline{\beta}\overline{\alpha}^2\overline{\beta}^2\overline{\alpha}^2]\}$ respectively n, p, q or s it follows that

$$\varphi_{n,p,q,r}(w) = n\varphi_{1,0,0,0}(w) + p\varphi_{0,1,0,0}(w) + q\varphi_{0,0,1,0}(w) + r\varphi_{0,0,0,1}(w)$$

In particular, by choosing $w = w_k(\overline{\alpha},\overline{\beta})^k$ we obtain that the coefficient of the Euler class $\chi(n,p,q,r)$ verifies

$$\chi(n,p,q,r) = n\chi(1,0,0,0) + pe(0,1,0,0) + qe(0,0,1,0) + r\chi(0,0,0,1) \pmod{k}$$

This equality holds for all k and thefore $\chi(n,p,q,r)$ is linear. □

We are able now to finish the proof of the proposition. By the definition of the extension class $\chi(n,p,q,0)$ takes the same value for those $T_{n,p,q,0}$ that are isomorphic by an isomorphism inducing identity on T and on the center. Such isomorphisms L have the form $L(\overline{\alpha}) = \overline{\alpha}z^x$ and $L(\overline{\beta}) = \overline{\beta}z^y$. Thus $T_{n,p,q,0}$ is isomorphic as extension to $T_{n',p',q',0}$ if and only if there exists integers $x, y \in Z$ such that

$$n = n' + 5x + 5y,\ p = p' + 4x,\ q = q' + 3y$$

In particular the linear form $\chi(n,p,q,0)$ should be invariant by the transforms corresponding to arbitrary $x, y \in \mathbb{Z}$ and thus it should be a multiple of $12n - 15p - 20q$.

Last, the extension $\widetilde{T} \subset \mathrm{Homeo}^+(\mathbb{R})$ is known to have Euler number 1. The central element z acts as the unit translation on the line and in order to identify it with an element of the family $T_{n,p,q}$ it suffices to compute the rotation index associated to the elements α^4, β^3 and $(\beta\alpha)^5$. One obtains that $\widetilde{T}$ is actually isomorphic to $T_{3,1,1}$. Thus $\chi(n,p,q,0)$ is precisely given by the claimed linear form. □

3.3 Computing $\alpha(n,p,q,r)$

Proposition 3.3. *We have $\alpha(n,p,q,r) = r$.*

Proof. Since the commutator $[\beta\alpha\beta, \alpha^2\beta\alpha\beta\alpha^2] = 1$ in T it gives rise to a 2-cycle in homology given by

$$\mu = (\beta\alpha\beta, \alpha^2\beta\alpha\beta\alpha^2) - (\alpha^2\beta\alpha\beta\alpha^2, \beta\alpha\beta)$$

and hence representing a class $[\mu] \in H_2(T)$. As in the proof of Proposition 3.1 $[\mu] = i_*([\eta])$ where

$$\eta = (AB^{-1}, A^{-1}BA) - (A^{-1}BA, AB^{-1})$$

is the 2-cycle on F associated to the commutator $[AB^{-1}, A^{-1}BA]$.

We claim that

Lemma 3.3. *$\langle \alpha, [\mu] \rangle = 1$, where $\langle , \rangle : H^2(T) \times H_2(T) \to \mathbb{Z}$ is the obvious pairing.*

Proof. Consider the discrete Godbillon-Vey 2-cocycle $\overline{gv} : T \times T \to \mathbb{Z}$ defined by the formula

$$\overline{gv}(g,h) = \sum_{x \in S^1} \det \begin{pmatrix} \log_2 h'_r(x) & \log_2 (g \circ h)'_r(x) \\ h''(x) & (g \circ h)''(x) \end{pmatrix}$$

Here γ'' stands for

$$\gamma''(x) = \log_2 \gamma'_r(x) - \log_2 \gamma'_l(x)$$

where γ'_r, γ'_l are the right respectively left derivatives of γ. Notice the derivative of $\gamma \in T$ is a locally constant function, having finitely many discontinuity points.

It is well-known (see [21]) that the 2-cocycle $\overline{gv}$ represent 2α in cohomology. A direct computation using this cocycle shows that $\overline{gv}(\mu) = 2$ and hence proving the claim. □

We have

$$\langle c_{T_{n,p,q,r}}, i_*([\eta]) \rangle = \langle c_{T_{n,p,q,r}}|_F, [\eta] \rangle = \langle c_{F_r}, [\eta] \rangle$$

where F_r is the central extension of F given by

$$F_r = \langle A, B, z; [AB^{-1}, A^{-1}BA] = z^r,\ [AB^{-1}, A^{-2}BA^2] = 1, [z, A] = [z, B] = 1 \rangle$$

According to the arguments from the proof of Proposition 3.1 these extensions exhaust the set of all central extensions of F that arise from central extensions of T.

On the other hand we have

$$\begin{aligned} \langle c_{T_{n,p,q,r}}, [\mu]) \rangle &= \langle \alpha(n,p,q,r)\alpha + \chi(n,p,q,r)\chi, i_*[\eta] \rangle = \langle \alpha(n,p,q,r)\alpha|_F + \chi(n,p,q,r)\chi|F, [\eta] \rangle \\ &= \alpha(n,p,q,r)\langle \alpha|F, [\eta] \rangle = \alpha(n,p,q,r) \end{aligned}$$

because the image of χ in $H^2(F)$ vanishes.

Observe that F_r does not depend on n, p, q and thus its class does not depend either. This shows that $\alpha(n,p,q,r)$ is a function on r. This function is computed through the choice of a setwise section in the projection $F_r \to F$, which is further evaluated to a fixed 2-cycle η and thus it should be a linear function with integer coefficients and without constant terms. This implies that $\alpha(n,p,q,r) = \lambda r$, for $\lambda \in \mathbb{Z}$.

Eventually the set of extensions $T_{n,p,q,r}$ exhaust all central extensions by $\mathbb{Z}$ and thus their isomorphisms types yield all classes in $H^2(T)$, in particular all classes of the form $n\alpha \in T$. Since an isomorphism type of extension lead to a unique value of the commutator, independently on the choice of lifts $\overline{\alpha}, \overline{\beta}$ it follows that $\lambda \in \{-1, 1\}$. □

Corollary 3.2. *The extensions $T_{n,p,q,r}$ describe all central extensions of T by $\mathbb{Z}$.*

The extensions $T_{n,p,q,0}$ describe all central extensions of T by $\mathbb{Z}$ associated to multiples of the Euler class, and all of them split over F.

Proof. The groups $T_{n,p,q,0}$ are extensions of T by a cyclic group. Moreover any central extension of T has the form $T_{n,p,q,r}$. The formula for $\alpha(n,p,q,r)$ shows that $r=0$ if we want that its class be a multiple of the Euler class. Further the set of central extensions of T by $\mathbb{Z}$ up to isomorphism over T are in one-to-one correspondence with the classes of $H^2(T)$. In particular any multiple of χ should be realized by one extension by $\mathbb{Z}$. Now, for given $m \in \mathbb{Z}$ there exists only one isomorphism type of $T_{n,p,q,0}$ for which n,p,q are solutions of $12n-15p-20q=m$. This follows from the uniqueness of solutions up to equivalence $p \sim p+4x$, $q \sim q+3y$, $n \sim n+5x+5y$. Thus the respective extension must have kernel $\mathbb{Z}$. □

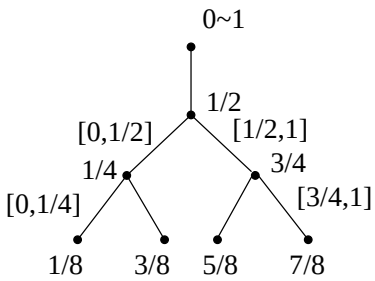
3.4 The Greenberg-Sergiescu extension $\mathcal{A}_T$

We know the value of $\chi(n,p,q,r)$ for $r=0$, but in order to find the coefficient of r we need to know explicitly central extensions with nontrivial Godbillon-Vey class. Fortunately, Greenberg and Sergiescu in [22] had constructed such an extension of T by B_∞. The main difficulty in analyzing this extension comes from the fact that there are several perspectives for analyzing the group T, either as a group of dyadic piecewise affine homeomorphisms of S^1 or else as a group $PPSL(2,\mathbb{Z})$ of piecewise $PSL(2,\mathbb{Z})$ automorphisms of the circle at infinity. If we plug in the discrete Godbillon-Vey then the formulas from [21, 22] use the first point of view. If we are seeking for the mapping class group perspective it is the second point of view which is manifest and there is no direct relationship between this and the former one. The keypoint in the calculations below is to pass from one perspective to the other. We have therefore to give a detalied account of the group $\mathcal{A}_T$, follwing Kapoudjian and Sergiescu [24].

We will use the mapping class group description of the Ptolemy-Thompson group but we will enlarge the surface. We follow closely [24, 17].

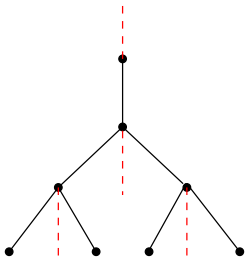
The surface D occurs in the process of understanding the almost action of T on the infinite binary tree. Recall that an almost automorphism of a tree is a map sending the complement of a finite tree isomorphically on the complement of a finite tree. The action of T acting as mapping class group of D induces an almost action on the binary tree. This point of view emphasize the realization of T as the group $PPSL(2,\mathbb{Z})$.

We can think of T as the group of dyadic piecewise affine homeomorphisms of S^1. This will lead to another, more subtle, way to construct an action of T on a regular tree. Let $\mathcal{T}$ be the rooted binary tree with one finite leaf, obtained by spliting one edge of the usual binary tree at some vertex (the root) and attaching one more edge. Label the vertices and edges (see [8]) of $\mathcal{T}$ inductively as follows. The leaf is labeled $0 \sim 1$ and the edge joining it with the root by $[0,1]$. The root is labeled $\frac{1}{2}$, its left descending edge is labeled $[0,\frac{1}{2}]$ and the right decending one $[\frac{1}{2},1]$. Further if an edge is labeled $[\frac{k}{2^n},\frac{k+1}{2^n}]$, then its bottom vertex is labeled $\frac{2k+1}{2^{n+1}}$ and the two edges issued from it are labeled, the left one $[\frac{k}{2^n},\frac{2k+1}{2^{n+1}}]$ and the right one $[\frac{2k+1}{2^{n+1}},\frac{k+1}{2^n}]$ respectively.



The almost action of T on $\mathcal{T}$ could be read off from the description of elements of T as pairs of stable binary trees. Alternatively, we identify elements of T as dyadic piecewise affine homeomorphisms of $S^1=[0,1]/0\sim 1$. The action of such a homeomorphism induces a bijection of the set of vertices of $\mathcal{T}$ (identified to their labels). This bijection is an almost automorphism of the tree $\mathcal{T}$.

Consider next the tree $\mathcal{E}T$ obtained from $\mathcal{T}$ by adjoining a pending line (with infinitely many vertices on it) to each vertex of $\mathcal{T}$.



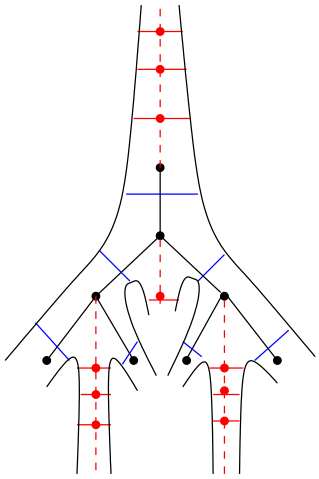
There is an obvious extension of the almost action of T from $\mathcal{T}$ to $\mathcal{E}T$. However there exists a more interesting one as it was discovered in [22].

Definition 3.2. *Let* $\frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}} \subset S^1$ *denote the set of images of dyadic numbers. Assume that* T *is identified with the subgroup of* $\mathrm{Homeo}^+(S^1)$ *of piecewise linear homeomorphisms. A* cocyle *is a map* $K : T \times \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}} \to \mathbb{Z}$ *satisfying the following conditions:*

1. *For any* $\gamma \in T$, $K(\gamma, x)$ *vanishes for all but finitely many points* $x \in \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}}$ *and*
$$\sum_{x \in \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}}} K(\gamma, x) = 0$$
2. $K(\gamma\delta, w) = K(\gamma, \delta(w)) + K(\delta, w)$, *for all* $\gamma, \delta \in T$, $w \in \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}}$.

For any cocycle $K : T \times \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}} \to \mathbb{Z}$ we can associate such an action. Let $\gamma \in T$. Then γ induces a bijection denoted by the same letter between the vertices of the rooted tree $\mathcal{T}$, which were identified with $\frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}}$. This bijection induces an almost automorphism of the tree $\mathcal{T}$. Moreover, let $n = \max_{x \in \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}}} |K(\gamma, x)|$. If $v \in \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}}$ is a vertex of $\mathcal{T}$ let f_v denote the pending line at v and $f_v^{\geq n} \subset f_v$ be the subtree of those points at distance at least n from v. We define the almost action of γ on $\mathcal{E}T$ as being the unique isometric bijection from $f_v^{\geq n}$ to $f_{\gamma(v)}^{\geq n}$.

Definition 3.3. *The enhanced ribbon tree* ED *is obtained by thickening* $\mathcal{E}T$ *in the plane and* ED^* *by puncturing* ED *along the vertices of the pending lines. Notice that the vertices of* $\mathcal{T}$ *are not among the punctures.*



The canonical rigid structures of ED *and* ED^* *are those from the picture above which decomposes* ED *into (punctured) octogons centered at vertices of* $\mathcal{T}$ *and (punctured) squares along the pending lines at vertices of* $\mathcal{T}$.

Admissible subsurfaces of ED *or* ED^* *are connected finite unions of elementary pieces.*

Given a cocycle K one finds that the almost action of T on $\mathcal{E}T$ induces an embedding of the group T into the asymptotic mapping class group $\mathcal{M}(ED)$ with the given structure. Specifically, define the content of an admisible subsurface $\Sigma \subset ED$ to be the number of squares it contains. Then T is the group of mapping classes that preserve the content i.e. those mapping classes of homeomorphisms φ for which $\varphi(\Sigma)$ and Σ have the same content for any admissible Σ.

We have an obvious exact sequence

$$1 \to B_\infty \to \mathcal{M}(ED^*) \to \mathcal{M}(ED) \to 1$$

Using the embedding $\iota_K : T \hookrightarrow \mathcal{M}(ED)$ we can restrict $\mathcal{M}(ED^*)$ at $\iota_K(T)$. This restriction is the group $\mathcal{A}_{T,K}$, which fits also in the exact sequence

$$1 \to B_\infty \to \mathcal{A}_{T,K} \to T \to 1$$

The main example of a nontrivial cocycle is the one associated to the discrete Godbillon-Vey class. Specifically, identify T with the group of dyadic piecewise affine homeomorphisms of $[0,1]/0 \sim 1$. For any $\gamma \in T$ and $v \in \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}}$ we set

$$\gamma''(v) = \log_2 \gamma_r'(v) - \log_2 \gamma_l'(v)$$

where γ_r', γ_l' are the right respectively left derivatives of γ. Notice the derivative is locally constant function, having finitely many discontinuity points. It is wellknown ([21, 22]) that $K(\gamma, v) = \gamma''(v)$ is a cocycle.

The extension $\mathcal{A}_{T,K}$ obtained when K is the Godbillon-Vey cocycle above is simply denoted $\mathcal{A}_T$.

Remark 3.2. The definition from [24] was slightly different because it used $n+1$ instead of n in the definition and we have punctures at the vertices of $\mathcal{T}$; in particular the vertex v of T was always sent into $\gamma(v)$. Nevertheless, the two groups $\mathcal{A}_T$ in [24] and the present paper coincide. Notice that there is a homeomorphism between the two differently punctured surfaces which slide all the punctures of the pending lines one unit such that their first punctures belong are now the vertices of T. This homeomorphism conjugates between the two versions of $\mathcal{A}_T$. Our version has the advantage of simplifying the already cumbersome computations of the next section.

3.5 The abelianized extension $\mathcal{A}_T^{ab}$

Proposition 3.4. *The class $c_{\mathcal{A}_T^{ab}} \in H^2(T)$ is given by $c_{\mathcal{A}_T^{ab}} = \alpha$.*

Proof. This is already stated in [22]. In fact $\mathcal{A}_T$ splits over the cyclic subgroups $\mathbb{Z}/2^m\mathbb{Z} \subset T$ for all m and this implies that the coefficient of χ vanishes. Moreover, the coefficient of α is shown in [22] to be one. □

Proposition 3.5. *There is an isomorphism of extensions between $\mathcal{A}_T^{ab}$ and $T_{30,16,3,1}$.*

Proof. We have to consider α and β as elements of the group of homeomorphisms preserving the dyadics of S^1. Recall that T has the standard generators A, B, C from [8], as described in section 2.1. Realizing A, B, C as (stable) couples of binary trees we can identify

$$\beta = C^{-1},\ \alpha = C^{-1}B$$

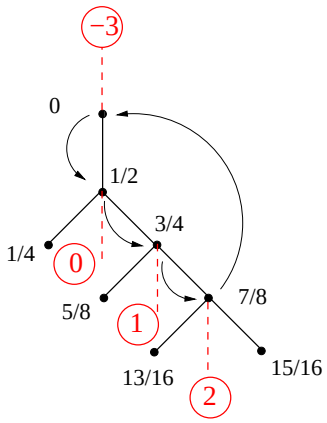
and thus

$$\beta(x) = \begin{cases} \frac{x}{2} + \frac{1}{2}, & \text{if} \quad x \in [0, \frac{1}{2}] \\ x + \frac{1}{4}, & \text{if} \quad x \in [\frac{1}{2}, \frac{3}{4}] \\ 2x - \frac{3}{2}, & \text{if} \quad x \in [\frac{3}{4}, 1] \end{cases}, \quad \alpha(x) = \begin{cases} \frac{x}{2} + \frac{1}{2}, & \text{if} \quad x \in [0, \frac{3}{4}] \\ x + \frac{1}{8}, & \text{if} \quad x \in [\frac{3}{4}, \frac{7}{8}] \\ 4x - \frac{7}{2}, & \text{if} \quad x \in [\frac{7}{8}, 1] \end{cases}$$

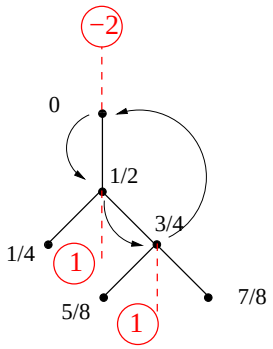
This implies that

$$\beta''(0) = -2, \beta''\left(\frac{1}{2}\right) = 1, \beta''\left(\frac{3}{4}\right) = 1,\ \alpha''(0) = -3, \alpha''\left(\frac{1}{2}\right) = 0, \alpha''\left(\frac{3}{4}\right) = 1, \alpha''\left(\frac{7}{8}\right) = 2$$

In the enhanced rooted tree model model we can therefore explain the action of α and β as in the pictures below.

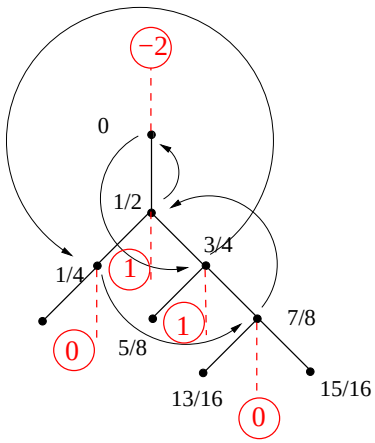


α



β

It follows by direct calculation that the action of $\beta\alpha$ is described by:



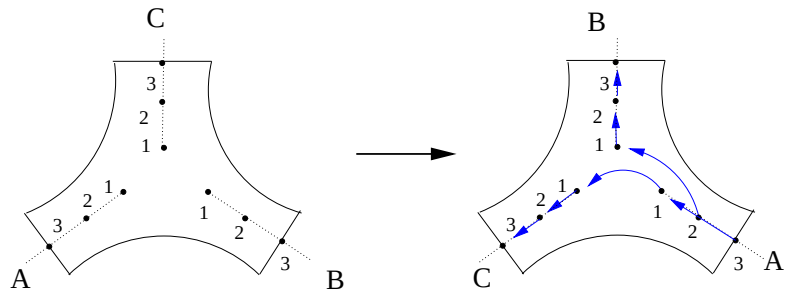
βα

The problem we face now is to consider lifts α^{**} and β^{**} of α and β as mapping classes of homeomorphisms of the enhanced surface ED^*.

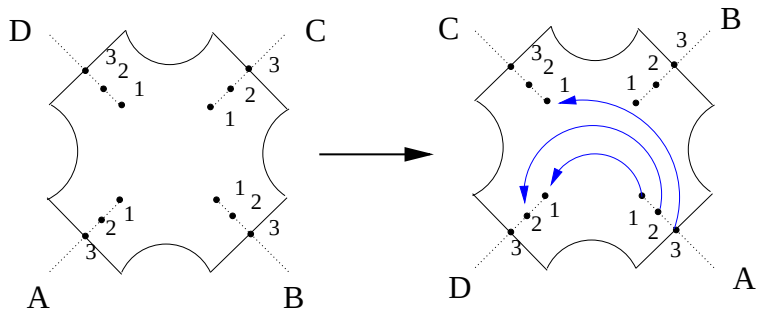
For the sake of simplicity we change the labels corresponding to the vertices of the rooted tree, as follows: A states for 0, B for 1/2, C for 3/4, D for 7/8 and E for 1/4. Moreover the pending line at A has its vertices labeled $A_1, A_2, \ldots$ and so on for all other vertices.

The supports of these classes homeomorphisms correspond to suitable disks around the vertices.

1. β^{**} has a support a disk embedded into ED^* containing A_i, B_i, C_i for $i \leq 3$. Moreover the action of β is described as follows: first β acts as a rotation of order 3 in the plane; next the vertices A_1 and A_2 are slided in counterclockwise direction towards the positions C_1 and B_1, respectively. In meantime the the punctures C_j (and B_j) are simultaneously translated one unit along their pending lines and hence C_j (respectively B_j) will arrive to the position formerly occupied by C_{j+1} (respectively B_{j+1}), for all $j \geq 1$. The punctures A_j (for $j \geq 3$) are translated simultaneously two negative units along their pending line and hence A_j will arrive to the position formerly occupied by A_{j-2}.



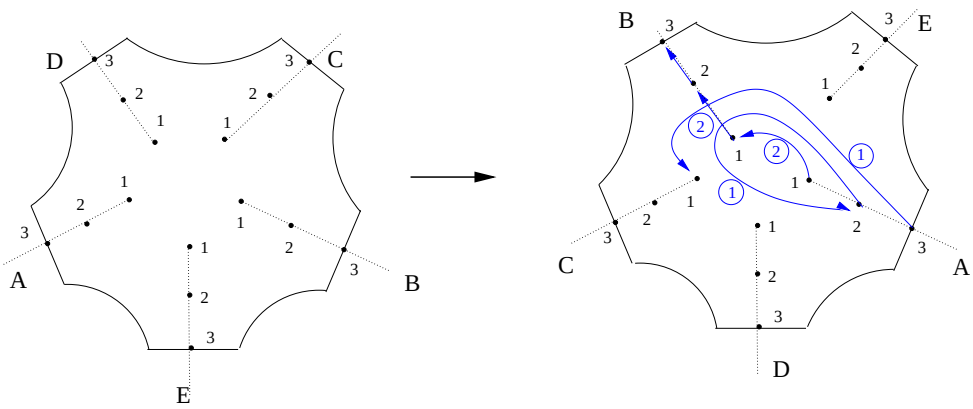
2. α^{**} has a support a disk embedded into ED^* containing A_i, B_i, C_i, D_i for $i \leq 4$. The action of α is described as follows: first α acts as a rotation of order 4 in the plane; further A_1 and A_2 are slided in counterclockiwse direction onto D_1 and respectively D_2, while A_3 is slided into C_1. The slidings occur simultaneously with the the translations of all punctures D_j $(j \geq 1)$ two units along their pending lines and the C_j $(j \geq 1)$ one unit along their pending line. Moreover the A_j $(j \geq 4)$ are translated three negative units along their pending line. The trajectories of the points are represented below (we did not figure the obvious translations along the pending lines):



We are able now to figure out the element $\beta^{**}\alpha^{**} \in \mathcal{A}_T$. Its support is now a disk embedded into ED^* containing A_i, B_i, C_i, D_i, E_i for $i \leq 4$.

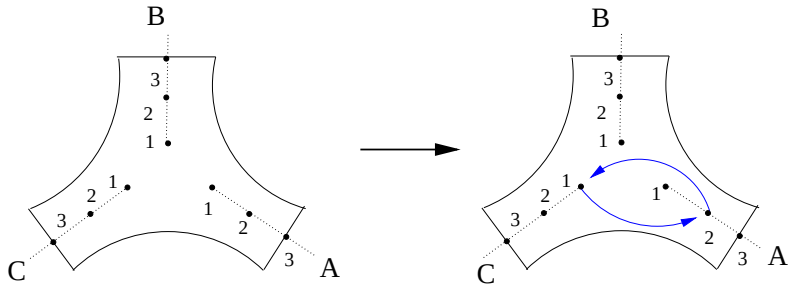
Notice however that the shape of the punctures trajectories is not sufficient for recovering the relative position of punctures. We have to specify somewhat the speed of each puncture along its trajectory, or equivalently, to specify a parameterization. There is a way to associate natural numbers to arcs of trajectories as follows, called discrete parameterization. The time interval is divided into N smaller intervals for some N. The arc λ is given the label $k \in \{1, 2, \ldots, N\}$ if the respective puncture travels along λ precisely in the k-th interval of time. Actually this says that whenever we have two arcs (disjoint or not) labeled j and k, with $j < k$ the respective punctures travel first along the arc j and next along the arc k. A mapping class group element written as a word in the α^{**} and β^{**} will lead naturally to a discrete parameterization. Moreover, N is chosed such that the mapping class of the respective homeomorphism is uniquely determined by the discrete parameterization.

As an example $\beta^{**}\alpha^{**}$ is completely described by the following picture (including parameterization):



Recall that lifts in $\mathcal{A}_T$ of the relations in T should give elements of the infinite braid group B_∞, the kernel of the projection $\mathcal{A}_T \to T$. Denote by $a : \mathcal{A}_T \to \mathcal{A}_T^{ab}$ the projection homomorphism. The restriction $a|_{B_\infty} : B_\infty \to Z$ is then identified to the abelianization homomorphism. We have therefore to compute the integers n, p, q so that $a(\beta^{**3}) = z^q$, $a(\alpha^{**4}) = z^p$, $a((\beta^{**}\alpha^{**})^5) = z^n$.

Lemma 3.4. *The braid $\beta^{**3} \in B_\infty$ is given by the picture below*



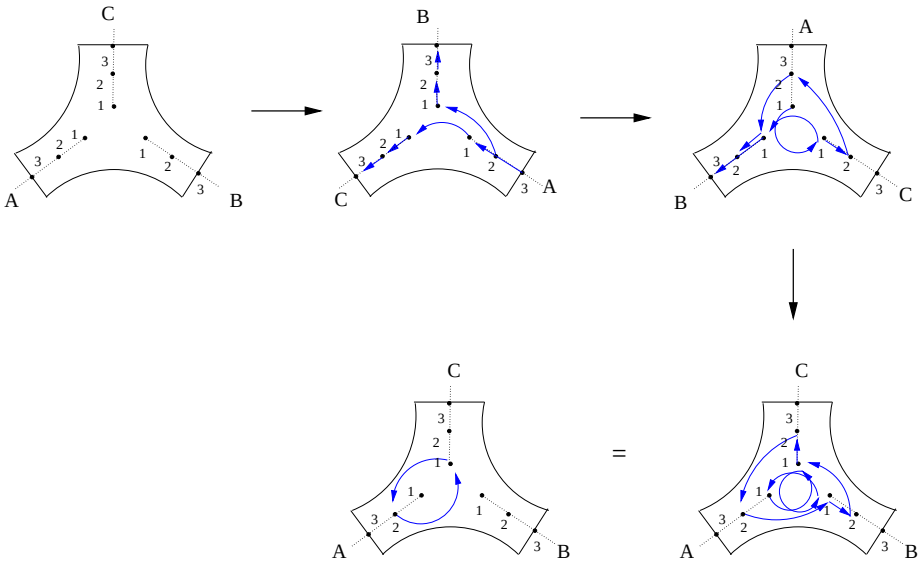
In particular $a(\beta^{**3}) = z^3$.

Proof. We will give a very explicit proof of what is going on in this (simplest) situation. The terminology is somewhat unconventional: we say that punctures "travel" from one location to another, in certain intervals of time; we choose the time intervals so that their trajectories do not intersect and so one can recover the class of the associated homeomorphism. It is of course sufficient to have a finite number of such intervals, which we also call steps, and the relative speed within each interval is not important, because any two speed values lead to isotopic homeomorphisms. Eventually we can compose the classes obtained by describing them step by step. The labels are assigned to the punctures and thus they travel around; at each step there are induced (infinite) permutations of the labels. At the end we find an element of a finite braid group inside B_∞ and trajectories of punctures are now viewed as strands of that braid. Strands are said trivial if they are trivial as braid strands. We will exemplify below with β^{**3}.

Each time we have an action of β there is a first step comprising an order 3 rotation in the plane and a second step, the sliding, where two punctures keep traveling in the counterclockwise direction, while the others are fixed, along a circle arc of angle $\frac{4\pi}{3}$ (a $\frac{2}{3}$-turn) and respectively $\frac{2\pi}{3}$, according to the picture of β^{**}: the puncture labeled A_1 is the one which travel faster while A_2 travels a shorter amount. There are also some translations along the pending lines so that, for instance, B_1 is sent into B_2 and C_1 onto C_2. In general, we will not bother to represent on the picture these translations, except when their final action is nontrivial.

Assume now that we want to compose two such classes β^{**}. Then, we draw first the result of the first β^{**} action and further we resume with the first step, namely an order 3 rotation. One applies next the second step sliding and observe that the puncture labeled A_1 is again located in the position from which it has to be slided. Its fellow traveler is this time the puncture labeled C_1, which has moved at the previous step from position C_1 onto C_2.

It is clear now how to add one more β^{**}: an order 3 rotation sends the puncture labeled A_1 again into the position to be slided and thus it completes one more $\frac{2}{3}$-turn to arrive into its initial position. Its fellow traveler in the sliding step is the puncture labeled A_2, which followed before the path from A_2 to B_1 and then from B_1 onto the location of B_2. The sliding will send it into the location of C_1.



The trajectory of A_1 is a strand of the braid β^{**3} which can be split off. This implies that, by means of an isotopy we can assume that A_1 is fixed. This isotopy corresponds to shriking the trajectory of A_1 to a point. This shrinking could be done without touching the others trajectories, which means that it lifts to an isotopy between braids in the three dimensional space. It follows that the only nontrivial part of the braid β^{**3} is the exchange between the punctures A_2 and C_1. The translations along the pending lines yield trivial strands for the remaining punctures. □

□

The remaining calculations are of the same sort, but involve more complicated braids. We were unable to find the braids (as the two dimensional picture is misleading) but we will make use of additional simplifications to help computing the images under the abelianization map.

First, one can find $a(\sigma)$ using only the winding numbers of the trajectories of the braid σ. Let σ be given as a geometrical braid in $\mathbb{R}^2 \times [0,1]$ by the parameterizations $(x_k(t), t)$, for $t \in [0,1]$, each subscript k corresponding to one strand. The (relative) winding number $\nu(j,k)$ of the strands j and k is the angle that the vector $x_j(t) - x_k(t)$ swept when t goes from 0 to 1. Turning counterclocwise yields positive angles. If the punctures sit all on the same line in $\mathbb{R}^2 \times 0$ then relative winding numbers are multiples of π. However, they make sense even when the punctures are given arbitrary positions in the plane. The (total) winding number $\nu(\sigma)$ is the sum of all relative winding numbers of its (distinct) strands.

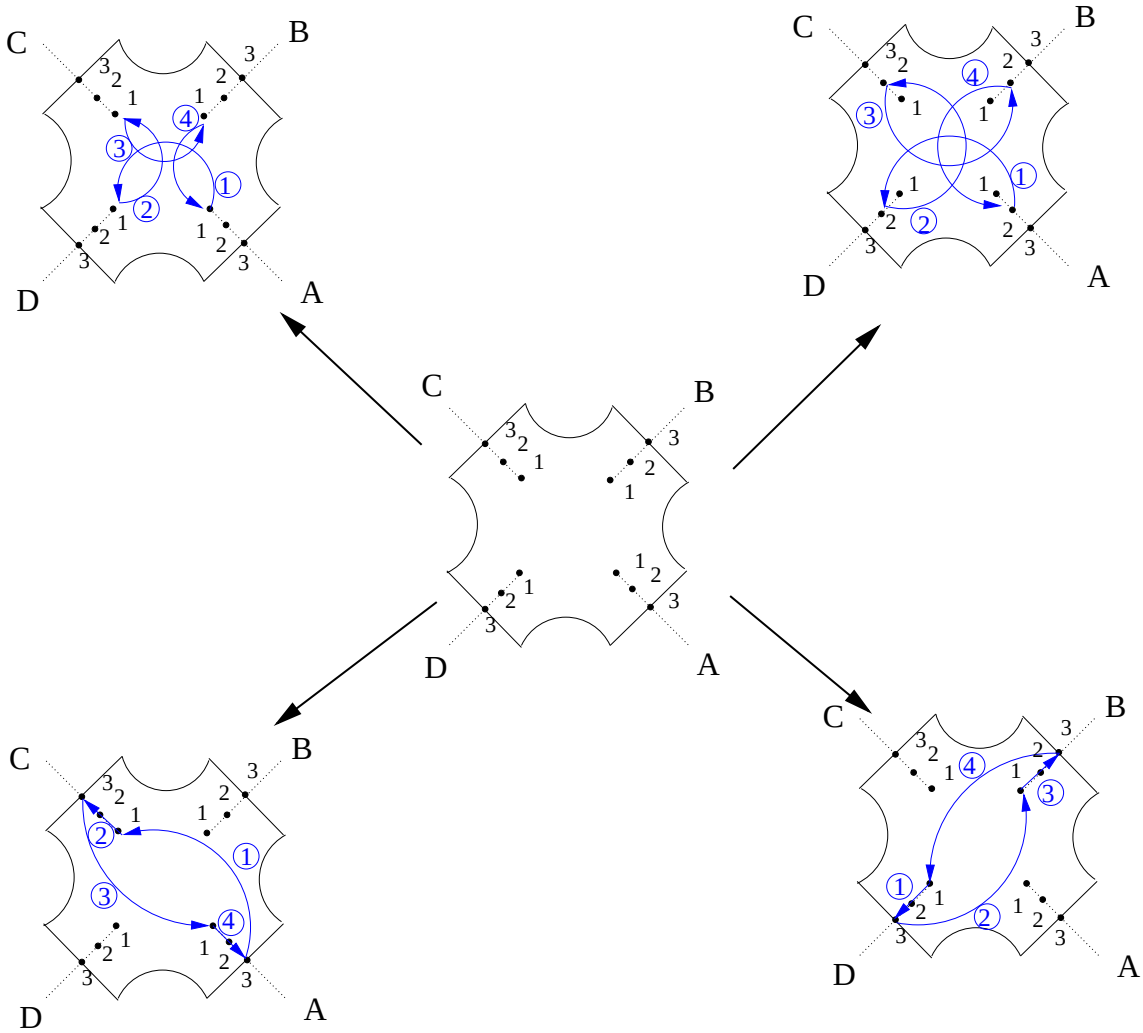
Lemma 3.5. *We have* $\pi a(\sigma) = \nu(\sigma)$.

Proof. Both sides are group homomorphisms and their values on the generators coincide. □

It is much simpler to compute winding numbers when trajectories are known.

Lemma 3.6. *We have* $\nu(\alpha^{**4}) = 16\pi$.

Proof. We draw each trajectory individually, along with its discrete parameterization. The only nontrivial trajectories are those of the strands starting at A_1, A_2, A_3 and D_1. The remaining ones are easily shown to be trivial and having zero winding numbers with all others.



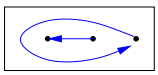
Then we compute easily from the picture above the relative winding numbers between strands (hereby identified to their startpoint):

$$\nu(A_1, A_2) = 6\pi,\ \nu(A_1, A_3) = \nu(A_2, A_3) = \nu(A_3, D_1) = \nu(A_2, D_1) = \nu(A_1, D_1) = 2\pi$$

This ends the proof of the Lemma. □

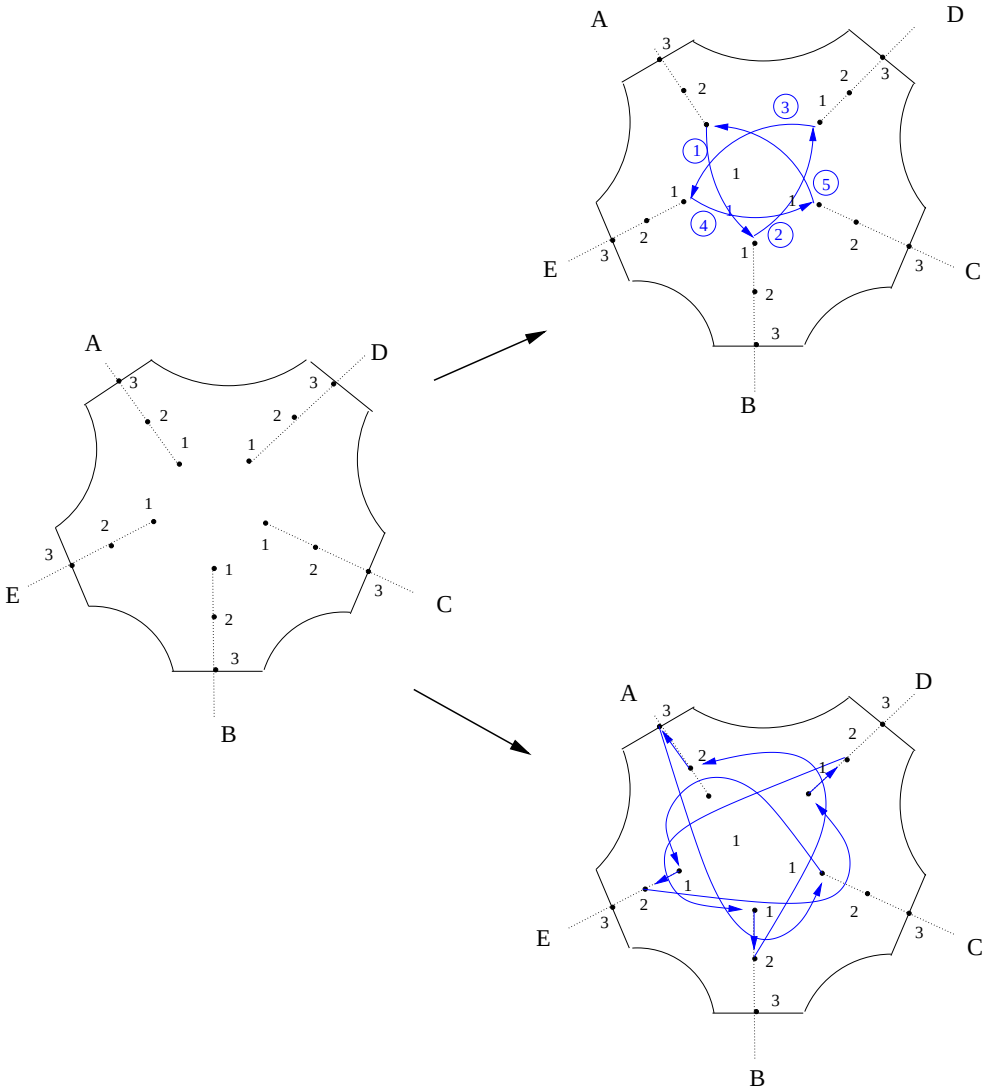
Lemma 3.7. *We have* $\nu((\beta^{**}\alpha^{**})^5) = 30\pi$.

Proof. Recall that pictures define mapping classes of homeomorphisms. Then, for any mapping classes H_1 and H_2 for which the compositions below make sense and represent elements of B_∞ we have the identity

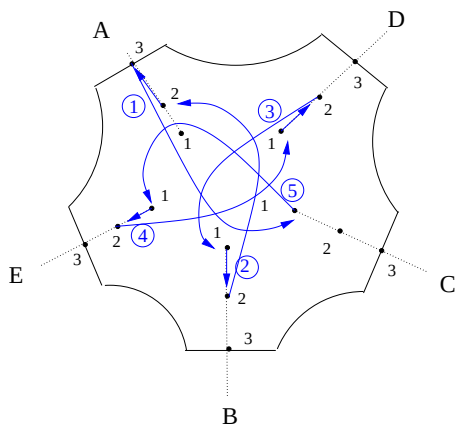
$a\Big(H_1 \circ$  $\circ H_2\Big) - a\Big(H_1 \circ$ $\circ H_2\Big) = 2$

In fact the first braid is obtained from the second one by inserting some σ^2, which braids two punctures twice. Replacing the first braid by the second one will be called a (direct) simplification of the braid diagram.

We can use a direct simplification within the picture of $\beta^{**}\alpha^{**}$ in order to remove the loop trajectory based at A_2. Thus after 5 simplifications within $(\beta^{**}\alpha^{**})^5$ we obtain the following trajectories.



We did not draw but the nontrivial trajectories which are of two types: a long trajectory of the strand starting at A_1 and a braid cycle permuting circularly the nine punctures $A_2, A_3, C_1, E_1, E_2, D_1, D_2, B_1$ and B_2. The braid cycle admits 4 direct simplifications by sliding C_1 outward to E_2D_1, B_1 outward A_3C_1, E_1 outward of D_2B_1 and D_1 outward of B_2A_2 and becomes the following braid b pictured below:



By direct calculation (taking care of the discrete parameterizations for both b and the trajectory of A_1) we

obtain

$$\sum_X \nu(A_1, X) = 4\pi$$

where the sum is over all strands X of the braid b. Since the braid b is now a cycle of length 9 we find that $\nu(b) = 8$. Summing up all contributions to $\nu((\beta^{**}\alpha^{**})^5)$ we obtain 30, as claimed. □

Remark 3.3. The elements $\beta, \alpha, \beta\alpha$ are respectively of order 3, 4 and 5. Since the Euler class of $\mathcal{A}_T$ is zero, it splits over cyclic subgroups of T and the same holds then for $\mathcal{A}_T^{ab}$. This implies that $q \equiv 0(\mathrm{mod}\, 3)$, $p \equiv 0(\mathrm{mod}\, 4)$, $n \equiv 0(\mathrm{mod}\, 5)$. Moreover, the permutation induced by $(\beta^{**}\alpha^{**})^5$ is a cycle of length 9 and thus is even. In particular $n \equiv 0(\mathrm{mod}\, 2)$ and so $n \equiv 0(\mathrm{mod}\, 10)$.

End of the proof of Theorem 3.1. The form $\chi(n, p, q, r)$ is linear by Lemma 3.2, $\chi(n, p, q, 0) = 12n - 15p - 20q$ by Proposition 3.2 and $\chi(30, 16, 3, 1) = 0$, by the previous two Propositions. This implies that χ has the required form, which togeher with Proposition 3.3 finishes the proof.

The analysis above can be used in establishing that

Proposition 3.6. *The group $\mathcal{A}_{T,K}$ is finitely generated.*

Proof. Consider the group $\mathcal{A}_T$, the general case following the same way. The elements α^{**} and β^{**} generate the quotient T and it suffices to add sufficiently many elements to be able to generate all of B_∞, which is the braid group on the punctures. Let σ be the braiding of the first two punctures of the pending line at A. Let denote by t the mapping class of the homeomorphism which translates one unit along the line made by gluing together the two (half)-lines pending at A and B. Observe now that t conjugate σ to any of the braid generators of the pending line at A and B. On the other hand the action of T by conjugation on $\mathcal{A}_T$ is tranzitive on the set of pending lines, as it is the action of $PSL(2, \mathbb{Z})$ on the binary tree. This means that for vertex of the rooted binary tree there exists a word $w = w(\alpha^{**}, \beta^{**})$ such that the corresponding element of $\mathcal{A}_T$ sends the pending line at v asymptotically onto the pending line at A. The meaning of the word asymptotically is that all but finitely many vertices of the respective line are sent into the other ones. The first few vertices might be sent onto other pending lines, as it happens with α^{**} and β^{**} which slide finitely many points. However sliding of punctures occur only at the supports of α^{**} and β^{**}. Thus the line pending at v is sent into the line pending at A and slidings of its vertices could appear only when its first vertex will reach the set $\{A_1, B_1, C_1, D_1, E_1\}$. Let then M be the mapping class group of an admissible subsurface containing the supports of α^{**} and β^{**}. It follows that adjoining the generators of M to $\alpha^{**}, \beta^{**}, \sigma, t$ we generate all of B_∞ and thus $\mathcal{A}_T$. □

4 Odds and ends

4.1 Geometric extensions

We would like to understand all extensions

$$1 \to B_\infty \to G \to T \to 1$$

coming out from *nature*. A tentative approach is say that such an extension is geometric if there exists a teselation of a a planar surface Σ with infinitely many punctures such that G is the asymptotic mapping class group of Σ with this extra structure (see also [19]). Then B_∞ is the braid group in the punctures.

In order to avoid trivial constructions we restrict to those examples which are *minimal* in some sense. The simplest minimality condistion is to ask that the natural homomorphism $T \to Out(B_\infty)$ has one orbit of generators of B_∞ i.e. T acts tranzitively on the first homology of the surface Σ. Alternatively, this amounts to require that the lifts of the generators α and β of T together with a stanadard braid generator σ form a generator system for G.

The groups $T_{1,0,0}$ and $T_{3,1,0}$ are geometric (see [17]) and minimal. Although $\mathcal{A}$ is also geometric one needs to modify the minimality condition above in order to be fit for it.

It seems that there are only finitely many such minimal geometric extensions, for appropriate minimality conditions.

4.2 Finite surfaces

It is known that the mapping class group $\mathcal{M}(\Sigma)$ of a punctured surface Σ embeds into the groupoid of flips acting on the triangulations of Σ with vertices at punctures. Quantization of the Teichmüller space of the surface Σ lead then by the technology of [15] to projective representations of the mapping class group $\mathcal{M}(\Sigma)$ and thus to a central extension $\widehat{\mathcal{M}(\Sigma)}$.

Recall also that $H^2(\mathcal{M}(\Sigma))$ is freely generated by the Euler class χ together with the classes corresponding to each one of the punctures.

It seems plausible that the class $c_{\widehat{\mathcal{M}(\Sigma)}}$ of the extension is actually equal to $12\chi \in H^2\mathcal{M}(\Sigma))$.

Notice that additional work is needed for obtaining this result because of our lack of knowledge of the Ptolemy groupoids and their associated groups (see [27]) for finite surfaces of positive genus.

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