Symmetry Groupoids and Admissible Vector Fields for Coupled Cell Networks

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Abstract

The space of admissible vector fields, consistent with the structure of a network of coupled dynamical systems, can be specified in terms of the network's symmetry groupoid. The symmetry groupoid also determines the robust patterns of synchrony in the network — those that arise because of the network topology. In particular, synchronous cells can be identified in a canonical manner to yield a quotient network. Admissible vector fields on the original network induce admissible vector fields on the quotient, and any dynamical state of such an induced vector field can be lifted to the original network, yielding an analogous state in which certain sets of cells are synchronized. We specify necessary and sufficient conditions for *all* admissible vector fields on the quotient to lift in this manner. These conditions are combinatorial in nature, and the proof uses invariant theory for the symmetric group. We also relate the symmetry groupoid of a quotient to that of the original network, and show that there is a close analogy with the usual normalizer symmetry that arises in group-equivariant dynamics.

1 Introduction

Coupled cell systems are finite sets of dynamical systems, called *cells*, which are coupled together. The topology or 'architecture' of the coupling is specified by a labelled graph or coupled cell network, [5, 6, 17]. Such systems arise in many areas of applied science, including communication via the Internet, the spread of epidemics, food webs in ecosystems, metabolic networks in the cell, neural circuits, networks of gene expression, animal locomotion, commercial supply chains, electrical power grids, transport networks, and crowd flow.

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Until recently the abstract theory of coupled cell systems has mainly focused on the effects of symmetry in the network [3, 4, 5, 6] and the consequent formation of spatial and spatiotemporal patterns. The formal setting for this theory centres upon the symmetry group of the network.

The analysis of robust patterns of synchrony in general coupled cell systems — that is, dynamics in which sets of cells behave identically as a consequence of the network topology — has led to the fruitful notion of the 'symmetry groupoid' of a coupled cell network [17]. A groupoid is a generalization of a group, in which products of elements are not always defined: see Higgins [9]. The symmetry groupoid of a coupled cell network is a natural algebraic formalization of the 'local symmetries' that relate subsets of the network to each other. In particular, the 'admissible' vector fields — those specified by the network topology — are precisely those that are equivariant under the action of the symmetry groupoid.

Robust patterns of synchrony correspond to the existence of a 'quotient' network, in which synchronous cells are identified. One of the main theorems of [17] is that if $\phi : G_1 \to G_2$ is a quotient map of networks, then it induces a map $\hat{\phi}$ between the spaces $\mathcal{F}^P_{G_1}$ and $\mathcal{F}^{\overline{P}}_{G_2}$ of admissible vector fields on phase spaces $P, \overline{P}$ for G_1, G_2. Examples in that paper show that for some networks $\hat{\phi}$ is surjective, but for others it is not. The surjectivity of $\hat{\phi}$ is important because it determines whether all dynamics on G_2 lifts to synchronous dynamics on G_1. The Surjectivity Problem for coupled cell networks asks for a characterization of those networks for which $\hat{\phi}$ is surjective, and we solve that problem in this paper.

Specifically, our main result is Theorem 7.6, which gives necessary and sufficient conditions for any admissible vector field on a quotient network to lift to an admissible vector field on the original network. These conditions are combinatorial in nature, and are determined by groupoid-theoretic conditions on the network topology. The proofs are algebraic, and make essential use of elementary invariant theory for direct products of symmetric groups.

The analysis is motivated by an analogy between the symmetry group of a symmetric network and the symmetry groupoid of a general one. The analogous surjectivity problem for symmetric networks is intimately related to the existence (or not) of 'hidden symmetries' [16].

Suppose that a finite (or, more generally, compact Lie) group acts linearly on a real vector space X. A smooth map $f : X \to X$ is Γ-*equivariant* if

$$f(\gamma x) = \gamma f(x) \qquad \forall x \in X, \gamma \in \Gamma$$

If Σ is a subgroup of Γ then the *fixed-point subspace* of Σ is

$$\mathrm{Fix}(\Sigma) = \{x \in X : \sigma x = x \qquad \forall \sigma \in \Sigma\}$$

It is well known and easy to prove that if f is Γ-equivariant then

$$f(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma)$$

so we can form the restriction $g = f|_{\mathrm{Fix}(\Sigma)}$. It can be shown that g is equivariant under the natural action of $N(\Sigma)/\Sigma$ on $\mathrm{Fix}(\Sigma)$, where $N(\Sigma)$ is the normalizer of Σ in Γ.

However, normalizer equivariance does not always characterize the possible maps g. Extra conditions may be required, known as 'hidden symmetries'. When no hidden symmetries are present, every smooth $N(\Sigma)/\Sigma$-equivariant map on $\mathrm{Fix}(\Sigma)$ can be extended to a smooth Γ-equivariant map on X.

In section 8 we show that the Surjectivity Problem for coupled cell networks is analogous to the hidden symmetry problem for symmetric networks. Associated with any quotient map of a coupled cell network there is a subgroupoid Σ and a normalizer groupoid $N(\Sigma)$. The map $\hat{\phi} : \mathcal{F}^{P}_{G_1} \to \mathcal{F}^{\overline{P}}_{G_2}$ is surjective if and only if G_2 is the 'natural' quotient determined by ϕ and the induced vector fields on G_2 are precisely the $N(\Sigma)/\Sigma$-equivariant vector fields. Thus $\hat{\phi}$ is surjective if and only if there are no 'hidden groupoid symmetries' associated with ϕ.

2 Coupled Cell Graphs

A coupled cell network is a network of dynamical systems, coupled together. Such systems can be represented schematically by a directed graph, whose nodes correspond to cells and whose edges represent couplings, and for this reason we will employ the alternative name 'coupled cell graph'. We start by defining what we mean by a coupled cell graph.

Definition 2.1 A *coupled cell graph* G consists of:

(a) A finite set $\mathcal{C} = \{1, \ldots, n\}$ of *nodes* (or *cells*).

(b) A finite set of ordered pairs $\mathcal{E} \subset \mathcal{C} \times \mathcal{C}$ of directed *edges* or *arrows*. If $(a,b) \in \mathcal{E}$ then a is the *tail* and b is the *head*. An edge of the form (a,a) is *internal*. All other edges are *external*.

(c) An equivalence relation $\sim_C$ on the nodes in $\mathcal{C}$.

(d) An equivalence relation $\sim_E$ on the edges in $\mathcal{E}$.

We also assume:

(e) $\{(c,c) : \ c \in \mathcal{C}\} \subset \mathcal{E}$.

(f) If $(i,c) \sim_E (j,d)$ then $i \sim_C j$ and $c \sim_C d$.

(g) $(c,c) \sim_E (d,d')$ if and only if $d = d'$ and $d \sim_C c$.

We write $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$, and refer to G as a 'graph'. ◇

The diagram $\mathrm{Diag}(G)$ of G is constructed in the following way: for each $\sim_C$-equivalence class of nodes we choose a distinct node symbol; for each $\sim_E$-equivalence class of external edges we choose a distinct arrow.

Definition 2.1(f) implies that arrows between nodes can be identical only when the nodes at the heads are identical and the nodes at the tails are identical. Node symbols can be interpreted as arrows from a node to itself — that is, internal edges. Condition (g) implies that an internal edge cannot be equivalent to an external edge.

Example 2.2 Figure 1 shows a graph $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$, where
$\mathcal{C} = \{1,2,3,4,5\}$
$\mathcal{E} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,3),(3,4),(4,5),(3,1)\}$
$\sim_C$ −equivalence class: $\{1,2,3,4,5\}$
$\sim_E$ −equivalence classes: $\{(1,1),(2,2),(3,3),(4,4),(5,5)\}$,
$\{(1,2),(2,3),(3,4),(4,5),(3,1)\}$ ◇

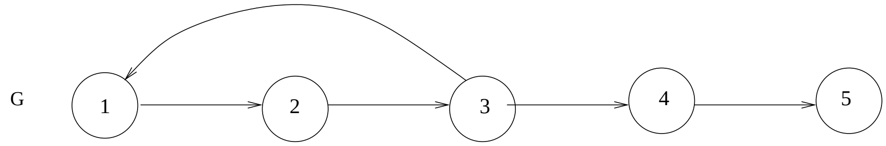

Figure 1: An example of a coupled cell graph G.

3 Symmetry Groupoids

Given a graph $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ as in Definition 2.1, we can define the 'symmetry groupoid' $\mathcal{B}_G$ of G. This definition is tailored to the dynamics of the network, and centres upon the notion of an 'input' set.

We start by reviewing some basic properties of groupoids.

3.1 Groupoids

A groupoid is a special kind of category, so we begin by defining a category, see MacLane [12], Herrlich and Stricker [8]. There are several equivalent formalizations of this concept. In this paper, a *category* $\mathcal{G}$ consists of

(a) A collection $\mathcal{O}$ of *objects* $a, b, \ldots$.

(b) A family of disjoint sets $G(a,b)$, one for each pair (a,b) of objects.

(c) A distinguished element ε_a of $G(a,a)$ for each a.

(d) A law of composition: if $\theta \in G(a,b)$ and $\phi \in G(b,c)$, then $\phi\theta \in G(a,c)$; otherwise $\phi\theta$ is not defined.

In addition we require two axioms:

1. *Associativity*: if $\theta \in G(a,b)$, $\phi \in G(b,c)$ and $\psi \in G(c,d)$, then $\psi(\phi\theta) = (\psi\phi)\theta$.

2. *Identity*: if $\theta \in G(a,b)$ then $\theta\varepsilon_a = \varepsilon_b\theta$.

A category is *small* if its objects form a set.

The members of $G(a,b)$ are called $\mathcal{G}$*-maps* or $\mathcal{G}$*-morphisms* from a to b. The element ε_a is called the *identity morphism* on $G(a,a)$.

A *groupoid* is a small category $\mathcal{G}$ consisting of objects and $\mathcal{G}$-morphisms, with the property that every $\mathcal{G}$-morphism has an inverse in $\mathcal{G}$.

The basic theory of groupoids can be found in Higgins [9], and is sketched in Brown [2]. Here we require the following concepts:

A groupoid $\mathcal{G}$ is *connected* if $G(a,b) \neq \emptyset$ for all objects a, b of $\mathcal{G}$. A *subgroupoid* $\mathcal{S}$ of a groupoid $\mathcal{G}$ is a subset of $\mathcal{G}$ that is closed under products (when defined) and taking the inverses. The *components* of a groupoid are its maximal connected subgroupoids. A groupoid is the disjoint union of its components ([9], Proposition 6, p.27).

3.2 Symmetry Groupoid of a Coupled Cell Graph

Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a graph in the sense of Definition 2.1.

Define the *input set* $I(c)$ of a node c to be

$$I(c) = \{i \in \mathcal{C} : (i,c) \in \mathcal{E}\}$$

Since we are assuming that $\{(c,c) : c \in \mathcal{C}\} \subset \mathcal{E}$ then we always have $c \in I(c)$.

Given $c,d \in \mathcal{C}$, if there is a bijection β from $I(c)$ to $I(d)$ such that: $\beta(c) = d$ and for all $i \in I(c)$ we have $(i,c) \sim_E (\beta(i), d)$, then we call β an *input isomorphism* from cell c to cell d. In that case we say that $c \sim_I d$. Thus $\sim_I$ is another equivalence relation on $\mathcal{C}$, which we call *input equivalence*.

Define:

$$B(c,d) = \{(\beta, c, d) : \beta \text{ is an input isomorphism from } c \text{ to } d\}$$

Observe that $B(c,d) \neq \emptyset$ if and only if $c \sim_I d$. As well as β, we include the head c and tail d in the formal definition of $B(c,d)$, to make those cells explicit. By doing this we ensure that distinct $B(c,d)$ are disjoint.

Example 3.1 We return to Example 2.2. Since $I(1) = \{1,3\}$ and $I(2) = \{2,1\}$, then $\beta_{12} : I(1) \to I(2)$ such that $\beta_{12}(1) = 2$, $\beta_{12}(3) = 1$ is an input isomorphism from cell 1 to cell 2. In fact $B(1,2) = \{(\beta_{12}, 1, 2)\}$. Also $B(1,1) = \{(\mathrm{id}_{\{1,3\}}, 1, 1)\}$. $\diamond$

Consider now

$$\mathcal{B}_G = \dot{\bigcup}_{c,d \in \mathcal{C}} B(c,d)$$

and define a product operation on $\mathcal{B}_G$. Elements $(\beta_2, c, d) \in B(c,d)$ and $(\beta_1, a, b) \in B(a,b)$ can be multiplied only when $b = c$, and in this case we define

$$(\beta_2, b, d)(\beta_1, a, b) = (\beta_2\beta_1, a, d) \in B(a,d) \tag{3.1}$$

where $\beta_2\beta_1$ denotes the usual composition of functions. We use $\dot{\cup}$ for disjoint union.

Theorem 3.2 *$\mathcal{B}_G$ is a groupoid whose objects are the nodes of G, and the $\mathcal{B}_G$-morphisms are the elements of the sets $B(c,d)$. The product operation between the morphisms is as defined in (3.1).*

Proof See [17] Definition 3.4. For consistency with our notation, note that $\varepsilon_c = (\mathrm{id}_{I(c)}, c, c)$ is the identity element of $B(c,c)$. Also the inverse of $(\beta, a, b) \in B(a,b)$ is $(\beta^{-1}, b, a) \in B(b,a)$. $\square$

Following [17], we call $\mathcal{B}_G$ the *symmetry groupoid* of the graph G. For any $c \in \mathcal{C}$, the set $B(c,c)$ is a group, called the *vertex group* corresponding to c.

Remark 3.3 The components of $\mathcal{B}_G$ are in one-to-one correspondence with the $\sim_I$-equivalence classes on $\mathcal{C}$ (and each component is a subgroupoid of $\mathcal{B}_G$). More precisely, if $\mathcal{A} \subseteq \mathcal{C}$ is an $\sim_I$-equivalence class, then $\dot{\cup}_{c,d \in \mathcal{A}} B(c,d)$ is a component of $\mathcal{B}_G$. We say that $c,d \in \mathcal{C}$ are *in the same (connected) component* of $\mathcal{B}_G$ if and only if $c \sim_I d$. $\diamond$

Structure of $B(c,d)$

Let $B(c,d) \subset \mathcal{B}_G$. We can specify the structure of the set

$$B'(c,d) = \{\beta : (\beta, c, d) \in B(c,d)\}$$

in terms of the structure of G. To simplify notation we write $B(c,d)$ instead of $B'(c,d)$. We can distinguish three cases:

1. If $c \not\sim_I d$ then $B(c,d) = \emptyset$.

2. If $c = d$ we can define an equivalence relation $\equiv_c$ on $I(c)$ by

$$j_1 \equiv_c j_2 \iff (j_1, c) \sim_E (j_2, c)$$

   If $K_1 = \{c\}, K_2, \ldots, K_{r(c)}$ are the $\equiv_c$-equivalence classes (on $I(c)$), then

$$B(c,c) = \mathbf{S}_{K_2} \times \cdots \times \mathbf{S}_{K_{r(c)}} \tag{3.2}$$

   where each $\mathbf{S}_{K_i}$ comprises all permutations of the set K_i, extended by the identity on $I(c) \setminus K_i$.

3. If $c \neq d$ and $c \sim_I d$ (and so $B(c,d) \neq \emptyset$), then for *any* $\beta \in B(c,d)$ we have

$$B(c,d) = \beta B(c,c) = B(d,d)\beta$$

4 Coupled Cell Systems

We make now precise the connection between coupled cell systems and coupled cell graphs. Recall that a coupled cell system is a network of dynamical systems coupled together. We represent such a system by a labelled directed graph G (that is, a coupled cell graph in the sense of Definition 2.1), whose nodes correspond to *cells*, and whose edges represent *couplings*. The term 'coupling' here is used in the sense that the output of certain cells affects the time-evolution of other cells.

4.1 Coupled Cell Systems and Coupled Cell Graphs

Again, we follow the treatment in [17]. Consider a coupled cell graph $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ with symmetry groupoid $\mathcal{B}_G$. We now define a space of vector fields associated with G. We say that these vector fields have symmetry groupoid $\mathcal{B}_G$. They are the vector fields that are compatible with the labelled graph structure.

Specifically, to each cell $c \in \mathcal{C}$ we associate a *cell phase space* P_c, which for simplicity we assume is a nonzero finite-dimensional vector space. The basic theory extends to the case when P_c is a smooth manifold, but the more sophisticated questions have not yet been explored in that generality.

If c, d are in the same $\sim_C$-equivalence class, then we suppose that $P_c = P_d$ and identify these spaces canonically. The *total phase space* is

$$P = \prod_{c \in \mathcal{C}} P_c$$

with coordinate system

$$x = (x_c)_{c \in \mathcal{C}}$$

on P. If $\mathcal{D}$ is any subset of $\mathcal{C}$ we define

$$P_{\mathcal{D}} = \prod_{c \in \mathcal{D}} P_c$$

and if $\pi_{\mathcal{D}} : P \to P_{\mathcal{D}}$ denotes the natural projection then

$$x_{\mathcal{D}} = \pi_{\mathcal{D}}(x)$$

Suppose that $\mathcal{D}_1, \mathcal{D}_2$ are subsets of $\mathcal{C}$ and that there is a bijection $\beta : \mathcal{D}_1 \to \mathcal{D}_2$. Define the *pullback map*

$$\beta^* : P_{\mathcal{D}_2} \to P_{\mathcal{D}_1}$$

by

$$(\beta^*(z))_j = z_{\beta(j)} \quad \forall j \in \mathcal{D}_1, z \in P_{\mathcal{D}_2}$$

If $x_{I(c)} = (x_c, x_{i_1}, \ldots, x_{i_r})$ and $\beta \in B(c,d)$, then $\beta^*(x_{I(d)}) = (x_d, x_{\beta(i_1)}, \ldots, x_{\beta(i_r)})$.

The class of vector fields determined by G is defined as follows:

Definition 4.1 A (smooth) vector field $f : P \to P$ is $\mathcal{B}_G$*-equivariant* or G*-admissible* if

(a) For any $c \in \mathcal{C}$ the component f_c depends only on $x_{I(c)}$. By abuse of notation we write $f_c(x) = f_c(x_{I(c)})$ to make this restriction on f_c explicit.

(b) For all $c,d \in \mathcal{C}$ and $\beta \in B(c,d)$ (so that in particular $d = \beta(c)$)

$$f_d(x_{I(d)}) = f_c(\beta^*(x_{I(d)})) \quad \forall x \in P$$

or, less explicitly,

$$f_d(x) = f_c(\beta^*(x))$$

$\diamond$

We say that $f_c : P_{I(c)} \to P_c$ is $B(c,c)$*-invariant* if

$$f_c(\beta^*(x_{I(c)})) = f_c(x_{I(c)}) \quad \forall x \in P$$

for all $\beta \in B(c,c)$. This property is the same as the usual invariance property under a group, if we consider $B(c,c)$ as acting on $P_{I(c)}$ as in (3.2). $\mathcal{B}_G$-equivariant maps can be specified in terms of $B(c,c)$-invariants:

Theorem 4.2 *A vector field $f : P \to P$ is $\mathcal{B}_G$-equivariant if and only if for each connected component Q of $\mathcal{B}_G$*

(a) f_c is $B(c,c)$-invariant for some $c \in Q$.

(b) For $d \in Q$ such that $d \neq c$, given (any) $\beta \in B(c,d)$, we have

$$f_d(x_{I(d)}) = f_c(\beta^*(x_{I(d)}))$$

Proof See [17] Lemma 4.5. □

Definition 4.3 For a given choice of the P_c we define $\mathcal{F}_G^P$ to consist of all smooth admissible vector fields on P. Clearly $\mathcal{F}_G^P$ is a vector space over $\mathbf{R}$. Like all function spaces, it can be equipped with a variety of topologies, but here only the vector space structure is relevant. ◇

Example 4.4 In Example 2.2, $\mathcal{B}_G$ is connected. All the vertex groups $B(i,i)$ consist only of the identity element $\varepsilon_i = (\mathrm{id}_{I(i)}, i, i)$. Moreover if $\beta_{ij} : I(i) = \{i, i_1\} \to I(j) = \{j, j_1\}$ denotes the isomorphism for which $\beta_{ij}(i) = j$ and $\beta_{ij}(i_1) = j_1$, then $B(i,j) = \{(\beta_{ij}, i, j)\}$. If P_1 corresponds to the phase space of cell 1, then the total phase space is $P = P_1^5$ since all the cells are identical. Using Theorem 4.2, given any $f_1 : P_{I(1)} \to P_1$, and setting $x_{I(1)} = (x_1, x_3)$, the $\mathcal{B}_G$-equivariance condition takes the form

$$f_j(x_j, x_{j_1}) = f_1(x_{\beta_{1j}(1)}, x_{\beta_{1j}(3)})$$

for $j = 2, \ldots, 5$. In other words, any $\mathcal{B}_G$-equivariant vector field $f : P \to P$ has the form

$$f(x_1,x_2,x_3,x_4,x_5) = (f_1(x_1,x_3), f_1(x_2,x_1), f_1(x_3,x_2), f_1(x_4,x_3), f_1(x_5,x_4))$$

◇

5 Balanced Equivalence Relations and Quotients

As explained in [17] synchrony in coupled cell systems may be a consequence of features that depend only on the given network architecture. That is, they are valid for any admissible vector field associated with a given coupled cell graph. Thus dynamical synchrony is related to purely combinatorial features of the network. To describe these features, we introduce the notion of a balanced equivalence relation $\bowtie$ on the nodes $\mathcal{C}$. Such equivalence relations can force the existence of certain flow-invariant spaces $\Delta_\bowtie$ for all $f \in \mathcal{F}_G^P$. Moreover the restriction of any f to $\Delta_\bowtie$ defines a new vector field associated with a new *quotient* coupled cell graph.

5.1 Balanced Equivalence Relations

An equivalence relation $\bowtie$ on $\mathcal{C}$ is *balanced* if for all $c, d \in \mathcal{C}$ with $c \bowtie d$ and $c \neq d$, there exists $\gamma \in B(c,d)$ such that $i \bowtie \gamma(i)$ for all $i \in I(c)$. Define the *polydiagonal*

$$\Delta_\bowtie = \{x \in P : \ x_c = x_d \text{ whenever } c \bowtie d, \ \forall c, d \in \mathcal{C}\}$$

which is a vector subspace of P.

Remark 5.1 A balanced equivalence relation refines $\sim_I$. That is, if $c \bowtie d$ then $c \sim_I d$. ◇

Theorem 5.2 *For any choice of total phase space P, an equivalence relation $\bowtie$ on $\mathcal{C}$ satisfies*

$$f(\Delta_\bowtie) \subseteq \Delta_\bowtie \quad \forall f \in \mathcal{F}_G^P$$

if and only if $\bowtie$ is balanced.

Proof See [17] Theorem 6.8, where $\bowtie$ is said to be 'robustly polysynchronous' if the above condition on $\Delta_\bowtie$ holds. □

5.2 Quotient maps

Quotient maps are a way to identify synchronous cells in a coupled cell system, while preserving the dynamics.

Definition 5.3 Let $G_i = (\mathcal{C}_i, \mathcal{E}_i, \sim_{C_i}, \sim_{E_i})$ be coupled cell graphs, for $i = 1,2$. A map $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ is a *quotient map* from G_1 to G_2 if

(a) ϕ is surjective.

(b) Input arrows lift: If $(i,c) \in \mathcal{E}_1$, then $(\phi(i), \phi(c)) \in \mathcal{E}_2$. Conversely, if $(j,d) \in \mathcal{E}_2$ and $c \in \mathcal{C}_1$ is such that $\phi(c) = d$, then there exists $i \in \mathcal{C}_1$ such that $\phi(i) = j$ and $(i,c) \in \mathcal{E}_1$.

(c) Input isomorphisms lift: Let $d, d' \in \mathcal{C}_2$ such that there exists $\beta_2 \in B_2(d,d')$. Choose $c, c' \in \mathcal{C}_1$ such that $\phi(c) = d$ and $\phi(c') = d'$. Then there exists $\beta_1 \in B_1(c,c')$ such that $\beta_2(\phi(i)) = \phi(\beta_1(i))$ for all $i \in I(c)$.

$\diamond$

Remark 5.4 Given a quotient map $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ between the two graphs, then the relation $\bowtie$ (more specifically, $\bowtie_\phi$) defined by

$$c \bowtie c' \Leftrightarrow \phi(c) = \phi(c')$$

is a balanced equivalence relation. Thus by Theorem 5.2 we have $f(\Delta_\bowtie) \subseteq \Delta_\bowtie$ for all $f \in \mathcal{F}_G^P$. $\diamond$

5.3 The Natural Quotient

Let $G_1 = (\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1})$ be a coupled cell graph and $\bowtie$ a balanced equivalence relation on $\mathcal{C}_1$. Following [17] we construct a coupled cell graph $G_2 = G_1/\bowtie$ called the 'natural quotient' of G_1 by $\bowtie$, whose cells are the equivalence classes $\mathcal{C}_2$ of $\bowtie$. Moreover, G_2 is a quotient of G_1, and is universal among such quotients. That is: non-isomorphic quotient graphs can correspond to the same balanced equivalence relation, but they can all be obtained from the natural quotient by leaving cells unchanged but refining the relation $\sim_E$ of edge-equivalence.

Consider $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ such that $\phi(c) = \overline{c}$ where $\overline{c}$ denotes the $\bowtie$-equivalence class of the cell $c \in \mathcal{C}_1$. Now $G_2 = (\mathcal{C}_2, \mathcal{E}_2, \sim_{C_2}, \sim_{E_2})$ is defined by:

1. $\mathcal{C}_2 = \{\overline{c} : c \in \mathcal{C}_1\}$
2. $\overline{c} \sim_{C_2} \overline{d} \Longleftrightarrow c \sim_{C_1} d$
3. $\mathcal{E}_2 = \{(\overline{i}, \overline{c}) : (i,c) \in \mathcal{E}_1\}$
4. If $(j,d) \in \mathcal{E}_2$ and $c \in \mathcal{C}_1$ is such that $\overline{c} = d$, define

$$\Omega_c(j) = \{i \in I(c) : \overline{i} = j\}$$

Given $(j_1,d_1), (j_2,d_2) \in \mathcal{E}_2$ then:

$$(j_1,d_1) \sim_{E_2} (j_2,d_2) \tag{5.3}$$

if and only if for some $c_1, c_2 \in \mathcal{C}_1$ such that $\overline{c_1} = d_1, \overline{c_2} = d_2$ there exists $\gamma \in B_1(c_1,c_2)$ such that

$$\gamma(\Omega_{c_1}(j_1)) = \Omega_{c_2}(j_2)$$

This definition does not depend on the choice of c_1, c_2.

Theorem 5.5 *The above map ϕ is a quotient map between G_1 and G_2. Moreover, it is universal.*

Proof See [17] Theorem 8.3. □

Example 5.6 Consider the graph G of Example 2.2, and the balanced equivalence relation on $\mathcal{C}$ with classes:

$$\{1,4\},\ \{2,5\},\ \{3\}$$

We obtain the quotient graph $G/\bowtie = (\mathcal{C}_2, \mathcal{E}_2, \sim_{C_2}, \sim_{E_2})$, where
$\mathcal{C}_2 = \{\overline{1},\overline{2},\overline{3}\}$
$\mathcal{E}_2 = \{(\overline{1},\overline{1}),(\overline{2},\overline{2}),(\overline{3},\overline{3}),(\overline{1},\overline{2}),(\overline{2},\overline{3}),(\overline{3},\overline{1})\}$
$\sim_{C_2}$ −equivalence class: $\{\overline{1},\overline{2},\overline{3}\}$
$\sim_{E_2}$ −equivalence classes: $\{(\overline{1},\overline{1}),(\overline{2},\overline{2}),(\overline{3},\overline{3})\},\ \{(\overline{1},\overline{2}),(\overline{2},\overline{3}),(\overline{3},\overline{1})\}$
(See Figure 2.)

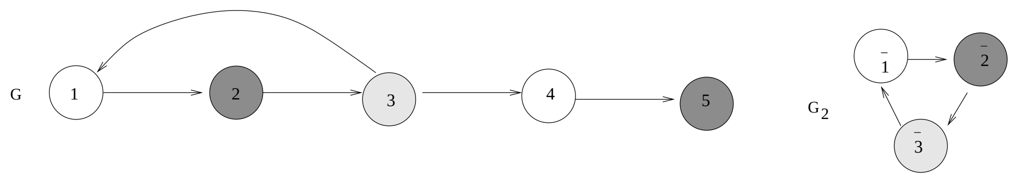

Figure 2: A coupled cell graph G and the corresponding quotient graph $G_2 = G/\bowtie$ given by the balanced equivalence relation $\bowtie$ with classes $\{1,4\}$, $\{2,5\}$ and $\{3\}$.

◇

6 Induced Vector Fields

In [17] it is shown that any quotient map $\phi:\ G_1 \to G_2$ converts G_1-admissible vector fields into G_2-admissible vector fields in a natural way. We present this procedure formally in the case when $\phi : G_1 \to G_1/\bowtie$ is the natural quotient constructed in Section 5.3. We also illustrate this construction with some examples, which are useful as motivation for our main theorem.

6.1 Induced Vector Fields are Admissible

Let G_1 be a coupled cell graph and let $\bowtie$ be a balanced equivalence relation on $\mathcal{C}_1$. Let $G_2 = G_1/\bowtie$ and consider $\phi:\ G_1 \to G_2$ as in Section 5.3. Recall that $f(\Delta_{\bowtie}) \subseteq \Delta_{\bowtie}$ for all admissible vector fields $f \in \mathcal{F}^P_{G_1}$ by Theorem 5.2.

Having chosen the cell phase spaces P_c for $c \in \mathcal{C}_1$, then for each $\overline{c} \in \mathcal{C}_2$ we define the corresponding cell phase space to be

$$\overline{P}_{\overline{c}} = P_c$$

If we choose a set of representatives $\mathcal{R}$ for ϕ (one for each $\bowtie$-equivalence class) we define

$$\overline{P} = \prod_{c \in \mathcal{R}} \overline{P}_{\overline{c}} = \prod_{c \in \mathcal{R}} P_c$$

to be the total phase space for G_2. If $x = (x_c)_{c \in \mathcal{C}_1}$ are coordinates on P, we can consider $y = (y_{\overline{c}})_{\overline{c} \in \mathcal{C}_2}$ as coordinates on $\overline{P}$. In other words, each cell $\overline{c}$ of $G_1/\bowtie$ inherits the phase space of any (hence every) cell that lies in the $\bowtie$-equivalence class $\overline{c}$.

Now define a map $\alpha: \overline{P} \to P$ by

$$(\alpha(y))_c = y_{\overline{c}} \quad \forall c \in \mathcal{C}_1,\ y \in \overline{P}$$

Since $f(\Delta_{\bowtie}) \subseteq \Delta_{\bowtie}$, $\forall f \in \mathcal{F}^P_{G_1}$, then as in [17] we may define

$$\begin{array}{rccc} \overline{f}: & \overline{P} & \to & \overline{P} \\ & y & \mapsto & \alpha^{-1}(f(\alpha(y))) \end{array}$$

and $\overline{f}$ is called the *induced vector field* corresponding to f. That is, $\overline{f}$ is the projection by α^{-1} onto $\overline{P}$ of f restricted to $\Delta_{\bowtie}$.

Theorem 6.1 *For any $f \in \mathcal{F}^P_{G_1}$, the induced vector field $\overline{f} \in \mathcal{F}^{\overline{P}}_{G_2}$. In another words, the function*

$$\begin{array}{rccc} \hat{\phi}: & \mathcal{F}^P_{G_1} & \to & \mathcal{F}^{\overline{P}}_{G_2} \\ & f & \mapsto & \overline{f} \end{array}$$

is well defined.

Proof See [17] Theorem 9.2. □

In [17] it is observed that $\hat{\phi}$ is surjective for some graphs, but not for others. The aim of this paper is to find necessary and sufficient conditions for the map $\hat{\phi}$ to be surjective. It is *never* surjective if G_2 is not the natural quotient, by [17] Corollary 8.7, so without loss of generality we may assume that $G_2 = G_1/\bowtie$. The dynamics of f on G_1 and that of $\overline{f}$ on G_2 are related. It is shown in [17] that any state of $\overline{f}$ 'lifts' to a corresponding state of f in which all $\bowtie$-equivalent cells are synchronous. The question we now address is the following: given any vector field $\overline{f}$ in $\mathcal{F}^{\overline{P}}_{G_2}$, is there always a vector field f in $\mathcal{F}^P_{G_1}$ which, when restricted to the polydiagonal $\Delta_{\bowtie}$, coincides with $\overline{f}$? We show (Theorem 7.6) that two combinatorial conditions are needed in order for $\hat{\phi}$ to be surjective.

6.2 Examples

We give some examples of graphs G and balanced equivalence relations to illustrate some situations where the map $\hat{\phi}$ is surjective, and others where it is not. These examples motivate all of our subsequent analysis.

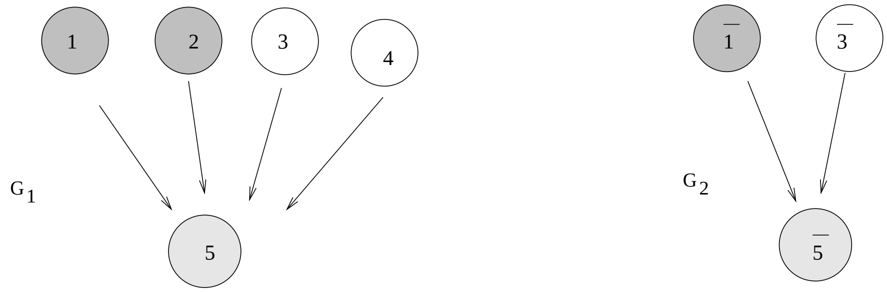

Figure 3: A coupled cell graph G_1 and the corresponding quotient graph G_2 given by the $\bowtie$-equivalence relation with classes $\{1,2\}$, $\{3,4\}$ and $\{5\}$.

Example 6.2 Let $G_1 = (\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1})$ where
$\mathcal{C}_1 = \{1,2,3,4,5\}$
$\mathcal{E}_1 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,5),(2,5),(3,5),(4,5)\}$
$\sim_{C_1}$ −equivalence class: $\{1,2,3,4,5\}$
$\sim_{E_1}$ −equivalence classes: $\{(1,1),(2,2),(3,3),(4,4),(5,5)\}$, $\{(1,5),(2,5),(3,5),(4,5)\}$

Suppose for simplicity that $P = \mathbf{R}^5$. (Similar considerations apply if $P = \mathbf{R}^{5k}$ for $k > 1$, but the calculations are more complicated.) Any admissible vector field $f \in \mathcal{F}^P_{G_1}$ has the following form:

$$f(x) = (f_1(x_1), f_1(x_2), f_1(x_3), f_1(x_4), f_5(x_5, \overline{x_1, x_2, x_3, x_4}))$$

where $\overline{x_1, x_2, x_3, x_4}$ means that f_5 is invariant under the permutations of the corresponding x_i. That is, it is a symmetric function of those x_i, Macdonald [11]. Observe that for this example, $B_1(1,2) = \{(\beta, 1, 2)\}$ where $\beta : I(1) = \{1\} \to I(2) = \{2\}$. The map β satisfies $\beta(1) = 2$, and is an input isomorphism since $(1,1) \sim_{E_1} (2,2)$ (which is equivalent to $1 \sim_{C_1} 2$). Thus $\mathcal{B}_{G_1}$-equivariance implies that $f_2(x_2) = f_1(x_2)$. Similarly for f_3, f_4.

(a) Consider the (balanced) equivalence relation $\bowtie$ on $\mathcal{C}_1$ with classes

$$\{1,2\},\ \{3,4\},\ \{5\}$$

and let $G_2 = (\mathcal{C}_2, \mathcal{E}_2, \sim_{C_2}, \sim_{E_2})$ be the corresponding natural quotient graph. See Figure 3. Any admissible $g \in \mathcal{F}^{\overline{P}}_{G_2}$ where $\overline{P} = \mathbf{R}^3$ has the form

$$g(y) = (g_1(y_1), g_1(y_3), g_5(y_5, \overline{y_1, y_3}))$$

Moreover any $g_5(y_5, \overline{y_1, y_3})$ is a restriction of $f_5(x_5, \overline{x_1, x_2, x_3, x_4})$ to the space

$$\Delta_1 = \{(y_1, y_1, y_3, y_3, y_5)\}$$

so $\hat{\phi}$ is surjective.

(b) We consider now the same graph G_1 but a different balanced equivalence relation $\bowtie$ on $\mathcal{C}_1$. This time the $\bowtie$-equivalence classes are

$$\{1\},\ \{2,3,4\},\ \{5\}$$

See Figure 4 for the natural quotient graph G_2. As before, $\overline{P} = \mathbf{R}^3$. Any admissible $h \in \mathcal{F}^{\overline{P}}_{G_2}$ has the form

$$h(y) = (h_1(y_1), h_1(y_2), h_5(y_5, y_1, y_2))$$

We can find $h_5(y_5,y_1,y_2)$ that is not a restriction of $f_5(x_5,\overline{x_1,x_2,x_3,x_4})$ to the space

$$\Delta_2 = \{(y_1,y_2,y_2,y_2,y_5)\}$$

For example $h_5(y_5,y_1,y_2) = y_1 + y_2$. Therefore $\hat{\phi}$ is not surjective in this case. $\diamond$

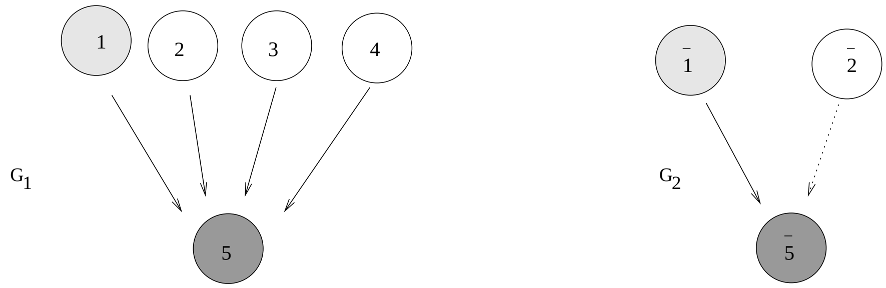

Figure 4: A coupled cell graph G_1 and the corresponding natural quotient graph G_2 given by the equivalence relation $\bowtie$ with classes $\{1\}$, $\{2,3,4\}$ and $\{5\}$.

Example 6.3 Let $G_1 = (\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1})$ where
$\mathcal{C}_1 = \{1,2,3,4,5\}$
$\mathcal{E}_1 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,5),(2,5),(3,5),(4,5)\}$
$\sim_{C_1}$ −equivalence class: $\{1,2,3,4,5\}$
$\sim_{E_1}$ −equivalence classes: $\{(1,1),(2,2),(3,3),(4,4),(5,5)\}$, $\{(1,5),(3,5)\}$, $\{(2,5),(4,5)\}$

Suppose that $P = \mathbf{R}^5$. Any admissible vector field $f \in \mathcal{F}^P_{G_1}$ has the form

$$f(x) = (f_1(x_1), f_1(x_2), f_1(x_3), f_1(x_4), f_5(x_5,\overline{x_1,x_3},\overline{x_2,x_4}))$$

(a) Consider the balanced equivalence relation $\bowtie$ on $\mathcal{C}_1$ with classes

$$\{1,2,3\},\ \{4\},\ \{5\}$$

and let $G_2 = (\mathcal{C}_2, \mathcal{E}_2, \sim_{C_2}, \sim_{E_2})$ be the corresponding natural quotient graph. See Figure 5. Any admissible $g \in \mathcal{F}^{\overline{P}}_{G_2}$ where $\overline{P} = \mathbf{R}^3$ has the form

$$g(y) = (g_1(y_1), g_1(y_4), g_5(y_5,y_1,y_4))$$

Moreover any $g_5(y_5,y_1,y_3)$ is a restriction of $f_5(x_5,\overline{x_1,x_3},\overline{x_2,x_4})$ to the space

$$\Delta_3 = \{(y_1,y_1,y_1,y_4,y_5)\}$$

In this example, $\hat{\phi}$ is surjective.

(b) We consider now the same graph G_1 but a different balanced equivalence relation $\bowtie$ on $\mathcal{C}_1$:

$$\{1,2\},\ \{3\},\ \{4\},\ \{5\}$$

See Figure 6 for the natural quotient graph G_2. Now $\overline{P} = \mathbf{R}^4$. Any admissible $h \in \mathcal{F}^{\overline{P}}_{G_2}$ has the form

$$h(y) = (h_1(y_1), h_1(y_3), h_1(y_4), h_5(y_5,y_1,y_3,y_4))$$

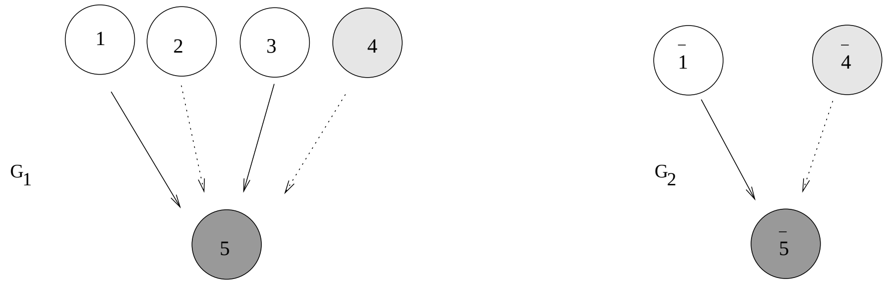

Figure 5: A coupled cell graph G_1 and the corresponding natural quotient graph G_2 given by the equivalence relation $\bowtie$ with classes $\{1,2,3\}$, $\{4\}$ and $\{5\}$.

Not every $h_5(y_5, y_1, y_3, y_4)$ is a restriction of $f_5(x_5, \overline{x_1, x_3}, \overline{x_2, x_4})$ to the space

$$\Delta_4 = \{(y_1, y_1, y_3, y_4, y_5)\}$$

For example $h_5(y_5, y_1, y_3, y_4) = y_1$ is not a restriction of this type. So $\hat{\phi}$ is not surjective in this case. $\diamond$

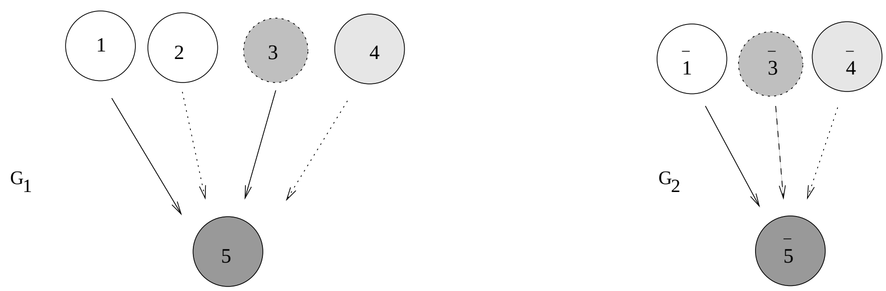

Figure 6: A coupled cell graph G_1 and the corresponding natural quotient graph G_2 given by the equivalence relation $\bowtie$ with classes $\{1,2\}$, $\{3\}$, $\{4\}$ and $\{5\}$.

7 Surjectivity of $\hat{\phi}$

We now come to the main result of this paper. We give necessary and suficient conditions for the map $\hat{\phi}$ to be surjective. As explained earlier, we restrict attention to the natural quotient since otherwise $\hat{\phi}$ cannot be surjective. The idea is the following: we derive necessary conditions for the surjectivity of $\hat{\phi}$ by considering vector fields with linear components. By applying invariant theory for $\mathcal{B}_{G_2}$-equivariant maps, we prove that these conditions are also sufficient. We first carry out the proof for polynomial vector fields, and then extend it to the smooth case by standard methods.

Consider a coupled cell graph $G_1 = (\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1})$ and a balanced equivalence relation $\bowtie$ on $\mathcal{C}_1$. Make a choice of phase space P (and so of $\overline{P}$). Let ϕ be the natural quotient map, and let $\hat{\phi}$ the corresponding map defined in Theorem 6.1. Recall that

$$\Delta_{\bowtie} = \{x \in P : \; x_c = x_d \text{ whenever } c \bowtie d, \; \forall c, d \in \mathcal{C}_1\}$$

Take any $c \in \mathcal{Q}$, where $\mathcal{Q}$ is a component of $\mathcal{B}_{G_1}$. Consider

$$I(c) = \{i \in \mathcal{C}_1 : (i,c) \in \mathcal{E}_1\}$$

with a partition into subsets that lie in distinct $\bowtie$-equivalence classes:

$$I(c) = O_1 \dot\cup \cdots \dot\cup O_{p(c)} \tag{7.4}$$

Thus if $a \in O_i$ and $b \in O_j$ where $i \neq j$, then $\overline{a} \neq \overline{b}$.

Definition 7.1 The sets O_i and O_j are *c-identical* or $B_1(c,c)$*-isomorphic* if there is $\gamma \in B_1(c,c)$ such that

$$\gamma O_i = O_j$$

If no γ exists that satisfies those conditions, then O_i, O_j are said to be *c-distinct*. $\diamond$

Remark 7.2 Let $a \in O_i$ and $b \in O_j$ where $i \neq j$. Then O_i, O_j are c-identical if and only if $(\overline{a},\overline{c}) \sim_{E_2} (\overline{b},\overline{c})$ (recall the definition of $\sim_{E_2}$ in (5.3)). In the notation of Section 5.3

$$O_i = \Omega_c(i_1)$$

for any $i_1 \in O_i$. $\diamond$

Consider the partition of the set $\{1,\ldots,p(c)\}$ into subsets where each contains the indices i, j such that O_i and O_j are c-identical. If we denote by i the $\bowtie$-equivalence class of O_i then we can write

$$I(\overline{c}) = \{1,\ldots,p(c)\} = L_1 \dot\cup \cdots \dot\cup L_{s(c)}$$

where $L_1,\ldots,L_{s(c)}$ are the $B_2(\overline{c},\overline{c})$-orbits on $I(\overline{c})$.

Let

$$B_1(c,c) = \mathbf{S}_{K_1} \times \cdots \times \mathbf{S}_{K_{r(c)}}$$

where $K_1 = \{c\}, K_2,\ldots,K_{r(c)}$ are the $\equiv_c$-equivalence classes (on $I(c)$).

Let

$$V_2 = \mathbf{R}[Y_1,\ldots,Y_{p(c)}]$$

be the real vector space of polynomials in the indeterminates $Y_1,\ldots,Y_{p(c)}$, and let

$$V_1 = \mathbf{R}[X_1,\ldots,X_{n(c)}]$$

be the real vector space of polynomials in the indeterminates $X_1,\ldots,X_{n(c)}$, where $n(c)$ denotes the cardinality of $I(c)$.

Consider the subspace S_2 of V_2 defined by

$$S_2 = \mathbf{R}\left\{\sum_{i\in L_1} Y_i,\ \ldots,\ \sum_{i\in L_{s(c)}} Y_i\right\}$$

and S_1 the subspace of V_1 defined by:

$$S_1 = \mathbf{R}\left\{\sum_{i\in K_1} X_i,\ \ldots,\ \sum_{i\in K_{r(c)}} X_i\right\}$$

Define

$$p_j(X) = \sum_{i \in K_j} X_i \equiv p_j(X_{K_j})$$

so that

$$S_1 = \mathbf{R}\left\{p_1(X), \ldots, p_{r(c)}(X)\right\}$$

Consider

$$S'_{2,c} = \mathbf{R}\left\{p_1(X'), \ldots, p_{r(c)}(X')\right\}$$

where X' is defined in the following way. Given $j = 1, \ldots, p(c)$, then set

$$X'_i = Y_j \quad \forall i \in O_j$$

Remark 7.3 By Theorem 6.1, $S'_{2,c}$ is clearly a subspace of S_2. Moreover, S_2 has dimension $s(c)$ where $s(c)$ is the number of c-distinct subsets of $\bowtie$-equivalence classes in $I(c)$. $\diamond$

Let $d \in Q$, so that $B_1(c,d) \neq \emptyset$. Consider the partition of $I(d)$ into subsets that lie in distinct $\bowtie$-equivalence classes

$$I(d) = O'_1 \,\dot{\cup}\, \cdots \,\dot{\cup}\, O'_{p(d)}$$

Definition 7.4 An input isomorphism $\gamma \in B_1(c,d)$ is $\bowtie$-*compatible* if

$$i \bowtie j \Longleftrightarrow \gamma(i) \bowtie \gamma(j)$$

for all $i, j \in I(c)$. $\diamond$

Remark 7.5 If there is a $\bowtie$-compatible $\gamma \in B_1(c,d)$, then $p(d) = p(c)$, and O_i, O_j are c-identical if and only if $\gamma(O_i), \gamma(O_j)$ are d-identical. Moreover, $B_2(\overline{c}, \overline{d}) \neq \emptyset$ if and only if there is a $\bowtie$-compatible $\gamma \in B_1(c,d)$. $\diamond$

Our main result is:

Theorem 7.6 *Choose a total phase space P and form the corresponding space $\overline{P}$. Consider a coupled cell graph $G_1 = (\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1})$ and a balanced equivalence relation $\bowtie$ on $\mathcal{C}_1$. Let ϕ be the natural quotient map and let $\hat{\phi}$ be the corresponding map on admissible vector fields defined in Theorem 6.1:*

$$\begin{array}{rccc} \hat{\phi}: & \mathcal{F}^P_{G_1} & \to & \mathcal{F}^{\overline{P}}_{G_2} \\ & f & \mapsto & \overline{f} \end{array}$$

Then $\hat{\phi}$ is surjective if and only for each component Q of $\mathcal{B}_{G_1}$:

1. *For some (hence all) $c \in Q$, the space $S'_{2,c}$ has dimension equal to the number of c-distinct subsets of $\bowtie$-equivalence classes in $I(c)$.*

2. *For $d \in Q$ such that $d \neq c$, there exists a $\bowtie$-compatible $\beta \in B_1(c,d)$.*

Note that the same combinatorial conditions hold for any choice of total phase space P.

The proof is accomplished in two steps. The main work goes into proving the result for polynomial vector fields. We then extend the theorem to smooth vector fields using the well known result of Schwarz [15].

We begin by proving Theorem 7.6 for vector fields with polynomial components. Let $\mathcal{P}^P_{G_1}$ and $\mathcal{P}^{\overline{P}}_{G_2}$ denote the classes of admissible polynomial vector fields for G_1 and G_2.

Proposition 7.7 *The function*

$$\begin{array}{rccc} \hat{\phi}: & \mathcal{P}^P_{G_1} & \to & \mathcal{P}^{\overline{P}}_{G_2} \\ & f & \mapsto & \overline{f} \end{array}$$

is surjective if and only if for each component Q of $\mathcal{B}_{G_1}$:

1. *For some (hence all) $c \in Q$, the space $S'_{2,c}$ has dimension equal to the number of c-distinct subsets of $\bowtie$-equivalence classes in $I(c)$.*

2. *For $d \in Q$ such that $d \neq c$, there exists a $\bowtie$-compatible $\beta \in B_1(c,d)$.*

Proof Let f be any admissible polynomial vector field in $\mathcal{P}^P_{G_1}$. By Theorem 4.2, for any component Q of $\mathcal{B}_{G_1}$ and $c \in Q$, $\mathcal{B}_{G_1}$-equivariance of f on the components $f_d : P_{I(d)} \to P_d = P_c$ with $d \in Q$ is equivalent to

(a) $B_1(c,c)$-invariance of $f_c : P_{I(c)} \to P_c$.

(b) $f_d(x_{I(d)}) = f_c(\beta^*(x_{I(d)}))$ for some $\beta \in B_1(c,d)$.

Suppose that $c,d \in Q$, and there is no $\bowtie$-compatible $\beta \in B_1(c,d)$. Then $B_2(\overline{c},\overline{d}) = \emptyset$ and $c \not\bowtie d$ since $\bowtie$ is balanced. In particular, $\mathcal{B}_{G_2}$-equivariance does not impose any relation between the $f_{\overline{c}}$ and $f_{\overline{d}}$ components. Thus $\hat{\phi}$ cannot be surjective in this case.

Alternatively, there exists some $\bowtie$-compatible $\beta \in B_1(c,d)$. Then

$$\overline{f}_{\overline{d}}(y_{I(\overline{d})}) = \overline{f}_{\overline{c}}(\beta'^*(y_{I(\overline{d})}))$$

for some $\beta' \in B_2(\overline{c},\overline{d})$ induced by β. In this case $\hat{\phi}$ is surjective if and only if

$$\begin{array}{c} \text{any}\, B_2(\overline{c},\overline{c})\text{-invariant}\, \overline{f}_{\overline{c}} : \overline{P}_{I(\overline{c})} \to \overline{P}_{\overline{c}}\, \text{is a restriction to}\ \Delta_{\bowtie} \\ \text{of some}\, B_1(c,c)\text{-invariant}\, f_c : P_{I(c)} \to P_c \end{array} \tag{7.5}$$

Here

$$P_{I(c)} = \prod_{i \in I(c)} P_i, \quad \overline{P}_{I(\overline{c})} = \prod_{j \in I(\overline{c})} \overline{P}_j$$

Moreover, (7.5) is valid if and only if the same condition is valid for each real component of $f_{\overline{c}}$. (Recall that the space $P_{\overline{c}} = P_c$ can be any finite-dimensional real vector space.) That is, all the real-valued $B_2(\overline{c},\overline{c})$-invariants on $\overline{P}_{I(\overline{c})}$ are restrictions to the space $\Delta_{\bowtie}$ of real-valued $B_1(c,c)$-invariants on $P_{I(c)}$.

Using the above notation, it follows that

$$B_2(\overline{c},\overline{c}) = \mathbf{S}_{L_1} \times \cdots \times \mathbf{S}_{L_{s(c)}}$$

and $s(c)$ is the number of c-distinct subsets of $\bowtie$-equivalence classes in $I(c)$. If the space $S'_{2,c}$ has dimension lower than $s(c)$, then trivially we can find linear $B_2(\bar{c},\bar{c})$-invariants that are not the restriction to $\Delta_{\bowtie}$ of (linear) $B_1(c,c)$-invariants. Thus in this case, the map $\hat{\phi}$ is not surjective.

We prove now that if the dimension of $S'_{2,c}$ equals $s(c)$, then $\hat{\phi}$ is surjective. Using Lemma 7.8 below, it is sufficient to prove that when the hypothesis is valid, any $\mathbf{S}_{L_i}$-invariant (depending only on the y_j with $j \in L_i$) is the restriction to $\Delta_{\bowtie}$ of a $B_1(c,c)$-invariant. Lemma 7.9 below then completes the proof. □

It remains to prove the two lemmas. The first is:

Lemma 7.8 *Consider $V_1^{d_1},\ldots,V_s^{d_s}$ where each V_i is a finite-dimensional vector space, say with dimension k_i, and denote by $x_i = (x_{i,1},\ldots,x_{i,d_i})$ coordinates on $V_i^{d_i}$. Thus each $x_{i,j}$ is a vector with k_i components. Let*

$$\Gamma = \mathbf{S}_{d_1} \times \cdots \times \mathbf{S}_{d_s}$$

and

$$V = V_1^{d_1} \times \cdots \times V_s^{d_s}$$

with a Γ-action on V defined in the following way: if $\sigma \in \mathbf{S}_{d_i}$, then

$$\sigma \cdot x = (x_1,\ldots,x_{i-1},\sigma\cdot x_i, x_{i+1},\ldots,x_s)$$

where

$$\sigma \cdot x_i = (x_{i,\sigma(1)},\ldots,x_{i,\sigma(d_i)})$$

Then any real polynomial Γ-invariant is a sum of polynomials of the form

$$q_1(x_1)q_2(x_2)\cdots q_s(x_s)$$

where for $j = 1,\ldots,s$, each $q_j(x_j)$ is $\mathbf{S}_{d_j}$-invariant.

Proof The idea of the proof is simple but the notation is complicated. Essentially, we use the fact that any invariant can be obtained as a linear combination of symmetrized monomials, so the proof reduces to computations with monomials.

In detail, recall that $p:\ V \to \mathbf{R}$ is Γ-invariant if and only if

$$p(\sigma\cdot x) = p(x) \quad \forall \sigma \in \Gamma,\ x \in V$$

This condition holds if and only if $p:\ V \to \mathbf{R}$ is $\mathbf{S}_{d_i}$-invariant, where $\mathbf{S}_{d_i}$ acts nontrivially only on $V_i^{d_i}$.

Monomials in x_1 have the form

$$x_{1,1}^{I_1}\cdots x_{1,d_1}^{I_{d_1}}$$

where $I_1,\ldots,I_{d_1} \in (\mathbf{Z}_0^+)^{k_1}$, and each $x_{1,j}^{I_j}$ is a monomial in the k_1 components of $x_{1,j}$.

Let $p:\ V \to \mathbf{R}$ be a Γ-invariant polynomial, and write it as linear combination of monomials in x_1. Suppose that $p(x)$ contains a term that is a scalar multiple of

$$x_{1,1}^{I_1}\cdots x_{1,d_1}^{I_{d_1}}\, q(x_2,\ldots,x_s)$$

Since p is $\mathbf{S}_{d_1}$-invariant and $\mathbf{S}_{d_1}$ acts trivially on $x_2,\ldots,x_s$, then $p(x)$ must also contain

$$x_{1,\sigma(1)}^{I_1}\cdots x_{1,\sigma(d_1)}^{I_{d_1}}\, q(x_2,\ldots,x_s)$$

for all $\sigma\in\mathbf{S}_{d_1}$. It follows that $p(x)$ contains a scalar multiple of

$$\left(\sum_{\sigma\in\mathbf{S}_{d_1}} x_{1,\sigma(1)}^{I_1}\cdots x_{1,\sigma(d_1)}^{I_{d_1}}\right) q(x_2,\ldots,x_s)$$

Now we repeat the same argument for $q(x_2,\ldots,x_s)$. □

Lemma 7.9 *Suppose that $S'_{2,c}$ has dimension $s(c)$. Consider $L_i=\{1,\ldots,s\}$. Thus $\mathbf{S}_{L_i}=\mathbf{S}_s$ and $O_1,\ldots,O_s$ have the same cardinality. Let $P_k=V$ for all $k\in O_1\cup\cdots\cup O_s$, where V is any finite-dimensional real vector space. Denote by o_i the cardinality of O_i. Then any real $\mathbf{S}_s$-invariant polynomial $p:\ V^s\to\mathbf{R}$ is a restriction to the space*

$$\Delta=\left\{\left(\underbrace{y_1,\ldots,y_1}_{o_1},\ldots,\underbrace{y_s,\ldots,y_s}_{o_1},\underbrace{y_{s+1},\ldots,y_{s+1}}_{o_{s+1}},\ldots,\underbrace{y_{p(c)},\ldots,y_{p(c)}}_{o_{p(c)}}\right)\right\}$$

of a real $B_1(c,c)$-invariant polynomial defined on

$$\underbrace{V^{o_1}\times\cdots\times V^{o_1}}_{s}\times P_{s+1}^{o_{s+1}}\times\cdots\times P_{p(c)}^{o_{p(c)}}$$

Proof Observe that $O_1,\ldots,O_s$ all have the same cardinality since they are c-identical ($L_i=\{1,\ldots,s\}$). Moreover we may (if necessary) reorder the cells x_i so that $O_1=\{1,\ldots,o_1\}$, $O_2=\{o_1+1,\ldots,2o_1\},\ldots$, and so

$$\Delta=\Delta_{\bowtie}\cap P_{I(c)}$$

Suppose that V has dimension d. Choose coordinates $(y_1,\ldots,y_s)$ on V^s, where $y_i=(y_{i,1},\ldots,y_{i,d})$. Thus, if $\sigma\in\mathbf{S}_s$,

$$\sigma\cdot(y_1,\ldots,y_s)=(y_{\sigma(1)},\ldots,y_{\sigma(s)})$$

where

$$y_{\sigma(i)}=(y_{\sigma(i),1},\ldots,y_{\sigma(i),d})$$

A real polynomial $\mathbf{S}_s$-invariant on V^s is a linear combination of $\mathbf{S}_s$-invariants of the form

$$\sum_{\sigma\in\mathbf{S}_s} y_{\sigma(1)}^{I_1}\cdots y_{\sigma(s)}^{I_s} \tag{7.6}$$

where $I_i\in(\mathbf{Z}_0^+)^d$.

We must prove:

any polynomial of the form (7.6) is the restriction to Δ of a $B_1(c,c)$-invariant. (7.7)

The proof is performed by induction, and makes use of:

Definition 7.10 A polynomial (7.6) is of *type* m, where $1 \leq m \leq s$, if only m sets of indices, without loss of generality, $I_1, \ldots, I_m$, are non-zero. That is, $I_{m+1} = \cdots = I_s = (0, \ldots, 0)$, and $I_j \neq (0, \ldots, 0)$ for $j = 1, \ldots, m$. $\diamond$

We prove (7.7) by induction on the type m. If $m = 1$, then given *any* $I_1 \in (\mathbf{Z}_0^+)^d$, an expression (7.6) of type 1 has the form

$$p_{I_1}(y) = \sum_{\sigma \in \mathbf{S}_s} y_{\sigma(1)}^{I_1} = y_1^{I_1} + \cdots + y_s^{I_1}$$

Since $S'_{2,c}$ has dimension $s(c)$, which is the dimension of S_2, the subspaces S_2 and $S'_{2,c}$ are equal (recall Remark 7.3). Therefore there exist real coefficients $\alpha_1, \alpha_2, \ldots$, such that

$$Y_1 + \cdots + Y_s = \alpha_1 p_1(X') + \alpha_2 p_2(X') + \cdots \tag{7.8}$$

since $Y_1 + \cdots + Y_s \in S_2$. Moreover, the $p_i(X')$ that appear in (7.8) can be chosen to depend only on Y_j, where $j \sim_{C_2} 1$. We know that $(1, \overline{c}) \sim_{E_2} (i, \overline{c})$, so $1 \sim_{C_2} i$ for all $i \in \{1, \ldots, s\}$. Also, all the cells in the same $\equiv_c$-equivalence class are $\sim_{C_1}$-equivalent. Thus if some $p_i(X')$ in (7.8) depends on Y_l, Y_j such that $l \in \{1, \ldots, s\}$ and $j \notin \{1, \ldots, s\}$, then $j \sim_{C_2} l$ since $j = \overline{k}$ and $l = \overline{k'}$ for some $k, k' \in K_i$, and so $k \sim_{C_1} k'$. Thus $P_l = P_j = V = P_k = P_{k'}$.

Therefore we can write

$$\begin{aligned} p_{I_1}(y) &= \sum_{\sigma \in \mathbf{S}_s} y_{\sigma(1)}^{I_1} = \alpha_1 \left(\sum_{i \in K_1} x_i^{I_1} \right) \Bigg|_\Delta + \alpha_2 \left(\sum_{i \in K_2} x_i^{I_1} \right) \Bigg|_\Delta + \cdots \\ &= \left[\alpha_1 \left(\sum_{i \in K_1} x_i^{I_1} \right) + \alpha_2 \left(\sum_{i \in K_2} x_i^{I_1} \right) + \cdots \right] \Bigg|_\Delta \end{aligned}$$

where

$$q_{I_1}(x) = \alpha_1 \left(\sum_{i \in K_1} x_i^{I_1} \right) + \alpha_2 \left(\sum_{i \in K_2} x_i^{I_1} \right) + \cdots$$

is a $B_1(c,c)$-invariant.

We suppose that any polynomial of the form (7.6) of type less than or equal to m is the restriction to Δ of a $B_1(c,c)$-invariant. We now prove that the same holds for polynomials of type $m+1$. Consider

$$p_{I_1, \ldots, I_{m+1}}(y) = \sum_{\sigma \in \mathbf{S}_s} y_{\sigma(1)}^{I_1} y_{\sigma(2)}^{I_2} \cdots y_{\sigma(m+1)}^{I_{m+1}}$$

Take the $\mathbf{S}_s$-invariant polynomial

$$p(y) = p_{I_1}(y) \cdots p_{I_{m+1}}(y)$$

where

$$p_{I_i}(y) = \sum_{\sigma \in \mathbf{S}_s} y_{\sigma(1)}^{I_i}$$

Note that

$$p(y) = \left[q_{I_1}(x) \cdots q_{I_{m+1}}(x) \right] \Big|_\Delta$$

where

$$q_{I_j}(x) = \alpha_1 \left(\sum_{i \in K_1} x_i^{I_j} \right) + \alpha_2 \left(\sum_{i \in K_2} x_i^{I_j} \right) + \cdots$$

is a $B_1(c,c)$-invariant. Moreover

$$p(y) = p_{I_1,\ldots,I_{m+1}}(y) + \sum_i \beta_i r_i(y)$$

where each $\beta_i \in \mathbf{R}$ and each $r_i(y)$ is an $\mathbf{S}_s$-invariant of the form (7.6) and of type less than or equal to m. By hypothesis

$$r_i(y) = s_i(x)|_\Delta$$

for some $B_1(c,c)$-invariant $s_i(x)$. Thus

$$p_{I_1,\ldots,I_{m+1}}(y) = \left(q_{I_1}(x) \cdots q_{I_{m+1}}(x) - \sum_i \beta_i s_i(x) \right) \Bigg|_\Delta$$

□

We now parlay Proposition 7.7 into the corresponding result for smooth vector fields:

Proof of Theorem 7.6: As shown in Proposition 7.7, if any of the two conditions fails, then $\hat{\phi}$ is not surjective when considered as a map of polynomial vector fields. It is clear that $\hat{\phi}$ preserves jets (Taylor series) of smooth mappings, so the theorem of Borel (Bröcker and Lander [1] Theorem 4.9) implies that $\hat{\phi}$ is not surjective when considered as a map of smooth vector fields.

Assuming the two conditions are valid, by Theorem 4.2, $\hat{\phi}$ is surjective on smooth vector fields if and only if it induces a surjective map on smooth $B(c,c)$-invariant functions for all $c \in \mathcal{C}$. We have already derived necessary and sufficient conditions for $\hat{\phi}$ to induce a surjective map on polynomial $B(c,c)$-invariant functions for all $c \in \mathcal{C}$. Schwarz [15] (see also Mather [13] and Luna [10]) proves that in general for any compact Lie group Γ with an orthogonal action on $\mathbf{R}^n$, if the algebra of Γ-invariant polynomials is generated by $\rho_1, \ldots, \rho_k$ (and by Hilbert's basis theorem such a basis always exist), then any Γ-invariant C^∞-function of n variables is a C^∞-function of the generators $\rho_1, \ldots, \rho_k$. Thus if $\hat{\phi}$ induces a surjective map on these polynomial generators, it must induce a surjective map on smooth equivariant vector fields. □

Example 7.11 Let $G_1 = (\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1})$ where
$\mathcal{C}_1 = \{1,2,3,4,c\}$
$\mathcal{E}_1 = \{(1,1),(2,2),(3,3),(4,4),(c,c),(1,c),(2,c),(3,c),(4,c)\}$
$\sim_{C_1}$ $-$equivalence class: $\{1,2,3,4,c\}$
$\sim_{E_1}$ $-$equivalence classes: $\{(1,1),(2,2),(3,3),(4,4),(c,c)\}$, $\{(1,c)\}$, $\{(2,c),(3,c),(4,c)\}$

Consider the (balanced) equivalence relation $\bowtie$ on $\mathcal{C}_1$ with classes

$$O_1 = \{1,2\},\ O_3 = \{3\},\ O_4 = \{4\},\ O_c = \{c\}$$

and let $G_2 = (\mathcal{C}_2, \mathcal{E}_2, \sim_{C_2}, \sim_{E_2})$ be the corresponding quotient graph. See Figure 7.

Consider

$$V_1 = \mathbf{R}[X_1, X_2, X_3, X_4, X_c], \quad V_2 = \mathbf{R}[Y_1, Y_3, Y_4, Y_c]$$

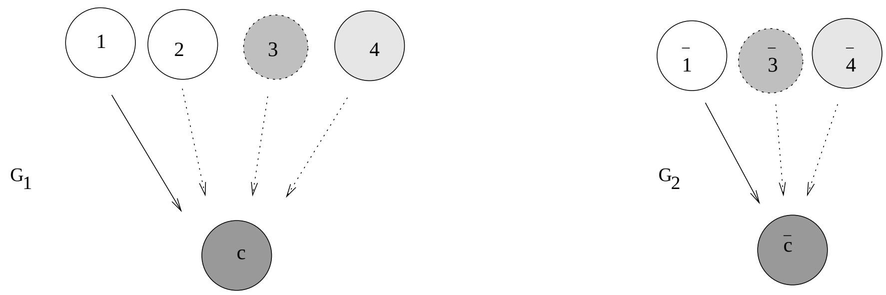

Figure 7: A coupled cell graph G_1 and the corresponding natural quotient graph G_2 given by the equivalence relation $\bowtie$ with classes $\{1,2\},\ \{3\},\ \{4\}$ and $\{c\}$.

Since
$$I(c) = O_1 \dot{\cup} O_3 \dot{\cup} O_4 \dot{\cup} O_c$$
and O_3, O_4 are c-identical, then
$$S_2 = \mathbf{R}\{Y_1, Y_3 + Y_4, Y_c\}$$
(and S_2 has dimension 3). Also $K_1 = \{1\},\ K_2 = \{2,3,4\},\ K_3 = \{c\}$ are the $\equiv_c$-equivalence classes:
$$I(c) = K_1 \dot{\cup} K_2 \dot{\cup} K_3, \quad B_1(c,c) = \mathbf{S}_{K_1} \times \mathbf{S}_{K_2} \times \mathbf{S}_{K_3}$$
Thus
$$p_1(X) = X_1, \quad p_2(X) = X_2 + X_3 + X_4, \quad p_3(X) = X_c$$
Recall that $p_j = \sum_{i \in K_j} X_i$. Let
$$X' = (Y_1, Y_1, Y_3, Y_4, Y_c)$$
Then
$$\begin{aligned} S'_{2,c} &= \mathbf{R}\{p_1(X'), p_2(X'), p_3(X')\} \\ &= \mathbf{R}\{Y_1, Y_1 + Y_3 + Y_4, Y_c\} \end{aligned}$$
and $S'_{2,c} = S_2$. By Theorem 7.6 the map $\hat{\phi}$ is surjective for any choice of P. ◇

8 Relation to Quotient Groupoids

Given a graph G_1 and a balanced equivalence relation $\bowtie$ on the nodes of G_1, Section 5.3 describes a method for constructing the natural quotient graph $G_2 = G_1/\bowtie$ and the associated quotient map ϕ. In [17] it is proved that ϕ is a quotient map between the graphs. We now prove that ϕ naturally induces a groupoid map $\phi' : \mathcal{T}^{\bowtie}_{G_1} \to \mathcal{B}_{G_2}$. Here $\mathcal{T}^{\bowtie}_{G_1}$ is the subgroupoid of $\mathcal{B}_{G_1}$ comprising the $\bowtie$-compatible input isomorphisms, and $\mathcal{B}_{G_2}$ is the symmetry groupoid of G_2.

Moreover, we prove that the map ϕ' is a groupoid quotient map, and deduce that $\mathcal{T}^{\bowtie}_{G_1}/\ker(\phi') \cong \mathcal{B}_{G_2}$. Indeed, we show that the groupoid situation is analogous to the 'normalizer quotient' property in the group-symmetric case, discussed in the introduction.

8.1 Background

We start by recalling from Higgins [9] the definitions of a quotient groupoid and a groupoid quotient map.

A subgroupoid $\mathcal{N} = \dot{\cup} N(a,b)$ of a groupoid $\mathcal{G} = \dot{\cup} G(a,b)$ is *normal* if

(a) $\mathcal{N}$ contains all the identity elements of $\mathcal{G}$, so in particular $\mathcal{G}$ and $\mathcal{N}$ have the same objects.

(b) If $\sigma \in N(a,a)$ and $\alpha \in G(b,a)$, then $\alpha^{-1}\sigma\alpha \in N(b,b)$.

Let $\mathcal{N} = \dot{\cup} N(a,b)$ be a normal subgroupoid of $\mathcal{G}$. Define an equivalence relation $\sim_{\mathcal{N}}$ on the objects of $\mathcal{G}$:

$$a \sim_{\mathcal{N}} b \Longleftrightarrow N(a,b) \neq \emptyset \qquad \text{for}\, a,b \in O$$

where O is the set of objects of $\mathcal{G}$ (and $\mathcal{N}$). Denote by $\overline{a}$ the equivalence class of $a \in O$, and let $\overline{O}$ be the set of classes. Define an equivalence relation (also denoted $\sim_{\mathcal{N}}$) on the maps of $\mathcal{G}$:

$$\alpha, \beta \in \mathcal{G},\ \alpha \sim_{\mathcal{N}} \beta \Longleftrightarrow \exists \mu, \nu \in \mathcal{N},\ \alpha = \mu\beta\nu$$

The equivalence classes are the cosets $\mathcal{N}\alpha\mathcal{N}$, which we denote by $\overline{\alpha}$. The product $\overline{\alpha}\overline{\beta}$ is defined if and only if there exists $\alpha' \in \overline{\alpha}$ and $\beta' \in \overline{\beta}$ such that $\alpha'\beta'$ is defined; in this case

$$\overline{\alpha}\,\overline{\beta} = \overline{\alpha\beta} \tag{8.9}$$

The *quotient groupoid* $\mathcal{G}/\mathcal{N}$ is the groupoid whose objects are $\overline{O}$, whose maps are the $\overline{\alpha}$, and having product operation (8.9).

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are categories, then a *functor* $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ assigns to each object a of $\mathcal{G}_1$ an object $\phi(a)$ of $\mathcal{G}_2$, and to each $\mathcal{G}_1$-morphism $\alpha \in G_1(a,b)$ a $\mathcal{G}_2$-morphism $\phi(\alpha)$ of $G_2(\phi(a),\phi(b))$ in such a manner that

(a) $\phi(\varepsilon_a) = \varepsilon_{\phi(a)}$ for each a.

(b) $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ whenever $\alpha\beta$ is defined.

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are groupoids, then a *groupoid map* $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is a functor from $\mathcal{G}_1$ to $\mathcal{G}_2$. Observe that ϕ then preserves inverses. The *kernel* of ϕ is defined by

$$\ker(\phi) = \{\alpha \in \mathcal{G}_1 :\ \phi(\alpha) = \varepsilon_a \text{ for some object } a \in \mathcal{G}_2\}$$

and it is a normal subgroupoid of $\mathcal{G}_1$.

A groupoid map $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ with kernel $\mathcal{N}$ is a *quotient map* if it induces a unique groupoid map $\phi^* : \mathcal{G}_1/\mathcal{N} \to \mathcal{G}_2$ which is an isomorphism ([9], Proposition 24, p.87).

A groupoid map $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is

(a) *Vertex-surjective* if $\phi : O_1 \to O_2$ is a surjection (where each O_i is the set of objects of $\mathcal{G}_i$).

(b) *Piecewise surjective* if $\phi : G_1(a,b) \to G_2(\phi(a),\phi(b))$ is surjective for each pair (a,b) of objects of $\mathcal{G}_1$.

Theorem 8.1 *A groupoid map $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is a quotient map if and only if ϕ is vertex-surjective and piecewise surjective.*

Proof See [9], Proposition 25, p.88. □

8.2 Quotient Groupoid Map

Define

$$\mathcal{T}^{\bowtie}_{G_1} = \dot{\bigcup} T(c,d)$$

where

$$T(c,d) = \{(\beta,c,d) \in B_1(c,d) : \beta \text{ is } \bowtie-\text{compatible}\}$$

Lemma 8.2 *$\mathcal{T}^{\bowtie}_{G_1}$ is a subgroupoid of $\mathcal{B}_{G_1}$.*

Proof Recall that

$$T(c,d) = \{(\beta,c,d) \in B_1(c,d) : i \bowtie j \Longleftrightarrow \beta(i) \bowtie \beta(j),\ \forall i,j \in I(c)\}$$

(Definition 7.4.) Thus if $(\beta_1,a,b) \in B_1(a,b)$, $(\beta_2,b,c) \in B_1(c,d)$ and β_1,β_2 are both $\bowtie$-compatible, then $(\beta_2\beta_1,a,c) \in B_1(a,c)$ and $\beta_2\beta_1$ is $\bowtie$-compatible. Also $(\beta_1^{-1},b,a) \in B_1(b,a)$ and β_1^{-1} is $\bowtie$-compatible. □

Definition 8.3 We define $\phi' : \mathcal{T}^{\bowtie}_{G_1} \to \mathcal{B}_{G_2}$ in the following way:
Objects
$\phi' : \mathcal{C}_1 \to \mathcal{C}_2$ is such that $\phi'(c) = \phi(c) = \overline{c}$. By the definition of $\mathcal{C}_2$ this map is surjective.
Morphisms
Given $c,d \in \mathcal{C}_1$ (and $T(c,d) \neq \emptyset$) then

$$\begin{aligned} \phi' : \quad & T(c,d) \to B_2(\overline{c},\overline{d}) \\ & (\beta,c,d) \mapsto (\beta',\overline{c},\overline{d}) \end{aligned} \tag{8.10}$$

where

$$\beta'(\bar{i}) = \overline{\beta(i)},\ i \in I(c) \tag{8.11}$$

◇

Along with $T^{\bowtie}_{G_1}$ we require:

Definition 8.4 Let $\mathcal{S}^{\bowtie}_{G_1} = \dot{\cup} S(c,d)$ be the subgroupoid of $\mathcal{B}_{G_1}$ where

$$S(c,d) = \{\gamma \in B_1(c,d) : i \bowtie \gamma(i),\ \forall i \in I(c)\}$$

◇

We may then state:

Theorem 8.5 *Consider a graph G_1, a balanced equivalence relation $\bowtie$ on the nodes of G_1, and $\phi : G_1 \to G_2$ the quotient map constructed in Section 5.3. Then the map $\phi' : \mathcal{T}^{\bowtie}_{G_1} \to \mathcal{B}_{G_2}$ constructed above (Definition 8.3) is a quotient map with kernel $\mathcal{S}^{\bowtie}_{G_1}$ (and $\mathcal{T}^{\bowtie}_{G_1}/\mathcal{S}^{\bowtie}_{G_1} \cong \mathcal{B}_{G_2}$).*

Proof

We discuss three cases:

(i) $c = d$. Partition $I(c)$ into subsets that lie in distinct $\bowtie$-equivalence classes:

$$I(c) = O_1 \dot{\cup} \cdots \dot{\cup}\, O_{p(c)}$$

Thus given $a,b \in I(c)$ then $\overline{a} \neq \overline{b}$ if and only if $a \in O_i,\ b \in O_j$, and $i \neq j$. Note that if $\beta \in T(c,c)$ then (8.11) is well defined since if $i, j \in O_i$, then $\bar{i} = \bar{j}$ and $\beta(i),\ \beta(j) \in O_k$ for some k, and so $\overline{\beta(i)} = \overline{\beta(j)}$. Moreover β' is an input isomorphism since

$$(\bar{i},\overline{c}) \sim_{E_2} (\overline{\beta(i)},\overline{c})$$

if and only if there exists $\gamma \in B_1(c,c)$ such that

$$\gamma(\Omega_c(\bar{i})) = \Omega_c(\overline{\beta(i)})$$

Suppose $i \in O_i$. Then $O_i = \Omega_c(\bar{i})$. We can take $\gamma = \beta \in T(c,c)$, so $\beta(\Omega_c(\bar{i})) = \beta O_i = O_k$ where $\beta(i) \in O_k$. Thus $O_k = \Omega_c(\overline{\beta(i)})$. The map (8.10) is surjective by Theorem 5.5: since $\phi : G_1 \to G_2$ is a quotient map, input isomorphisms lift (property (c) of Definition 5.3). Also note that $(\mathrm{id}_{I(c)}, c, c) \in T(c,c)$ and $\phi'(\mathrm{id}_{I(c)}, c, c) = (\mathrm{id}_{I(\overline{c})}, \overline{c}, \overline{c})$.

(ii) $c \neq d$ and $c \bowtie d$. Since $\bowtie$ is balanced, there exists $\gamma \in B_1(c,d)$ such that $i \bowtie \gamma(i)$ for all $i \in I(c)$, and so $\gamma \in T(c,d)$. Moreover any $\beta \in T(c,d)$ is of the form

$$\beta = \gamma\beta_1$$

for some $\beta_1 \in T(c,c)$. Note that in this case

$$(\gamma\beta_1)'(\bar{i}) = \overline{\gamma\beta_1(i)} = \overline{\beta_1(i)}$$

That is, for any $\gamma \in T(c,d)$ such that $i \bowtie \gamma(i)$ for all $i \in I(c)$, then $\gamma' = \mathrm{id}_{I(\overline{c})}$ and so $(\gamma, c, d) \in \ker(\phi')$.

(iii) $c \neq d,\ c \not\bowtie d$ and $T(c,d) \neq \emptyset$. Consider as before

$$I(c) = O_1 \dot{\cup} \cdots \dot{\cup}\, O_{p(c)}$$

and

$$I(d) = O'_1 \dot{\cup} \cdots \dot{\cup}\, O'_{p(d)}$$

Since $T(c,d) \neq \emptyset$ it follows than $p(c) = p(d)$. As in (i) the map β' in (8.11) is well defined and it is an input isomorphism. Also the map (8.10) is surjective by Theorem 5.5. □

Now we develop the analogy with the group-symmetric case. The *fixed-point* subspace of $\mathcal{S}_{G_1}^{\bowtie}$ is

$$\mathrm{Fix}(\mathcal{S}_{G_1}^{\bowtie}) = \{x \in P :\ x_c = x_d \text{ whenever } S(c,d) \neq \emptyset\}$$

Remark 8.6 Note that $c \bowtie d$ if and only if $S(c,d) \neq \emptyset$ since $\bowtie$ is balanced. Thus

$$\mathrm{Fix}(\mathcal{S}_{G_1}^{\bowtie}) = \Delta_{\bowtie}$$

◇

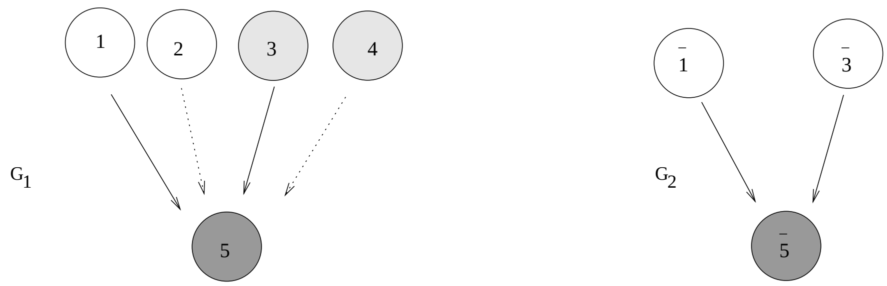

Figure 8: A coupled cell graph G_1 and the corresponding natural quotient graph G_2 given by the equivalence relation $\bowtie$ with classes $\{1,2\}$, $\{3,4\}$ and $\{5\}$.

Example 8.7 Consider the graph G_1 of Example 6.3 and the (balanced) equivalence relation $\bowtie$ on $\mathcal{C}_1$ with classes

$$\mathcal{O}_1 = \{1,2\},\ \mathcal{O}_3 = \{3,4\},\ \mathcal{O}_5 = \{5\}$$

Let $G_2 = (\mathcal{C}_2, \mathcal{E}_2, \sim_{C_2}, \sim_{E_2})$ be the corresponding quotient graph. See Figure 8.

The classes $\mathcal{O}_1$ and $\mathcal{O}_3$ are 5-identical and

$$T(5,5) = \{(\mathrm{id}_{I(5)}, 5, 5),\ ((13)(24), 5, 5)\}$$

Then $\phi' :\ T(5,5) \to B_2(\overline{5}, \overline{5})$ is such that

$$\phi'(\mathrm{id}_{I(5)}, 5, 5) = (\mathrm{id}_{I(\overline{5})}, \overline{5}, \overline{5})$$

$$\phi'((13)(24), 5, 5) = ((\overline{1}\ \overline{3}), \overline{5}, \overline{5})$$

$\diamond$

8.3 The Normalizer Viewpoint

How does the symmetry groupoid of the natural quotient graph G_2 relate to that of G_1? As mentioned in the introduction, there is an analogy here with a question in equivariant bifurcation theory. Suppose that Γ is a group acting on V, that $f : V \to V$ is Γ-equivariant, and let Σ be a subgroup of Γ. Then f leaves $\mathrm{Fix}(\Sigma)$ invariant, and we can ask which conditions characterize the restriction $f|_{\mathrm{Fix}(\Sigma)}$. The most obvious such condition is normalizer-equivariance: $f|_{\mathrm{Fix}(\Sigma)}$ is $N(\Sigma)/\Sigma$-equivariant. See [7] Chapter XIII Exercise 2.2. In some cases, this is the only condition required, but in others, 'hidden symmetries' impose more complicated conditions.

We now show that something closely analogous happens in the groupoid case. We begin by defining the groupoid analogue of the normalizer of a subgroup:

Definition 8.8 Let $\mathcal{G}$ be a groupoid. The *normalizer* of a subgroupoid $\mathcal{S}$ of $\mathcal{G}$ is the largest subgroupoid $\mathcal{H}$ such that $\mathcal{S}$ is a normal subgroupoid of $\mathcal{H}$. $\diamond$

Lemma 8.9 *$\mathcal{T}^{\bowtie}_{G_1}$ is the normalizer of $\mathcal{S}^{\bowtie}_{G_1}$ in $\mathcal{B}_{G_1}$.*

Proof Suppose that $\theta \in B_1(c,d)$ normalizes $\mathcal{S}^{\bowtie}_{G_1}$. Thus

$$\theta\sigma\theta^{-1} \in S(d,d) \quad \forall \sigma \in S(c,c) \tag{8.12}$$

Let L be any set of the form $L = K \cap U$ where K is a $\equiv_d$-equivalence class in $I(d)$ and U is a $\bowtie$-equivalence class in $I(d)$, and let $l \in L$. By (8.12)

$$\theta\sigma\theta^{-1}(l) \bowtie l \quad \forall \sigma \in S(c,c) \tag{8.13}$$

Let K' be the intersection of the $\equiv_c$-equivalence class of $\theta^{-1}(l)$ in $I(c)$ with the $\bowtie$-equivalence class of $\theta^{-1}(l)$ in $I(c)$. Given any $k \in K'$, there exists $\sigma \in S(c,c)$ with

$$k = \sigma\theta^{-1}(l)$$

since $\bowtie$ is balanced (and $k \bowtie \theta^{-1}(l)$ because $k, \theta^{-1}(l) \in K'$). By (8.13)

$$\theta(k) \bowtie l$$

and so

$$\theta(K') \subseteq L$$

This is true for every L, and $I(c), I(d)$ have the same cardinality. Thus we must have $\theta(K') = L$. This shows that $\theta \in T(c,d)$. □

Before providing an example it is convenient to discuss a technical issue: the relation between the symmetry groupoid and the symmetry group of a symmetric graph. When the graph G has symmetry, the symmetry groupoid $\mathcal{B}_G$ is *not* the same as the symmetry group Γ_G. However, the two are closely related. In particular, the symmetry group can be interpreted as a subgroupoid of the symmetry groupoid.

Recall that $\Gamma = \Gamma_G$ acts as a group of permutations of $\mathcal{C}$. Define subsets $\Gamma(c,d) \subseteq \Gamma$ by:

$$\Gamma(c,d) = \{\gamma \in \Gamma : \gamma(c) = d\}$$

so that in particular $\Gamma(c,c)$ is the *stabilizer* of c in Γ, which is a subgroup. See Neumann *et al.* [14].

Proposition 8.10

$$\Gamma(c,d)|_{I(c)} \subseteq B(c,d)$$

Proof Check definitions. Note that technically we must restrict the permutations in $\Gamma(c,d)$ to the input set $I(c)$. □

Note that $\Gamma(c,d)|_{I(c)}$ may not equal $B(c,d)$. If G has only trivial symmetry, $\mathcal{B}_G$ may still contain nontrivial $B(c,d)$.

The sets of maps $\Gamma(c,d)|_{I(c)}$, for all $c,d \in \mathcal{C}$, define a groupoid $\hat{\Gamma}$. It is possible for $\mathcal{B}_G$ to be larger than $\hat{\Gamma}$. Indeed this is the interesting case for us.

The main feature of this reformulation of symmetry in terms of groupoid structure is:

Proposition 8.11 *A vector field f on P is Γ-equivariant, in the usual sense, if and only if it is $\hat{\Gamma}$-equivariant.*

Proof This is a simple computation. □

Thus the groupoid formulation encodes the same symmetry information as the symmetry group, but in a different way.

Example 8.12 Let G_1 be a coupled cell consisting of 5 identical cells with all-to-all identical coupling. Say $\mathcal{C}_1 = \{1,2,3,4,5\}$. Thus the symmetry groupoid of G_1 is effectively the group $\mathbf{S}_5$. More precisely: $B(c,d)$ is the set of permutations $\sigma \in \mathbf{S}_5$ such that $\sigma(c) = d$.

Consider the balanced equivalence relation $\bowtie$ on $\mathcal{C}_1$ with classes

$$\{1,2\},\ \{3,4\},\ \{5\}$$

and let $G_2 = (\mathcal{C}_2, \mathcal{E}_2, \sim_{C_2}, \sim_{E_2})$ be the corresponding quotient graph. See Figure 9.

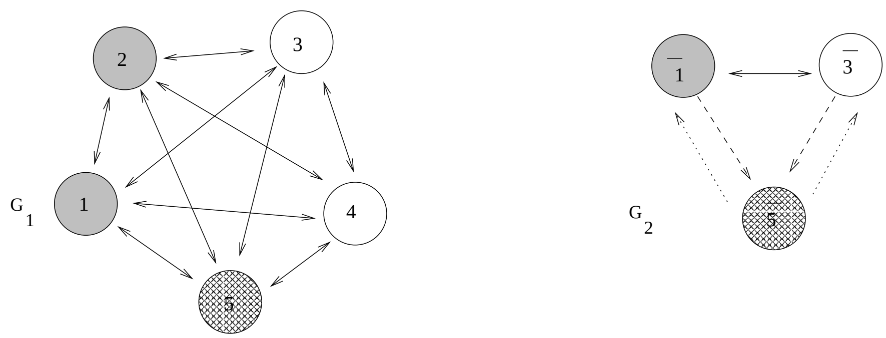

Figure 9: A coupled cell graph G_1 with $\mathbf{S}_5$-symmetry and the corresponding natural quotient graph G_2 with $\mathbf{Z}_2$-symmetry given by the equivalence relation $\bowtie$ with classes $\{1,2\}$, $\{3,4\}$ and $\{5\}$.

Suppose that $\overline{P} = V^5$ where V is any finite-dimensional vector space. Then

$$\Delta_{\bowtie} = \{(y_1, y_1, y_3, y_3, y_5)\} = \mathrm{Fix}(\mathcal{S}^{\bowtie}_{G_1})$$

where

$$\mathcal{S}^{\bowtie}_{G_1} = \mathbf{S}_{\{1,2\}} \times \mathbf{S}_{\{3,4\}}$$

and $\mathcal{T}^{\bowtie}_{G_1}$ is the group generated by $\mathcal{S}^{\bowtie}_{G_1}$ and $(13)(24)$. In fact $\mathcal{T}^{\bowtie}_{G_1}$ is the groupoid corresponding to the the normalizer of $\mathcal{S}^{\bowtie}_{G_1}$ in $\mathbf{S}_5$, and

$$\mathcal{T}^{\bowtie}_{G_1} / \mathcal{S}^{\bowtie}_{G_1} \cong \mathbf{Z}_2$$

where $\mathbf{Z}_2$ is the symmetry group of G_2, interpreted as a groupoid as explained above. ◇

Remark 8.13 Theorem 7.6 cannot be specialized to the case of Γ-symmetric networks, to provide an analogous theorem for the group-symmetric case. The proof of Proposition 7.7 (which Theorem 7.6 depends on) relies on the direct product structure of symmetric groups of the vertex groups $B_1(c,c)$. (Recall the end of Section 3.2.) Moreover, in general, the groups $\Gamma(c,c)|_{I(c)}$ are not of that type.

◇

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