Uniform stabilization in weighted Sobolev spaces for the KdV equation posed on the half-line

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Abstract

Studied here is the large-time behavior of solutions of the Korteweg-de Vries equation posed on the right half-line under the effect of a localized damping. Assuming as in [20] that the damping is active on a set $(a_0, +\infty)$ with $a_0 > 0$, we establish the exponential decay of the solutions in the weighted spaces $L^2((x+1)^m dx)$ for $m \in \mathbb{N}^*$ and $L^2(e^{2bx}dx)$ for $b > 0$ by a Lyapunov approach. The decay of the spatial derivatives of the solution is also derived.

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Key words. Exponential Decay, Korteweg-de Vries equation, Stabilization.

1 Introduction

The Korteweg-de Vries (KdV) equation was first derived as a model for the propagation of small amplitude long water waves along a channel [9, 16, 17]. It has been intensively studied from various aspects for both mathematics and physics since the 1960s when solitons were discovered through solving the KdV equation, and the inverse scattering method, a so-called nonlinear Fourier transform, was invented to seek solitons [14, 22]. It is now well known that the KdV equation is not only a good model for water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance weak nonlinear and dispersive effects.

The initial boundary value problems (IBVP) arise naturally in modeling small-amplitude long waves in a channel with a wavemaker mounted at one end [1, 2, 3, 29]. Such mathematical formulations have received considerable attention in the past, and a satisfactory theory of global well-posedness is available for initial and boundary conditions satisfying physically relevant smoothness and consistency assumptions (see e.g. [1, 4, 6, 7, 11, 12, 13] and the references therein).

The analysis of the long-time behavior of IBVP on the quarter-plane for KdV has also received considerable attention over recent years, and a review of some of the results related to the issues we address here can be found in [5, 7, 19]. For stabilization and controllability issues on the half line, we refer the reader to [20] and [27, 28], respectively.

In this work, we are concerned with the asymptotic behavior of the solutions of the IBVP for the KdV equation posed on the positive half line under the presence of a localized damping represented by the function a; that is,

$$(1)\qquad \begin{cases} u_t + u_x + u_{xxx} + uu_x + a(x)u = 0, \quad x,\, t \in \mathbb{R}^+, \\ u(0,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad x > 0. \end{cases}$$

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Assuming $a(x) \geq 0$ a.e. and that $u(.,t) \in H^3(\mathbb{R}^+)$, it follows from a simple computation that

$$\frac{dE}{dt} = -\int_0^\infty a(x)|u(x,t)|^2 dx - \frac{1}{2}|u_x(0,t)|^2 \tag{2}$$

where

$$E(t) = \frac{1}{2}\int_0^\infty |u(x,t)|^2 dx \tag{3}$$

is the total energy associated with (1). Then, we see that the term $a(x)u$ plays the role of a feedback damping mechanism and, consequently, it is natural to wonder whether the solutions of (1) tend to zero as $t \to \infty$ and under what rate they decay. When $a(x) > a_0 > 0$ almost everywhere in $\mathbb{R}^+$, it is very simple to prove that $E(t)$ converges to zero as t tends to infinity. The problem of stabilization when the damping is effective only in a subset of the domain is much more subtle. The following result was obtained in [20].

Theorem 1.1 *Assume that the function $a = a(x)$ satisfies the following property*

$$a \in L^\infty(\mathbb{R}^+),\ a \geq 0 \text{ a.e. in } \mathbb{R}^+ \text{ and } a(x) \geq a_0 > 0 \text{ a.e. in } (x_0, +\infty) \tag{4}$$

for some numbers $a_0, x_0 > 0$. Then for all $R > 0$ there exist two numbers $C > 0$ and $\nu > 0$ such that for all $u_0 \in L^2(\mathbb{R}^+)$ with $||u_0||_{L^2(\mathbb{R}^+)} \leq R$, the solution u of (1) satisfies

$$||u(t)||_{L^2(\mathbb{R}^+)} \leq Ce^{-\nu t}||u_0||_{L^2(\mathbb{R}^+)}. \tag{5}$$

Actually, Theorem 1.1 was proved in [20] under the additional hypothesis that

$$a(x) \geq a_0 \text{ a.e. in } (0,\delta) \tag{6}$$

for some $\delta > 0$, but (6) may be dropped by replacing the unique continuation property [20, Lemma 2.4] by [30, Theorem 1.6]. The exponential decay of $E(t)$ is obtained following the methods in [23, 25, 26] which combine multiplier techniques and compactness arguments to reduce the problem to some unique continuation property for weak solutions of KdV.

Along this work we assume that the real-valued function $a = a(x)$ satisfies the condition (4) for some given positive numbers a_0, x_0. In this paper we investigate the stability properties of (1) in the weighted spaces introduced by Kato in [15]. More precisely, for $b > 0$ and $m \in \mathbb{N}$, we prove that the solution u exponentially decays to 0 in L^2_b and $L^2_{(x+1)^m dx}$ (if $u(0)$ belongs to one of these spaces), where

$$L^2_b = \{u : \mathbb{R}^+ \to \mathbb{R};\ \int_0^\infty |u(x)|^2 e^{2bx} dx < \infty\},$$

$$L^2_{(x+1)^m dx} = \{u : \mathbb{R}^+ \to \mathbb{R};\ \int_0^\infty |u(x)|^2 (x+1)^m dx < \infty\}.$$

The following weighted Sobolev spaces

$$H^s_b = \{u : \mathbb{R}^+ \to \mathbb{R};\ \partial^i_x u \in L^2_b \text{ for } 0 \leq i \leq s;\ u(0) = 0 \text{ if } s \geq 1\}$$

and

$$H^s_{(x+1)^m dx} = \{u : \mathbb{R}^+ \to \mathbb{R};\ \partial^i_x u \in L^2_{(x+1)^{m-i} dx} \text{ for } 0 \leq i \leq s;\ u(0) = 0 \text{ if } s \geq 1\},$$

endowed with their usual inner products, will be used thereafter. Note that $H^0_b = L^2_b$ and that $H^0_{(x+1)^m dx} = L^2_{(x+1)^m dx}$.

The exponential decay in $L^2_{(x+1)^m dx}$ is obtained by constructing a convenient Lyapunov function (which actually decreases strictly on the sequence of times $\{kT\}_{k\ge 0}$) by induction on m. For $u_0 \in L^2_{(x+1)^m dx}$, we also prove the following estimate

$$||u(t)||_{H^1_{(x+1)^m dx}} \le C\frac{e^{-\mu t}}{\sqrt{t}}||u_0||_{L^2_{(x+1)^m dx}} \tag{7}$$

in two situations: (i) $m = 1$ and $||u_0||_{L^2_{(x+1)^m dx}}$ is arbitrarily large; (ii) $m \ge 2$ and $||u_0||_{L^2_{(x+1)^m dx}}$ is small enough. In the situation (ii), we first establish a similar estimate for the linearized system and next apply the contraction mapping principle in a space of functions fulfilling the exponential decay. Note that (7) combines the (global) Kato smoothing effect to the exponential decay.

The exponential decay in L^2_b is established for any initial data $u_0 \in L^2_b$ under the additional assumption that $4b^3 + b < a_0$. Next, we can derive estimates of the form

$$||u(t)||_{H^s_b} \le C\frac{e^{-\mu t}}{t^{s/2}}||u_0||_{L^2_b}$$

for any $s \ge 1$, revealing that $u(t)$ decays exponentially to 0 in strong norms.

It would be interesting to see if such results are still true when the function a has a smaller support. It seems reasonable to conjecture that similar positive results can be derived when the support of a contains a set of the form $\cup_{k\ge 1}[ka_0, ka_0 + b_0]$ where $0 < b_0 < a_0$, while a negative result probably holds when the support of a is a finite interval, as the L^2 norm of a soliton-like initial data may not be sufficiently dissipated over time. Such issues will be discussed elsewhere.

The plan of this paper is as follows. Section 2 is devoted to global well-posedness results in the weighted spaces L^2_b and $L^2_{(x+1)^2 dx}$. In section 3, we prove the exponential decay in $L^2_{(x+1)^m dx}$ and L^2_b, and establish the exponential decay of the derivatives as well.

2 Global well-posedness

2.1 Global well-posedness in $\mathbf{L^2_b}$

Fix any $b > 0$. To begin with, we apply the classical semigroup theory to the linearized system

$$\begin{cases} u_t + u_x + u_{xxx} + a(x)u = 0, & x,\, t \in \mathbb{R}^+, \\ u(0,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x > 0. \end{cases} \tag{8}$$

Let us consider the operator

$$A : D(A) \subset L^2_b \to L^2_b$$

with domain

$$D(A) = \{u \in L^2_b;\ \partial^i_x u \in L^2_b \text{ for } 1 \le i \le 3 \text{ and } u(0) = 0\}$$

defined by

$$Au = -u_{xxx} - u_x - a(x)u.$$

Then, the following result holds.

Lemma 2.1 *The operator A defined above generates a continuous semigroup of operators $(S(t))_{t\ge 0}$ in L^2_b.*

Proof. We first introduce the new variable $v = e^{bx}u$ and consider the following (IBVP)

$$\text{(9)} \qquad \begin{cases} v_t + (\partial_x - b)v + (\partial_x - b)^3 v + a(x)v = 0, & x,\, t \in \mathbb{R}^+, \\ v(0,t) = 0, & t > 0, \\ v(x,0) = v_0(x) = e^{bx}u_0(x), & x > 0. \end{cases}$$

Clearly, the operator $B : D(B) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ with domain

$$D(B) = \{u \in H^3(\mathbb{R}^+);\ u(0) = 0\}$$

defined by

$$Bv = -(\partial_x - b)v - (\partial_x - b)^3 v - a(x)v$$

is densely defined and closed. So, we are done if we prove that for some real number λ the operator $B - \lambda$ and its adjoint $B^* - \lambda$ are both dissipative in $L^2(\mathbb{R}^+)$. It is readily seen that $B^* : D(B^*) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ is given by $B^*v = (\partial_x + b)v + (\partial_x + b)^3 v - a(x)v$ with domain

$$D(B^*) = \{v \in H^3(\mathbb{R}^+);\ v(0) = v'(0) = 0\}.$$

Pick any $v \in D(B)$. After some integration by parts, we obtain that

$$(Bv, v)_{L^2} = -\frac{1}{2}v_x^2(0) - 3b\int_0^\infty v_x^2 dx + (b + b^3)\int_0^\infty v^2 dx - \int_0^\infty a(x)v^2 dx,$$

that is,

$$([B - (b^3 + b)]v, v)_{L^2} \leq 0.$$

Analogously, we deduce that for any $v \in D(B^*)$

$$(v, [B^* - (b^3 + b)]v)_{L^2} \leq 0$$

which completes the proof. ■

The following linear estimates will be needed.

Lemma 2.2 *Let $u_0 \in L_b^2$ and $u = S(\cdot)u_0$. Then, for any $T > 0$*

$$\text{(10)} \qquad \frac{1}{2}\int_0^\infty |u(x,T)|^2 dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2 dx + \int_0^T\int_0^\infty a(x)|u|^2 dxdt + \frac{1}{2}\int_0^T u_x^2(0,t)dt = 0$$

$$\text{(11)} \qquad \begin{aligned} &\frac{1}{2}\int_0^\infty |u(x,T)|^2 e^{2bx} dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2 e^{2bx} dx + 3b\int_0^T\int_0^\infty u_x^2 e^{2bx} dxdt \\ &-(4b^3 + b)\int_0^T\int_0^\infty u^2 e^{2bx} dxdt + \int_0^T\int_0^\infty a(x)|u|^2 e^{2bx} dxdt + \frac{1}{2}\int_0^T u_x^2(0,t)dt = 0. \end{aligned}$$

As a consequence,

$$\text{(12)} \qquad ||u||_{L^\infty(0,T;L_b^2)} + ||u_x||_{L^2(0,T;L_b^2)} \leq C\, ||u_0||_{L_b^2},$$

where $C = C(T)$ is a positive constant.

Proof. Pick any $u_0 \in D(A)$. Multiplying the equation in (1) by u and integrating over $(0,+\infty) \times (0,T)$, we obtain (10). Then, the identity may be extended to any initial state $u_0 \in L^2_b$ by a density argument. To derive (11) we first multiply the equation by $(e^{2bx}-1)u$ and integrate by parts over $(0,+\infty) \times (0,T)$ to deduce that

$$\frac{1}{2}\int_0^\infty |u(x,T)|^2(e^{2bx}-1)dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2(e^{2bx}-1)dx +$$
$$+3b\int_0^T\int_0^\infty u_x^2 e^{2bx}dxdt - (4b^3+b)\int_0^T\int_0^\infty u^2 e^{2bx}dxdt +$$
$$+\int_0^T\int_0^\infty a(x)|u|^2(e^{2bx}-1)dxdt = 0.$$

Adding the above equality and (10) hand to hand, we obtain (11) using the same density argument. Then, Gronwall inequality, (4) and (11) imply that

$$||u||_{L^\infty(0,T;L^2_b)} \leq C\,||u_0||_{L^2_b},$$

with $C = C(T) > 0$. This estimate together with (11) gives us

$$||u_x||_{L^2(0,T;L^2_b)} \leq C\,||u_0||_{L^2_b},$$

where $C = C(T)$ is a positive constant. ■

The global well-posedness result reads as follows:

Theorem 2.3 *For any $u_0 \in L^2_b$ and any $T > 0$, there exists a unique solution $u \in C([0,T];L^2_b) \cap L^2(0,T;H^1_b)$ of (1).*

Proof. By computations similar to those performed in the proof of Lemma 2.2, we obtain that for any $f \in C^1([0,T];L^2_b)$ and any $u_0 \in D(A)$, the solution u of the system

$$\begin{cases} u_t + u_x + u_{xxx} + a(x)u = f, & x \in \mathbb{R}^+,\ t \in (0,T), \\ u(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^+, \end{cases}$$

fulfills

$$\sup_{0\leq t\leq T} ||u(t)||_{L^2_b} + \left(\int_0^T\int_0^\infty |u_x|^2 e^{2bx}dxdt\right)^{\frac{1}{2}} \leq C\left(||u_0||_{L^2_b} + \int_0^T ||f||_{L^2_b}dt\right) \tag{13}$$

for some constant $C = C(T)$ nondecreasing in T. A density argument yields that $u \in C([0,T];L^2_b)$ when $f \in L^1(0,T;L^2_b)$ and $u_0 \in L^2_b$.

Let $u_0 \in L^2_b$ be given. To prove the existence of a solution of (1) we introduce the map Γ defined by

$$(\Gamma u)(t) = S(t)u_0 + \int_0^t S(t-s)N(u(s))\,ds$$

where $N(u) = -uu_x$, and the space

$$F = C([0,T];L^2_b) \cap L^2(0,T;H^1_b)$$

endowed with its natural norm. We shall prove that Γ has a fixed-point in some ball $B_R(0)$ of F. We need the following

Claim 1. If $u \in H_b^1$ then

$$||u^2 e^{2bx}||_{L^\infty(\mathbb{R}^+)} \leq (2+2b)\, ||u||_{L_b^2} ||u||_{H_b^1}.$$

From Cauchy-Schwarz inequality, we get for any $\overline{x} \in \mathbb{R}^+$

$$\begin{aligned} u^2(\overline{x})e^{2b\overline{x}} &= \int_0^{\overline{x}} [u^2 e^{2bx}]_x dx = \int_0^{\overline{x}} [2uu_x e^{2bx} + 2bu^2 e^{2bx}]dx \\ &\leq 2(\int_0^\infty u^2 e^{2bx} dx)^{\frac{1}{2}} (\int_0^\infty u_x^2 e^{2bx} dx)^{\frac{1}{2}} + 2b \int_0^\infty u^2 e^{2bx} dx \leq (2+2b)||u||_{L_b^2} ||u||_{H_b^1} \end{aligned}$$

which guarantees that Claim 1 holds.

Claim 2. There exists a constant $K > 0$ such that for $0 < T \leq 1$

$$||\Gamma(u) - \Gamma(v)||_F \leq K T^{\frac{1}{4}} (||u||_F + ||v||_F) ||u - v||_F, \quad \forall\, u, v \in F.$$

According to the previous analysis,

$$||\Gamma(u) - \Gamma(v)||_F \leq C ||uu_x - vv_x||_{L^1(0,T;L_b^2)}.$$

So, applying triangular inequality and Hölder inequality, we have

$$\begin{aligned} ||\Gamma(u) - \Gamma(v)||_F \leq C\{ &||u - v||_{L^2(0,T;L^\infty(0,\infty))} ||u||_{L^2(0,T;H_b^1)} + \\ &+ ||v||_{L^2(0,T;L^\infty(0,\infty))} ||u - v||_{L^2(0,T;H_b^1)} \}. \end{aligned} \tag{14}$$

Now, by Claim 1, we have

$$||u||_{L^2(0,T;L^\infty(0,\infty))} \leq C\, T^{\frac{1}{4}} ||u||^{\frac{1}{2}}_{L^\infty(0,T;L_b^2)} ||u||^{\frac{1}{2}}_{L^2(0,T;H_b^1)}. \tag{15}$$

Then, combining (14) and (15), we deduce that

$$||\Gamma(u) - \Gamma(v)||_F \leq C\, T^{\frac{1}{4}} \{\, ||u||_F + ||v||_F \,\} ||u - v||_F. \tag{16}$$

Let $T > 0$, $R > 0$ be numbers whose values will be specified later, and let $u \in B_R(0) \subset F$ be given. Then, by Claim 2 and Lemma 2.2, $\Gamma u \in F$ and

$$||\Gamma u||_F \leq C\, (\, ||u_0||_{L_b^2} + T^{\frac{1}{4}} ||u||_F^2 \,).$$

Consequently, for $R = 2C||u_0||_{L_b^2}$ and $T > 0$ small enough, Γ maps $B_R(0)$ into itself. Moreover, we infer from (16) that this mapping contracts if T is small enough. Then, by the contraction mapping theorem, there exists a unique solution $u \in B_R(0) \subset F$ to the problem (1) for T small enough.

In order to prove that this solution is global, we need some a priori estimates. So, we proceed as in the proof of Lemma 2.2 to obtain for the solution u of (1)

$$\frac{1}{2}\int_0^\infty |u(x,T)|^2 dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2 dx + \int_0^T \int_0^\infty a(x)|u|^2 dx dt + \frac{1}{2}\int_0^T u_x^2(0,t) dt = 0 \tag{17}$$

and

$$\begin{aligned} &\frac{1}{2}\int_0^\infty |u(x,T)|^2 e^{2bx} dx - \frac{1}{2}\int_0^\infty |u_0(x)|^2 e^{2bx} dx + \frac{1}{2}\int_0^T u_x^2(0,t) dt \\ &\quad + 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt - (4b^3 + b) \int_0^T \int_0^\infty u^2 e^{2bx} dx dt \\ &\quad + \int_0^T \int_0^\infty a(x)|u|^2 e^{2bx} dx dt - \frac{2b}{3} \int_0^T \int_0^\infty u^3 e^{2bx} dx dt = 0. \end{aligned} \tag{18}$$

First, observe that

$$|\int_0^\infty u^2 e^{2bx}dx| = |-\frac{1}{b}\int_0^\infty uu_x e^{2bx}dx| \le \frac{1}{b}(\int_0^\infty u^2e^{2bx}dx)^{\frac{1}{2}}(\int_0^\infty u_x^2 e^{2bx}dx)^{\frac{1}{2}},$$

therefore,

$$\int_0^\infty u^2e^{2bx}dx \le \frac{1}{b^2}\int_0^\infty u_x^2 e^{2bx}dx.$$

Combined to Claim 1, this yields

$$||u(x)e^{bx}||_{L^\infty(\mathbb{R}^+)} \le C||u_x||_{L^2_b}.$$

On the other hand, it follows from (17) that

$$||u(t)||_{L^2(\mathbb{R}^+)} \le ||u_0||_{L^2(\mathbb{R}^+)},$$

hence

$$\begin{aligned}\int_0^T\int_0^\infty |u|^3 e^{2bx}dxdt &\le \int_0^T ||ue^{bx}||_{L^\infty(\mathbb{R}^+)}(\int_0^\infty |u|^2e^{bx}dx)dt \\ &\le C\int_0^T ||u_x||_{L^2_b}||u||_{L^2_b}||u||_{L^2}dt \\ &\le \delta||u_x||^2_{L^2(0,T;L^2_b)} + C_\delta||u||^2_{L^2(0,T;L^2_b)},\end{aligned}$$

where $\delta > 0$ is arbitrarily chosen and $C = C(b,\delta,||u_0||_{L^2(\mathbb{R}^+)})$ is a positive constant. Combining this inequality (with $\delta < 9/2$) to (18) results in

$$||u(T)||^2_{L^2_b} \le ||u_0||^2_{L^2_b} + C\int_0^T ||u||^2_{L^2_b}dt$$

where $C = C(b, ||u_0||_{L^2(\mathbb{R}^+)})$ does not depend on T. It follows from Gronwall lemma that

$$||u(T)||^2_{L^2_b} \le ||u_0||^2_{L^2_b}e^{CT}$$

for all $T > 0$, which gives the global well-posedness. ■

2.2 Global well-posedness in $L^2_{(x+1)^2dx}$

Definition 2.4 *For $u_0 \in L^2_{(x+1)^2dx}$ and $T > 0$, we denote by a* mild solution *of* (1) *any function $u \in C([0,T]; L^2_{(x+1)^2dx}) \cap L^2(0,T;H^1_{(x+1)^2dx})$ which solves* (1)*, and such that for some $b > 0$ and some sequence $\{u_{n,0}\} \subset L^2_b$ we have*

$$\begin{aligned}&u_{n,0} \to u_0 \quad \textit{strongly in} \quad L^2_{(x+1)^2dx},\\ &u_n \to u \textit{ weakly}* \quad \textit{in } L^\infty(0,T;L^2_{(x+1)^2dx}),\\ &u_n \to u \textit{ weakly in } L^2(0,T;H^1_{(x+1)^2dx}),\end{aligned}$$

u_n denoting the solution of (1) *emanating from $u_{n,0}$ at $t = 0$.*

Theorem 2.5 *For any $u_0 \in L^2_{(x+1)^2dx}$ and any $T > 0$, there exists a unique mild solution $u \in C([0,T]; L^2_{(x+1)^2dx}) \cap L^2(0,T;H^1_{(x+1)^2dx})$ of* (1).

Proof. We prove the existence and the uniqueness in two steps.

Step 1. Existence

Since the embedding $L^2_b \subset L^2_{(x+1)^2dx}$ is dense, for any given $u_0 \in L^2_{(x+1)^2dx}$ we may construct a sequence $\{u_{n,0}\} \subset L^2_b$ such that $u_{n,0} \to u_0$ in $L^2_{(x+1)^2dx}$ as $n \to \infty$. For each n, let u_n denote the solution of (1) emanating from $u_{n,0}$ at $t = 0$, which is given by Theorem 2.3. Then $u_n \in C([0,T]; L^2_b) \cap L^2(0,T;H^1_b)$ and it solves

$$u_{n,t} + u_{n,x} + u_{n,xxx} + u_n u_{n,x} + a(x)u_n = 0, \tag{19}$$

$$u_n(0,t) = 0 \tag{20}$$

$$u_n(x,0) = u_{n,0}(x). \tag{21}$$

Multiplying (19) by $(x+1)^2 u_n$ and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty (x+1)^2|u_n(x,T)|^2dx + 3\int_0^T\int_0^\infty (x+1)|u_{n,x}|^2dxdt + \frac{1}{2}\int_0^T |u_{n,x}(0,t)|^2dt \\
&- \int_0^T\int_0^\infty (x+1)|u_n|^2dxdt - \frac{2}{3}\int_0^T\int_0^\infty (x+1)u_n^3\,dxdt + \int_0^T\int_0^\infty (x+1)^2u_n^2a(x)dx \\
&\quad = \frac{1}{2}\int_0^\infty (x+1)^2|u_{n,0}(x)|^2dx.
\end{aligned} \tag{22}$$

Scaling in (19) by u_n gives

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty |u_n(x,T)|^2dx + \frac{1}{2}\int_0^T |u_{n,x}(0,t)|^2dt + \int_0^T\int_0^\infty a(x)|u_n(x,t)|^2dxdt \\
&= \frac{1}{2}\int_0^\infty |u_{n,0}(x)|^2dx,
\end{aligned}$$

hence

$$||u_n||_{L^2(\mathbb{R}^+)} \le ||u_{n,0}||_{L^2(\mathbb{R}^+)} \le C \tag{23}$$

where $C = C(||u_0||_{L^2(\mathbb{R}^+)})$. It follows that

$$\begin{aligned}
\frac{2}{3}\int_0^\infty (x+1)|u_n|^3dx &\le \frac{2\sqrt{2}}{3}||u_{n,x}||^{\frac{1}{2}}_{L^2(\mathbb{R}^+)}||u_n||^{\frac{3}{2}}_{L^2(\mathbb{R}^+)}||(x+1)u_n||_{L^2(\mathbb{R}^+)} \\
&\le \int_0^\infty (x+1)|u_{n,x}|^2dx + C\int_0^\infty (x+1)^2|u_n|^2dx
\end{aligned} \tag{24}$$

which, combined to (22), gives

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty (x+1)^2|u_n(x,T)|^2dx + 2\int_0^T\int_0^\infty (x+1)|u_{n,x}|^2dxdt + \frac{1}{2}\int_0^T |u_{n,x}(0,t)|^2dt \\
&\quad \le \frac{1}{2}\int_0^\infty (x+1)^2|u_{n,0}(x)|^2dx + C\int_0^T\int_0^\infty (x+1)^2|u_n(x,t)|^2dxdt.
\end{aligned}$$

An application of Gronwall's lemma yields

$$\begin{aligned}
||u_n||_{L^\infty(0,T;L^2_{(x+1)^2dx})} &\le C(T, ||u_{n,0}||_{L^2_{(x+1)^2dx}}), \\
||u_{n,x}||_{L^2(0,T;H^1_{(x+1)^2dx})} &\le C(T, ||u_{n,0}||_{L^2_{(x+1)^2dx}}), \\
||u_{n,x}(0,.)||_{L^2(0,T)} &\le C(T, ||u_{n,0}||_{L^2_{(x+1)^2dx}}).
\end{aligned}$$

Therefore, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly } * \text{ in } L^\infty(0,T;L^2_{(x+1)^2dx}), \\ u_n \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_{(x+1)^2dx}), \\ u_{n,x}(0,.) \rightharpoonup u_x(0,.) \text{ weakly in } L^2(0,T). \end{cases}$$

Note that, for all $L>0$, $\{u_n\}$ is bounded in $L^2(0,T;H^1(0,L)) \cap H^1(0,T;H^{-2}(0,L))$, hence by Aubin's lemma, we have (after extracting a subsequence if needed)

$$u_n \to u \text{ strongly in } L^2(0,T;L^2(0,L)) \text{ for all } L>0.$$

This gives that $u_n u_{n,x} \to u u_x$ in the sense of distributions, hence the limit $u \in L^\infty(0,T;L^2_{(x+1)^2dx}) \cap L^2(0,T;H^1_{(x+1)^2dx})$ is a solution of (1). Let us check that $u \in C([0,T];L^2_{(x+1)^2dx})$. Since $u \in C([0,T];H^{-2}(\mathbb{R}^+)) \cap L^\infty(0,T;L^2_{(x+1)^2dx})$, we have that $u \in C_w([0,T];L^2_{(x+1)^2dx})$ (see e.g. [21]), where $C_w([0,T];L^2_{(x+1)^2dx})$ denotes the space of sequentially weakly continuous functions from $[0,T]$ into $L^2_{(x+1)^2dx}$.

We claim that $u \in L^3(0,T;L^3(\mathbb{R}^+))$. Indeed, from Moser estimate (see [31])

$$||u||_{L^\infty(\mathbb{R}^+)} \le \sqrt{2}||u_x||^{\frac{1}{2}}_{L^2(\mathbb{R}^+)}||u||^{\frac{1}{2}}_{L^2(\mathbb{R}^+)} \tag{25}$$

and Young inequality we get

$$\int_0^\infty |u|^3dx \le ||u||_{L^\infty}||u||^2_{L^2} \le \sqrt{2}||u_x||^{\frac{1}{2}}_{L^2}||u||^{\frac{5}{2}}_{L^2} \le \varepsilon||u_x||^2_{L^2} + c_\varepsilon||u||^{\frac{10}{3}}_{L^2} \tag{26}$$

where $\varepsilon>0$ is arbitrarily chosen and c_ε denotes some positive constant. Since $u \in C_w([0,T];L^2_{(x+1)^2dx}) \cap L^2(0,T;H^1_{(x+1)^2dx})$, it follows that $u \in L^3(0,T;L^3(\mathbb{R}^+))$. On the other hand, $u(0,t)=0$ for $t \in (0,T)$ and $u_x(0,.) \in L^2(0,T)$. Scaling in (1) by $(x+1)^2u$ yields for all $t_1,t_2 \in (0,T)$

$$\begin{aligned} &\frac{1}{2}\int_0^\infty (x+1)^2|u(x,t_2)|^2dx - \frac{1}{2}\int_0^\infty (x+1)^2|u(x,t_1)|^2dx \\ &= -3\int_{t_1}^{t_2}\int_0^\infty (x+1)|u_x|^2dxdt - \frac{1}{2}\int_{t_1}^{t_2}|u_x(0,t)|^2dt + \int_{t_1}^{t_2}\int_0^\infty (x+1)|u|^2dxdt \\ &+\frac{2}{3}\int_{t_1}^{t_2}\int_0^\infty (x+1)u^3dxdt - \int_{t_1}^{t_2}\int_0^\infty (x+1)^2a(x)|u|^2dxdt. \end{aligned} \tag{27}$$

Therefore $\lim_{t_1\to t_2}\left| ||u(t_2)||^2_{L^2_{(x+1)^2dx}} - ||u(t_1)||^2_{L^2_{(x+1)^2dx}}\right| = 0$. Combined to the fact that $u \in C_w([0,T];L^2_{(x+1)^2dx})$, this yields $u \in C([0,T],L^2_{(x+1)^2dx})$.

Step 2. Uniqueness

Here, C will denote a universal constant which may vary from line to line. Pick $u_0 \in L^2_{(x+1)^2dx}$, and let $u,v \in C([0,T];L^2_{(x+1)^2dx}) \cap L^2(0,T;H^1_{(x+1)^2dx})$ be two mild solutions of (1). Pick two sequences $\{u_{n,0}\}$, $\{v_{n,0}\}$ in L^2_b for some $b>0$ such that

$$u_{n,0} \to u_0 \text{ strongly in } L^2_{(x+1)^2dx}, \tag{28}$$

$$u_n \to u \text{ weakly} * \text{ in } L^\infty(0,T;L^2_{(x+1)^2dx}), \tag{29}$$

$$u_n \to u \text{ weakly in } L^2(0,T;H^1_{(x+1)^2dx}) \tag{30}$$

and also

$$v_{n,0} \to u_0 \text{ strongly in } L^2_{(x+1)^2dx}, \tag{31}$$

$$v_n \to v \text{ weakly} * \text{ in } L^\infty(0,T;L^2_{(x+1)^2dx}), \tag{32}$$

$$v_n \to v \text{ weakly in } L^2(0,T;H^1_{(x+1)^2dx}). \tag{33}$$

We shall prove that $w = u - v$ vanishes on $\mathbb{R}^+ \times [0,T]$ by providing some estimate for $w_n = u_n - v_n$. Note first that w_n solves the system

$$w_{n,t} + w_{n,x} + w_{n,xxx} + aw_n = f_n = v_n v_{n,x} - u_n u_{n,x}, \tag{34}$$

$$w_n(0,t) = 0, \tag{35}$$

$$w_n(x,0) = w_{n,0}(x) = u_{n,0}(x) - v_{n,0}(x). \tag{36}$$

Scaling in (34) by $(x+1)w_n$ yields

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty (x+1)|w_n(x,t)|^2dx + \frac{3}{2}\int_0^t\int_0^\infty |w_{n,x}|^2dxd\tau - \frac{1}{2}\int_0^t\int_0^\infty |w_n|^2dxd\tau \\
&\le \frac{1}{2}\int_0^\infty (x+1)|w_{n,0}|^2dx + \int_0^t (\int_0^\infty (x+1)|w_n|^2dx)^{\frac{1}{2}}(\int_0^\infty (x+1)|f_n|^2dx)^{\frac{1}{2}}d\tau \\
&\le \frac{1}{2}\int_0^\infty (x+1)|w_{n,0}|^2dx + \frac{1}{4}\sup_{0<\tau<t}\int_0^\infty (x+1)|w_n(x,\tau)|^2dx \\
&\quad + [\int_0^T (\int_0^\infty (x+1)|f_n|^2dx)^{\frac{1}{2}}d\tau]^2.
\end{aligned}$$

Since $||w_n(t)||_{L^2(\mathbb{R}^+)} \le ||w_n(t)||_{L^2_{(x+1)dx}}$, this yields for $T < 1/10$

$$\begin{aligned}
&\sup_{0<t<T}\int_0^\infty (x+1)|w_n(x,t)|^2dx + \int_0^T\int_0^\infty |w_{n,x}|^2dxdt \\
&\le C[\int_0^\infty (x+1)|w_{n,0}(x)|^2dx + \left(\int_0^T (\int_0^\infty (x+1)|f_n|^2dx)^{\frac{1}{2}}d\tau\right)^2].
\end{aligned} \tag{37}$$

It remains to estimate $\int_0^T (\int_0^\infty (x+1)|f_n|^2dx)^{\frac{1}{2}}dt$. We split f_n into

$$f_n = (v_n - u_n)v_{n,x} + u_n(v_{n,x} - u_{n,x}) = f_n^1 + f_n^2.$$

We have that

$$\begin{aligned}
\int_0^T (\int_0^\infty (x+1)|f_n^1|^2dx)^{\frac{1}{2}}dt &= \int_0^T (\int_0^\infty (x+1)|w_n|^2|v_{n,x}|^2dx)^{\frac{1}{2}}dt \\
&\le \int_0^T ||w_n||_{L^\infty(\mathbb{R}^+)}(\int_0^\infty (x+1)|v_{n,x}|^2dx)^{\frac{1}{2}}dt \\
&\le (\int_0^T ||w_n||^2_{L^\infty(\mathbb{R}^+)}dt)^{\frac{1}{2}}(\int_0^T\int_0^\infty (x+1)|v_{n,x}|^2dxdt)^{\frac{1}{2}}.
\end{aligned}$$

By Sobolev embedding, we have that

$$\begin{aligned}
(\int_0^T ||w_n||^2_{L^\infty(\mathbb{R}^+)}dt)^{\frac{1}{2}} &\le (\int_0^T ||w_n||^2_{H^1(\mathbb{R}^+)}dt)^{\frac{1}{2}} \\
&\le \sqrt{T}\sup_{0<t<T}||w_n||_{L^2(\mathbb{R}^+)} + ||w_{n,x}||_{L^2(0,T;L^2(\mathbb{R}^+))}.
\end{aligned}$$

Thus

$$\int_0^T \Big(\int_0^\infty (x+1)|f_n^1|^2 dx\Big)^{\frac12} dt \le ||v_{n,x}||_{L^2(0,T;L^2_{(x+1)dx})}\big(\sqrt{T}\sup_{0<t<T}||w_n||_{L^2(\mathbb{R}^+)} \tag{38}$$
$$+||w_{n,x}||_{L^2(0,T;L^2(\mathbb{R}^+))}\big)$$

On the other hand, we have that

$$\begin{aligned}
&\int_0^T \Big(\int_0^\infty (x+1)|f_n^2|^2 dx\Big)^{\frac12} dt \\
&= \int_0^T \Big(\int_0^\infty (x+1)|u_n|^2|w_{n,x}|^2 dx\Big)^{\frac12} dt \\
&\le \int_0^T ||(x+1)^{\frac12}u_n||_{L^\infty(\mathbb{R}^+)}||w_{n,x}||_{L^2(\mathbb{R}^+)}dt \\
&\le C\int_0^T \big(||(x+1)^{\frac12}u_n||_{L^2(\mathbb{R}^+)} + ||(x+1)^{\frac12}u_{n,x}||_{L^2(\mathbb{R}^+)}\big)||w_{n,x}||_{L^2(\mathbb{R}^+)}dt \\
&\le C\Big(\sqrt{T}||(x+1)u_n||_{L^\infty(0,T;L^2(\mathbb{R}^+))} \\
&\qquad +||(x+1)^{\frac12}u_{n,x}||_{L^2(0,T,L^2(\mathbb{R}^+))}\Big)||w_{n,x}||_{L^2(0,T;L^2(\mathbb{R}^+))}. \qquad (39)
\end{aligned}$$

Gathering together (37), (38) and (39), we conclude that for $T < 1/10$

$$h_n(T) \le K_n(T)h_n(T) + C||w_{n,0}||^2_{L^2_{(x+1)dx}}$$

where

$$h_n(t) := \sup_{0<\tau<T}\int_0^\infty (x+1)|w_n(x,\tau)|^2dx + \int_0^T\int_0^\infty |w_{n,x}|^2dxdt \tag{40}$$

$$\begin{aligned}
K_n(T) \le C\Big(&\int_0^T\int_0^\infty (x+1)|v_{n,x}|^2dxdt + T||(x+1)u_n||^2_{L^\infty(0,T;L^2(\mathbb{R}^+))} \\
&+\int_0^T\int_0^\infty (x+1)|u_{n,x}|^2dxdt\Big) \qquad (41)
\end{aligned}$$

and C denotes a universal constant. The following claim is needed.
Claim 3.

$$\lim_{T\to 0}\limsup_{n\to\infty}\int_0^T\int_0^\infty (x+1)|u_{n,x}|^2dxdt = 0, \quad \lim_{T\to 0}\limsup_{n\to\infty}\int_0^T\int_0^\infty (x+1)|v_{n,x}|^2dxdt = 0.$$

Clearly, it is sufficient to prove the claim for the sequence $\{u_n\}$ only. From (27) applied with $u=u_n$ on $[0,T]$, we obtain

$$\begin{aligned}
&\frac12\int_0^\infty (x+1)^2|u_n(x,T)|^2dx + 3\int_0^T\int_0^\infty (x+1)|u_{n,x}|^2dxdt \\
&\le \frac12\int_0^\infty (x+1)^2|u_{n,0}|^2dx + \int_0^T\int_0^\infty (x+1)|u_n|^2dxdt + \frac23\int_0^T\int_0^\infty (x+1)|u_n|^3dxdt.
\end{aligned}$$

Combined to (23)-(24), this gives

$$\begin{aligned}
&||u_n(T)||^2_{L^2_{(x+1)^2dx}} + \int_0^T\int_0^\infty (x+1)|u_{n,x}|^2dxdt \\
&\le ||u_{n,0}||^2_{L^2_{(x+1)^2dx}} + C\int_0^T ||u_n||^2_{L^2_{(x+1)^2dx}}dt. \qquad (42)
\end{aligned}$$

It follows from Gronwall lemma that

$$||u_n(t)||^2_{L^2_{(x+1)^2dx}} \le ||u_{n,0}||^2_{L^2_{(x+1)^2dx}} e^{Ct} \tag{43}$$

Using (43) in (42) and taking the limit sup as $n \to \infty$ gives for a.e. T

$$||u(T)||^2_{L^2_{(x+1)^2dx}} + \limsup_{n\to\infty} \int_0^T \int_0^\infty |u_{n,x}|^2 dxdt \le e^{CT} ||u_0||^2_{L^2_{(x+1)^2dx}}$$

As u is continuous from $\mathbb{R}^+$ to $L^2_{(x+1)^2dx}$, we infer that

$$\lim_{T\to 0} \limsup_{n\to\infty} \int_0^T \int_0^\infty |u_{n,x}|^2 dxdt = 0.$$

The claim is proved. Therefore, we have that for $T > 0$ small enough and n large enough, $K_n(T) < \frac{1}{2}$, and hence

$$h_n(T) \le 2C ||w_n(0)||^2_{L^2_{(x+1)dx}}.$$

This yields

$$||u - v||^2_{L^\infty(0,T;L^2_{(x+1)dx})} \le \liminf_{n\to\infty} h_n(T) \le 2C \liminf_{n\to\infty} ||w_n(0)||^2_{L^2_{(x+1)dx}} = 0$$

and $u = v$ for $0 < t < T$. This proves the uniqueness for T small enough. The general case follows by a classical argument. ■

Remark 2.6 *1. If we assume only that $u_0 \in L^2_{(x+1)dx}$, then a proof similar to Step 1 gives the existence of a mild solution $u \in C([0,T]; L^2_{(x+1)dx}) \cap L^2(0,T; H^1_{(x+1)dx})$ of (1). The uniqueness of such a solution is open. The existence and uniqueness of a solution issuing from $u_0 \in L^2_{(x+1)dx}$ in a class of functions involving a Bourgain norm has been given in [13].*

2. If $u_0 \in L^2_{(x+1)^m dx}$ with $m \ge 3$, then $u \in C([0,T]; L^2_{(x+1)^m dx}) \cap L^2(0,T; H^1_{(x+1)^m dx})$ for all $T > 0$ (see below Theorem 3.1).

3 Asymptotic Behavior

3.1 Decay in $L^2_{(x+1)^m dx}$

Theorem 3.1 *Assume that the function $a = a(x)$ satisfies (4). Then, for all $R > 0$ and $m \ge 1$, there exist numbers $C > 0$ and $\nu > 0$ such that*

$$||u(t)||_{L^2_{(x+1)^m dx}} \le C\, e^{-\nu t} ||u_0||_{L^2_{(x+1)^m dx}}$$

for any solution given by Theorem 2.5, whenever $||u_0||_{L^2_{(x+1)^m dx}} \le R$.

Proof. The proof will be done by induction in m. We set

$$V_0(u) = E(u) = \frac{1}{2} \int_0^\infty u^2 dx \tag{44}$$

and define the Lyapunov function V_m for $m \ge 1$ in an inductive way

$$V_m(u) = \frac{1}{2} \int_0^\infty (x+1)^m u^2 dx + d_{m-1} V_{m-1}(u), \tag{45}$$

where $d_{m-1} > 0$ is chosen sufficiently large (see below).

Suppose first that $m = 1$ and put $V = V_1$. Multiplying the first equation in (1) by u and integrating by parts over $\mathbb{R}^+ \times (0,T)$, we obtain

$$\frac{1}{2}\int_0^\infty |u(x,T)|^2 dx = \frac{1}{2}\int_0^\infty |u_0(x)|^2 dx - \int_0^T \int_0^\infty a(x)|u|^2 dxdt - \frac{1}{2}\int_0^T u_x^2(0,t)dt. \tag{46}$$

Now, multiplying the equation by xu, we deduce that

$$\begin{aligned}&\frac{1}{2}\int_0^\infty x|u(x,T)|^2 dx - \frac{1}{2}\int_0^\infty x|u_0(x)|^2 dx + \frac{3}{2}\int_0^T\int_0^\infty u_x^2 dxdt\\ &-\frac{1}{2}\int_0^T\int_0^\infty u^2 dxdt - \frac{1}{3}\int_0^T\int_0^\infty u^3 dxdt + \int_0^T\int_0^\infty xa(x)|u|^2 dxdt = 0.\end{aligned} \tag{47}$$

Combining (46) and (47) it follows that

$$\begin{aligned}&V(u) - V(u_0) + (d_0+1)\left(\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)|u|^2 dxdt\right)\\ &+\frac{3}{2}\int_0^T\int_0^\infty u_x^2 dxdt - \frac{1}{2}\int_0^T\int_0^\infty u^2 dxdt - \frac{1}{3}\int_0^T\int_0^\infty u^3 dxdt\\ &\qquad\qquad + \int_0^T\int_0^\infty xa(x)|u|^2 dxdt = 0.\end{aligned} \tag{48}$$

The next step is devoted to estimate the nonlinear term in the left hand side of (48). To do that, we first assume that $||u_0||_{L^2} \le 1$.

By (26) we have that

$$\int_0^\infty |u|^3 dx \le \varepsilon ||u_x||_{L^2}^2 + c_\varepsilon ||u||_{L^2}^{\frac{10}{3}}$$

for any $\varepsilon > 0$ and some constant $c_\varepsilon > 0$. Thus, if $||u_0||_{L^2} \le 1$, we have $||u||_{L^2}^{\frac{10}{3}} \le ||u||_{L^2}^2$ and

$$\int_0^T\int_0^\infty |u|^3 dxdt \le \varepsilon \int_0^T\int_0^\infty u_x^2 dxdt + c_\varepsilon \int_0^T\int_0^\infty u^2 dxdt. \tag{49}$$

Moreover, according to [20], there exists $c_1 > 0$, satisfying

$$\int_0^T\int_0^\infty u^2 dxdt \le c_1\{\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)u^2 dxdt\}. \tag{50}$$

Now, combining (48)-(50) and taking $\varepsilon < \frac{1}{2}$ and $d_0 := 2c_1(\frac{1}{2} + \frac{c_\varepsilon}{3})$ we obtain

$$\begin{aligned}&V(u(T)) - V(u_0) + \frac{d_0+1}{2}(\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)|u|^2 dxdt)\\ &\quad +(\frac{3}{2} - \frac{\varepsilon}{3})\int_0^T\int_0^\infty u_x^2 dxdt + \int_0^T\int_0^\infty xa(x)|u|^2 dxdt \le 0\end{aligned} \tag{51}$$

or

$$V(u(T)) - V(u_0) \le -\widetilde{c}\{\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty (x+1)a(x)|u|^2 dxdt + \int_0^T\int_0^\infty u_x^2 dxdt\} \tag{52}$$

where $\widetilde{c} > 0$. We aim to prove the existence of a constant $c > 0$ satisfying

$$V(u(T)) - V(u_0) \le -cV(u_0) \tag{53}$$

Indeed, such an inequality gives at once the decay $V(u(t)) \le ce^{-\nu t}V(u_0)$. To this end, we need to establish two claims.

CLAIM 4. There exists $c > 0$ such that

$$\int_0^T V(u)dt \le c\{\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty (x+1)a(x)u^2 dxdt\}.$$

Since $u_0 \in L^2_{(x+1)dx} \subset L^2$, from (4) and (50) we get

$$\begin{aligned}
\int_0^T V(u)dt &= \frac{1}{2}\int_0^T \int_0^\infty (x+1)u^2 dxdt + \frac{d_0}{2}\int_0^T \int_0^\infty u^2 dxdt \\
&\le \frac{c_1 d_0}{2}\{\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty a(x)u^2 dxdt\} \\
&\qquad + \frac{1}{2}\int_0^T \int_0^{x_0} (x+1)u^2 dxdt + \frac{1}{2}\int_0^T \int_{x_0}^\infty (x+1)u^2 dxdt \\
&\le \frac{c_1 d_0}{2}\{\frac{1}{2}\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty a(x)u^2 dxdt\} \\
&\qquad + \frac{1}{2}(x_0+1)\int_0^T \int_0^{x_0} u^2 dxdt + \frac{1}{2}\int_0^T \int_{x_0}^\infty (x+1)\frac{a(x)}{a_0}u^2 dxdt \\
&\le c\{\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty (x+1)a(x)u^2 dxdt\}.
\end{aligned}$$

CLAIM 5.

$$V(u_0) \le C(\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty (x+1)a(x)u^2 dxdt + \int_0^T \int_0^\infty u_x^2 dxdt) \tag{54}$$

where $C > 0$.

Multiplying the first equation in (1) by $(T-t)u$ and integrating by parts in $(0,\infty)\times(0,T)$, we obtain

$$\begin{aligned}
&\frac{T}{2}\int_0^\infty |u_0(x)|^2 dx = \\
&\frac{1}{2}\int_0^T \int_0^\infty |u|^2 dxdt + \int_0^T \int_0^\infty (T-t)a(x)|u|^2 dxdt + \frac{1}{2}\int_0^T (T-t)u_x^2(0,t)dt,
\end{aligned} \tag{55}$$

and therefore, using (50)

$$\int_0^\infty |u_0(x)|^2 dx \le C\left(\int_0^T \int_0^\infty a(x)|u|^2 dxdt + \int_0^T u_x^2(0,t)dt\right). \tag{56}$$

Now, multiplying by $(T-t)xu$, it follows that

$$\begin{aligned}
&-\frac{T}{2}\int_0^\infty x|u_0(x)|^2 dx + \frac{1}{2}\int_0^T \int_0^\infty x|u|^2 dxdt + \frac{3}{2}\int_0^T \int_0^\infty (T-t)u_x^2 dxdt \\
&-\frac{1}{2}\int_0^T \int_0^\infty (T-t)u^2 dxdt + \int_0^T \int_0^\infty (T-t)xa(x)|u|^2 dxdt - \\
&\qquad\qquad -\frac{1}{3}\int_0^T \int_0^\infty (T-t)u^3 dxdt = 0.
\end{aligned}$$

The identity above and (49) allow us to conclude that

$$\int_0^\infty x|u_0(x)|^2dx$$
$$\le C\{\int_0^T\int_0^\infty (x+1)|u|^2dxdt + \int_0^T\int_0^\infty u_x^2dxdt + \int_0^T\int_0^\infty xa(x)|u|^2dxdt+ \tag{57}$$
$$+\int_0^T\int_0^\infty |u|^3dxdt\} \le C\{\int_0^T V(u(t))dt + \int_0^T\int_0^\infty xa(x)u^2dxdt + \int_0^T\int_0^\infty u_x^2dxdt\}$$

for some $C>0$. Claim 5 follows from Claim 4 and (56)-(57). ■

The previous computations give us (53) (and the exponential decay) when $||u_0||_{L^2}\le 1$. The general case is proved as follows. Let $u_0\in L^2_{(x+1)dx}\subset L^2$ be such that $||u_0||_{L^2}\le R$. Since $u\in C(\mathbb{R}^+;L^2(\mathbb{R}^+))$ and $||u(t)||_{L^2}\le \alpha e^{-\beta t}||u_0||_{L^2}$, where $\alpha=\alpha(R)$ and $\beta=\beta(R)$ are positive constants, $||u(T)||_{L^2}\le 1$ if we pick T satisfying $\alpha e^{-\beta T}R<1$. Then, it follows from (48)-(26) and (53) that for some constants $\nu>0,\ c>0,\ C>0$

$$V(u(t+T))\le ce^{-\nu t}V(u(T))\le c(T||u_0||^2_{L^2}+T||u_0||^{\frac{10}{3}}_{L^2}+V(u_0))e^{-\nu t},$$

hence

$$V(u(t))\le Ce^{-\nu t}V(u_0),$$

where $C=C(R)$, which concludes the proof when $m=1$.

Induction Hypothesis: There exist $c>0$ and $\rho>0$ such that if $V_{m-1}(u_0)\le\rho$, we have

$$V_m(u)-V_m(u_0) \qquad (*)_m$$
$$\le -c\{\int_0^T u_x^2(0,t)dt+\int_0^T\int_0^\infty (x+1)^{m-1}u_x^2dxdt+\int_0^T\int_0^\infty (x+1)^m a(x)u^2dxdt\}$$
$$V_m(u_0) \qquad (**)_m$$
$$\le c\{\int_0^T u_x^2(0,t)dt+\int_0^T\int_0^\infty (x+1)^{m-1}u_x^2dxdt+\int_0^T\int_0^\infty (x+1)^m a(x)u^2dxdt\}.$$

By (52)-(54), the induction hypothesis is true for $m=1$. Pick now an index $m\ge 2$ and assume that $d_0,...,d_{m-2}$ have been constructed so that $(*)_k-(**)_k$ are fulfilled for $1\le k\le m-1$. We aim to prove that for a convenient choice of the constant d_{m-1} in (45), the properties $(*)_m-(**)_m$ hold true.

Let us investigate first $(*)_m$. We multiply the first equation in (1) by $(x+1)^m u$ to obtain

$$\begin{aligned}&V_m(u)-V_m(u_0)-d_{m-1}(V_{m-1}(u)-V_{m-1}(u_0))\\&-\frac{m(m-1)(m-2)}{2}\int_0^T\int_0^\infty (x+1)^{m-3}u^2dxdt+\frac{1}{2}\int_0^T u_x^2(0,t)dt\\&+\frac{3m}{2}\int_0^T\int_0^\infty (x+1)^{m-1}u_x^2dxdt-\frac{m}{2}\int_0^T\int_0^\infty (x+1)^{m-1}u^2dxdt\\&-\frac{m}{3}\int_0^T\int_0^\infty (x+1)^{m-1}u^3dxdt+\int_0^T\int_0^\infty (x+1)^m a(x)u^2dxdt=0.\end{aligned}\tag{58}$$

The next steps are devoted to estimate the terms in the above identity. First, combining (4) and (50) we infer the existence of a positive constant $c > 0$ such that

$$\begin{aligned}
&\int_0^T \int_0^\infty (x+1)^{m-1} u^2 dxdt \\
&= \int_0^T \int_0^{x_0} (x+1)^{m-1} u^2 dxdt + \int_0^T \int_{x_0}^\infty (x+1)^{m-1} u^2 dxdt \\
&\le (x_0+1)^{m-1} \int_0^T \int_0^\infty u^2 dxdt + \frac{1}{a_0} \int_0^T \int_0^\infty a(x)(x+1)^{m-1} u^2 dxdt \\
&\le c\,\{\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty (x+1)^{m-1} a(x) u^2 dxdt\} \\
&\le -c\,\{V_{m-1}(u) - V_{m-1}(u_0)\}
\end{aligned} \tag{59}$$

where we used $(*)_{m-1}$. In the same way

$$\begin{aligned}
&\int_0^T \int_0^\infty (x+1)^{m-3} u^2 dxdt \\
&\le \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dxdt \le -c\,\{V_{m-1}(u) - V_{m-1}(u_0)\}
\end{aligned} \tag{60}$$

where $c > 0$ is a positive constant. Moreover, assuming $V_{m-1}(u_0) \le \rho$ with $\rho > 0$ small enough (so that by exponential decay of $V_{m-1}(u(t))$ we have $\int_0^\infty (x+1)^{m-1} |u(x,t)|^2 dx \le 1$ for all $t \ge 0$) and proceeding as in the case $m = 1$, we obtain the existence of $\varepsilon > 0$ and $c_\varepsilon > 0$ satisfying

$$\begin{aligned}
&\int_0^T \int_0^\infty (x+1)^{m-1} |u|^3 dxdt \\
&\le \varepsilon \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dxdt + c_\varepsilon \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dxdt.
\end{aligned} \tag{61}$$

Indeed,

$$\begin{aligned}
&\int_0^\infty (x+1)^{m-1} |u|^3 dx \\
&\le ||u||_{L^\infty} \int_0^\infty (x+1)^{m-1} u^2 dx \le \sqrt{2} ||u_x||_{L^2}^{\frac{1}{2}} ||u||_{L^2}^{\frac{1}{2}} \int_0^\infty (x+1)^{m-1} u^2 dx \\
&\le \varepsilon \int_0^\infty (x+1)^{m-1} u_x^2 dx + c_\varepsilon \int_0^\infty u^2 dx + c_\varepsilon \left(\int_0^\infty (x+1)^{m-1} u^2 dx \right)^2.
\end{aligned} \tag{62}$$

Then, if we return to (58) and take $\varepsilon < 9/2$ and $d_{m-1} > 0$ large enough, from (59)-(61) if follows that

$$\begin{aligned}
&V_m(u) - V_m(u_0) \\
&\le -c\,\{\int_0^T u_x^2(0,t)dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dxdt + \int_0^T \int_0^\infty a(x)(x+1)^m u^2 dxdt\} \\
&+ \frac{d_{m-1}}{2} (V_{m-1}(u) - V_{m-1}(u_0)).
\end{aligned} \tag{63}$$

This yields $(*)_m$, by $(*)_{m-1}$. Let us now check $(**)_m$. It remains to estimate the terms in the right hand side of (63). We multiply the first equation in (1) by $(T-t)(x+1)^m u$ to obtain

$$\frac{T}{2}\int_0^\infty (x+1)^m u_0^2 dx = \frac{1}{2}\int_0^T\int_0^\infty (x+1)^m u^2 dxdt$$
$$-\frac{m(m-1)(m-2)}{2}\int_0^T\int_0^\infty (T-t)(x+1)^{m-3}u^2 dxdt + \frac{1}{2}\int_0^T (T-t)u_x^2(0,t)dt$$
$$+\frac{3m}{2}\int_0^T\int_0^\infty (T-t)(x+1)^{m-1}u_x^2 dxdt - \frac{m}{2}\int_0^T\int_0^\infty (T-t)(x+1)^{m-1}u^2 dxdt$$
$$-\frac{m}{3}\int_0^T\int_0^\infty (T-t)(x+1)^{m-1}u^3 dxdt + \int_0^T\int_0^\infty (T-t)(x+1)^m a(x)u^2 dxdt.$$

Then, proceeding as above, we deduce that

$$\int_0^T (x+1)^m u_0^2 dx$$
$$\leq c\{\int_0^T\int_0^\infty (x+1)^{m-1}u^2 dxdt + \int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty (x+1)^{m-1}u_x^2 dxdt$$
$$+\int_0^T\int_0^\infty (x+1)^m a(x)u^2 dxdt\}$$
$$\leq c\{\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty (x+1)^{m-1}u_x^2 dxdt + \int_0^T\int_0^\infty (x+1)^m a(x)u^2 dxdt\}.$$

Combined to $(**)_{m-1}$, this yields $(**)_m$. This completes the construction of the sequence $\{V_m\}_{m\geq 1}$ by induction.

Let us now check the exponential decay of V_m for $m \geq 2$. It follows from $(*)_m - (**)_m$ that

$$V_m(u) - V_m(u_0) \leq -c\, V_m(u_0)$$

where $c > 0$, which completes the proof when $V_{m-1}(u_0) \leq \rho$. The global result $(V_{m-1}(u_0) \leq R)$ is obtained as above for $m = 1$. ∎

Corollary 3.2 *Let $a = a(x)$ fulfilling (4) and $a \in W^{2,\infty}(0,\infty)$. Then for any $R > 0$, there exist positive constants $c = c(R)$ and $\mu = \mu(R)$ such that*

$$||u_x(t)||_{L^2(\mathbb{R}^+)} \leq c\frac{e^{-\mu t}}{\sqrt{t}}||u_0||_{L^2_{(x+1)dx}} \tag{64}$$

for all $t > 0$ and all $u_0 \in L^2_{(x+1)dx}$ satisfying $||u_0||_{L^2_{(x+1)dx}} \leq R$.

Proof. Pick any $R > 0$ and any $u_0 \in L^2_{(x+1)dx}$ with $||u_0||_{L^2_{(x+1)dx}} \leq R$. By Theorem 3.1 there are some constants $C = C(R)$ and $\nu = \nu(R)$ such that

$$||u(t)||_{L^2_{(x+1)dx}} \leq Ce^{-\nu t}||u_0||_{L^2_{(x+1)dx}}. \tag{65}$$

Using the multiplier $t(u^2 + 2u_{xx})$ we obtain after some integrations by parts that for all $0 < t_1 < t_2$

$$t_2\int_0^\infty u_x^2(x,t_2)dx + \int_{t_1}^{t_2} tu_x^2(0,t)dt + 2\int_{t_1}^{t_2}\int_0^\infty ta(x)u_x^2 dxdt + \int_{t_1}^{t_2} tu_{xx}^2(0,t)dt$$
$$= -\frac{1}{3}\int_{t_1}^{t_2}\int_0^\infty u^3 dxdt + \frac{t_2}{3}\int_0^\infty u^3(x,t_2)dx + \int_{t_1}^{t_2}\int_0^\infty tu^3 a(x)dxdt$$
$$+\int_{t_1}^{t_2}\int_0^\infty u_x^2 dxdt + \int_{t_1}^{t_2}\int_0^\infty ta''(x)u^2 dxdt. \tag{66}$$

1. Let us assume first that $T > 1$. Applying (66) on the time interval $[T-1, T]$, we infer that

$$\int_0^\infty |u_x(x,T)|^2 dx \le c\left(\int_{T-1}^T \int_0^\infty |u|^3 dxdt + ||u(T)||^3_{L^3(\mathbb{R}^+)} + \int_{T-1}^T ||u||^2_{H^1(\mathbb{R}^+)} dt\right). \tag{67}$$

To estimate the cubic terms in (67), we use (26) to obtain

$$\begin{aligned}\int_0^\infty |u_x(x,T)|^2 dx \le \varepsilon \int_0^\infty |u_x(x,T)|^2 dx \\ +c_\varepsilon\big(||u(T)||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)} + \int_{T-1}^T (||u||^2_{H^1(\mathbb{R}^+)} + ||u||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)}) dt\big).\end{aligned} \tag{68}$$

Note that by (65)

$$||u(T)||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)} \le (Ce^{-\nu T}||u_0||_{L^2_{(x+1)dx}})^{\frac{10}{3}} \le C^{\frac{10}{3}} R^{\frac{4}{3}} e^{-\nu T}||u_0||^2_{L^2_{(x+1)dx}}.$$

It follows from (48), (26), and (65) that

$$\begin{aligned}&\int_{T-1}^T (||u||^2_{H^1(\mathbb{R}^+)} + ||u||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)}) dt \\ &\quad \le C\left(V_1(u(T-1)) + \int_{T-1}^T \big(||u||^2_{L^2(\mathbb{R}^+)} + ||u||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)}\big) dt\right) \\ &\quad \le Ce^{-\nu T}||u_0||^2_{L^2_{(x+1)dx}}\end{aligned} \tag{69}$$

where $C = C(R,\nu)$. (64) for $T \ge 1$ follows from (68) and (69) by choosing $\varepsilon < 1$ and $\mu < \nu$.
2. Assume now that $T \le 1$. Estimating again the cubic terms in (66) (with $[t_1, t_2] = [0, T]$) by using (26), we obtain

$$\begin{aligned}T\int_0^\infty u_x^2(x,T)dx \;&\le\; \frac{T}{3}\left(\varepsilon||u_x(T)||^2_{L^2(\mathbb{R}^+)} + C_\varepsilon||u(T)||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)}\right) \\ &\quad + C_\varepsilon \int_0^T (||u||^2_{H^1(\mathbb{R}^+)} + ||u||^{\frac{10}{3}}_{L^2(\mathbb{R}^+)}) dt.\end{aligned} \tag{70}$$

By (48), (26) and (65), we have that

$$\int_0^1\int_0^\infty |u_x|^2 dxdt \le C(R)||u_0||^2_{L^2_{(x+1)dx}} \tag{71}$$

which, combined to (70) with $\varepsilon = 1$ and (65), gives

$$||u_x(T)||^2_{L^2(\mathbb{R}^+)} \le C(R)T^{-1}||u_0||^2_{L^2_{(x+1)dx}}$$

for all $T < 1$. This gives (64) for $T < 1$. ■
Corollary 3.2 may be extended (locally) to the weighted space $L^2_{(x+1)^m dx}$ ($m \ge 2$) in following the method of proof of [24, Theorem 1.1].

Corollary 3.3 *Let $a = a(x)$ fulfilling (4) and $m \ge 2$. Then there exist some constants $\rho > 0$, $C > 0$ and $\mu > 0$ such that*

$$||u(t)||_{H^1_{(x+1)^m dx}} \le C\frac{e^{-\mu t}}{\sqrt{t}}||u_0||_{L^2_{(x+1)^m dx}}$$

for all $t > 0$ and all $u_0 \in L^2_{(x+1)^m dx}$ satisfying $||u_0||_{L^2_{(x+1)^m dx}} \le \rho$.

Proof. We first prove estimates for the linearized problem

$$u_t + u_x + u_{xxx} + au = 0 \tag{72}$$

$$u(0,t) = 0 \tag{73}$$

$$u(x,0) = u_0(x) \tag{74}$$

and next apply a perturbation argument to extend them to the nonlinear problem (1). Let us denote by $W(t)u_0 = u(t)$ the solution of (72)-(74). By computations similar to those performed in the proof of Theorem 3.1, we have that

$$||W(t)u_0||_{L^2_{(x+1)^m dx}} \le C_0 e^{-\nu t} ||u_0||_{L^2_{(x+1)^m dx}}.$$

We need the

CLAIM 6. Let $k \in \{0, ..., 3\}$. Then there exists a constant $C_k > 0$ such that for any $u_0 \in H^k_{(x+1)^m dx}$,

$$||W(t)u_0||_{H^k_{(x+1)^m dx}} \le C_k e^{-\nu t} ||u_0||_{H^k_{(x+1)^m dx}}. \tag{75}$$

Indeed, if $u_0 \in H^3_{(x+1)^m dx}$, then $u_t(.,0) \in L^2_{(x+1)^{m-3} dx}$, and since $v = u_t$ solves (72)-(73), we also have that

$$||u_t(.,t)||_{L^2_{(x+1)^{m-3} dx}} \le C_0 e^{-\nu t} ||u_t(.,0)||_{L^2_{(x+1)^{m-3} dx}}.$$

Using (72), this gives

$$||W(t)u_0||_{H^3_{(x+1)^m dx}} \le C_3 e^{-\nu t} ||u_0||_{H^3_{(x+1)^m dx}}.$$

This proves (75) for $k = 3$. The fact that (75) is valid for $k = 1, 2$ follows from a standard interpolation argument, for $H^k_{(x+1)^m dx} = [H^0_{(x+1)^m dx}, H^3_{(x+1)^m dx}]_{\frac{k}{3}}$.

Lemma 3.4 *Pick any number $\mu \in (0, \nu)$. Then there exists some constant $C = C(\mu) > 0$ such that for any $u_0 \in L^2_{(x+1)^m dx}$*

$$||W(t)u_0||_{H^1_{(x+1)^m dx}} \le C \frac{e^{-\mu t}}{\sqrt{t}} ||u_0||_{L^2_{(x+1)^m dx}}. \tag{76}$$

Proof. Let $u_0 \in L^2_{(x+1)^m dx}$ and set $u(t) = W(t)u_0$ for all $t \ge 0$. By scaling in (72) by $(x+1)^m u$, we see that for some constant $C_K = C_K(T)$

$$||u||_{L^2(0,1;H^1_{(x+1)^m dx})} \le C_K ||u_0||_{L^2_{(x+1)^m dx}}.$$

This implies that $u(t) \in H^1_{(x+1)^m dx}$ for a.e. $t \in (0,1)$ which, combined to (75), gives that $u(t) \in H^1_{(x+1)^m dx}$ for all $t > 0$. Pick any $T \in (0,1]$. Note that, by (75),

$$||u(T)||_{H^1_{(x+1)^m dx}} \le C_1 e^{-\nu(T-t)} ||u(t)||_{H^1_{(x+1)^m dx}}, \quad \forall t \in (0,T). \tag{77}$$

Integrating with respect to t in (77) yields

$$[C_1^{-1} ||u(T)||_{H^1_{(x+1)^m dx}}]^2 \int_0^T e^{2\nu(T-t)} dt \le \int_0^T ||u(t)||^2_{H^1_{(x+1)^m dx}} dt,$$

and hence

$$\begin{aligned} ||u(T)||_{H^1_{(x+1)^m dx}} &\le C_K\, C_1 \sqrt{\frac{2\nu}{e^{2\nu T} - 1}} ||u_0||_{L^2_{(x+1)^m dx}} \\ &\le \frac{C_K\, C_1}{\sqrt{T}} ||u_0||_{L^2_{(x+1)^m dx}} \end{aligned}$$

for $0 < T \leq 1$. Therefore

$$(78) \qquad ||u(t)||_{H^1_{(x+1)^m dx}} \leq C_K\, C_1 e^{\nu} \frac{e^{-\nu t}}{\sqrt{t}} ||u_0||_{L^2_{(x+1)^m dx}} \qquad \forall t \in (0,1).$$

(76) follows from (78) and (75), since $\mu < \nu$. ■

Let us return to the proof of Corollary 3.3. Fix a number $\mu \in (0,\nu)$, where ν is as in (75), and let us introduce the space

$$F = \{u \in C(\mathbb{R}^+; H^1_{(x+1)^m dx}); \quad ||e^{\mu t} u(t)||_{L^\infty(\mathbb{R}^+; H^1_{(x+1)^m dx})} < \infty\}$$

endowed with its natural norm. Note that (1) may be recast in the following integral form

$$(79) \qquad u(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s))\, ds$$

where $N(u) = -uu_x$. We first show that (79) has a solution in F provided that $u_0 \in H^1_{(x+1)^m dx}$ with $||u_0||_{H^1_{(x+1)^m dx}}$ small enough. Let $u_0 \in H^1_{(x+1)^m dx}$ and $u \in F$ with $||u_0||_{H^1_{(x+1)^m dx}} \leq r_0$ and $||u||_F \leq R$, r_0 and R being chosen later. We introduce the map Γ defined by

$$(80) \qquad (\Gamma u)(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s))\, ds \qquad \forall t \geq 0.$$

We shall prove that Γ has a fixed point in the closed ball $B_R(0) \subset F$ provided that $r_0 > 0$ is small enough.

For the forcing problem

$$\begin{cases} u_t + u_x + u_{xxx} + au = f \\ u(0,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

we have the following estimate

$$\sup_{0\leq t\leq T} ||u(t)||^2_{L^2_{(x+1)^m dx}} + \int_0^T\int_0^\infty (x+1)^{m-1} u_x^2 dxdt \leq C\left(||u_0||^2_{L^2_{(x+1)^m dx}} + ||f||^2_{L^1(0,T;L^2_{(x+1)^m dx})}\right).$$

Let us take $f = N(u) = -uu_x$. Observe that for all $x > 0$

$$\begin{aligned} (x+1)u^2(x) &= \left|\int_0^\infty \frac{d}{dx}[(x+1)u^2(x)]dx\right| \\ &\leq C\left(\int_0^\infty (x+1)^m |u|^2 dx + \int_0^\infty (x+1)^{m-1}|u_x|^2 dx\right) \end{aligned}$$

whenever $m \geq 2$. It follows that for some constant $K > 0$

$$\begin{aligned} ||uu_x||^2_{L^2_{(x+1)^m dx}} &\leq ||(x+1)u^2||_{L^\infty(\mathbb{R}^+)} \int_0^\infty (x+1)^{m-1}|u_x|^2 dx \\ &\leq K||u||^4_{H^1_{(x+1)^m dx}}. \end{aligned}$$

Therefore, for any $T > 0$,

$$\sup_{0\leq t\leq T} ||(\Gamma u)(t)||^2_{L^2_{(x+1)^m dx}} + \int_0^T\int_0^\infty (x+1)^{m-1}|(\Gamma u)_x|^2 dxdt \leq C\left(||u_0||^2_{L^2_{(x+1)^m dx}} + (\int_0^T ||u(t)||^2_{H^1_{(x+1)^m dx}} dt)^2\right) < \infty.$$

Thus $\Gamma u \in C(\mathbb{R}^+, L^2_{(x+1)^m dx}) \cap L^2_{loc}(\mathbb{R}^+; H^1_{(x+1)^m dx})$ with $(\Gamma u)(0) = u_0$. We claim that $\Gamma u \in F$. Indeed, by (75),

$$||e^{\mu t} W(t) u_0||_{H^1_{(x+1)^m dx}} \le C_1 ||u_0||_{H^1_{(x+1)^m dx}}$$

and for all $t \ge 0$

$$\begin{aligned} ||e^{\mu t} \int_0^t W(t-s) N(u(s)) ds||_{H^1_{(x+1)^m dx}} &\le C e^{\mu t} \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{t-s}} ||N(u(s))||_{L^2_{(x+1)^m dx}} ds \\ &\le C \int_0^t \frac{e^{\mu s}}{\sqrt{t-s}} K (e^{-\mu s} ||u||_F)^2 ds \\ &\le CK ||u||_F^2 \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{s}} ds \\ &\le CK(2 + \mu^{-1}) ||u||_F^2 \end{aligned}$$

where we used Lemma 3.4. Pick $R > 0$ such that $CK(2 + \mu^{-1}) R \le \frac{1}{2}$, and r_0 such that $C_1 r_0 = \frac{R}{2}$. Then, for $||u_0||_{H^1_{(x+1)^m dx}} \le r_0$ and $||u||_F \le R$, we obtain that

$$||e^{\mu t} (\Gamma u)(t)||_{H^1_{(x+1)^m dx}} \le C_1 r_0 + CK(2 + \mu^{-1}) R^2 \le R, \quad t \ge 0.$$

Hence Γ maps the ball $B_R(0) \subset F$ into itself. Similar computations show that Γ contracts. By the contraction mapping theorem, Γ has a unique fixed point u in $B_R(0)$. Thus $||u(t)||_{H^1_{(x+1)^m dx}} \le Ce^{-\mu t} ||u_0||_{H^1_{(x+1)^m dx}}$ provided that $||u_0||_{H^1_{(x+1)^m dx}} \le r_0$ with r_0 small enough. Proceeding as in the proof of Lemma 3.4, we have that

$$||u(t)||_{H^1_{(x+1)^m dx}} \le C \frac{e^{-\mu t}}{\sqrt{t}} ||u_0||_{L^2_{(x+1)^m dx}} \qquad \text{for } 0 < t < 1,$$

provided that $||u_0||_{L^2_{(x+1)^m dx}} \le \rho_0$ with $\rho_0 < 1$ small enough. The proof is complete with a decay rate $\mu' < \mu$. ■

Corollary 3.5 *Assume that $a(x)$ satisfies (4) and that $\partial_x^k a \in L^\infty(\mathbb{R}^+)$ for all $k \ge 0$. Pick any $u_0 \in L^2_{(x+1)^m dx}$. Then for all $\varepsilon > 0$, all $T > \varepsilon$, and all $k \in \{1, ..., m\}$, there exists a constant $C = C(\varepsilon, T, k) > 0$ such that*

$$\int_\varepsilon^\infty (x+1)^{m-k} |\partial_x^k u(x,t)|^2 dx \le C ||u_0||^2_{L^2_{(x+1)^m dx}} \qquad \forall t \in [\varepsilon, T]. \tag{81}$$

Proof. The proof is very similar to the one in [18, Lemma 5.1] and so we only point out the small changes. First, it should be noticed that the presence in the KdV equation of the extra terms u_x and $a(x)u$ does not cause any serious trouble. On the other hand, choosing a cut-off function in x of the form $\eta(x) = \psi_0(x/\varepsilon)$ (instead of $\eta(x) = \psi_0(x - x_0 + 2)$ as in [18]) where $\psi_0 \in C^\infty(\mathbb{R}, [0, 1])$ satisfies $\psi_0(x) = 0$ for $x \le 1/2$ and $\psi_0(x) = 1$ for $x \ge 1$, allows to overcome the fact that u is a solution of (1) on the half-line only. ■

3.2 Decay in L^2_b

This section is devoted to the exponential decay in L^2_b. Our result reads as follows:

Theorem 3.6 *Assume that the function $a = a(x)$ satisfies (4) with $4b^3 + b < a_0$. Then, for all $R > 0$, there exist $C > 0$ and $\nu > 0$, such that*

$$||u(t)||_{L^2_b} \leq Ce^{-\nu t}||u_0||_{L^2_b} \qquad t \geq 0$$

for any solution u given by Theorem 2.3.

Proof. We introduce the Lyapunov function

$$V(u) = \frac{1}{2}\int_0^\infty u^2 e^{2bx}dx + c_b \int_0^\infty u^2 dx, \tag{82}$$

where c_b is a positive constant that will be chosen later. Then, adding (17) and (18) hand by hand we obtain

$$\begin{aligned} V(u) - V(u_0) = (4b^3 + b)&\int_0^T \int_{x_0}^\infty u^2 e^{2bx}dxdt + (4b^3+b)\int_0^T\int_0^{x_0} u^2 e^{2bx}dxdt \\ &- 3b\int_0^\infty\int_0^\infty u_x^2 e^{2bx}dxdt + \frac{2b}{3}\int_0^T\int_0^\infty u^3 e^{2bx}dxdt \\ &- (c_b + \frac{1}{2})\int_0^T u_x^2(0,t)dt - \int_0^T\int_0^\infty a(x)|u|^2(e^{2bx} + 2c_b)dxdt, \end{aligned} \tag{83}$$

where x_0 is the number introduced in (4). On the other hand, since $L^2_b \subset L^2_{(x+1)dx}$, $||u(t)||_{L^2(0,\infty)}$ and $||u_x(t)||_{L^2(0,\infty)}$ decays to zero exponentially. Consequently, from Moser estimate we deduce that $||u(t)||_{L^\infty(0,\infty)} \to 0$. We may assume that $(2b/3)||u(t)||_{L^\infty} < \varepsilon = a_0 - (4b^3 + b)$ for all $t \geq 0$, by changing u_0 into $u(t_0)$ for t_0 large enough. Therefore

$$\begin{aligned} &\frac{2b}{3}\int_0^T\int_0^\infty |u|^3 e^{2bx}dxdt \\ &\leq \frac{2b}{3}\int_0^T ||u(t)||_{L^\infty(0,\infty)}\left(\int_0^\infty |u|^2 e^{2bx}dx\right)dt \leq \varepsilon\int_0^T\int_0^\infty u^2 e^{2bx}dxdt. \end{aligned} \tag{84}$$

So, returning to (83), the following holds

$$\begin{aligned} &V(u) - V(u_0) - (4b^3 + b + \varepsilon)\int_0^T\int_0^{x_0} u^2 e^{2bx}dxdt \\ &+3b\int_0^T\int_0^\infty u_x^2 e^{2bx}dxdt + (c_b + \frac{1}{2})\int_0^T u_x^2(0,t)dt + 2c_b\int_0^T\int_0^\infty a(x)|u|^2 dxdt \leq 0. \end{aligned} \tag{85}$$

Moreover, according to [20] there exists $C > 0$ satisfying

$$\begin{aligned} &\int_0^T\int_0^{x_0} u^2 e^{2bx}dxdt \\ &\leq e^{2bx_0}\int_0^T\int_0^{x_0} u^2 dxdt \leq C\{\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)u^2 dxdt\} \end{aligned}$$

since $L^2_b \subset L^2(\mathbb{R}^+)$. Then, choosing c_b sufficiently large, the above estimate and (85) give us that

$$\begin{aligned} V(u) - V(u_0) \leq -C\{&\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)u^2 dxdt \\ &+ \int_0^T\int_0^\infty u_x^2 e^{2bx}dxdt\} \leq -C\,V(u_0), \end{aligned} \tag{86}$$

which allows to conclude that $V(u)$ decays exponentially. The last inequality is a consequence of the following results:

CLAIM 7. There exists a positive constant $C > 0$, such that

$$\int_0^T V(u(t))dt \le C\int_0^T\int_0^\infty u_x^2 e^{2bx}dxdt.$$

First, observe that

$$|\int_0^\infty u^2 e^{2bx}dx| = |-\frac{1}{b}\int_0^\infty uu_x e^{2bx}dx| \le \frac{1}{b}(\int_0^\infty u^2 e^{2bx}dx)^{\frac{1}{2}}(\int_0^\infty u_x^2 e^{2bx}dx)^{\frac{1}{2}},$$

therefore,

$$\int_0^\infty u^2 e^{2bx}dx \le \frac{1}{b^2}\int_0^\infty u_x^2 e^{2bx}dx. \tag{87}$$

Then, from (4) and (87) we have

$$V(u(t)) \le (\frac{1}{2}+c_b)\int_0^\infty u^2 e^{2bx}dx \le (\frac{1}{2}+c_b)b^{-2}\int_0^\infty u_x^2 e^{2bx}dx$$

which gives us Claim 7.

CLAIM 8.

$$V(u_0) \le C\{\int_0^T u_x^2(0,t)dt + \int_0^T\int_0^\infty u_x^2 e^{2bx}dxdt + \int_0^T V(u(t))dt\},$$

where C is a positive constant.

Multiplying the first equation in (1) by $(T-t)ue^{2bx}$ and integrating by parts in $(0,\infty)\times(0,T)$, we obtain

$$\begin{aligned} &-\frac{T}{2}\int_0^\infty |u_0(x)|^2 e^{2bx}dx + \frac{1}{2}\int_0^T\int_0^\infty |u|^2 e^{2bx}dxdt + 3b\int_0^T\int_0^\infty (T-t)u_x^2 e^{2bx}dxdt \\ &+\frac{1}{2}\int_0^T (T-t)u_x^2(0,t)dt - (4b^3+b)\int_0^T\int_0^\infty (T-t)u^2 e^{2bx}dxdt \\ &+\int_0^T\int_0^\infty (T-t)a(x)|u|^2 e^{2bx}dxdt - \frac{2b}{3}\int_0^T\int_0^\infty (T-t)u^3 e^{2bx}dxdt = 0 \end{aligned} \tag{88}$$

and therefore,

$$\begin{aligned} \int_0^\infty |u_0(x)|^2 e^{2bx}dx \le C(&\int_0^T u_x^2(0,t)dt + \frac{1}{2}\int_0^T\int_0^\infty u^2 e^{2bx}dxdt \\ &+ \int_0^T\int_0^\infty u_x^2 e^{2bx}dxdt + \int_0^T\int_0^\infty |u|^3 e^{2bx}dxdt). \end{aligned} \tag{89}$$

Then, combining (87) and (84), we derive Claim 8. (86) follows at once. This proves the exponential decay when $||u(t)||_{L^\infty} \le 3\varepsilon/(2b)$. The general case is obtained as in Theorem 3.1 ■

Corollary 3.7 *Assume that the function $a = a(x)$ satisfies (4) with $4b^3 + b < a_0$. Then for any $R > 0$, there exist positive constants $c = c(R)$ and $\mu = \mu(R)$ such that*

$$||u_x(t)||_{L_b^2} \le c\frac{e^{-\mu t}}{\sqrt{t}}||u_0||_{L_b^2} \tag{90}$$

for all $t > 0$ and all $u_0 \in L_b^2$ satisfying $||u_0||_{L_b^2} \le R$.

Corollary 3.8 *Assume that the function $a = a(x)$ satisfies (4) with $4b^3 + b < a_0$, and let $s \geq 2$. Then there exist some constants $\rho > 0$, $C > 0$ and $\mu > 0$ such that*

$$||u(t)||_{H_b^s} \leq C\frac{e^{-\mu t}}{t^{\frac{s}{2}}}||u_0||_{L_b^2}$$

for all $t > 0$ and all $u_0 \in L_b^2$ satisfying $||u_0||_{L_b^2} \leq \rho$.

The proof of Corollary 3.7 (resp. 3.8) is very similar to the proof of Corollary 3.2 (resp. 3.3), so it is omitted.

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