Tangential symmetries of Darboux integrable systems

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Abstract

In this paper we analyze the tangential symmetries of Darboux integrable decomposable exterior differential systems. Our definition of a decomposable exterior differential system is a generalization of the hyperbolic systems defined in [3, §1.1]. This generalization of hyperbolic exterior differential systems includes the classic notion of Darboux integrability for first order systems and second order scalar equations. For Darboux integrable systems the general solution can be found by integration (solving ordinary differential equations). We show that this property holds for our generalized systems as well.

We give a geometric construction of the Lie algebras of tangential symmetries associated to the Darboux integrable systems. This construction has the advantage over previous constructions [14, 15], [12, 13] that our construction does not require the use of adapted coordinates and works for arbitrary dimension of the underlying manifold. In particular it works for the prolongations of decomposable exterior differential systems.

Independently of the author, Anderson, Fels and Vassiliou [1] have developed a theory similar to the one described in this article. Their method has a more algebraic flavor and uses adapted coframes. Their approach allows the use of symbolic software to calculate numerous examples of Darboux integrable systems.

Introduction

Darboux integrability of second order scalar equations was introduced by Gaston Darboux [5, 6]. Generalizations to many other systems are known, such as first order systems [13], hyperbolic exterior differential systems [3] and elliptic equations [10].

An important step in the classification of Darboux integrable equations was given by Vessiot in the papers [14, 15]. Vessiot associates to each second order equation Darboux integrable at order 2 a Lie algebra of dimension 3 or less. He then uses the classification of 3-dimensional Lie algebras and some further computations to give a complete classification of the Darboux integrable equations up to contact transformations. Vassiliou [12, 13] gives a more geometric interpretation of the Lie algebras, which he calls Lie algebras of tangential symmetries. The Lie algebras of Vessiot and Vassiliou are constructed using calculations in local coordinates.

In this article we introduce the concept of decomposable exterior differential systems. The decomposable systems include all classical Darboux integrable equations as well as hyperbolic exterior differential systems. We give a geometric construction of the Lie algebras of tangential symmetries. This geometric construction works for all dimensions of the systems. This is in contrast with the theory developed by Vessiot and Vassiliou who only give their results for specific dimensions.

The geometric construction has some limitations as well. The constructions requires solving ordinary differential equations, which makes it difficult to perform the construction in concrete examples. For small dimensions Vessiot used the classification of finite-dimensional Lie algebras to classify the Darboux integrable equations. There are two reasons why this approach fails for more our more general systems, but even for equations that are Darboux integrable at higher order. The first reason is that there is no tractable classification of general Lie algebras of high dimension. The second reason is that there is no explicit construction of a Darboux integrable system from a given finite-dimensional Lie algebra.

In this paper we assume that, unless stated otherwise, all objects are smooth and (if applicable) of constant rank. If it is necessary to restrict to open neighborhoods to make our constructions, then we will not always mention this explicitly. Part of the results in this paper have been published already, with proofs and more examples, in [8, Chapter 10].

1 Decomposable exterior differential systems

In this section we give the definition of Darboux integrability for decomposable exterior differential systems. In Section 1.1 we give a brief review of distributions, in Section 1.2 we give the main definition of Darboux integrable decomposable systems and finally in section Section 1.5 we compare our definition to the classic definitions and give some examples.

1.1 Distributions

The objects dual to (constant-rank) Pfaffian systems are distributions. In this paper we will assume, unless stated otherwise, that all distributions are smooth and locally of constant rank.

Definition 1.1. A *distribution* on a smooth manifold M of rank k is a subbundle of the tangent bundle TM of rank k.

The distribution spanned by the vector fields $X_1, \ldots, X_n$ is denoted by $\mathrm{span}(X_1, \ldots, X_n)$.

For a distribution $\mathcal{V}$ and a vector field X we say that X is *contained in* $\mathcal{V}$ *pointwise* (or just *contained in* $\mathcal{V}$) and write $X \subset \mathcal{V}$ if $X_m \in \mathcal{V}_m$ for all points m. If X is not contained in $\mathcal{V}$ this means that there exists a point m such that $X_m \notin \mathcal{V}_m$. This does not imply that $X_m \notin \mathcal{V}_m$ for all points m. We will say that X is *pointwise not contained* in $\mathcal{V}$ if the stronger statement, that $X_m \notin \mathcal{V}_m$ for all m, holds.

Invariants An *invariant* for a distribution $\mathcal{V}$ is a function I on M such that $X(I) = 0$ for all $X \subset \mathcal{V}$. This is equivalent to $\mathcal{V} \subset \ker(\mathrm{d}I)$. Classically, the invariants of a distribution are called *first integrals* [9, p. 95], [11, pp. 10, 289]. We say that k invariants $I^1, \ldots, I^k$ are *functionally independent* at a point x if the rank of the Pfaffian system $\mathrm{span}(\mathrm{d}I^1, \ldots, \mathrm{d}I^k)$ is equal to k at x. By the Frobenius theorem an integrable rank k distribution on an n-dimensional manifold has locally precisely $n - k$ functionally independent invariants.

Projections and lifting Let $\phi : M \to B$ be a smooth map. If ϕ is a diffeomorphism, then we can define the push forward $\phi_* X$ of a vector field X at $y = \phi(x)$ as $(\phi_* X)_y = (T_x\phi)X_x$. Locally we can define the push forward of a vector field under an immersion in the same way. If ϕ is a smooth map, then in general there is no push forward of a vector field X. If for all points x^1, x^2 with $\phi(x^1) = \phi(x^2)$ the vectors $(T_{x^1}\phi)(X)$ and $(T_{x^2}\phi)(X)$ are equal, we say that X *projects down* to B and we write $\phi_* X$ for the projected vector field. In a similar way, we can project distributions $\mathcal{V}$ on M to B if for all points x in the fiber $\phi^{-1}(y)$ the image $(T_x\phi)(\mathcal{V})$ is equal to a fixed linear subspace $\mathcal{W}_{\phi(x)}$ of $T_{\phi(x)}B$.

Example 1.2. Let $\phi : \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the first component. On $\mathbb{R}^2$ take coordinates x, y and define the vector fields

$$X = x\partial_x, \quad Y = x\partial_x + y\partial_y, \quad Z = (1 + y^2)\partial_x.$$

Under the map ϕ the vector fields X and Y project to the base manifold, the vector field Z does not project. The bundle $\mathcal{Z} = \mathrm{span}(Z)$ does project to $\mathbb{R}$.

Lemma 1.3 (Lie brackets of projected vector fields)**.** *Let $\pi : M \to B$ be a smooth map. Let X, Y be two vector fields on M that project to vector fields $\tilde{X} = \pi_* X$ and $\tilde{Y} = \pi_* Y$ on B, respectively. Then the commutator $[X, Y]$ projects down to B and $\pi_*[X, Y] = [\tilde{X}, \tilde{Y}]$.*

1.2 Darboux integrable decomposable exterior differential systems

Definition 1.4. Let M be a manifold of dimension $n = s+k+l$ with an exterior differential system $\mathcal{I}$ such that locally there exists a coframing of M of the form $\theta^1, \dots, \theta^s$, $\omega^1, \dots, \omega^k$, $\omega^{k+1}, \dots, \omega^{k+l}$ such that $\mathcal{I}$ is generated algebraically by the forms

$$\theta^1, \dots, \theta^s, \tag{1a}$$

$$\omega^i \wedge \omega^j, \quad 1 \leq i, j \leq k, \tag{1b}$$

$$\omega^\alpha \wedge \omega^\beta, \quad k+1 \leq \alpha, \beta \leq k+l. \tag{1c}$$

We call $\mathcal{M} = (M, \mathcal{I})$ a *decomposable exterior differential system* or a *decomposable system*. We define $I = \operatorname{span}(\theta^1, \dots, \theta^s)$ and $\mathcal{V} = I^\perp$. A coframing satisfying the structure equations (1) is called an *admissible local coframing*.

Note that for $k = l = 2$ this definition corresponds to the definition in [3, p. 29]. We say the system has *(extended) class* (s, k, l). The systems in [3] are all of class $(s, 2, 2)$ for $s \geq 0$.

Every decomposable system invariantly defines two distributions

$$\begin{aligned} \mathcal{F} &= \left(\operatorname{span}\left(\theta^1, \dots, \theta^s, \omega^{k+1}, \dots, \omega^{k+l}\right)\right)^\perp, \\ \mathcal{G} &= \left(\operatorname{span}\left(\theta^1, \dots, \theta^s, \omega^1, \dots, \omega^k\right)\right)^\perp. \end{aligned} \tag{2}$$

This motivates the following alternative definition of a decomposable exterior differential system. [1]

Definition 1.5. Let M be an n-dimensional manifold with two distributions $\mathcal{F}, \mathcal{G}$. We call the triple $\mathcal{M} = (M, \mathcal{F}, \mathcal{G})$ *decomposable system* on M if $\mathcal{F}, \mathcal{G}$ are of constant rank and the following conditions hold:

1. The intersection of $\mathcal{F}$ and $\mathcal{G}$ is empty.
2. $[\mathcal{F}, \mathcal{G}] \equiv 0 \mod \mathcal{F} \oplus \mathcal{G}$.

It is not difficult to check that with the correspondence (2) the two definitions 1.4 and 1.5 are locally equivalent. Globally the definitions are equivalent if we allow to interchange the two distributions $\mathcal{F}$ and $\mathcal{G}$. The last condition in Definition 1.5 above corresponds to the corresponding exterior differential system being closed.

The distributions $\mathcal{F}, \mathcal{G}$ are called the *characteristic systems* of the decomposable system. We denote by $n_\mathcal{F}$ and $n_\mathcal{G}$ the number of invariants of the bundles $\mathcal{F}$ and $\mathcal{G}$, respectively. We define $\mathcal{V} = \mathcal{F} \oplus \mathcal{G}$. In terms of exterior differential systems $\mathcal{V} = I^\perp$. If the distribution $\mathcal{V}$ has invariants, then the completion of $\mathcal{V}$ is an integrable distribution and we can restrict our structures to the leaves of $\operatorname{compl}(\mathcal{V})$. The leaves of the integrable distribution $\operatorname{compl}(\mathcal{V})$ each carry the

[1] Anderson calls $\mathcal{F}^\perp$ and $\mathcal{G}^\perp$ the *singular Pfaffian systems.*

structure of a decomposable system. For this reason we will assume from here on that $\mathcal{V}$ has no invariants.

The Darboux integrability of a decomposable system is determined by the invariants of the characteristic systems.

Definition 1.6. Let $(M, \mathcal{F}, \mathcal{G})$ be a decomposable system without invariants. The system is *Darboux integrable* if $\mathcal{F}$ has at least $\mathrm{rank}(\mathcal{G})$ functionally independent invariants $I^1, \dots I^{\mathrm{rank}(\mathcal{G})}$ and $\mathcal{G}$ has at least $\mathrm{rank}(\mathcal{F})$ functionally independent invariants $J^1, \dots J^{\mathrm{rank}(\mathcal{F})}$ such that $\mathrm{span}(\mathrm{d}I^1, \dots, \mathrm{d}I^{\mathrm{rank}(\mathcal{G})}, \mathrm{d}J^1, \dots, \mathrm{d}J^{\mathrm{rank}(\mathcal{F})}) \cap I = 0$.

In the case that the decomposable system is given by the contact structure of a system of partial differential equations, then the Darboux integrability corresponds to the classical notion of Darboux integrability. One of the main properties of Darboux integrable equations, namely the construction of integral manifolds by integration (see Section 1.4), is also present for our Darboux integrable hyperboloc systems. The definition above corresponds to the definition of Darboux integrability in [3] in the case that $\mathrm{rank}(\mathcal{F}) = \mathrm{rank}(\mathcal{G}) = 2$. In Section 2 we will see that the Darboux integrability property leads to a very rigid structure on the manifold.

Example 1.7. Let $M = \mathbb{R}^2$ with coordinates x, y. Then we can take $\mathcal{F}$ to be spanned by the vector field ∂_x and $\mathcal{G}$ spanned by ∂_y. The bundle $\mathcal{F}$ has y as an invariant, the bundle $\mathcal{G}$ has x as an invariant. The triple $(M, \mathcal{F}, \mathcal{G})$ is a Darboux integrable decomposable system.

In fact for every direct product $M_1 \times M_2$ the distributions $\mathcal{F} = TM_1$ and $\mathcal{G} = TM_2$ define a Darboux integrable decomposable system on $M_1 \times M_2$.

In this paper we will require the slightly more restrictive condition that the number of invariants for each of the bundles $\mathcal{F}, \mathcal{G}$ is *equal* to the rank of the other bundle. This together with the assumption of no invariants leads to the following 2 conditions.

$$\dim \mathrm{compl}(\mathcal{V}) = \dim M \quad \text{(no invariants)}, \tag{3a}$$

$$n_{\mathcal{F}} = \mathrm{rank}(\mathcal{G}) \quad \text{and} \quad n_{\mathcal{G}} = \mathrm{rank}(\mathcal{F}). \tag{3b}$$

In the case that $n_{\mathcal{F}} > \mathrm{rank}(\mathcal{G})$ or $n_{\mathcal{G}} > \mathrm{rank}(\mathcal{F})$ we can still carry out the constructions described below, but with some technical modifications.

Given a Darboux integrable system $\mathcal{F}$, $\mathcal{G}$ there is a natural projection onto the space of invariants. The completions of $\mathcal{F}$ and $\mathcal{G}$ are integrable and hence they define a foliation of M of codimension $n_{\mathcal{F}}$ and $n_{\mathcal{G}}$, respectively.

Definition 1.8. Let $(M, \mathcal{F}, \mathcal{G})$ be a decomposable system satisfying (3). Locally define B_1 to be the quotient of M by the completion of $\mathcal{G}$ and B_2 to be the quotient of M by the completion of $\mathcal{F}$. Let π_1 and π_2 be the projection of M on B_1 and B_2, respectively. The projection $\pi = \pi_1 \times \pi_2 : M \to B = B_1 \times B_2$ is a natural projection onto the manifold $B = B_1 \times B_2$ of dimension $n_{\mathcal{G}} + n_{\mathcal{F}}$. We call such a projection a *Darboux projection*.

The tangent spaces to the fibers of the Darboux projection are equal to the integrable distribution $\mathcal{Z} = \mathrm{compl}(\mathcal{F}) \cap \mathrm{compl}(G) = \ker \pi$. We write $\mathfrak{z}$ for the Lie algebra of vector fields tangent to the projection. The vector fields in $\mathfrak{z}$ are precisely the vector fields in the distribution $\mathcal{Z}$. The invariants of $\mathcal{F}$ and $\mathcal{G}$ are precisely the functions in $\pi_1^*(C^\infty(B_1))$ and $\pi_2^*(C^\infty(B_2))$, respectively.

There is a natural isomorphism of $T(B_1 \times B_2)$ with $TB_1 \otimes TB_2$. Using this identification we have the following lemma.

Lemma 1.9. *The distributions $\mathcal{F}$ and $\mathcal{G}$ project to B. The image of $\mathcal{F}$ is equal to $TB_1 \times \{0\}$ and the image of $\mathcal{G}$ is equal to $\{0\} \times TB_2$.*

Proof. The condition in (3b) implies that $\dim B = \mathrm{rank}\, \mathcal{V}$. The together with the fact that $\mathcal{V}$ has no invariants implies that the projection of $\mathcal{V}$ is onto TB. Since $\mathcal{F}$ is contained in compl($\mathcal{F}$) and B_2 is defined locally as the foliation of M by the leaves of the completion of $\mathcal{F}$, the projection of $\mathcal{F}$ is contained in $TB_1 \times \{0\}$. Since $\mathcal{F}$ has rank $n_\mathcal{G}$ and $\mathcal{F}$ is transversal to the projection π (since $\mathcal{V}$ is transversal), the image of $\mathcal{F}$ under $T_m\pi$ has rank $n_\mathcal{G}$ and must be equal to $TB_1 \times \{0\}$. For $\mathcal{G}$ a similar argument works. □

Since the bundles $\mathcal{F}$ and $\mathcal{G}$ project nicely onto B, we can lift vectors and vector fields on B to vectors and vector fields on M. Another way of saying this is that $\mathcal{V} = \mathcal{F} \oplus \mathcal{G}$ provides a connection for the bundle $M \to B$. We have

$$\begin{aligned} TM &= \mathrm{compl}(\mathcal{F}) \oplus \mathcal{G} = \mathcal{F} \oplus \mathrm{compl}(\mathcal{G}) \\ &= \mathcal{F} \oplus (\mathrm{compl}(\mathcal{F}) \cap \mathrm{compl}(\mathcal{G})) \oplus \mathcal{G} \\ &= \mathcal{F} \oplus \mathcal{Z} \oplus \mathcal{G}. \end{aligned}$$

1.3 Integral elements and prolongations

Definition 1.10. Let $\mathcal{M} = (M, \mathcal{F}, \mathcal{G})$ be a decomposable system. We define a 2-dimensional linear subspace E of $\mathcal{V}_m$ to be an *integral element* of $\mathcal{M}$ if $\dim E \cap \mathcal{F}_m = \dim E \cap \mathcal{G}_m = 1$.

For a decomposable system an integral element in the sense of Definition 1.10 above corresponds to the definition of an integral element in the ordinary sense of the corresponding exterior differential system $\mathcal{I}$ [2]. The space of 2-dimensional integral elements of a decomposable system has a very simple structure.

Lemma 1.11. *Let $(M, \mathcal{F}, \mathcal{G})$ be a decomposable system. The map*

$$\mathbb{P}\mathcal{F} \oplus \mathbb{P}\mathcal{G} \to \mathrm{Gr}_2(TM) : (f, g) \mapsto f + g$$

defines an isomorphism from $\mathbb{P}\mathcal{F} \oplus \mathbb{P}\mathcal{G}$ to the space of 2-dimensional integral elements of the decomposable system.

For our decomposable systems we can define prolongations in a similar way to [3, §1.3]. Recall that a decomposable system is of class (s, k, l) if $\dim M = s + k + l$, $\mathrm{rank}\, \mathcal{F} = k$ and $\mathrm{rank}\, \mathcal{G} = l$. The prolongation of a system of class (s, k, l) is a decomposable system of class $(n + (k-1) + (l-1), k, l)$. For $k = l = 2$ this corresponds to [3, Proposition, p. 53].

1.4 Lifting solutions

We define an integral manifold (a "solution") of a decomposable system $(M, \mathcal{F}, \mathcal{G})$ to be a 2-dimensional submanifold S of M such that for all points $s \in S$ the tangent space $T_s S$ is an integral element of the system. The integral manifolds of a decomposable system are precisely the 2-dimensional integral manifolds of the bundle $\mathcal{V} = \mathcal{F} \oplus \mathcal{G}$ that satisfy the non-degeneracy condition that at each point the tangent space of the integral manifold intersected with both $\mathcal{F}$ and $\mathcal{G}$ is non-empty. In this section we will show that for a Darboux integrable decomposable system the integral manifolds can be found by solving *ordinary differential equations*. In contrast, for a general decomposable system the integral manifolds are the solutions of a system of *partial differential equations*. In the case $k = l = 2$ this system of partial differential equations is decomposable (in the sense of partial differential equations), hence the name *decomposable* systems.

Assume $(M, \mathcal{F}, \mathcal{G})$ is a Darboux integrable decomposable system satisfying (3). Let $\pi : M \to B$ be the Darboux projection of the system. We can parameterize the integral manifolds of this system in the following way. Select a curve γ_1 in B_1 and a curve γ_2 in B_2. The product of these two curves is a surface U in B. Let $S = \pi^{-1}(U)$ be the inverse image of U. The distribution $\mathcal{V}$ restricts on the inverse image S to an integrable distribution $\mathcal{W}$ of rank 2. The leaves of this distribution are integral manifolds of the decomposable system. Since $\mathcal{W}$ is integrable, finding the integral manifolds can be done by solving ordinary differential equations. The integral manifolds obtained in this way depend on two arbitrary functions of $k-1$ and $l-1$ variables (to determine the curves γ_1 and γ_2) and s integration constants. It is not difficult to show that locally every integral manifold of the decomposable system is given in this way as the *lift* of two curves γ_1, γ_2.

1.5 Examples

Hyperbolic exterior differential systems are discussed in detail in [3, 4]. It is shown that the equation manifolds with the contact structure of Monge-Ampère equations, first order systems and second order scalar equations can all be realized as decomposable exterior differential systems of class $s = 1$, 2 and 3, respectively, see [3, Example 5, 6, 7]. The solutions of the equations correspond to the integral manifolds of the decomposable exterior differential systems. All Darboux integrable equations in the classes mentioned above provide examples of Darboux integrable decomposable systems.

Vassiliou defines in [13, Definition 2.10] the concept of a manifold of (p, q)-decomposable type. In terms of our Definition 1.5 a manifold of (p, q)-decomposable type is a decomposable exterior differential system without invariants, but with the additional conditions that $\mathcal{F}$ and $\mathcal{G}$ are non-integrable and

$$\operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{G} = 2, \ \dim M \geq 6, \ n_{\mathcal{F}} = p, \ n_{\mathcal{G}} = q. \tag{4}$$

Condition 2b) in the definition of Vassiliou corresponds to the assumption of no invariants. Under the additional assumptions (4) one can easily show that condition 2c) corresponds to condition 2. The final condition 2d) defines the number of invariants of the distributions $\mathcal{F}$ and $\mathcal{G}$. The main case in the work of Vassiliou are the manifolds of $(2,2)$-decomposable type and these correspond to our definition of a Darboux integrable decomposable system.

Example 1.12. Consider the equation manifold associated to the Liouville equation $z_{xy} = \exp(z)$. For the equation manifold in the second order jet bundle we use the coordinates x, y, z, $p = z_x$, $q = z_y$, $r = z_{xx}$, $t = z_{yy}$. On this manifold M we have two natural distributions defined by the characteristic systems. In coordinates x, y, z, p, q, r, t we have

$$\begin{aligned} \mathcal{F} &= \operatorname{span}\left(\partial_x + p\partial_z + r\partial_p + \exp(z)\partial_q + q\exp(z)\partial_t, \partial_r\right), \\ \mathcal{G} &= \operatorname{span}\left(\partial_x + q\partial_z + \exp(z)\partial_p + t\partial_q + p\exp(z)\partial_r, \partial_t\right). \end{aligned}$$

The contact structure on the equation manifold is given by $\mathcal{V} = \mathcal{F} \oplus \mathcal{G}$. The bundle $\mathcal{F}$ has two invariants $y, t - q^2/2$ and the bundle $\mathcal{G}$ has two invariants $x, r - p^2/2$. The triple $(M, \mathcal{F}, \mathcal{G})$ is a Darboux integrable decomposable system. Integration of this system leads to the general solution of the Liouville equation

$$z(x, y) = \ln\left(\frac{2\phi'(x)\psi'(y)}{(\phi(x) + \psi(y))^2}\right).$$

Here ϕ, ψ are 2 arbitrary functions.

Finally we note that for many elliptic systems of partial differential equations we can define the equivalent notion of an “elliptic exterior differential system”. For these systems we can also define the Darboux integrability property and construct solutions using ordinary integration. See [8, §8.1.4] for more details.

2 Lie algebras of tangential symmetries

In this section we will define and construct the tangential symmetries of Darboux integrable systems. The constructions here are *local* and work in the smooth setting. We will assume all decomposable systems have no invariants. We start in Section 2.1 with some results on reciprocal Lie algebras necessary later on.

Definition 2.1. Let $(M, \mathcal{F}, \mathcal{G})$ be a Darboux integrable decomposable system with Darboux projection $\pi : M \to B_1 \times B_2$. We define the *tangential symmetries* of $\mathcal{F}$ and $\mathcal{G}$ as the space of all vector fields in $\mathfrak{z}$ that are symmetries of the distributions $\mathcal{F}$ and $\mathcal{G}$, respectively. We write $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$ for the tangential symmetries of $\mathcal{G}$ and $\mathcal{F}$, respectively. The vector fields in $\mathfrak{z}$ that are symmetries of both $\mathcal{F}$ and $\mathcal{G}$ are called *tangential symmetries* of the decomposable system.

The name tangential characteristic symmetries was introduced by Vassiliou [13]. The main results of this paper are the two theorems below. They are both proved by the constructions in Section 2.2.

Theorem 2.2. *Let $(M, \mathcal{F}, \mathcal{G})$ be a Darboux integrable decomposable system with projection $\pi : M \to B_1 \times B_2$. The space of tangential symmetries of $\mathcal{F}$ ($\mathcal{G}$) is a finite-dimensional module over $\pi_1^*(C^\infty(B_1))$ (over $\pi_1^*(C^\infty(B_2))$) of dimension s.*

Theorem 2.3 (Main theorem)**.** *Let $(M, \mathcal{F}, \mathcal{G})$ be a Darboux integrable decomposable system with projection $\pi : M \to B_1 \times B_2$.*

There exist finite-dimensional Lie algebras of vector field L, R on M tangential to the projection π such that L and R commute and the elements in L and R are tangential symmetries of $\mathcal{G}$ and $\mathcal{F}$, respectively.

2.1 Reciprocal Lie algebras

Let M be a smooth manifold. We denote the space of smooth vector fields on M by $\mathcal{X}(M)$. Let $\mathfrak{g}$ be an n-dimensional Lie algebra and let $\alpha : \mathfrak{g} \to \mathcal{X}(M)$ be a representation of $\mathfrak{g}$ in the space of vector fields on M. In the theory to be developed below we will work locally and M will often be of the same dimension as $\mathfrak{g}$, so we can think of M as an open subset of $\mathbb{R}^n$. We say the representation is *locally transitive* if $\alpha(\mathfrak{g})$ locally generates as a $C^\infty(M)$-module the space of vector fields $\mathcal{X}(M)$. If $\dim M = \dim \mathfrak{g} = n$, then a transitive representation defines an injective map $\mathfrak{g} \to \mathcal{X}(M)$ and we can identify $\mathfrak{g}$ with its representation as vector fields on M.

A Lie algebra of vector fields on M is a Lie subalgebra of $\mathcal{X}(M)$. For any Lie algebra $\mathfrak{g}$ of vector fields on M and a point m in M we can define the *evaluation map*

$$\mathrm{ev}(\mathfrak{g})_m : \mathfrak{g} \to T_m M : X \mapsto X(m). \tag{5}$$

A Lie algebra $\mathfrak{g}$ of vector fields on M is *locally transitive* at $m \in M$ if and only if the evaluation map $\mathrm{ev}(\mathfrak{g})_m$ is a linear isomorphism from $\mathfrak{g}$ onto $T_m M$. From here on we will identify a Lie algebra with its representation as vector fields whenever the representation is injective.

The following theorem and lemma are both proved in [8, pp. 242–243].

Theorem 2.4. *Let M be a smooth n-dimensional manifold. Let $\mathfrak{g}$ be an n-dimensional locally transitive Lie subalgebra of $\mathcal{X}(M)$. Then the centralizer $\mathfrak{h}$ of $\mathfrak{g}$ is an n-dimensional locally transitive Lie algebra of vector fields. The Lie algebra $\mathfrak{g}$ is anti-isomorphic with $\mathfrak{h}$ in the sense that for every point $x \in M$ the invertible linear map $\alpha_x = (\mathrm{ev}(\mathfrak{h})_x)^{-1} \circ \mathrm{ev}(\mathfrak{g})_x$ is a Lie algebra anti-homomorphism. In particular for all vector fields $X, Y \in \mathfrak{g}$ we have $\alpha_x([X, Y]) = -[\alpha_x(X), \alpha_x(Y)]$.*

We call a pair of commutative, locally transitive Lie algebras a pair of *reciprocal Lie algebras*. For a pair of reciprocal Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ the center of $\mathfrak{g}$ is equal to the center of $\mathfrak{h}$.

Example 2.5 (Reciprocal Lie algebra). Consider the affine Lie algebra $\mathfrak{aff}(1)$, represented by the two vector fields

$$e_1 = \partial_{x^1} - x^2 \partial_{x^2}, \quad e_2 = \partial_{x^2}.$$

Then the reciprocal Lie algebra $\mathfrak{h}$ is generated by

$$f_1 = \partial_{x^1}, \quad f_2 = \exp(-x^1)\partial_{x^2}.$$

Note that $[e_1, e_2] = e_2$ and $[f_1, f_2] = -f_2$ so the structure constants for both Lie algebras are related by a minus sign. The Lie algebras $\mathfrak{aff}(1)$ and $\mathfrak{h}$ commute in the sense that $[e_i, f_j] = 0$, $1 \leq i, j \leq 2$.

We will end this section with a lemma on reciprocal Lie algebras. We will use this lemma in Section 2.2.

Lemma 2.6. *Let M be a smooth connected manifold with Lie algebras of vector fields $\mathfrak{g}$, $\mathfrak{h}$. We assume that i) $\mathfrak{g}$ and $\mathfrak{h}$ commute, i.e., for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ we have $[X, Y] = 0$ and ii) for all $x \in M$ the evaluation maps $\mathrm{ev}(\mathfrak{g})_x : \mathfrak{g} \to T_x M$ and $\mathrm{ev}(\mathfrak{h})_x : \mathfrak{h} \to T_x M$ are surjective. Then for all $x \in M$ the maps $\mathrm{ev}(\mathfrak{g})_x$ and $\mathrm{ev}(\mathfrak{h})_x$ are injective and hence $\mathfrak{g}$, $\mathfrak{h}$ are reciprocal Lie algebras of dimension equal to the dimension of M.*

The Lie algebra of a Lie group G is equal to the tangent space at the identity element $\mathfrak{g} = T_e G$. The Lie algebra can be identified with both the space of left-invariant vector fields on G and the space of right-invariant vector fields on G. The spaces of left- and right-invariant vector fields on a Lie group are reciprocal Lie algebras [7, pp. 41,42].

2.2 Geometric construction

In this section we will show how to construct the tangential symmetries of Darboux integrable decomposable systems.

Commuting vector fields Let $(M, \mathcal{F}, \mathcal{G})$ be a Darboux integrable decomposable system with Darboux projection $\pi : M \to B$. We assume that $\mathcal{V} = \mathcal{F} \oplus \mathcal{G}$ has no invariants and condition (3b) holds. Select locally a basis of commuting vector fields $\tilde{F}_1, \ldots, \tilde{F}_{n_\mathcal{G}}$, $n_\mathcal{G} = \operatorname{rank} \mathcal{F}$ for B_1 and $\tilde{G}_1, \ldots, \tilde{G}_{n_\mathcal{F}}$ for B_2, i.e., $[\tilde{F}_i, \tilde{F}_j] = 0$, $[\tilde{G}_i, \tilde{G}_j] = 0$. As vector fields on $B = B_1 \times B_2$ we then automatically have $[\tilde{F}_i, \tilde{G}_j] = 0$. We can lift these vector fields to unique vector fields in M by requiring that the lifted vector fields are contained in $\mathcal{V}$. We write F_i and G_j for the lift of $\tilde{F}_i$ and $\tilde{G}_j$, respectively. Since the vector fields F_i and G_j are contained in $\mathcal{F}$ and $\mathcal{G}$, respectively, their Lie brackets $[F_i, G_j]$ must be contained in $\mathcal{V}$. On the other hand, the projections have Lie bracket $[\tilde{F}_i, \tilde{G}_j]$ equal to zero and therefore $[F_i, G_j]$ must be contained in the tangent space of the fibers of the projection. But $\mathcal{V}$ is transversal to the fibers of the projection and it follows that $[F_i, G_j] = 0$.

We will use the vector fields F_i and G_j to construct various Lie algebras on the fibers of the projection. We only make a *choice* of vector fields to make the constructions and the proofs easier, most of the Lie algebras that we construct are *independent* of the particular choice of F_i and G_j.

We define $\mathfrak{f}$ as the Lie algebra of vector fields (over $\mathbb{R}$, not over $\mathbb{C}^\infty(M)$) generated by F_i, $1 \leq i \leq \operatorname{rank} \mathcal{F}$. This Lie algebra is contained in the Lie algebra of vector fields in the completion of $\mathcal{F}$ and is not necessarily finite-dimensional. We define $\mathfrak{g}$ as the Lie algebra of vector fields (over $\mathbb{R}$) generated by G_i, $1 \le i \le \operatorname{rank} \mathcal{G}$.

For two vector fields F_i, F_j the Lie bracket $[F_i, F_j]$ is tangent to the fibers of the projection π. This follows from the fact that the F_i are lifts of commuting vector fields and hence $T\pi([F_i, F_j]) = [\tilde{F}_i, \tilde{F}_j] = 0$. This implies that the derived Lie algebras $\mathfrak{f}'$ and $\mathfrak{g}'$ consist of vector fields that are tangential to the projection. Since the generators F_i for $\mathfrak{f}$ are not tangential, we conclude $\mathfrak{f}' = \mathfrak{f} \cap \mathfrak{z}$. The elements of $\mathfrak{f}'$ commute with the elements in $\mathfrak{g}$ and hence the elements of $\mathfrak{f}'$ are symmetries of $\mathcal{G}$ that are tangential to the projection π.

The discussion above shows that the vector fields in $\mathfrak{f}'$ are all tangential symmetries of $\mathcal{G}$. We will see below (also see Lemma 2.7) that the tangential symmetries can be expressed in terms of the Lie algebras $\mathfrak{f}'$ and $\mathfrak{g}'$.

Restriction to fibers For every point $b \in B = B_1 \times B_2$ we write M_b for the fiber $\pi^{-1}(b)$. We write $\mathfrak{z}_b$ for the vector fields on M_b. For the tangential vector fields $\mathfrak{z}$ we define the restriction map

$$\rho_b : \mathfrak{z} \longrightarrow \mathfrak{z}_b. \tag{6}$$

The restriction map is a Lie algebra homomorphism.

Since the vector fields in $\mathfrak{f}'$ and $\mathfrak{g}'$ are tangential we can define the restriction maps $\rho_b : \mathfrak{f}' \to \mathfrak{z}_b$ and $\rho_b : \mathfrak{g}' \to \mathfrak{z}_b$ as well. We denote the image of $\mathfrak{f}'$ under ρ_b by $\mathfrak{f}'_b$ and the image of $\mathfrak{g}'$ under ρ_b by $\mathfrak{g}'_b$. Since the Lie algebras $\mathfrak{f}'$ and $\mathfrak{g}'$ commute, the Lie algebras $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ are commuting Lie algebras of vector fields on M_b.

Since $\mathcal{F}$ has $n_{\mathcal{F}}$ invariants, the codimension of the completion of $\mathcal{F}$ is equal to $n_{\mathcal{F}}$ on an open dense subset. The same is true for $\mathcal{G}$. In the constructions that follow we will always assume that we have restricted to such open dense subsets. Under this assumption the vector fields in the Lie algebra $\mathfrak{f}'$ span the tangent space to the fiber. We can choose a set of vector fields $X_1, \ldots, X_s$ (s is the dimension of the fibers) in $\mathfrak{f}'$ such that the restriction of these vector fields to a fiber M_b is a basis for $\mathfrak{f}'_b$. Let X be a tangential symmetry of $\mathcal{G}$. Since the vector field X is tangential we can write $X = \sum_{j=1}^{s} c^j X_j$ for certain functions c^j. Since X is a tangential symmetry, the commutator of X with $\mathcal{G}$ is contained in $\mathcal{G}$ and hence for all $Y \subset \mathcal{G}$

$$[X, Y] \equiv \sum_{j=1}^{s} [c^j X_j, Y] \equiv \sum_{j=1}^{s} c_j [X_j, Y] - Y(c^j) X_j \equiv \sum_{j=1}^{s} Y(c_j) X_j \equiv 0 \mod \mathcal{G}.$$

This implies $Y(c_j) = 0$ for all $Y \subset \mathcal{G}$ and hence the c^j are functions of the invariants of $\mathcal{G}$ only, so $c^j \in \pi_1^*(C^\infty(B_1))$. This proves that the tangential symmetries of $\mathcal{G}$ are a $\pi_1^*(C^\infty(B_1))$-module over the Lie algebra $\mathfrak{f}'$. We have proved

Lemma 2.7. *The tangential symmetries of $\mathcal{G}$ are a $\pi_1^*(C^\infty(B_1))$-module over $\mathfrak{f}'$. The tangential symmetries of $\mathcal{F}$ are a $\pi_2^*(C^\infty(B_2))$-module over $\mathfrak{g}'$.*

The lemma above also proves Theorem 2.2.

The Lie algebras $\mathfrak{f}'$ and $\mathfrak{g}'$ depend on the choice of commuting vector fields $\tilde{F}_i$ and $\tilde{G}_j$, respectively. The Lie algebras of tangential symmetries $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$ are independent of this choice (this is clear from Definition 2.1). In the lemma below we show that the Lie algebras $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ are also independent of the choice of commuting vector fields.

Lemma 2.8. *For every point $b \in B$ the Lie algebras $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ on M_b are invariantly defined reciprocal Lie algebras. The type of the Lie algebra does not depend on the point $b \in B$.*

Proof. The Lie algebra $\mathfrak{f}'$ is tangential to the fibers of the projection. Since $\mathcal{F}$ has only $n_{\mathcal{F}} = \operatorname{rank} \mathcal{G}$ invariants, it follows that for y in an open dense subset of M the image of the evaluation map $\operatorname{ev}(\mathfrak{f}')_y : \mathfrak{f}' \to T_y M_{\pi(y)}$ spans the tangent space $T_y M_{\pi(y)}$. This in turn implies that for all points x in an open subset of M_b the evaluation map $\operatorname{ev}(\mathfrak{f}'_b)_x : \mathfrak{f}'_b \to T_x M_b$ is surjective. The same is true for $\mathfrak{g}'_b$. Hence we can apply Lemma 2.6 to $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ and conclude that $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$ are reciprocal Lie algebras on M_b. From the definitions it follows directly that the Lie algebra $\mathfrak{f}'_b$ only depends on the choice of vector fields F_i, and not on the choice of the vector fields G_j. In the same way $\mathfrak{g}'_b$ does only depend on the choice of G_j. On the other hand, $\mathfrak{f}'_b$ is the centralizer of $\mathfrak{g}'_b$ in the fiber M_b and hence $\mathfrak{f}'_b$ only depends on the choice of G_j. This implies that $\mathfrak{f}'_b$ is invariantly defined. For the same reason $\mathfrak{g}'_b$ is invariantly defined.

Make a choice of s vector fields X_i in $\mathfrak{f}'$ such that at each point the vector fields span the tangent space to the fibers of the projection. Locally, near a point $x \in M_b$, we can think of the vector fields X_i as defining a section of the homomorphism $\rho_b : \mathfrak{f}' \to \mathfrak{f}'_b$.

Since the vector fields X_i span the tangent space to the fiber and the commutator of two tangential vector fields is tangential again, we have $[X_i, X_j] = \sum_{k=1}^{s} c_{ij}^k X_k$ for certain functions c_{ij}^k. Since the X_i commute with $\mathfrak{g}$ it follows that the functions c_{ij}^k depend only on the invariants of $\mathcal{G}$. If we restrict to one of the leaves of the completion of $\mathcal{G}$, then the invariants of $\mathcal{G}$ are constant and hence the coefficients c_{ij}^k will be constant. Locally, the leaves are foliated by the fibers of the projection and the fact that the coefficients c_{ij}^k depend only on the invariants of $\mathcal{G}$ shows that all fibers M_b in the same leaf of $\operatorname{compl}(\mathcal{G})$ have an isomorphic Lie algebra $\mathfrak{f}'_b$. So if we move in the direction of the completion of $\mathcal{G}$, then the type of $\mathfrak{f}'_b$ does not change. For the same reason the Lie algebras $\mathfrak{g}'_b$ for all fibers M_b in a leaf of the completion of $\mathcal{F}$ have the same type.

The type of $\mathfrak{f}'_b$ is equal to the type of $\mathfrak{g}'_b$ (reciprocal Lie algebras are anti-isomorphic). Therefore if we move in the directions of $\mathcal{F}$ and $\mathcal{G}$ the type of both $\mathfrak{g}'_b$ and $\mathfrak{f}'_b$ does not change. Hence the type of the reciprocal Lie algebras on the fibers is independent of the choice of fiber M_b. □

We conclude that the fibers of the projection carry an invariant structure of two reciprocal Lie algebras. Since the type is locally constant, the type of the Lie algebra is an *invariant* of Darboux integrable decomposable systems.

Back to local vector fields The next step is to extend the Lie algebras on the fibers to Lie algebras on M. The following lemma proves Theorem 2.3 by constructing L and R as subalgebras of $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$, respectively.

Lemma 2.9. *On M there exist finite-dimensional Lie subalgebras L, R of $\tilde{\mathfrak{f}}$ and $\tilde{\mathfrak{g}}$, respectively, such that for all fibers M_b the restriction map ρ_b defines a Lie algebra isomorphism to the Lie algebras on the fibers. The Lie algebras L and R are commuting.*

Proof. Choose a basis of vector fields X_j for $\mathfrak{z}$ contained in $\mathfrak{f}'$. The Lie brackets of the X_j define the structure coefficients

$$[X_i, X_j] = \sum_{k=1}^{s} c_{ij}^k X_k. \tag{7}$$

We already know that these structure coefficients are functions in $\pi_1^*(C^\infty(B_1))$. Choose a fiber M_{b_0}. Since the type of the Lie algebra is constant for each fiber, there exists for every point $b_1 \in B_1$ a linear transformation $\mu(b_1) \in \mathrm{GL}(s, \mathbb{R})$ such that the vector fields $Y_i = \mu(b_1)_i^j X_j$ have the same structure constants as the restrictions of X_j to the fiber M_{b_0}. [2]Locally we can arrange that $\mu : B_1 \to \mathrm{GL}(s, \mathbb{R})$ is a smooth map, see Remark 2.10.

The new vector fields Y_j have the structure of a finite-dimensional Lie algebra L and this Lie algebra consists of tangential symmetries of $\mathcal{G}$. In the same way we can construct a finite-dimensional Lie subalgebra R of $\tilde{\mathfrak{g}}$. □

We call L and R *tangential Lie algebras* of symmetries of $\mathcal{G}$ and $\mathcal{F}$, respectively. The Lie algebras L and R are not-unique. We can for example multiply a basis X_j of L with a matrix $\mu_j^k \in \mathrm{GL}(s, \mathbb{R}) \otimes \pi_1^*(C^\infty(B_1))$. The new vector fields $Y_j = \sum_{l=1}^s \mu_j^k X_k$ have structure coefficients $d_{ij}^k(x) = \sum_{\alpha,\beta,l=1}^s \mu_i^\alpha(x)\mu_j^\beta(x)c_{\alpha\beta}^l(\mu^{-1})_l^k(x)$. If the new structure coefficients are constants (for example if for all points x the matrix $\mu_j^k(x)$ is in the stabilizer of the structure constants), then the vector fields Y_j define a tangential Lie algebra of tangential symmetries as well.

Remark 2.10. Let $\mathcal{L}$ be the space of Lie algebra structures on $\mathbb{R}^s$. This space is an algebraic variety in $C_s = \Lambda^2(\mathbb{R}^s)^* \otimes \mathbb{R}^s$ defined the Jacobi identity. The group $G = \mathrm{GL}(s, \mathbb{R})$ acts on C_s. The vector fields X_j from Lemma 2.9 define a map $\tilde{\mu} : B_1 \to C_s$ by assigning to a point $b_1 \in B_1$ the structure constants of X_j in a fiber above b_1 (the structure coefficients are independent of the point in B_2). Since the vector fields X_i are smooth, the map $\tilde{\mu}$ is smooth as well. By assumption the image of $\tilde{\mu}$ is contained in a single orbit A of the action of G.

Let a be equal to $\tilde{\mu}(x)$. Then the orbit A is equal to G/G_a, where G_a is the stabilizer subgroup of the point a in the orbit. We want to prove that there is a smooth lift of the map $\tilde{\mu}$ to a map $\mu : B_1 \to \mathrm{GL}(s, \mathbb{R})$ such that the diagram below is commutative.

[2]Since the vector fields X_i do not act on the functions in $\pi_1^*(C^\infty(B_1))$!

$$
\begin{tikzcd}
 & \mathrm{GL}(s,\mathbb{R}) \arrow[d] & \\
B_1 \arrow[ur, dashed, "\mu"] \arrow[r, "\tilde{\mu}"] & A \subset C_s \\
 & G/G_a \arrow[u, phantom, "\|"]
\end{tikzcd}
$$

We have to be carefull here because the map $\tilde{\mu}$ is continuous with respect to the topology on A induced from the surrounding space C_s. The structure on A as the homogeneous space G/G_a might be different. From the theory in [8, §A.3] it follows that $\tilde{\mu}$ is a smooth map to G/G_a. The projection $G \to G/G_a$ is a principal fiber bundle and hence there is a smooth lift of $\tilde{\mu}$.

Using the Lie algebras L and R we can construct a normal form for any Darboux integrable system.

Theorem 2.11. *Let $(M, \mathcal{F}, \mathcal{G})$ be a Darboux integrable system satisfying* (3)*. Then locally there is a unique local Lie group H such that the manifold M is of the form*

$$B_1 \times B_2 \times H,$$

with $B_1 \subset \mathbb{R}^{n_\mathcal{G}}$, $B_2 \subset \mathbb{R}^{n_\mathcal{F}}$. The Darboux projection π is given by the projection on $B_1 \times B_2$. The tangential symmetries of $\mathcal{F}$ and $\mathcal{G}$ are tangent to the fibers of π and restricted to each fiber they form reciprocal Lie algebras. The left- and right-invariant vector fields on H define the tangential Lie algebras of symmetries on M.

Commuting part In each fiber M_b of the projection, we can consider the centers of the Lie algebras on the fiber. The center of $\mathfrak{f}'_b$ is equal to the center of $\mathfrak{g}'_b$. We write $\mathfrak{c}_b$ for the center of the Lie algebra on M_b. The distribution $\mathcal{C}$ spanned by $\mathfrak{c}_b$, $b \in B$, is invariantly defined and is integrable and hence defines a local foliation of the fibers.

Lemma 2.12. *The tangential vector fields that are symmetries of both $\mathcal{F}$ and $\mathcal{G}$ are tangential vector fields for which the restriction to a fiber is in the center of the Lie algebra of tangential symmetries. In particular, these vector fields are contained in $\mathcal{C}$. The tangential symmetries of a decomposable system are equal to the center of L (which is equal to the center of R).*

The space of tangential symmetries of the system is equal to the center of L (which is equal to the center of R).

Proof. Restricted to each fiber M_b a tangential symmetry is invariant under both $\mathfrak{f}'_b$ and $\mathfrak{g}'_b$. Since the restriction is invariant under $\mathfrak{g}'_b$ it must be contained in $\mathfrak{f}'_b$. Since the restriction commutes with $\mathfrak{f}'_b$ it is by definition contained in the center of $\mathfrak{f}'_b$.

From the definitions if is clear that any element from the center of L is a tangential symmetry. Using the fact that L spans $\mathcal{C}$ we can show that the center of L contains all tangential symmetries. □

The converse is not true. Not every vector field contained in $\mathcal{C}$ is a tangential symmetry of $\mathcal{F}$ or $\mathcal{G}$. This lemma shows that, unless the Lie group associated to a decomposable system is abelian, the method of Darboux does not coincide with the symmetry methods for partial differential equations introduced by Lie.

Remark 2.13. Given a Lie group there is no systematic way of constructing a corresponding Darboux integrable system. Even if we can construct such a system, there is no guarantee that this system corresponds to a prolonged first order system or second order equation. Despite this, Vessiot [14, 15] succeeded in using the classification of 3-dimensional Lie algebras to construct a classification of all Darboux integrable decomposable Goursat equations.

2.3 Examples

Example 2.14 (Wave equation). On $M = \mathbb{R}^5$ with coordinates x, y, z, p, q introduce the following two distributions:

$$\mathcal{F} = \operatorname{span}\left(\partial_x + p\partial_z, \partial_p\right), \quad \mathcal{G} = \operatorname{span}\left(\partial_y + q\partial_z, \partial_q\right).$$

The triple $(M, \mathcal{F}, \mathcal{G})$ defines a Darboux integrable decomposable system. The contact structure on the first order equation manifold of the wave equation $\partial^2 z/\partial x \partial y = 0$ is given precisely by this structure.

The invariants of $\mathcal{F}$ are y, q and the invariants of $\mathcal{G}$ are x, p. The Darboux projection is given by

$$\pi : M \to \mathbb{R}^2 \times \mathbb{R}^2 : (x, y, z, p, q) \mapsto (x, p) \times (y, q).$$

As a set of commuting vector fields on $\mathbb{R}^2 \times \mathbb{R}^2$ we take $\tilde{F}_1 = \partial_x$, $\tilde{F}_2 = \partial_p$, $\tilde{G}_1 = \partial_y$, $\tilde{G}_2 = \partial_y$. The lifts to vector fields on M are given by

$$\begin{aligned} F_1 &= \partial_x + p\partial_z, \quad F_2 = \partial_p, \\ G_1 &= \partial_y + q\partial_z, \quad G_2 = \partial_q. \end{aligned}$$

Define $F_3 = [F_1, F_2] = -\partial_z$ and $G_3 = [G_1, G_2] = -\partial_z$. The fibers of the projection π are isomorphic to $\mathbb{R}$ are we can use z as a coordinate on the fibers. On the leaves of the completion of $\mathcal{G}$ (i.e., x and p constant) we have a 3-dimensional Lie algebra with structure equations

$$[F_1, F_2] = F_3, \quad [F_1, F_3] = 0, \quad [F_2, F_3] = 0.$$

The same holds for the leaves of the completion of $\mathcal{F}$. The fibers of the projection are 1-dimensional and have the structure of a 1-dimensional abelian Lie algebra (generated by the vector field ∂_z).

Example 2.15. Consider the decomposable second order equation

$$z_{xy} = \frac{az}{(x+y)^2}. \tag{8}$$

This equation is Darboux integrable on the $k+1$-jets if $a = k(k+1)$. Hence for $a = k(k+1)$ the $(k-1)$-th prolonged equation manifold has dimension $5+2k$ and the prolonged Monge systems define a Darboux integrable decomposable system.

We work out the case $k = 2$ in detail. The prolonged equation manifold has coordinates x, y, z, $p = z_x$, $q = z_y$, $r = z_{xx}$, $t = z_{yy}$, z_{xxx}, z_{yyy}. The prolonged characteristic systems are $\mathcal{F} = \operatorname{span}(F_1, F_2)$, $\mathcal{G} = \operatorname{span}(G_1, G_2)$ with

$$
\begin{aligned}
F_1 &= \partial_x + p\partial_z + r\partial_p + \sigma\partial_q + z_{xxx}\partial_r + Y(\sigma)\partial_t + Y(Y(\sigma))\partial_{z_{yyy}},\\
F_2 &= \partial_{z_{xxx}},\\
G_1 &= \partial_y + q\partial_z + \sigma\partial_p + t\partial_q + X(\sigma)\partial_r + z_{yyy}\partial_t + X(X(\sigma))\partial_{z_{xxx}},\\
G_2 &= \partial_{z_{yyy}}.
\end{aligned}
$$

Here $\sigma = 6z/(x+y)^2$, $X = \partial_x + p\partial_z + r\partial_p + \sigma\partial_q$ and $Y = \partial_y + q\partial_z + \sigma\partial_p + t\partial_q$. Both characteristic systems have two invariants,

$$
\begin{aligned}
I_{\mathcal{F}} &= \{\, y, z_{yyy} + 6\frac{q + t(x+y)}{(x+y)^2} \,\},\\
I_{\mathcal{G}} &= \{\, x, z_{xxx} + 6\frac{p + r(x+y)}{(x+y)^2} \,\}.
\end{aligned}
$$

If we make a transformation to the variables $\tilde{x} = x$, $\tilde{y} = y$, $\tilde{z} = z$, $\tilde{p} = p$, $\tilde{q} = q$, $\tilde{r} = r$, $\tilde{t} = t$, $\tilde{a} = z_{xxx} + 6(p + r(x+y))/(x+y)^2$, $\tilde{b} = z_{yyy} + 6(q + t(x+y))/(x+y)^2$, then we can easily construct the Darboux projection, choose commuting vector fields, lifts these vector fields to M and calculate the tangential symmetries. Translated back to the original variables we find that the tangential symmetries of $\mathcal{F}$ are spanned by

$$
\begin{aligned}
L_1 &= \partial_t - 6H\partial_{z_{yyy}},\\
L_2 &= \partial_q - 6H\partial_t + 30H^2\partial_{z_{yyy}},\\
L_3 &= \partial_z - 6H\partial_q + 24H^2\partial_b - 108H^3\partial_{z_{yyy}},\\
L_4 &= H\partial_z - H^2\partial_p - 3H^2\partial_q + 2H^3\partial_r\\
&\quad + 10H^3\partial_t + 6H^4\partial_{z_{xxx}} + 42H^4\partial_{z_{yyy}},\\
L_5 &= H^2\partial_z - 2H^3(\partial_p + \partial_q)\\
&\quad + 6H^4(\partial_r + \partial_t) - 24H^4(\partial_{z_{xxx}} + \partial_{z_{yyy}}),
\end{aligned}
$$

with $H = (x+y)^{-1}$. The tangential symmetries of $\mathcal{G}$ are given by the same expressions, but with x, p, r and z_{xxx} replaced by y, q, t and z_{yyy}, respectively. The tangential symmetries commute and the Lie group associated to this Darboux integrable system is the 5-dimensional abelian Lie group.

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