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SUPERCONFORMAL INDICES OF THREE-DIMENSIONAL THEORIES RELATED BY MIRROR SYMMETRY

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Abstract. Recently, Kim, and Imamura and Yokoyama derived an exact formula for superconformal indices in three-dimensional field theories. Using their results, we prove analytically the equality of superconformal indices in some $U(1)$-gauge group theories related by mirror symmetry. The proofs are based on well-known identities in the theory of q-special functions. We also suggest a general index formula taking into account the $U(1)_J$ global symmetry present for abelian theories.

1. Introduction

The superconformal index (SCI) technique is a very useful tool for testing supersymmetric dualities. Initially, this technique was introduced for four-dimensional supersymmetric field theories [1, 2] in the context of $\mathcal{N}=1$ SYM Seiberg dualities and AdS/CFT correspondence for $\mathcal{N}=4$ SYM field theories. In this case, the indices are described by the elliptic hypergeometric integrals [3] as observed first by Dolan and Osborn [4]. Various developments and applications of this technique are described in [5, 6, 7, 8, 9].

Later, in [10], the superconformal index for three-dimensional supersymmetric Chern–Simons theories with large rank of the gauge group N was introduced, and the coincidence with the gravitational background index in the context of AdS/CFT correspondence [11] has been established. In [12] superconformal characters of three-dimensional supersymmetric theories have been constructed and, after taking restrictions for parameters in them, SCIs for theories considered in [10] can be obtained in a different way. In [13, 14, 15, 16, 17], various three-dimensional SCIs were calculated in the large N limit for comparison with their gravity duals. The partition functions of three-dimensional supersymmetric field theories are studied in [18, 19, 20, 21, 22, 23, 24, 25].

The superconformal index for $\mathcal{N}=6$ Chern–Simons theory with *finite* N was derived by Kim in [14]. The contribution to SCIs of chiral fields with arbitrary R-charge was found recently by Imamura and Yokoyama in [26]. After combining everything, this gives an exact formula for $3d$ SCIs analogous to Römelsberger's result for $4d$ $\mathcal{N}=1$ SYM theories [2]. SCIs of some $3d$ $\mathcal{N}=2$ supersymmetric field theories and their mirror partners [27] (see also [28, 29] for a general discussion of such theories) were computed in [26] and their coincidence was confirmed up to the first several terms of the corresponding series expansions in chemical potentials. The main goal of the present work consists in the analytic proof of exact coincidence of SCIs for these mirror symmetric $3d$ theories.

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Let us describe the SCI derived in [14, 26]. For two particular supercharges $\mathcal{Q}$ and $\mathcal{Q}^\dagger = \mathcal{S}$, the superconformal partner of $\mathcal{Q}$, the following algebraic relation holds:

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} \;=\; 2\mathcal{H} = \Delta - R - J_3, \tag{1}$$

where the space-time symmetry generators Δ, R, J_3 are the Hamiltonian, the R-charge, and the third component of the angular momentum, respectively. One defines an extension of the Witten index for the theories compactified on the two-sphere S^2 as

$$I \;=\; \operatorname{Tr}\left((-1)^{\mathcal{F}} e^{-\beta\mathcal{H}} x^{\Delta+J_3} \prod_{i=1}^{\operatorname{rank} F} t_i^{F_i}\right), \tag{2}$$

where $\mathcal{F}$ is the fermion number operator and x is the chemical potential associated with the operator $\Delta + J_3$ commuting with the chosen supercharges. Chemical potentials t_i are associated with commuting generators of the group F describing other global symmetries of the theory. Analogously to the $4d$ case one computes the trace over the space of states defined by the kernel of $\mathcal{Q}$ and $\mathcal{Q}^\dagger$, since contributions of the states with non-zero eigenvalues of $\mathcal{H}$ cancel each other. In other words, one works with the gauge invariant BPS states protected by at least one pair of supercharges which do not form long multiplets. As a result, the index does not depend on β.

Similar to the $4d$ case, the $3d$ SCI is computed in two steps by using the localization procedure in $\mathbf{S}^2 \times \mathbf{S}^1$. One constructs first the so-called single-particle state index and then computes the full SCI. The single-particle state index is defined by the formula [14, 26]

$$\begin{aligned} \operatorname{ind}(e^{\mathrm{i}g_j}, s, x, t) = &- \sum_{\alpha\in G} e^{\mathrm{i}\alpha(g)} x^{2|\alpha(s)|} \\ &+ \sum_{\Phi} \sum_{\rho\in R_\Phi} \left[e^{\mathrm{i}\rho(g)} t_i^{f_i} \frac{x^{2|\rho(s)|+\Delta_\Phi}}{1-x^2} - e^{-\mathrm{i}\rho(g)} t_i^{-f_i} \frac{x^{2|\rho(s)|+2-\Delta_\Phi}}{1-x^2} \right], \end{aligned} \tag{3}$$

where the first term describes contributions of the gauge fields, and the rest comes from the matter fields with flavour charges f_i. The chemical potentials $g = \{g_1, \dots, g_{\operatorname{rank} G}\}$ are associated with the generators of the maximal torus of the gauge group G. Similarly we write $s = \{s_1, \dots, s_{\operatorname{rank} G}\}$, where s_j are some half-integers associated with the magnetic monopole fluxes. The sum $\sum_{\alpha\in G}$ is taken over the roots of the Lie algebra of G. Without the term $x^{2|\alpha(s)|}$ it would yield essentially the character for the adjoint representation. This contribution was computed in [14].

The second term was also computed in case of $\mathcal{N} = 6$ superconformal Chern–Simons theory in [14] for the matter fields with the specific R-charges $\Delta_\Phi = 1/2$. In [26], the contribution of chiral fields with general R-charges is determined. Here Δ_Φ is the Weyl weight of a chiral multiplet Φ lying in the representation R_Φ of the gauge group G. Similar to the $4d$ case, the scalar component of the chiral superfield Φ has R-charge equal to Δ_Φ, and the fermion component has the R-charge $\Delta_\Phi - 1$. The sum $\sum_{\rho\in R_\Phi}$ is the sum over all terms with the weight $\rho(g)$ for a given chiral field Φ lying in the R_Φ representation of the gauge group G. The symbols $\alpha(g)$ and $\alpha(s)$ in the first term are used for a separate presentation of the gauge field and the monopole contributions coming from the adjoint representation of G.

Having the single-particle state index, one derives a full SCI using the plethystic exponent [26]

$$I(x,t) = \sum_s \frac{1}{\mathrm{Sym}} \int e^{-S_{\mathrm{CS}}^{(0)}} e^{\mathrm{i}b_0(g)} x^{\epsilon_0} \prod_{i=1}^{\mathrm{rank}\, F} t_i^{q_{0i}} \cdot \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{ind}(z_j^n, s, x^n, t_i^n)\right] \prod_{j=1}^{\mathrm{rank}\, G} \frac{dz_j}{2\pi\mathrm{i} z_j}, \quad (4)$$

where $z_j = e^{\mathrm{i}g_j}$,

$$\epsilon_0 = \sum_\Phi (1-\Delta_\Phi) \sum_{\rho \in R_\Phi} |\rho(s)| - \sum_{\alpha \in G} |\alpha(s)| \quad (5)$$

is the zero-point contribution to the energy,

$$q_{0i} = -\sum_\Phi \sum_{\rho \in R_\Phi} |\rho(s)| f_i \quad (6)$$

is the zero-point contribution to the flavour charges, and, finally,

$$b_0(g) = -\sum_\Phi \sum_{\rho \in R_\Phi} |\rho(s)| \rho(g), \quad (7)$$

which can be regarded as the one-loop correction to the Chern–Simons term.

In the presence of the Chern–Simons term, there is a contribution $e^{-S_{\mathrm{CS}}^{(0)}}$, where

$$S_{\mathrm{CS}}^{(0)} = 2\mathrm{i}\, \mathrm{Tr}_{CS}(gs), \quad (8)$$

with Tr_{CS} standing for the trace including Chern–Simons levels [14, 26]. For $U(N)_k$ theory with Chern–Simons level k, this factor equals $2\mathrm{i}k \sum_{j=1}^{\mathrm{rank}\, G} s_j g_j$ [14].

In contrast to the four-dimensional case, where the index contains integration over the gauge group only [1, 2, 4], here one has additionally the sum over the fluxes of rank G independent monopoles. Monopoles appear as solutions of the classical field equation associated with the saddle points in the localization procedure. Effectively, this leads to the shifted spin j and the shifted value of eigenvalue for J_3 due to the contribution from the background fluxes m_j, and the variable

$$s_j = \frac{1}{2} m_j, \quad m_j \in \mathbb{Z},$$

is introduced for convenience. In the above formulas, the quantities $\rho(s)$ represent the same Weyl weights in this background flux for a taken chiral field Φ. For example, the chiral superfields — gauge group singlets have $\rho(g) = \rho(s) = 0$.

As pointed out in [14], the integration over the gauge group is a little bit tricky, because of the presence of monopoles. It was shown in [14] that the contribution coming from the vector multiplet for $s_i \neq s_j$ and for $s_i = s_j$ is different, since in the first case the monopole spherical harmonics and in the latter case the usual spherical harmonics are used. As suggested in [26], this fact is already included in the term for the contribution of the vector multiplet. The term $\frac{1}{\mathrm{Sym}}$ in (4) appears because of the same reason, it is connected with the fact that the initial gauge group G is 'broken' by the monopoles into the product $G_1 \times \cdots \times G_k$, which gives

$\mathrm{Sym} = \prod_{i=1}^{k} (\operatorname{rank} G_i)!$ [14], which can also be written in the form

$$\mathrm{Sym} = \prod_{i=1}^{\operatorname{rank} G} \left(\sum_{j=i}^{\operatorname{rank} G} \delta_{s_i, s_j} \right), \tag{9}$$

where $\delta_{a,b}$ is the usual Kronecker delta-function.

In [26], SCIs for mirror symmetric theories were calculated with some restrictions — the same chemical potentials were used for both quark fields belonging to one flavour. Moreover, corresponding formulas did not contain the chemical potential associated with the abelian symmetry group $U(1)_J$ [29] resolving degeneracies. Shifting the scalar component of vector multiplet by an arbitrary constant, one can take into account this extra global symmetry. In the Appendix, we present SCIs with the most general set of chemical potentials. Surprisingly, we found that the SCI terms related to the symmetry group $U(1)_J$ can not be obtained directly from the results of [26], suggesting that they may be incomplete.

In contrast to the four-dimensional case, where in most cases the condition of the anomaly absence fixes the R-charge, in the three-dimensional case the R-charge in (4) is not fixed. Moreover, it can be arbitrary, since adding to it any combination of the abelian global charges represents again the R-charge. The latter fact is reflected by the appearance of the free parameter h associated with the R-charge in the SCIs. Recently the Z-extremization procedure was suggested in [21] for obtaining the exact R-charges of matter fields in the IR fixed points of $3d$ theories.

Formulas (3) and (4) resemble to some extent the procedure of calculating SCIs in $4d$ supersymmetric field theories [1, 4], but they are much more involved. In particular, there are the terms in addition to the plethystic exponent, which is a new structural element. The building block of SCIs for $3d$ theories is given by the infinite q-product

$$(z;q)_\infty = \prod_{j=0}^{\infty} (1 - zq^j), \qquad |q| < 1.$$

Let us consider the chiral superfield Φ with arbitrary R-charge Δ_Φ in the fundamental representation of $U(N_c)$. Then the single-particle state index is

$$\operatorname{ind}_C(x, e^{\mathrm{i}g_j}, s_j) \;=\; \sum_{j=1}^{N_c} \frac{x^{\Delta_\Phi + 2|s_j|} e^{\mathrm{i}g_j} - x^{2-\Delta_\Phi+2|s_j|} e^{-\mathrm{i}g_j}}{1 - x^2}, \tag{10}$$

and the full SCI is obtained as

$$\begin{aligned} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{ind}_C(x^n, e^{\mathrm{i}ng_j}, s_j) \right) & \\ = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{N_c} \frac{x^{n(\Delta_\Phi + 2|s_j|)} e^{\mathrm{i}ng_j} - x^{n(2-\Delta_\Phi+2|s_j|)} e^{-\mathrm{i}ng_j}}{1 - x^{2n}} \right) & \\ = \prod_{j=1}^{N_c} \frac{(x^{2-\Delta_\Phi+2|s_j|} e^{-\mathrm{i}g_j}; x^2)_\infty}{(x^{\Delta_\Phi+2|s_j|} e^{\mathrm{i}g_j}; x^2)_\infty}. & \end{aligned} \tag{11}$$

The contribution of gauge fields looks fundamentally different. When the gauge group is $U(N_c)$, the contribution of the vector multiplet is [14]

$$\mathrm{ind}_V(x, e^{\mathrm{i}g_j}, s_j) = -\sum_{i,j=1,\, i\neq j}^{N_c} x^{|s_i - s_j|} e^{\mathrm{i}(g_i - g_j)}, \tag{12}$$

and the contribution to the full SCI is given by

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\,\mathrm{ind}_V(x^n, e^{\mathrm{i}ng_j}, s_j)\right) = \prod_{i,j=1,\, i\neq j}^{N_c} (1 - x^{|s_i - s_j|} e^{\mathrm{i}(g_i - g_j)}). \tag{13}$$

We need also some mathematical definitions from the theory of q-special functions (see, e.g., [30]). The basic hypergeometric series ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r\left[\begin{array}{c} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{array}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, b_2, \ldots, b_r; q)_n} z^n, \tag{14}$$

where

$$(a_1, a_2, \ldots, a_{r+1}; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_{r+1}; q)_n,$$

with the q-shifted factorial $(z; q)_n$ being given by

$$(z; q)_n = \begin{cases} 1, & \text{for } n = 0, \\ \prod_{j=0}^{n-1} (1 - zq^j), & \text{for } n > 0, \\ \prod_{j=1}^{-n} (1 - zq^{-j})^{-1}, & \text{for } n < 0. \end{cases} \tag{15}$$

For $|q| < 1$, one can write

$$(z; q)_n \; = \; \frac{(z; q)_\infty}{(zq^n; q)_\infty}. \tag{16}$$

It is also convenient to use the notation

$$(az^{\pm 1}; q)_n := (az; q)_n (az^{-1}; q)_n.$$

The bilateral basic hypergeometric series ${}_r\psi_r$ is defined by

$${}_r\psi_r\left[\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{array}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} z^n. \tag{17}$$

2. Mirror symmetry for $d = 3$ $\mathcal{N} = 2$ field theory with $U(1)$-gauge group and $N_f = 1$

Here we discuss consequences of the mirror symmetry for three-dimensional $\mathcal{N} = 2$ supersymmetric field theory with $U(1)$ gauge group and $N_f = 1$ in its IR fixed point, whose mirror partner is the free Wess–Zumino theory [27, 28, 29]. The initial electric theory has one flavour or two quark superfields Q and $\tilde{Q}$ of charges $+1$ and -1 and R-charges $\Delta_Q = \Delta_{\widetilde{Q}} = 1/3$. The superconformal index is given by the expression

$$I_{e,N_f=1} = \sum_{k\in\mathbb{Z}} q^{|k|/3} \int_{\mathbb{T}} \frac{(q^{5/6+|k|/2} z^{\pm 1}; q)_\infty}{(q^{1/6+|k|/2} z^{\pm 1}; q)_\infty} \frac{dz}{2\pi \mathrm{i} z}, \tag{18}$$

where $\mathbb{T}$ is the unit circle with positive orientation and $|q| < 1$. It coincides with the function which appeared in [26] as formula (67) after the change of the chemical potential, the integration variable, and the summation variable, given by

$$q = x^2, \quad z = e^{ia}, \quad k = 2s.$$

After evaluating the integral as a sum of residues, we can write

$$I_{e,N_f=1} = \sum_{k\in\mathbb{Z}} q^{|k|/3} \frac{(q^{1+|k|}, q^{2/3}; q)_\infty}{(q, q^{1/3+|k|}; q)_\infty} \, {}_2\phi_1\left[\begin{matrix} q^{1/3+|k|}, q^{1/3} \\ q^{1+|k|} \end{matrix}; q, q^{2/3}\right]. \tag{19}$$

The superconformal index of the mirror partner theory containing one meson $M = Q\widetilde{Q}$ and two singlets $V_\pm$ [29] with R-charge $\Delta_{M,V_\pm} = 2/3$ has the form

$$I_{m,N_f=1} = \frac{(q^{2/3}; q)_\infty^3}{(q^{1/3}; q)_\infty^3}. \tag{20}$$

Actually, in both of these theories there exists an additional $U(1)_J$-symmetry group [29], which is not taken into account in [26]. In general, it is necessary to introduce an additional chemical potential w for it. In the Appendix, we present these more general SCIs. The expressions given above correspond to the choice $w = 1$.

Mirror symmetry is supposed to lead to equal superconformal indices of two theories. In [26], coincidence of only the first several terms of the series expansions in q of (18) and (20) was checked. Our goal is to give an analytic proof of the equality of these indices, which we formulate as a mathematical theorem.

Theorem 1. *The equality* $I_{e,N_f=1} = I_{m,N_f=1}$ *holds true.*

We shall present the proof of a more general statement containing an additional free parameter describing the R-charge of a chiral superfield $\Delta_\Phi = h$. For this theory, the SCI is given by the expression

$$I_{e,N_f=1;h} = \sum_{k\in\mathbb{Z}} a^{|k|/2} \int_{\mathbb{T}} \frac{(a^{1/2} q^{1/2+|k|/2} z^{\pm 1}; q)_\infty}{(a^{-1/2} q^{1/2+|k|/2} z^{\pm 1}; q)_\infty} \frac{dz}{2\pi \mathrm{i} z}, \tag{21}$$

where we write $a = q^{1-h}$ and assume the constraint $|q/a| < 1$. Note that from first glance one cannot replace in the series summand the modulus $|k|$ by k since then, for any a, there will be a negative value of k such that the contour $\mathbb{T}$ stops to separate the geometric sequences of poles converging to zero from their reciprocals.

After evaluating this integral as a sum of residues, one finds

$$I_{e,N_f=1;h} = \sum_{k\in\mathbb{Z}} a^{|k|/2} \frac{(q^{1+|k|}, a; q)_\infty}{(q, q^{1+|k|}/a; q)_\infty} \, {}_2\phi_1\left[\begin{matrix} q^{1+|k|}/a, q/a \\ q^{1+|k|} \end{matrix}; q, a\right]. \tag{22}$$

The superconformal index of the mirror theory (which is again a free theory of chiral superfields) will be

$$I_{m,N_f=1;h} = \frac{(a; q)_\infty}{(q/a; q)_\infty} \frac{(q/a^{1/2}; q)_\infty^2}{(a^{1/2}; q)_\infty^2}. \tag{23}$$

Again these expressions correspond to the choice $w = 1$ of the chemical potential associated with the $U(1)_J$-group (see the Appendix).

Theorem 2. *The equality* $I_{e,N_f=1;h} = I_{m,N_f=1;h}$ *holds true.*

We start by evaluating the double sum

$$\sum_{k=-\infty}^{\infty} \frac{a^{k/2}\,(a; q)_\infty\,(q^{1+k}; q)_\infty}{(q; q)_\infty\,(q^{1+k}/a; q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a; q)_n\,(q^{1+k}/a; q)_n}{(q; q)_n\,(q^{1+k}; q)_n} a^n. \tag{24}$$

In order to do this, we interchange the sums over k and n and write the (now) inner sum over k in bilateral series notation:

$$\sum_{n=0}^{\infty} \frac{a^n\,(q/a;q)_n^2\,(a;q)_\infty}{(q;q)_n^2\,(q/a;q)_\infty}\,{}_1\psi_1\left[\begin{matrix} q^{1+n}/a \\ q^{1+n} \end{matrix}; q, a^{1/2}\right].$$

The ${}_1\psi_1$-series can be summed by means of Ramanujan's summation (see [30, (5.2.1)])

$${}_1\psi_1\left[\begin{matrix} A \\ B \end{matrix}; q, Z\right] = \frac{(q, B/A, AZ, q/AZ;q)_\infty}{(B, q/A, Z, B/AZ;q)_\infty}. \tag{25}$$

If we apply this formula and then write the sum over n in basic hypergeometric notation, then we obtain the expression

$$\frac{(a;q)_\infty\,(q/a^{1/2};q)_\infty}{(a^{1/2};q)_\infty\,(q/a;q)_\infty}\,{}_1\phi_0\left[\begin{matrix} q/a \\ - \end{matrix}; q, a^{1/2}\right].$$

After application of the q-binomial theorem (see [30, (1.3.2)])

$${}_1\phi_0\left[\begin{matrix} A \\ - \end{matrix}; q, Z\right] = \frac{(AZ;q)_\infty}{(Z;q)_\infty},$$

our expression simplifies to (23).

Now, the double sum (24) was not exactly what we wanted. We want

$$\sum_{k=-\infty}^{\infty} \frac{a^{\frac{|k|}{2}}\,(a;q)_\infty\,(q^{1+|k|};q)_\infty}{(q;q)_\infty\,(q^{1+|k|}/a;q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a;q)_n\,(q^{1+|k|}/a;q)_n}{(q;q)_n\,(q^{1+|k|};q)_n} a^n, \tag{26}$$

which is just the expression in (22), where the ${}_2\phi_1$-series is written out explicitly.

So, in view of the above, it suffices to prove that the sum over non-negative k in (24) equals the sum over non-positive k in (24), that is,

$$\sum_{k=0}^{\infty} \frac{a^{\frac{k}{2}}\,(a;q)_\infty\,(q^{1+k};q)_\infty}{(q;q)_\infty\,(q^{1+k}/a;q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a;q)_n\,(q^{1+k}/a;q)_n}{(q;q)_n\,(q^{1+k};q)_n} a^n$$
$$= \sum_{k=-\infty}^{0} \frac{a^{\frac{k}{2}}\,(a;q)_\infty\,(q^{1+k};q)_\infty}{(q;q)_\infty\,(q^{1+k}/a;q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a;q)_n\,(q^{1+k}/a;q)_n}{(q;q)_n\,(q^{1+k};q)_n} a^n. \tag{27}$$

This is not too difficult to see: we take the right-hand side of (27) and replace k by $-k$:

$$\sum_{k=0}^{\infty} \frac{a^{-\frac{k}{2}}\,(a;q)_\infty\,(q^{1-k};q)_\infty}{(q;q)_\infty\,(q^{1-k}/a;q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a;q)_n\,(q^{1-k}/a;q)_n}{(q;q)_n\,(q^{1-k};q)_n} a^n.$$

We see now that, actually, this is an undefined expression, due to the terms $(q^{1-k};q)_\infty$ and $(q^{1-k};q)_n$ in numerator and denominator, respectively. We have to interpret this as an appropriate limit. In particular, the terms in the second sum for $n = 0, 1, \dots, k-1$ do not contribute anything. We may therefore start the sum over n at $n = k$. This leads to the expression

$$\sum_{k=0}^{\infty} \frac{a^{-\frac{k}{2}}\,(a;q)_\infty\,(q^{1-k};q)_\infty}{(q;q)_\infty\,(q^{1-k}/a;q)_\infty} \sum_{n=k}^{\infty} \frac{(q/a;q)_n\,(q^{1-k}/a;q)_n}{(q;q)_n\,(q^{1-k};q)_n} a^n$$
$$= \sum_{k=0}^{\infty} \frac{a^{\frac{k}{2}}\,(a;q)_\infty\,(q^{1+k};q)_\infty}{(q;q)_\infty\,(q^{1+k}/a;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{1+k}/a;q)_n\,(q/a;q)_n}{(q^{1+k};q)_n\,(q;q)_n} a^n,$$

where we replaced n by $n+k$ and did some simplification. This is exactly the left-hand side of (27). This proves equality of (24) and (26), and thus that the double sum (26) equals the product in (23).

Remark. The possibility to replace $|k|$ by k in the bilateral series requires some physical explanation. The variable k is the quantized magnetic flux associated with the Dirac monopole solution in this model. It is related somehow with the spin variable $j=|k|$ of the spherical harmonics, and the situation looks like that after replacing $|k|$ by k we come to the sum over $j\in\mathbb{Z}/2$ related with non-negative values of the $SO(3)$ Casimir operator proportional to $(j+1/2)^2$.

Note that, in contrast to $4d$ $\mathcal{N}=1$ SYM theories, where one can often fix $U(1)_R$ group hypercharges in the infrared fixed point, in $3d$ theories this is not the case [29]. The proven equality of SCIs remains true for arbitrary h (or a) although the physically acceptable theory corresponds to the value $h=1/3$ (such a possibility was pointed out already in [26]). The exact superconformal R-symmetry charges are found with the help of techniques described in [21]. Thus, similar to the $4d$ situation [6], mathematical properties of SCIs do not distinguish unitary theories from the non-unitary ones. SCIs count topological objects of the theories irrespective whether the cohomology space of $\mathcal{Q}$ and $\mathcal{Q}^\dagger=\mathcal{S}$ contains ghost fields violating unitarity or not. The equality of SCIs does not correspond necessarily to duality/mirror symmetry between physically acceptable theories, one should analyze separately their physical content.

3. Mirror symmetry for $d=3$ $\mathcal{N}=2$ theory with $U(1)$ gauge group and $N_f=2$.

We consider now SCIs for $3d\,\mathcal{N}=2$ field theory with $U(1)$ gauge group and $N_f=2$ flavours and its magnetic partner which are dual to each other in their respective IR fixed points [27, 28, 29]. The electric theory has $F=SU(2)_l\times SU(2)_r\times U(1)_J$ flavour symmetry for superfields with R-charges $\Delta_{Q_i}=\Delta_{\widetilde{Q_i}}=h$, $i=1,2$ [29]. For each symmetry group one should introduce chemical potentials: u for $SU(2)_l$, v for $SU(2)_r$ and w for $U(1)_J$, but we consider first the restricted region of these parameters where $u=v$ and $w=1$, which was analyzed perturbatively in [26]. In the Appendix we present general expressions for indices. The restricted SCI has the form

$$I_{e,N_f=2;h}=\sum_{k\in\mathbb{Z}}a^{|k|}\int_{\mathbb{T}}\frac{(a^{1/2}q^{1/2+|k|/2}v^{\pm1}z^{\pm1};q)_\infty}{(a^{-1/2}q^{1/2+|k|/2}v^{\pm1}z^{\pm1};q)_\infty}\frac{dz}{2\pi\mathrm{i}z}, \tag{28}$$

where the chemical potential v is associated with the identified $SU(2)$ flavour groups. After residue calculus, we obtain

$$\begin{aligned}I_{e,N_f=2;h}=\sum_{k\in\mathbb{Z}}a^{|k|}&\frac{(q^{1+|k|},q^{1+|k|}v^2,a,av^{-2};q)_\infty}{(v^{-2},q^{1+|k|}/a,q^{1+|k|}v^2/a,q;q)_\infty}\\&\times{}_4\phi_3\left[\begin{matrix}q^{1+|k|}/a,q^{1+|k|}v^2/a,q/a,qv^2/a\\q^{1+|k|},q^{1+|k|}v^2,qv^2\end{matrix};q,a^2\right]+\left(v\mapsto\frac{1}{v}\right),\end{aligned} \tag{29}$$

where $(v\mapsto\frac{1}{v})$ stands for the preceding expression in which v was replaced by $1/v$. The mirror partner of this theory has again the same $U(1)$ gauge group and two flavours with R-charges $\Delta_{q_i}=\Delta_{\widetilde{q}_i}=1-h$, $i=1,2$, and meson fields $M_{ij}=Q_i\widetilde{Q}_j, i,j=1,2$, with R-charge $\Delta_{M_{ij}}=\Delta_{Q_i}+\Delta_{\widetilde{Q}_j}=2h$, and two singlet superfields $V_\pm$ with R-charge $\Delta_{V_\pm} = 2(1-h)$. Its restricted superconformal index has the form

$$I_{m,N_f=2;h} = \frac{(av^{\pm 2};q)_\infty}{(qv^{\pm 2}/a;q)_\infty} \sum_{k\in\mathbb{Z}} (q/a)^{|k|} \int_{\mathbb{T}} \frac{(a^{-1/2}q^{1+|k|/2}v^{\pm 1}z^{\pm 1};q)_\infty}{(a^{1/2}q^{|k|/2}v^{\pm 1}z^{\pm 1};q)_\infty} \frac{dz}{2\pi \mathrm{i} z}. \tag{30}$$

Note that in the generic case considered in the Appendix the prefactor in front of the sum is considerably more complicated. Residue calculus leads to

$$\begin{aligned} I_{m,N_f=2;h} = \frac{(av^{\pm 2};q)_\infty}{(qv^{\pm 2}/a;q)_\infty} \times \Bigg(\sum_{k\in\mathbb{Z}} (q/a)^{|k|} \frac{(q^{1+|k|}, q^{1+|k|}v^2, q/a, qv^{-2}/a;q)_\infty}{(v^{-2}, aq^{|k|}, aq^{|k|}v^2, q;q)_\infty} \\ \times {}_4\phi_3 \left[\begin{matrix} aq^{|k|}, aq^{|k|}v^2, a, av^2 \\ qv^2, q^{1+|k|}, q^{1+|k|}v^2 \end{matrix} ; q, (q/a)^2 \right] + \left(v \mapsto \frac{1}{v} \right) \Bigg). \end{aligned} \tag{31}$$

Theorem 3. *The equality* $I_{e,N_f=2;h} = I_{m,N_f=2;h}$ *holds true.*

To prove this statement, we start with the double sum

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{a^{|k|}\,(q^{1+|k|}, q^{1+|k|}v^2, a, a/v^2;q)_\infty}{(1/v^2, q^{1+|k|}/a, q^{1+|k|}v^2/a, q;q)_\infty} \\ \cdot \sum_{n=0}^{\infty} \frac{(q^{1+|k|}/a, q^{1+|k|}v^2/a, q/a, qv^2/a;q)_n}{(q^{1+|k|}, q^{1+|k|}v^2, qv^2, q;q)_n} a^{2n}, \end{aligned} \tag{32}$$

representing the first term in (29) (the second being obtained by reflection $v \to 1/v$). It is again easy to see that, for $k<0$, we have

$$\begin{aligned} &\frac{a^{-k}\,(q^{1-k}, q^{1-k}v^2, a, a/v^2;q)_\infty}{(1/v^2, q^{1-k}/a, q^{1-k}v^2/a, q;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{1-k}/a, q^{1-k}v^2/a, q/a, qv^2/a;q)_n}{(q^{1-k}, q^{1-k}v^2, qv^2, q;q)_n} a^{2n} \\ &\quad = \frac{a^{k}\,(q^{1+k}, q^{1+k}v^2, a, a/v^2;q)_\infty}{(1/v^2, q^{1+k}/a, q^{1+k}v^2/a, q;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{1+k}/a, q^{1+k}v^2/a, q/a, qv^2/a;q)_n}{(q^{1+k}, q^{1+k}v^2, qv^2, q;q)_n} a^{2n}, \end{aligned}$$

if one interprets the left-hand side as the appropriate limit (namely that $(q^{1-k};q)_\infty/(q^{1-k};q)_n = (q^{1+n-k};q)_\infty$), by observing that the terms for $n = 0, 1, \ldots, k-1$ do not contribute to the sum on the left-hand side so that one can replace n by $n+k$ there. Consequently, we may rewrite (32) as

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{k=-n}^{\infty} \frac{a^{k}\,(q^{1+k+n}, q^{1+k}v^2, a, a/v^2;q)_\infty}{(1/v^2, q^{1+k}/a, q^{1+k}v^2/a, q;q)_\infty} \frac{(q^{1+k}/a, q^{1+k}v^2/a, q/a, qv^2/a;q)_n}{(q^{1+k}v^2, qv^2, q;q)_n} a^{2n} \\ &\qquad = \sum_{n=0}^{\infty} \frac{a^{n}\,(q^{1+n}, q^{1+n}v^2, a, a/v^2;q)_\infty}{(1/v^2, q^{1+n}/a, q^{1+n}v^2/a, q;q)_\infty} {}_2\phi_1 \left[\begin{matrix} qv^2/a, q/a \\ qv^2 \end{matrix} ; q, a \right] \\ &\qquad = \frac{(qv^2, a, a/v^2;q)_\infty}{(1/v^2, q/a, qv^2/a;q)_\infty} {}_2\phi_1 \left[\begin{matrix} qv^2/a, q/a \\ qv^2 \end{matrix} ; q, a \right]^2. \end{aligned} \tag{33}$$

Next we consider the first term in expression (31), namely

$$\frac{(av^2,a/v^2;q)_\infty}{(qv^2/a,q/av^2;q)_\infty}\sum_{k=-\infty}^{\infty}\frac{(q/a)^{|k|}\,(q^{1+|k|},q^{1+|k|}v^2,q/a,q/av^2;q)_\infty}{(1/v^2,q^{|k|}a,q^{|k|}av^2,q;q)_\infty}\times\sum_{n=0}^{\infty}\frac{(q^{|k|}a,q^{|k|}av^2,a,av^2;q)_n}{(q^{1+|k|},q^{1+|k|}v^2,qv^2,q;q)_n}(q/a)^{2n}. \quad (34)$$

Proceeding in the same manner as before, we see that this expression equals

$$\frac{(a/v^2,qv^2,q/a;q)_\infty}{(qv^2/a,1/v^2,a;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}av^2,a\\ qv^2\end{matrix};q,\frac{q}{a}\right]^2. \quad (35)$$

The equality between (33) and (35) (and, hence, between (32) and (34)) follows immediately from Heine's ${}_2\phi_1$-transformation formula (see [30, (1.4.1)])

$${}_2\phi_1\left[\begin{matrix}A,B\\ C\end{matrix};q,Z\right]=\frac{(B,AZ;q)_\infty}{(C,Z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}C/B,Z\\ AZ\end{matrix};q,B\right] \quad (36)$$

after substituting $A=qv^2/a, B=q/a, C=qv^2, Z=a$. Since the second terms in (29) and (31) arise from the respective first terms by the substitution $v\mapsto 1/v$, this completes the proof of the claimed equality between SCIs.

The full symmetry group of the ${}_2\phi_1$-series is generated by repeated applications of this transformation together with the permutation of its numerator parameters A and B, and it is isomorphic to the dihedral group D_6 [31]. This leads to two further ${}_2\phi_1$-transformation formulas, namely (see [30, (1.4.5)])

$${}_2\phi_1\left[\begin{matrix}A,B\\ C\end{matrix};q,Z\right]=\frac{(C/B,BZ;q)_\infty}{(C,Z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}ABZ/C,B\\ BZ\end{matrix};q,\frac{C}{B}\right] \quad (37)$$

and (see [30, (1.4.6)])

$${}_2\phi_1\left[\begin{matrix}A,B\\ C\end{matrix};q,Z\right]=\frac{(ABZ/C;q)_\infty}{(Z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}C/A,C/B\\ C\end{matrix};q,\frac{ABZ}{C}\right]. \quad (38)$$

Therefore we have two more different representations of the SCI:

$$\begin{aligned}I_{e,N_f=2;h}&=\frac{(qv^2/a,a/v^2,q,q;q)_\infty}{(1/v^2,qv^2,a,q/a;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}a,a\\ q\end{matrix};q,\frac{qv^2}{a}\right]^2+\left(v\mapsto\frac{1}{v}\right)\\&=\frac{(a/v^2;q)_\infty(av^2,q;q)_\infty^2}{(1/v^2,qv^2,a,q/a,qv^2/a;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}q/a,q/a\\ q\end{matrix};q,av^2\right]^2+\left(v\mapsto\frac{1}{v}\right), \quad (39)\end{aligned}$$

following from the first transformation with $A=q/a, B=qv^2/a$ and the second one with $A=qv^2/a, B=q/a$, where we choose $C=qv^2, Z=a$ in both cases. The third transformation does not yield new results. It is necessary to clarify whether these two new cases can attain a proper interpretation as SCIs of some new mirror field theories.

Expressions (33) and (35) are given by the product of two terms coming from the sum over the monopoles and the integration over the gauge group. This resembles a product of the holomorphic and anti-holomorphic parts of the four-point function of $2d$ conformal field theory, and the mirror maps resemble the transformation formulas for these functions in different channels.

One can consider the equality $I_{e,N_f=2;h}=I_{m,N_f=2;h}$ in the limit $v=\mathrm{i}$, which can be interpreted as the reduction of the number of flavours from $N_f=2$ to

$N_f = 1$. For $v = \mathrm{i}$, the expression $I_{e,N_f=2;h}$ reduces to $I_{e,N_f=1;h}$, where q is replaced by q^2 and the integration variable z is replaced by $-z^2$. The integral $I_{m,N_f=2;h}$ reduces also to $I_{e,N_f=1;h}$ with q/a replaced by a and with the additional multiplier $(-a;q)^2_\infty/(-q/a;q)^2_\infty$. Both integrals can be evaluated exactly, as described in the previous section, and we find that the equality $I_{e,N_f=2;h}|_{v=i} = I_{m,N_f=2;h}|_{v=i}$ is reduced to the identity

$$\frac{(a^2, q^2/a, q^2/a^2; q^2)_\infty}{(q^2/a^2, a, a; q^2)_\infty} = \frac{(q^2/a^2, aq, aq; q^2)_\infty}{(a^2, q/a, q/a; q^2)_\infty}\frac{(-a;q)^2_\infty}{(-q/a;q)^2_\infty}. \tag{40}$$

Multiplying both sides by $(a^2;q^2)_\infty/(q^2/a^2;q^2)_\infty$, one can see that it reduces to the square of the relation

$$\frac{(a^2, q^2/a; q^2)_\infty}{(q^2/a^2, a; q^2)_\infty} = \frac{(aq;q^2)_\infty}{(q/a;q^2)_\infty}\frac{(-a;q)_\infty}{(-q/a;q)_\infty},$$

which is easy to verify.

From the physical point of view, this is very similar to the reduction $t_j t_k = pq$ of the elliptic hypergeometric integral describing the superconformal index of a $4d\,\mathcal{N} = 1$ SYM theory with $SU(2)$ gauge group and 8 chiral superfields [5]. The latter theory has many dual partners, and the $N_f = 4$ to $N_f = 3$ reduction leads to four different theories — the original interacting electric theory with reduced number of flavours, two interacting magnetic theories with reduced number of chiral superfields, some additional mesons, and different flavour symmetries. The fourth theory was in the confined phase with free mesons, and its index was given by an explicit infinite product. All four forms correspond to the $W(E_6)$ Weyl group symmetry of the elliptic beta integral evaluation formula [3]. Our $3d$ mirror symmetric theories look similar to the first pair of the described $4d$ dual theories, and we expect existence of other mirror partners, whose SCIs would reduce for $v = \mathrm{i}$ directly to expression (23). The final comment concerns the parameter h related to the arbitrariness of the R-charges discussed in the Introduction. As we see, equality of SCIs holds true for arbitrary h, i.e., one does not need the exact physical value of the R-charge of the quarks in the IR fixed point.

4. Generalization to arbitrary N_f

Now we consider the electric $3d\,\mathcal{N} = 2$ supersymmetric theory with $U(1)$ gauge group and arbitrary number of flavours N_f with R-charges $\Delta_{Q_i} = \Delta_{\widetilde{Q}_i} = h$, $i = 1, \dots, N_f$. The global symmetry group is $SU(N_f)_l \times SU(N_f)_r \times U(1)_J$, so we should introduce chemical potentials for each subgroup s_i, t_i, $i = 1, \dots, N_f$, and w. However, we restrict to $s_i = t_i$, $i = 1, \dots, N_f$, $w = 1$. First we present the single-particle state index of this theory,

$$\mathrm{ind}(a, q, t_i) = \frac{q^{1/2+|s|/2}(z + z^{-1})}{1 - q}\sum_{i=1}^{N_f}\left(a^{-1/2}t_i - a^{1/2}t_i^{-1}\right), \tag{41}$$

from which one can easily compute the full SCI:

$$I_{e,N_f;h} = \sum_{k\in\mathbb{Z}} a^{N_f|k|/2}\int_{\mathbb{T}}\prod_{i=1}^{N_f}\frac{(a^{1/2}q^{1/2+|k|/2}t_i^{-1}z^{\pm1};q)_\infty}{(a^{-1/2}q^{1/2+|k|/2}t_i z^{\pm1};q)_\infty}\frac{dz}{2\pi \mathrm{i} z}, \tag{42}$$

where $\prod_{i=1}^{N_f} t_i = 1$. On the mirror symmetric side we shall have an (N_f-1)-fold multiple sum over the N_f-1 analogues of the variable s and an (N_f-1)-dimensional integral over the N_f-1 analogues of the variable z. After the substitution $t_k = e^{2\pi \mathrm{i} k/N_f}$ in the electric theory SCI, we come to the initial $N_f=1$ SCI with different choice of the parameters. Therefore we expect that the (N_f-1)-multiple sums/integrals of the mirror side also reduce appropriately.

The $N_f \to N_f - 1$ reduction in SCIs is realized similar to the $4d$ indices case [6]. It can be summarized as follows. First one should substitute $t_{N_f} = a^{1/2}$ so that, as is easily seen from (42), the contribution from the N_f-th flavour drops out (physically this means that we have integrated this field out by giving large mass to it). After having done this, one should renormalize the parameters t_i, $i=1,\dots,N_f-1$, and a as follows:

$$t_i \to a^{-1/2(N_f-1)} t_i,\ i=1,\dots,N_f-1, \quad a \to a^{(N_f-1)/N_f}.$$

This brings the SCI back to expression (42) with the replacement of N_f by N_f-1.

After residue calculus, we arrive at the following sum of well-poised ${}_{2N_f}\phi_{2N_f-1}$-series:

$$I_{e,N_f;h} = \sum_{k\in\mathbb{Z}} a^{N_f|k|/2} \prod_{i=2}^{N_f} \frac{(q^{1+|k|}t_1/t_i, a/t_it_1;q)_\infty}{(t_i/t_1, q^{1+|k|}/at_1t_i;q)_\infty} \frac{(q^{1+|k|}, a/t_1^2;q)_\infty}{(q^{1+|k|}t_1^2/a, q;q)_\infty}$$
$$\times {}_{2N_f}\phi_{2N_f-1}\left[\begin{matrix} q^{1+|k|}\frac{t_1^2}{a}, q^{1+|k|}\frac{t_1t_2}{a},\dots,q^{1+|k|}\frac{t_1t_{N_f}}{a}, \frac{qt_1^2}{a}, \frac{qt_1t_2}{a},\dots,\frac{qt_1t_{N_f}}{a} \\ \frac{qt_1}{t_2},\dots,\frac{qt_1}{t_{N_f}}, q^{1+|k|}, q^{1+|k|}\frac{t_1}{t_2},\dots,q^{1+|k|}\frac{t_1}{t_{N_f}} \end{matrix}; q, a^{N_f}\right]$$
$$+ \operatorname{idem}[1;2,\dots,N_f], \quad (43)$$

where idem $[1;2,\dots,N_f]$ means that one has to add the sum of all expressions arising from the previous expression by interchanging the index 1 with the indices $2,\dots,N_f$, one by one.

As in the previous section, we can rewrite the above expression in the form

$$I_{e,N_f;h} = \frac{(a/t_1^2, qt_1^2/a;q)_\infty}{(q,q;q)_\infty} \prod_{i=2}^{N_f} \frac{(a/t_1t_i, qt_1t_i/a;q)_\infty}{(t_i/t_1, qt_1/t_i;q)_\infty} \sum_{k=0}^{\infty} a^{N_f k} \frac{(q^{1+k};q)_\infty^2}{(q^{1+k}t_1^2/a;q)_\infty^2}$$
$$\times \prod_{i=2}^{N_f} \frac{(q^{1+k}t_1/t_i;q)_\infty^2}{(q^{1+k}t_1t_i/a;q)_\infty^2} {}_{N_f}\psi_{N_f}\left[\begin{matrix} q^{1+k}t_1^2/a, q^{1+k}t_1t_2/a,\dots,q^{1+k}t_1t_{N_f}/a \\ q^{1+k}, q^{1+k}t_1/t_2,\dots,q^{1+k}t_1/t_{N_f} \end{matrix}; q, a^{N_f/2}\right]$$
$$+ \operatorname{idem}[1;2,\dots,N_f], \quad (44)$$

which finally is rewritten as

$$I_{e,N_f;h} = \frac{(a/t_1^2;q)_\infty}{(qt_1^2/a;q)_\infty} \prod_{i=2}^{N_f} \frac{(a/t_1t_i, qt_1/t_i;q)_\infty}{(t_i/t_1, qt_1t_i/a;q)_\infty}$$
$$\times {}_{N_f}\phi_{N_f-1}\left[\begin{matrix} qt_1^2/a, qt_1t_2/a,\dots,qt_1t_{N_f}/a \\ qt_1/t_2,\dots,qt_1/t_{N_f} \end{matrix}; q, a^{N_f/2}\right]^2$$
$$+ \operatorname{idem}[1;2,\dots,N_f]. \quad (45)$$

Here, we see again the separation of the sum and the integral into N_f terms, each being the square of an ${}_{N_f}\phi_{N_f-1}$-series.

One can check that the term $|k|$ in the summand can be replaced by $k\in\mathbb{Z}$ (both before and after residue calculus), which automatically leads to the termination of the k-series from below. Indeed, when $k<0$ we have

$$\begin{aligned} & a^{N_f k/2}\prod_{i=1}^{N_f}\frac{(a^{1/2}q^{1/2+k/2}t_i^{-1}z^{\pm1};q)_\infty}{(a^{-1/2}q^{1/2+k/2}t_iz^{\pm1};q)_\infty} \\ &\qquad = a^{N_f k/2}\prod_{i=1}^{N_f}\frac{(a^{1/2}q^{1/2+k/2}t_i^{-1}z^{\pm1};q)_{-k}(a^{1/2}q^{1/2-k/2}t_i^{-1}z^{\pm1};q)_\infty}{(a^{-1/2}q^{1/2+k/2}t_iz^{\pm1};q)_{-k}(a^{-1/2}q^{1/2-k/2}t_iz^{\pm1};q)_\infty} \\ &\qquad = a^{-N_f k/2}\prod_{i=1}^{N_f}\frac{(a^{1/2}q^{1/2-k/2}t_i^{-1}z^{\pm1};q)_\infty}{(a^{-1/2}q^{1/2-k/2}t_iz^{\pm1};q)_\infty}, \end{aligned} \tag{46}$$

after taking into account the constraint $\prod_{i=1}^{N_f}t_i=1$. Thus we come to the following form of the SCI:

$$I_{e,N_f;h}=\sum_{k\in\mathbb{Z}}a^{N_f k/2}\int_{\mathbb{T}}\prod_{i=1}^{N_f}\frac{(a^{1/2}q^{1/2+k/2}t_i^{-1}z^{\pm1};q)_\infty}{(a^{-1/2}q^{1/2+k/2}t_iz^{\pm1};q)_\infty}\frac{dz}{2\pi \mathrm{i} z}, \tag{47}$$

with the same contour of integration $\mathbb{T}$ as before.

Now we would like to discuss another representation of SCIs. Interchanging the sum over k and integration over z in (47), we can write

$$\begin{aligned} I_{e,N_f;h} = & \int_{\mathbb{T}}\prod_{i=1}^{N_f}\frac{(\sqrt{aq}t_i^{-1}z^{\pm1};q)_\infty}{(\sqrt{q/a}t_iz^{\pm1};q)_\infty}\, {}_{2N_f}\psi_{2N_f}\left[\begin{matrix}\sqrt{q/a}t_iz^{\pm1}\\ \sqrt{aq}t_i^{-1}z^{\pm1}\end{matrix};q,a^{N_f}\right]\frac{dz}{2\pi \mathrm{i} z} \\ & + a^{N_f/2}\int_{\mathbb{T}}\prod_{i=1}^{N_f}\frac{(q\sqrt{a}t_i^{-1}z^{\pm1};q)_\infty}{(qt_iz^{\pm1}/\sqrt{a};q)_\infty}\, {}_{2N_f}\psi_{2N_f}\left[\begin{matrix}qt_iz^{\pm1}/\sqrt{a}\\ q\sqrt{a}t_i^{-1}z^{\pm1}\end{matrix};q,a^{N_f}\right]\frac{dz}{2\pi \mathrm{i} z}. \end{aligned}$$

It would interesting to find a pure integral representation of this expression, which may help in understanding a connection with elliptic hypergeometric integrals. We were able to represent the SCI as a sum of two integrals only in the case $N_f=1$. For that purpose, we use the following compact contour integral representation for a general ${}_2\psi_2$-series:

$${}_2\psi_2\left[\begin{matrix}a,q/d\\ b,q/c\end{matrix};q,\frac{\alpha d}{\beta c}\right]=\frac{(q,q,b/a,d/c;q)_\infty}{(b,d,q/a,q/c;q)_\infty}\int_{\mathbb{T}}\frac{(az\alpha,cz\beta,q/az\alpha,q/cz\beta;q)_\infty}{(z\alpha,z\beta,b/az\alpha,d/cz\beta;q)_\infty}\frac{dz}{2\pi i z}, \tag{48}$$

where $|b/a|<|\alpha/\beta|<|c/d|$. It is obtained after multiplying two ${}_1\psi_1$-series (25) with different choices of parameters (A,B), say (a,b) respectively (c,d), and $Z_1=\alpha z$, $Z_2=\beta z$, with the subsequent integration $\int_{\mathbb{T}}dz/z$. In this way we find the expression

$$\begin{aligned} \frac{I_{e,N_f=1;h}}{(a,q;q)_\infty^2} = & \int_{\mathbb{T}^2}\frac{(zw\sqrt{q/a},\sqrt{aq}/zw;q)_\infty^2}{(\sqrt{aq}z^{\pm1},\sqrt{q/a}z^{\pm1};q)_\infty(w,a/w;q)_\infty^2}\frac{dz}{2\pi \mathrm{i} z}\frac{dw}{2\pi \mathrm{i} w} \\ & + \sqrt{a}\int_{\mathbb{T}^2}\frac{(q(zw/\sqrt{a})^{\pm1},(\sqrt{a}/zw)^{\pm1};q)_\infty}{(\sqrt{a}z^{\pm1},qz^{\pm1}/\sqrt{a};q)_\infty(w,a/w;q)_\infty^2}\frac{dz}{2\pi \mathrm{i} z}\frac{dw}{2\pi \mathrm{i} w}. \end{aligned} \tag{49}$$

The usefulness of such a representation is not clear at present.

5. Concluding remarks

We described an interplay between mirror-symmetric three-dimensional superconformal field theories and the theory of q-special functions. This is similar to the relation between SCIs in four dimensions and the theory of elliptic hypergeometric integrals [4, 6]. Actually, three-dimensional $\mathcal{N}=2$ supersymmetric field theories can be obtained by dimensional reduction from four-dimensional $\mathcal{N}=1$ theories and thus the SCIs of these theories should be related by some reduction procedure. So far this question, which was the main motivation for our work, is not understood and requires a clarification. The calculation of $3d$ SCIs is not so straightforward as in $4d$ theories, and the sums over $3d$ monopole fluxes require proper interpretation in terms of the contour integrals. An understanding of this connection would allow us to apply the full power of $4d$ dualities together with the theory of elliptic hypergeometric integrals defining SCIs [3]. Another related problem is the problem of extending our results to gauge groups other than $U(1)$ which we have considered here.

q-Special functions have a much poorer structure than elliptic hypergeometric functions. For example, for $3d$ SCIs the kernels of integral-sums satisfy first order q-difference equations with rational coefficients, which is the only substitute for the $4d$ notion of total ellipticity, which is interpreted from the physical point of view as the 't Hooft anomaly matching conditions [6]. The only somewhat analogous notion is the parity anomaly [29] (see also [32] for early references and a perturbative analysis of this phenomenon), which is associated with the classical Chern–Simons term breaking the parity.

As to reduction of the number of flavours, in the $4d$ case there was a very simple procedure for doing this at the level of SCIs — it was sufficient to restrict parameters to $s_k t_k = pq$ for removing the k-th flavour [6]. In the $3d$ case, the analogous $N_f = 2 \ \to \ N_f = 1$ result is reached in $\mathcal{N}=2$ supersymmetric theory with $U(1)$ gauge group by the parameter restriction $v = \mathrm{i}$, and a similar reduction from arbitrary N_f to $N_f = 1$ should be valid after using higher roots of unity. The reduction $N_f \to N_f - 1$ in the general case is realized by the special choice of one parameter, namely $t_{N_f} = a^{1/2}$, with a subsequent renormalization of the remaining t_j's and a.

Another interesting fact we observe is the factorization of the sum over the monopoles and the integration over the gauge group in $3d$ SCIs of $\mathcal{N}=2$ supersymmetric field theory with $U(1)$ gauge group and N_f flavours; see (45). It resembles much the separation into holomorphic and antiholomorphic parts of the correlation functions in $2d$ CFT.

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Note added. After finishing this work, we became aware of [33], where SCIs induced by the topological charge of the group $U(1)_J$ are considered for $3d$ $\mathcal{N}=8$ theories, which partially overlaps with our considerations. Furthermore, the revised version of [16], which appeared after publishing the preprint version of our paper, includes perturbative consideration of SCIs related to the group $U(1)_J$, and its authors affirm an overlap with our results.

Appendix A. General SCIs

A.1. $N_f = 1$ **case.** Let us present the formulas for $3d$ supersymmetric $U(1)$ gauge group with one flavour for general chemical potentials:

$$I_{e,N_f=1;h;w} = \sum_{k\in\mathbb{Z}} a^{|k|/2} w^k \int_{\mathbb{T}} \frac{(a^{1/2}q^{1/2+|k|/2}z^{\pm1};q)_\infty}{(a^{-1/2}q^{1/2+|k|/2}z^{\pm1};q)_\infty} \frac{dz}{2\pi \mathrm{i} z}. \tag{50}$$

Here, a crucial difference from the $w = 1$ index considered in the main text is that the sum over s contains both $|k|$ and k, which, actually, requires a modification of the general formula found in [14, 16]. Namely, we suggest that this term appears after the substitution of the term w^Q into the general trace (2), where Q is the conserved charge of the global symmetry group $U(1)_J$ generated by the topological current $J^\mu = \epsilon^{\mu\nu\sigma}F_{\nu\sigma}$ [29] which is related to the monopole flux.

After residue calculus, we get

$$I_{e,N_f=1;h;w} = \sum_{k\in\mathbb{Z}} a^{|k|/2} w^k \frac{(q^{1+|k|}, a;q)_\infty}{(q, q^{1+|k|}/a;q)_\infty} {}_2\phi_1\left[\begin{matrix} q^{1+|k|}/a, q/a \\ q^{1+|k|} \end{matrix} ;q, a\right]. \tag{51}$$

The superconformal index of the mirror theory (which is again a free theory of chiral superfields) is

$$I_{m,N_f=1;h;w} = \frac{(a;q)_\infty}{(q/a;q)_\infty} \frac{(qw^{\pm1}/a^{1/2};q)_\infty}{(a^{1/2}w^{\pm1};q)_\infty}. \tag{52}$$

The equality $I_{e,N_f=1;h;w} = I_{m,N_f=1;h;w}$ of SCIs is proven completely analogously to the case $w = 1$. We leave the details to the reader.

A.2. $N_f = 2$ **case.** Now we consider the most general SCI for dual theories with $N_f = 2$. The initial theory index has the form

$$I_{e,N_f=2;h;w} = \sum_{k\in\mathbb{Z}} a^{|k|} w^k \int_{\mathbb{T}} \frac{(a^{1/2}q^{1/2+|k|/2}u^{\pm1}z^{-1}, a^{1/2}q^{1/2+|k|/2}v^{\pm1}z;q)_\infty}{(a^{-1/2}q^{1/2+|k|/2}u^{\pm1}z, a^{-1/2}q^{1/2+|k|/2}v^{\pm1}z^{-1};q)_\infty} \frac{dz}{2\pi \mathrm{i} z}, \tag{53}$$

where the chemical potentials u and v are associated with the $SU(2)_l \times SU(2)_r$ flavour group and w corresponds to the $U(1)_J$-group discussed in [29]. After residue calculus, we obtain

$$\begin{aligned} I_{e,N_f=2;h;w} = &\sum_{k\in\mathbb{Z}} a^{|k|} w^k \frac{(q^{1+|k|}, q^{1+|k|}v^2, au/v, a/uv;q)_\infty}{(v^{-2}, q^{1+|k|}v/au, q^{1+|k|}uv/a, q;q)_\infty} \\ &\times {}_4\phi_3\left[\begin{matrix} q^{1+|k|}v/au, q^{1+|k|}uv/a, qv/au, quv/a \\ q^{1+|k|}, q^{1+|k|}v^2, qv^2 \end{matrix} ;q, a^2\right] + \left(v \mapsto \frac{1}{v}\right), \end{aligned} \tag{54}$$

which gives

$$I_{e,N_f=2;h;w} = \frac{(qv^2, au/v, a/uv;q)_\infty}{(1/v^2, qv/aa, quv/a;q)_\infty} {}_2\phi_1\left[\begin{matrix} quv/a, qv/au \\ qv^2 \end{matrix} ;q, aw^{\pm1}\right] + \left(v \mapsto \frac{1}{v}\right). \tag{55}$$

Here, we use again a short notation: the term $w^{\pm1}$ indicates that we actually mean the product of two ${}_2\phi_1$-series, one with w in place of $w^{\pm1}$, the other with w^{-1} in place of $w^{\pm1}$.

The mirror partner has SCI of the form

$$
\begin{aligned}
I_{m,N_f=2;h;w} &= \frac{(qw^{\pm 1}/a;q)_\infty}{(aw^{\pm 1};q)_\infty}\frac{(au^{\pm 1}v^{\pm 1};q)_\infty}{(qu^{\pm 1}v^{\pm 1}/a;q)_\infty}\\
&\times \sum_{k\in\mathbb{Z}}(q/a)^{|k|}w^k\int_{\mathbb{T}}\frac{(a^{-1/2}q^{1+|k|/2}v^{\pm 1}z^{-1},a^{-1/2}q^{1+|k|/2}u^{\pm 1}z;q)_\infty}{(a^{1/2}q^{|k|/2}v^{\pm 1}z,a^{1/2}q^{|k|/2}u^{\pm 1}z^{-1};q)_\infty}\frac{dz}{2\pi \mathrm{i} z}. \qquad (56)
\end{aligned}
$$

The position of some of the poles of the integrand are proportional to u. To consider the poles with position proportional to v one can make the change $z\mapsto 1/z$, and then we have a situation similar to the one discussed in the main part of the text. Residue calculus leads to

$$
\begin{aligned}
I_{m,N_f=2;h;w} &= \frac{(qw^{\pm 1}/a;q)_\infty}{(aw^{\pm 1};q)_\infty}\frac{(au^{\pm 1}v^{\pm 1};q)_\infty}{(qu^{\pm 1}v^{\pm 1}/a;q)_\infty}\\
&\times\Bigg(\sum_{k\in\mathbb{Z}}(q/a)^{|k|}w^k\frac{(q^{1+|k|},q^{1+|k|}v^2,qu/av,q/auv;q)_\infty}{(v^{-2},aq^{|k|}v/u,aq^{|k|}uv,q;q)_\infty}\\
&\qquad\times {}_4\phi_3\left[\begin{matrix}aq^{|k|}v/u,aq^{|k|}uv,av/u,auv\\ q^{1+|k|},q^{1+|k|}v^2,qv^2\end{matrix};q,(q/a)^2\right]+\left(v\mapsto\frac{1}{v}\right)\Bigg), \qquad (57)
\end{aligned}
$$

and, finally, to

$$
\begin{aligned}
I_{m,N_f=2;h;w} &= \frac{(qw^{\pm 1}/a;q)_\infty}{(aw^{\pm 1};q)_\infty}\frac{(au^{\pm 1}v^{\pm 1};q)_\infty}{(qu^{\pm 1}v^{\pm 1}/a;q)_\infty}\\
&\times\left(\frac{(qv^2,qu/av,q/auv;q)_\infty}{(1/v^2,av/u,auv;q)_\infty}{}_2\phi_1\left[\begin{matrix}auv,av/u\\ qv^2\end{matrix};q,qw^{\pm 1}/a\right]+\left(v\mapsto\frac{1}{v}\right)\right), \qquad (58)
\end{aligned}
$$

with the same short notation as in (55).

One can easily see that (55) and (58) coincide by using the transformation formulas (36) and (38) in the text. Thus we established the equality of SCIs in the most general possible setting.

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