

AN EXACT RENORMALIZATION FORMULA FOR THE MARYLAND MODEL

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ABSTRACT. We discuss the difference Schrödinger equation $\psi_{k+1} + \psi_{k-1} + \lambda \cot(\pi\omega k + \theta)\psi_k = E\psi_k$, $k \in \mathbb{Z}$, where λ , ω , θ and E are parameters. We obtain explicit renormalization formulas relating its solutions for large $|k|$ to solutions of the equation with new parameters λ , ω , θ and E for bounded $|k|$. These formulas are similar to the renormalization formulas from the theory of Gaussian exponential sums.

1. INTRODUCTION

We consider the difference Schrödinger equation

$$\psi_{k+1} + \psi_{k-1} + \lambda \cot(\pi(\omega k + \theta))\psi_k = E\psi_k, \quad k \in \mathbb{Z}, \quad (1.1)$$

where $\omega \in (0, 1) \setminus \mathbb{Q}$, $\theta \in [0, 1)$, $\lambda > 0$ and $E \in \mathbb{R}$ are parameters; E is called the *spectral parameter*.

The Schrödinger operator in $l^2(\mathbb{Z})$ corresponding to (1.1) is referred to as the Maryland model. It is one of the popular models of spectral theory [4, 10]: being a non-trivial almost periodic operator, many of its important spectral properties can be explicitly described. There are interesting open problems related to the behavior of solutions of (1.1) for large $|k|$. For example, one can mention the study of the spectrum of the Maryland model for frequencies that are neither well nor badly approximable by rational numbers, e.g. [10], the investigation of the multiscale behavior of its (generalized) eigenfunctions, and the explanation of the time evolution generated by the Maryland model, e.g. [11].

In this paper, for sake of brevity we call (1.1) the Maryland equation.

The central result of this paper is a renormalization formula expressing solutions of (1.1) in terms of solutions of the Maryland equation with new parameters $\omega, \theta, \lambda, E$ for smaller $|k|$. This formula is similar to the well-known renormalization formula from the theory of Gaussian exponential sums, see, for example, [9].

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To describe the main result, define the parameters $l > 0$ and $-\pi < \eta < \pi$ in terms of E and λ so that $E + i\lambda = 2 \cos(\eta + il)$. Then

$$\lambda = -2\text{sh } l \sin \eta, \quad E = 2\text{ch } l \cos \eta. \quad (1.2)$$

Put

$$\mathcal{F}(z, \eta, l) = \begin{pmatrix} 2\text{ch } l \cos \eta + 2\text{sh } l \sin \eta \cot(\pi z) & -1 \\ 1 & 0 \end{pmatrix}. \quad (1.3)$$

The Maryland equation (1.1) is equivalent to the equation

$$\Psi_{k+1} = \mathcal{F}(k\omega + \theta, \eta, l) \Psi_k, \quad k \in \mathbb{Z} \quad (1.4)$$

(write down the equation for the first component of a vector solution of (1.4) !). Let $\mathcal{P}_k(\omega, \theta, \eta, l)$ be the matrix solution of (1.4) that is equal to the identity matrix for $k = 0$. Note that

$$\begin{aligned} \mathcal{P}_k(\omega, \theta, \eta, l) &= \mathcal{F}(\theta + (k-1)\omega, \eta, l) \dots \mathcal{F}(\theta + \omega, \eta, l) \mathcal{F}(\theta, \eta, l), \quad k \geq 1, \\ \mathcal{P}_k(\omega, \theta, \eta, l) &= \mathcal{F}(\theta + k\omega, \eta, l)^{-1} \dots \mathcal{F}(\theta - 2\omega, \eta, l)^{-1} \mathcal{F}(\theta - \omega, \eta, l)^{-1}, \quad k \leq -1. \end{aligned}$$

The main result is described by

Theorem 1.1. *For any $N \in \mathbb{Z}$, one has*

$$\mathcal{P}_N(\omega, \theta, \eta, l) = \Psi(\{\theta + N\omega\}, \eta, l) \sigma_2 \mathcal{P}_{N_1}(\omega_1, \theta_1, \eta_1, l_1) \sigma_2 \Psi^{-1}(\theta, \eta, l), \quad (1.5)$$

where

$$N_1 = -[\theta + N\omega], \quad \omega_1 = \left\{ \frac{1}{\omega} \right\}, \quad \theta_1 = \left\{ \frac{\theta}{\omega} \right\}, \quad \eta_1 = \frac{\eta}{\omega} \bmod 2\pi, \quad l_1 = \frac{l}{\omega}, \quad (1.6)$$

$[x]$ and $\{x\}$ denote the integer and the fractional parts of $x \in \mathbb{R}$,

$$\Psi(z, \eta, l) = \begin{pmatrix} \psi(z, \eta, l) & \psi(z-1, \eta, l) \\ \psi(z-\omega, \eta, l) & \psi(z-1-\omega, \eta, l) \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (1.7)$$

and ψ is the minimal meromorphic solution of the “complex Maryland equation”

$$\psi(z+\omega) + \psi(z-\omega) + \lambda \cot(\pi z) \psi(z) = E \psi(z), \quad z \in \mathbb{C}. \quad (1.8)$$

The importance of the minimal entire solutions, i.e., the solutions having the slowest possible growth for $\text{Im } z \rightarrow \pm\infty$, for the study of difference equations with entire periodic coefficients was revealed in [3]. For equation (1.8), the definition of the minimal meromorphic solution is formulated in Section 2. In the same section, we find out that this solution satisfies one more complex Maryland equation with new parameters. This is one of the key observations leading to the renormalization formula (1.5). The minimal solution is constructed in Section 4, where we obtain integral representations for it.

The above renormalization formula for the matrix product is as explicit as the renormalization formula obtained in [9] for the Gaussian exponential sums. These formulas have a similar structure. In (1.5), $N_1 \sim -\omega N$ for large N , and, as $0 < \omega < 1$, the analysis of the matrix product $\mathcal{P}_N(\omega, \theta, \eta, l)$ with a large number of factors is reduced to analysis of an analogous product with a smaller one. It is important to note that the factors $\Psi(\dots)$ in the right-hand side of (1.5) have to be controlled only on the interval $[0, 1)$.

As in [9], one can easily show that, after a finite number (of order of $\log N$) of the renormalizations applied consequently to the matrix products $\mathcal{P}_N(\omega, \theta, \eta, l)$, $\mathcal{P}_{N_1}(\omega_1, \theta_1, \eta_1, l_1)$ etc., one can reduce the number of factors to one. In the analysis of the Gaussian sums, the main role was played by quasiclassical effects arising

when the frequency $\omega_L = \{1/\omega_{L-1}\}$, $L \geq 1$, $\omega_0 = \omega$, is small. In the case of the Maryland equation, there is an additional effect. It is well known that the product $\omega_0\omega_1 \dots \omega_L$ exponentially decreases when L grows. Therefore, after many renormalizations, one encounters the large parameter $l/(\omega_0\omega_1 \dots \omega_L)$. So, one can expect that the analysis of the behavior of $\mathcal{P}_N(\omega, \theta, \eta, l)$ for large N can be very effective. We plan to employ this idea in our next publication.

Theorem 1.1 is obtained in Section 3. Its proof is based on monodromization method ideas. This method is a general renormalization approach suggested by V. Buslaev and A. Fedotov for studying difference equations on $\mathbb{R}$ with periodic coefficients. It was developed further in papers of A. Fedotov and F. Klopp, see the review article [6]. In Section 3, we describe the monodromization idea and give a proof of a general renormalization formula for the case of difference equations on $\mathbb{Z}$ with coefficients being restrictions to $\mathbb{Z}$ of functions defined and periodic on $\mathbb{R}$. Note that a similar formula was stated without proof in [8]. Formula (1.5) is a corollary from the general one and from the observation that the minimal meromorphic solution of the complex Maryland equation satisfies one more complex Maryland equation (with new parameters). This observation is equivalent to the fact that *the complex Maryland equation is invariant with respect to monodromization*. The reader finds more details in Section 3.

2. MINIMAL SOLUTIONS

In this section, we discuss the difference equations on the complex plane only. Equation (1.8) is invariant with respect to multiplication by $e^{2\pi iz/\omega}$. Therefore, if it has a meromorphic solution, it has meromorphic solutions growing as quickly as desired when $\text{Im } z \rightarrow \pm\infty$. To define the minimal meromorphic solution, i.e., the solution having the slowest growth for $\text{Im } z \rightarrow \pm\infty$, one has to impose some natural conditions on the set of its poles. To give the precise definition, we need to discuss the set of solutions of (1.8).

2.1. Solutions of difference equations. Let us list well-known elementary properties of the solutions of the equation

$$\psi(z + \omega) + \psi(z - \omega) + v(z)\psi(z) = 0, \quad z \in \mathbb{C}, \quad (2.1)$$

where v is a given function, and $\omega > 0$ is a given number.

Let ψ and $\tilde{\psi}$ be two solutions to (2.1). It can be easily seen that the expression

$$w(\psi(z), \tilde{\psi}(z)) = \psi(z)\tilde{\psi}(z - \omega) - \psi(z - \omega)\tilde{\psi}(z) \quad (2.2)$$

is ω -periodic in z . It is called the Wronskian of ψ and $\tilde{\psi}$.

If $w(\psi(z), \tilde{\psi}(z)) \neq 0$ for all z , one can show that any other solution ϕ admits the representation

$$\phi(z) = a(z)\psi(z) + b(z)\tilde{\psi}(z), \quad z \in \mathbb{C}, \quad (2.3)$$

with some ω -periodic a and b . This implies that the solution space of (2.1) is a two-dimensional module over the ring of ω -periodic functions. Note that (2.3) and the Wronskian definition imply that

$$a(z) = \frac{w(\phi(z), \tilde{\psi}(z))}{w(\psi(z), \tilde{\psi}(z))}, \quad b(z) = \frac{w(\psi(z), \phi(z))}{w(\psi(z), \tilde{\psi}(z))}. \quad (2.4)$$

2.2. The simplest solutions to the complex Maryland equation in a neighborhood of $\pm i\infty$. The periodicity of the potential in the complex Maryland equation allows to consider $+i\infty$ and $-i\infty$ as two singular points. For $Y \in \mathbb{R}$, we call the half-plane $\mathbb{C}_+(Y) = \{z \in \mathbb{C} : \text{Im } z > Y\}$ a neighborhood of $+i\infty$, and we call $\mathbb{C}_-(Y) = \{z \in \mathbb{C} : \text{Im } z < Y\}$ a neighborhood of $-i\infty$.

The minimal meromorphic solutions of the complex Maryland equation are defined in terms of the solutions having the “simplest” behavior in neighborhoods of $\pm i\infty$. The latter are described in

Theorem 2.1. *For sufficiently large $Y > 0$, in $\mathbb{C}_+(Y)$, there exist analytic solutions $u_\pm$ to the complex Maryland equation such that*

$$u_\pm(z) = e^{\pm \frac{l-i\eta}{\omega} z} (1 + o(1)), \quad \text{Im } z \rightarrow +i\infty, \quad (2.5)$$

uniformly in $z \in K_C = \{z \in \mathbb{C} : |\text{Im } z| \geq C|\text{Re } z|\}$, where $C > 0$ is an arbitrary fixed constant. In the terminology of [2], $u_\pm$ are Bloch solutions, i.e., $u_\pm(z+1) = \alpha_\pm(z)u_\pm(z)$ with some ω -periodic factors $\alpha_\pm$.

This theorem is proved in Section 4.4.

In a neighborhood of $-i\infty$, one can construct solutions $d_\pm$ similar to $u_\pm$. It is convenient to define them by the formulas

$$d_\pm(z) = \overline{u_\pm(\bar{z})}. \quad (2.6)$$

Representations (2.5) imply that

$$w(u_+, u_-) = e^{l-i\eta} - e^{-l+i\eta} + o(1), \quad \text{Im } z \rightarrow +\infty. \quad (2.7)$$

Therefore, for sufficiently large Y , the solutions $u_\pm$ form a basis for the space of the solutions defined on $\mathbb{C}_+(Y)$. Similarly, $d_\pm$ form a basis for the space of the solutions defined on $\mathbb{C}_-(-Y)$. We call the couples $(u_\pm)$ and $(d_\pm)$ the *canonical bases* for neighborhoods of $+i\infty$ and $-i\infty$, respectively.

We define $\alpha_\pm(z) = u_\pm(z+1)/u_\pm(z)$ and $\beta_\pm(z) = d_\pm(z+1)/d_\pm(z)$. The functions $\alpha_\pm$ and $\beta_\pm$ are ω -periodic. Representations (2.5) and (2.6) imply that

$$\alpha_\pm(z) = e^{\pm \frac{l-i\eta}{\omega}} (1 + o(1)), \quad \text{Im } z \rightarrow +i\infty, \quad (2.8)$$

$$\beta_\pm(z) = e^{\pm \frac{l+i\eta}{\omega}} (1 + o(1)), \quad \text{Im } z \rightarrow -i\infty. \quad (2.9)$$

2.3. Minimal meromorphic solution of the complex Maryland equation.

Let ψ be a solution of (1.8) analytic in the strip $S_0 = \{z \in \mathbb{C} : |\text{Re } z| \leq \omega\}$.

Remark 2.1. Equation (1.8) implies that ψ can be continued to a meromorphic function that can have poles only at the points $\pm(n + m\omega)$, $n, m \in \mathbb{N}$. Moreover, the poles located at the points $\pm(1 + \omega)$ are simple. For this new function, we keep the old notation ψ .

Let Y be chosen as in Theorem 2.1. The solution ψ admits the representations

$$\psi(z) = A_+(z)u_+(z) + A_-(z)u_-(z), \quad z \in \mathbb{C}_+(Y), \quad (2.10)$$

$$\psi(z) = B_+(z)d_+(z) + B_-(z)d_-(z), \quad z \in \mathbb{C}_-(Y), \quad (2.11)$$

with some ω -periodic analytic coefficients $A_\pm$ and $B_\pm$.

Definition 2.1. The solution ψ is called a *minimal meromorphic solution* to (1.8) if the coefficients $A_\pm$ and $B_\pm$ are bounded in $\mathbb{C}_\pm(Y)$.

For a minimal meromorphic solution ψ , the limits $a_{\pm}$ of $A_{\pm}$ for $\text{Im } z \rightarrow +\infty$ and the limits $b_{\pm}$ of $B_{\pm}$ for $\text{Im } z \rightarrow -\infty$ exist and are equal to the zeroth Fourier coefficients of $A_{\pm}$ and $B_{\pm}$, respectively. We call $a_{\pm}$ and $b_{\pm}$ the asymptotic coefficients of the minimal solution ψ .

In Section 4, we prove

Theorem 2.2. *For $|\eta| < \pi(1 + \omega)$, there exists a minimal meromorphic solution ψ of the complex Maryland equation. It is analytic in η , and its asymptotic coefficients do not vanish at $\eta \notin \omega\mathbb{Z}$.*

Remark 2.2. *The minimal meromorphic solution described in this theorem can be continued to a meromorphic function of η and l .*

The term “minimal” is explained by

Theorem 2.3. *Let ψ be a minimal meromorphic solution, and let its asymptotic coefficients $a_{\pm}$ (or $b_{\pm}$) be non-zero. Then*

- *any other minimal solution coincides with ψ up to a constant factor;*
- *if ϕ is a minimal solution, and one of its asymptotic coefficients is zero, then $\phi \equiv 0$.*

When proving this theorem, we use

Lemma 2.1. *Let ψ be a minimal solution. Then*

$$\begin{aligned} w(\psi(z+1), \psi(z)) &= a_+ a_- \left(e^{\frac{l-i\eta}{\omega}} - e^{-\frac{l-i\eta}{\omega}} \right) (e^{l-i\eta} - e^{-l+i\eta}) \\ &= b_+ b_- \left(e^{\frac{l+i\eta}{\omega}} - e^{-\frac{l+i\eta}{\omega}} \right) (e^{l+i\eta} - e^{-l-i\eta}). \end{aligned}$$

Proof. Remark 2.1 implies that the Wronskian of ψ and $\psi(\cdot + 1)$ is analytic in the strip $\{z \in \mathbb{C} : -1 < \text{Re } z < \omega\}$. As the Wronskian is ω -periodic, it is an entire function.

Let us study the behavior of the Wronskian for $\text{Im } z \rightarrow +\infty$. Recall that $u_{\pm}$ are Bloch solutions, $u_{\pm}(z+1) = \alpha_{\pm}(z)u_{\pm}(z)$, where $\alpha_{\pm}$ are ω -periodic. By means of (2.10), we get

$$\psi(z+1) = A_+(z+1)\alpha_+(z)u_+(z) + A_-(z+1)\alpha_-(z)u_-(z).$$

From this formula, representation (2.10) for ψ , and the ω -periodicity of $\alpha_{\pm}$ and $A_{\pm}$, we deduce that

$$\begin{aligned} w(\psi(z+1), \psi(z)) &= w(u_+(z), u_-(z)) \cdot \\ &\quad \cdot (A_+(z+1)A_-(z)\alpha_+(z) - A_-(z+1)A_+(z)\alpha_-(z)). \end{aligned} \tag{2.12}$$

Finally, the asymptotics (2.7), the definition of the asymptotic coefficients, and (2.8) imply that, as $\text{Im } z \rightarrow +\infty$, the right-hand side in (2.12) tends to the first expression for the Wronskian given in Lemma 2.1.

One similarly proves that, as $\text{Im } z \rightarrow -\infty$, $w(\psi(z+1), \psi(z))$ tends to the second expression for the Wronskian described in this lemma.

As the Wronskian is an entire periodic function that has finite limits as $\text{Im } z \rightarrow \pm\infty$, it is bounded. Being a bounded entire function, the Wronskian is independent of z , and

$$w(\psi(z+1), \psi(z)) = \lim_{\text{Im } z \rightarrow -\infty} w(\psi(z+1), \psi(z)) = \lim_{\text{Im } z \rightarrow +\infty} w(\psi(z+1), \psi(z)).$$

This leads to the statement of the lemma. $\square$

Let us turn to the proof of Theorem 2.3.

Proof. Let ϕ be one more minimal meromorphic solution of the complex Maryland equation. Assume that the asymptotic coefficients $a_{\pm}$ of ψ are non-zero (the case of $b_{\pm} \neq 0$ is treated in the same way). The above lemma implies that the solutions $z \rightarrow \psi(z)$ and $z \rightarrow \psi(z+1)$ form a basis for the space of solutions of the complex Maryland equation. Therefore, ϕ admits representation (2.3) with $\tilde{\psi}(z) = \psi(z+1)$. Recall that the coefficients in this representation are described in (2.4). As when proving Lemma 2.1, one shows that the Wronskians $w(\phi(z), \tilde{\psi}(z))$ and $w(\psi(z), \phi(z))$ in (2.4) are independent of z . As $w(\psi(z+1), \psi(z))$ is also independent of z , see Lemma 2.1, to check the first statement of the theorem, it suffices to check that $w(\phi(z), \psi(z))$ vanishes at some z . Since ψ and ϕ are analytic in $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq \omega\}$, the complex Maryland equation implies that both these solutions vanish at $z = 0$. Therefore, $w(\phi(z), \psi(z))|_{z=0} = 0$. This completes the proof of the first statement of the theorem.

Let us prove the second one. Lemma 2.1 shows that all the asymptotic coefficients of ψ are non-zero. By the first statement, $\phi = C\psi$ with a constant C . Therefore, asymptotic coefficients of the solutions ϕ and ψ are proportional with the same constant C . As one of the asymptotic coefficients of ϕ is zero, we have $C = 0$. Thus, $\phi = 0$. $\square$

2.4. Second difference equation for the minimal solutions. A central property of the minimal solutions is described by

Theorem 2.4. *Let ψ be a minimal meromorphic solution of the complex Maryland equation. Assume that its asymptotic coefficients are non-zero. Then it solves the equation*

$$\psi(z+1) + \psi(z-1) + \lambda_1 \cot(\pi z/\omega) \psi(z) = E_1 \psi(z), \quad z \in \mathbb{C}, \quad (2.13)$$

where

$$\lambda_1 = -2 \sin \eta_1 \operatorname{sh} l_1, \quad E_1 = 2 \cos \eta_1 \operatorname{ch} l_1, \quad (2.14)$$

and l_1 and η_1 are related to l and η by the formulas in (1.6).

Remark 2.3. As the minimal solution described in Theorem 2.2 is analytic in η , it solves (2.13) even if $\eta \in \omega\mathbb{Z}$, i.e., if its asymptotic coefficients vanish.

Proof. Lemma 2.1 implies that ψ and $\tilde{\psi} = \psi(\cdot + 1)$ form a basis for the solution space of the complex Maryland equation. The function $\phi = \psi(\cdot - 1)$ also solves this equation and, therefore, admits representation (2.3) with the periodic coefficients described by (2.4). As $w(\psi(z+1), \psi(z))$ is independent of z , see Lemma 2.1, the coefficient b in this representation is identically equal to -1 . So, to prove the theorem, it suffices to calculate the coefficient

$$a(z) = \frac{w(\psi(z-1), \psi(z+1))}{w(\psi(z), \psi(z+1))}. \quad (2.15)$$

The Wronskian in the denominator in this formula is described by Lemma 2.1. Let us discuss the Wronskian in the numerator. Remark 2.1 implies that $z \rightarrow w(\psi(z-1), \psi(z+1))$ is a meromorphic ω -periodic function analytic in the strip $0 < \operatorname{Re} z < \omega$ and that, on the boundary of the strip, it may have poles only at the

points $z = 0$ and $z = \omega$. By a reasoning similar to one from the proof of Lemma 2.1, one shows that, as $\text{Im } z \rightarrow +\infty$, the Wronskian tends to

$$a_+ a_- \left(e^{-\frac{2(l-i\eta)}{\omega}} - e^{\frac{2(l-i\eta)}{\omega}} \right) (e^{l-i\eta} - e^{-l+i\eta}),$$

and, as $\text{Im } z \rightarrow -\infty$, it tends to

$$b_+ b_- \left(e^{-\frac{2(l+i\eta)}{\omega}} - e^{\frac{2(l+i\eta)}{\omega}} \right) (e^{l+i\eta} - e^{-l-i\eta}).$$

We see that a is a meromorphic ω -periodic function, it may have poles only at $z \in \omega\mathbb{Z}$, these poles are simple, and a tends to constants as $\text{Im } z \rightarrow \pm\infty$. This implies that $a(z) = E_1 - \lambda_1 \cot(\pi z/\omega)$ with some constants E_1 and λ_1 .

The above asymptotics for $w(\psi(z-1), \psi(z+1))$ and the asymptotics for $w(\psi(z+1), \psi(z))$ in Lemma 2.1 imply that

$$a(z) \rightarrow \begin{cases} e^{\frac{l-i\eta}{\omega}} + e^{-\frac{l-i\eta}{\omega}}, & \text{Im } z \rightarrow +\infty, \\ e^{\frac{l+i\eta}{\omega}} + e^{-\frac{l+i\eta}{\omega}}, & \text{Im } z \rightarrow -\infty. \end{cases}$$

Therefore, $E_1 = 2 \cos(\eta/\omega) \text{ch}(l/\omega)$ and $\lambda_1 = -2 \sin(\eta/\omega) \text{sh}(l/\omega)$. This completes the proof of the theorem. $\square$

3. MONODROMIZATION AND RENORMALIZATION FORMULAS

In this section, first, following [6], we recall basic ideas of the monodromization theory, next, we prove a general renormalization formula for matrix cocycles, then, we describe some corollaries from these constructions for the Maryland equation and, after that, prove Theorem 1.1.

3.1. Monodromization.

3.1.1. *Monodromy matrix.* Consider the matrix solutions of the equation

$$\Psi(x + \omega) = M(x) \Psi(x), \quad x \in \mathbb{R}, \quad (3.1)$$

where M is a given 1-periodic $\text{SL}(2, \mathbb{C})$ -valued function and $0 < \omega < 1$ is a fixed number.

For any solution Ψ of equation (3.1), $\det \Psi$ is an ω -periodic function. We call a solution Ψ *fundamental*, if $\det \Psi$ is independent of x and does not vanish. Below, we assume that Ψ is a fundamental solution.

A function $\tilde{\Psi} : \mathbb{R} \rightarrow \text{GL}(2, \mathbb{C})$ solves (3.1) if and only if

$$\tilde{\Psi}(x) = \Psi(x) \cdot p(x), \quad \forall x \in \mathbb{R}, \quad (3.2)$$

where p is an ω -periodic matrix function.

The function $x \rightarrow \Psi(x+1)$ is a solution of (3.1) together with Ψ . Therefore,

$$\Psi(x+1) = \Psi(x) \cdot p(x), \quad p(x+\omega) = p(x), \quad \forall x \in \mathbb{R}.$$

The matrix

$$M_1(x) = p^t(\omega x),$$

where t denotes transposition, is the *monodromy* matrix corresponding to the fundamental solution Ψ . Like the matrix M from the input equation, the monodromy matrix is ω -periodic and unimodular.

3.1.2. *Very short introduction to the monodromization theory.* Let ω_1 be the Gauss transform of ω , i.e. $\omega_1 = \{\frac{1}{\omega}\}$. Consider the equation

$$\Psi_1(x + \omega_1) = M_1(x) \Psi_1(x), \quad x \in \mathbb{R}, \quad (3.3)$$

where M_1 is the monodromy matrix corresponding to a fundamental solution Ψ of (3.1). We say that (3.3) is a *monodromy equation* obtained from (3.1) via *monodromization*.

The monodromy equation (3.3) is similar to the input one: the matrices M and M_1 are both unimodular and 1-periodic. Therefore, the *monodromization procedure* can be continued: one can consider the monodromy matrix corresponding to a fundamental solution of (3.3) and the corresponding monodromy equation and so on. In result, one arrives to an infinite sequence of difference equations similar to the input one. There are deep relationships between these equations (see, for example, Theorem 3.1). The leading idea of the monodromization method is to analyze solutions of the input equation by analyzing properties of the dynamical system that defines the coefficients of each equation in the sequence in terms of the coefficients of the previous one.

3.1.3. *Renormalization of matrix cocycles.* Together with (3.1), consider the family of difference equations on $\mathbb{Z}$

$$\Psi_{k+1} = M(\omega k + \theta) \Psi_k, \quad k \in \mathbb{Z}, \quad (3.4)$$

where $0 \leq \theta < 1$ is the parameter indexing the equations. Let $k \rightarrow P_k(M, \omega, \theta)$ be the solution of (3.4) equal to the identity matrix when $k = 0$. It is obvious that $P_k(M, \omega, \theta) = M(\omega(k-1) + \theta) \dots M(\omega + \theta) M(\theta)$, when $k > 0$, and $P_k(M, \omega, \theta) = M^{-1}(\omega k + \theta) \dots M^{-1}(\omega - 2\theta) M(\omega - \theta)$, when $k < 0$.

Theorem 3.1 (on renormalizations of matrix cocycles). *Let Ψ be a fundamental solution of (3.1), and let M_1 be the corresponding monodromy matrix. Then, for all $N \in \mathbb{Z}$,*

$$P_N(M, \omega, \theta) = \Psi(\{\theta + N\omega\}) \sigma_2 P_{N_1}(M_1, \omega_1, \theta_1) \sigma_2 \Psi^{-1}(\theta), \quad (3.5)$$

$$N_1 = -[\theta + N\omega], \quad \omega_1 = \{1/\omega\}, \quad \theta_1 = \{\theta/\omega\}, \quad (3.6)$$

where σ_2 is the matrix defined in (1.7).

A renormalization formula similar to (3.5) was stated without proof in [8]. Formula (3.5) relates the solution $P(M, \omega, \theta)$ of equation (3.4) to the solution $P(M_1, \omega_1, \theta_1)$ of the equation of the same form but with the matrix M_1 and the parameters ω_1 and θ_1 instead of M , ω and θ .

Proof. In the case of $N = 0$, the statement is obvious. Assume that $N > 0$ (the case of $N < 0$ is treated similarly). Equation (3.1) implies that

$$\Psi(\theta + N\omega) = M(\theta + (N-1)\omega) M(\theta + (N-2)\omega) \dots M(\theta) \Psi(\theta) = P_N(M, \omega, \theta) \Psi(\theta).$$

The solution Ψ being fundamental, the matrix $\Psi(\theta)$ is invertible. Therefore,

$$P_N(M, \omega, \theta) = \Psi(\theta + N\omega) \Psi^{-1}(\theta). \quad (3.7)$$

The definition of the monodromy matrix M_1 implies that $\Psi(x) = \Psi(x-1)M_1^t\left(\frac{x-1}{\omega}\right)$. Using this relation to express $\Psi(\theta + N\omega)$ in terms of $\Psi(\{\theta + N\omega\})$, we get

$$\begin{aligned} P_N(M, \omega, \theta) &= \Psi(\{\theta + N\omega\}) M_1^t \left(\frac{\theta + N\omega - [\theta + N\omega]}{\omega} \right) \dots \\ &\dots M_1^t \left(\frac{\theta + N\omega - 2}{\omega} \right) M_1^t \left(\frac{\theta + N\omega - 1}{\omega} \right) \Psi^{-1}(\theta). \end{aligned}$$

Taking into account the 1-periodicity of M_1 and using (3.6), we arrive at the formula

$$P_N(M, \omega, \theta) = \Psi(\{\theta + N\omega\}) M_1^t(\theta_1 + N_1\omega_1) \dots M_1^t(\theta_1 - 2\omega_1) M_1^t(\theta_1 - \omega_1) \Psi^{-1}(\theta).$$

For any $A \in SL(2, \mathbb{C})$, one has $A^t = \sigma_2 A^{-1} \sigma_2$. Therefore,

$$\begin{aligned} \Psi(\{\theta + N\omega\}) \sigma_2 M_1^{-1}(\theta_1 + N_1\omega_1) \dots M_1^{-1}(\theta_1 - 2\omega_1) M_1^{-1}(\theta_1 - \omega_1) \sigma_2 \Psi^{-1}(\theta) &= \\ = \Psi(\{\theta + N\omega\}) \sigma_2 P_{N_1}(M_1, \omega_1, \theta_1) \sigma_2^{-1} \Psi^{-1}(\theta). \end{aligned}$$

This implies the statement of the theorem. $\square$

3.2. Monodromization and the Maryland equation.

3.2.1. Monodromy matrix for the complex Maryland equation. Let ψ be the minimal meromorphic solution of the complex Maryland equation described in Theorem 2.2. In terms of ψ , we construct the matrix Ψ as in (1.7). One has

Lemma 3.1. *The function $z \rightarrow \Psi(z, \eta, l)$ solves the equation*

$$\Psi(z + \omega) = \mathcal{F}(z, \eta, l) \Psi(z), \quad z \in \mathbb{C}. \quad (3.8)$$

If $\eta \notin \omega\mathbb{Z}$, the solution Ψ is fundamental.

Proof. The first statement is obvious. As $\det \Psi$ equals the Wronskian of ψ and $\psi(\cdot - 1)$, the second statement follows from Lemma 2.1. $\square$

Remark 3.1. *By Theorem 2.2, the solution Ψ analytically depends on η .*

The definition of the monodromy matrix and Theorem 2.4 imply

Theorem 3.2. *If $\eta \notin \omega\mathbb{Z}$, the monodromy matrix corresponding to the fundamental solution $\Psi(\cdot, \eta, l)$ equals $\mathcal{F}(\cdot, \eta_1, l_1)$, where η_1 and l_1 are defined by the formulas in (1.6).*

3.2.2. Invariance with respect to monodromization. Theorem 3.2 means that the matrix Maryland equation (3.8) is invariant with respect to monodromization: after monodromization, it appears to be transformed into the matrix Maryland equation with new parameters. Like (3.8), the latter is equivalent to a complex Maryland equation (1.8) (the equation with new parameters), and one can say that the complex Maryland equation is invariant with respect to monodromization. Actually, it is this invariance that leads to the renormalization formula (1.5).

In [3], the authors consider difference equations on $\mathbb{C}$ the coefficients of which are trigonometric polynomials, and describe equation families invariant with respect to monodromization. One of these families contains an equation related to the famous Almost Mathieu equation (in the same way as the Maryland equation is related to the complex Maryland equation). However, in general case, investigation of the trigonometric polynomial coefficients transformation, which occurs in result of monodromization, appears to be a very non-trivial problem. Only for very special trigonometric polynomials, this transformation is known to be elementary [8]. The

explicit transformation of the complex Maryland equation coefficients proved in this paper looks to be a very rare phenomenon.

3.2.3. Renormalization formula for the Maryland equation. For $\eta \notin \omega\mathbb{Z}$, Theorem 1.1 immediately follows Theorems 3.1 and 3.2. As the renormalization formula (1.5) is an equality of two functions analytic in η , the statement of Theorem 1.1 remains valid also for $\eta \in \omega\mathbb{Z}$.

4. CONSTRUCTION OF THE MINIMAL MEROMORPHIC SOLUTION

In this section, we construct a minimal meromorphic solution of the complex Maryland equation (1.8). First, we describe a solution analytic in $\mathbb{C} \setminus \mathbb{R}$. Next, we check that it can be continued to a meromorphic function. Then, we compute the asymptotics of this function for $\text{Im } z \rightarrow \pm\infty$ and show that it is minimal. Finally, we prove Theorem 2.1.

Below, C denotes different positive constants (independent of z), and $K_C = \{z \in \mathbb{C} : |\text{Im } z| \geq C|\text{Re } z|\}$.

4.1. Solution analytic in $\mathbb{C} \setminus \mathbb{R}$. We begin with a short description of a special function we use in this section.

4.1.1. σ -function. An analogous function was introduced and systematically studied in the diffraction theory [1]. Later it appeared in other domains, e.g. [5] and [3]. Below, we rely on the last paper.

The special function σ can be uniquely defined as a meromorphic solution of the difference equation

$$\sigma(z + \pi\omega) = (1 + e^{-iz})\sigma(z - \pi\omega) \quad (4.1)$$

which is analytic in the strip $S = \{z \in \mathbb{C} : |\text{Re } z| < \pi(1 + \omega)\}$, does not vanish there and admits in S the following uniform asymptotics:

$$\sigma(z) = 1 + o(1), \quad \text{Im } z \rightarrow -\infty, \quad (4.2)$$

$$\sigma(z) = e^{-\frac{iz^2}{4\pi\omega} + \frac{i\pi}{12\omega} + \frac{i\pi\omega}{12}}(1 + o(1)), \quad \text{Im } z \rightarrow \infty. \quad (4.3)$$

The asymptotics (4.2) and (4.3) appear to be uniform in K_C for any fixed C . The poles of σ are located at the points

$$z = -(\pi(1 + \omega) + 2\pi\omega k + 2\pi m), \quad k, m \in \mathbb{N} \cup \{0\}, \quad (4.4)$$

and its zeros are described by the formulas

$$z = \pi(1 + \omega) + 2\pi\omega k + 2\pi m, \quad k, m \in \mathbb{N} \cup \{0\}; \quad (4.5)$$

the zero at $z = \pi(1 + \omega)$ and the pole at $z = \pi(1 + \omega)$ are simple. We note that

$$\text{res}_{z=\pi(1+\omega)} \frac{1}{\sigma(z)} = -\sqrt{\omega} e^{\frac{i\pi}{12\omega} + \frac{i\pi\omega}{12} + \frac{i\pi}{4}}.$$

The σ function solves one more difference equation

$$\sigma(z + \pi) = (1 + e^{-iz/\omega})\sigma(z - \pi) \quad (4.6)$$

and satisfies the relations

$$\sigma(z) = e^{-\frac{iz^2}{4\pi\omega} + \frac{i\pi}{12\omega} + \frac{i\pi\omega}{12}} / \sigma(-z) \quad \text{and} \quad \overline{\sigma(\bar{z})} = 1 / \sigma(-z). \quad (4.7)$$

4.1.2. *Analytic solution of the complex Maryland equation.* Here, we always assume that $\text{Im } z \neq 0$. Instead of E and λ , we use the parameters $\eta \in \mathbb{R}$ and $l > 0$, see (1.2). It is convenient to consider $|\eta| < \pi + \omega$. We construct a solution represented by a contour integral, and we begin by describing the contour.

Put

$$\mathbb{C}' = \mathbb{C} \setminus ((\eta - il - \pi(1 + \omega) - \mathbb{R}_+) \cup (-\eta - il + \pi(1 + \omega) + \mathbb{R}_+) \cup (-\eta + il - \pi(1 + \omega) - \mathbb{R}_+) \cup (\eta + il + \pi(1 + \omega) + \mathbb{R}_+)).$$

For $z \in \mathbb{C}$, denote by $D(z)$ the set of rays going in $\mathbb{C}$ to infinity in parallel to the vectors corresponding to the complex numbers

$$\tau = e^{i\alpha}, \quad \alpha \in (-\arg z, -\arg z + \pi); \quad -\pi < \arg z < \pi. \quad (4.8)$$

We assume that $\gamma = \gamma(z)$ is a curve in $\mathbb{C}'$ and that, first, it goes from $-i\infty$ to $-2il$ along a ray of $D(z)$, then, it goes from $-2il$ to $2il$ along $i\mathbb{R}$ and, finally, it goes to $+i\infty$ along one more ray of $D(z)$.

Proposition 4.1. *If $|\eta| < \pi(1 + \omega)$, the formula*

$$\Upsilon(z) = \sin(\pi z) \sin\left(\frac{\pi}{\omega} z\right) \int_{\gamma(z)} e^{\frac{ipz}{\omega}} \frac{\sigma(p + \eta - il)\sigma(p - \eta + il)}{\sigma(p - \eta - il)\sigma(p + \eta + il)} dp \quad (4.9)$$

defines a solution of (1.8) analytic in $z \in \mathbb{C} \setminus \mathbb{R}$. This solution is also analytic in η .

Proof. The description of the poles and zeros of the σ -function implies the analyticity of the integrand in $p \in \mathbb{C}'$. The convergence and analyticity of the integral in (4.9) follow from estimates (4.2) and (4.3) and from the definition of the curve $\gamma(z)$. Let us check that Υ solves (1.8). Denote the contour integral by $X(z)$, and denote the integrand by $e^{\frac{ipz}{\omega}} \hat{X}(p)$. Equation (1.8) for Υ is equivalent to the equation

$$\begin{aligned} & \sin(\pi(z + \omega))X(z + \omega) + \sin(\pi(z - \omega))X(z - \omega) + \\ & + 2(\cos \eta \operatorname{ch} l \sin(\pi z) + \sin \eta \operatorname{sh} l \cos(\pi z))X(z) = 0. \end{aligned} \quad (4.10)$$

Assume that $\gamma = \gamma(z)$ goes to $\pm i\infty$ along rays from the set $D(z - \omega) \cap D(z + \omega)$. Then this curve can be used as the integration contour in the representations for each of the functions $X(z)$, $X(z - \omega)$ and $X(z + \omega)$. This allows to transform (4.10) to the equation

$$\begin{aligned} & \int_{\gamma + \pi\omega} e^{\frac{i(p+\omega)z}{\omega}} (1 + e^{-i(p+\eta-il)})(1 + e^{-i(p-\eta+il)}) \hat{X}(p - \pi\omega) dp - \\ & - \int_{\gamma - \pi\omega} e^{\frac{i(p+\omega)z}{\omega}} (1 + e^{-i(p-\eta-il)})(1 + e^{-i(p+\eta+il)}) \hat{X}(p + \pi\omega) dp = 0. \end{aligned} \quad (4.11)$$

Equation (4.1) and the definition of $\hat{X}$ imply that

$$\hat{X}(p + \pi\omega) = \frac{(1 + e^{-i(p+\eta-il)})(1 + e^{-i(p-\eta+il)})}{(1 + e^{-i(p-\eta-il)})(1 + e^{-i(p+\eta+il)})} \hat{X}(p - \pi\omega).$$

So, it suffices to check that, in (4.11), one can replace $\gamma \pm \pi\omega$ by γ . Consider the first integral in (4.11), the second one can be treated similarly. Translate $\gamma + \pi\omega$ to γ along the real line. Asymptotics (4.2) and (4.3) imply that, translating the integration contour, we do not break the convergence of the integral. So, we need only to check that, when being translated, the integration contour does not cross

any poles of the integrand. The description of the zeros and poles of σ shows that the contour can cross only the poles of $\hat{X}(\cdot - \pi\omega)$ located at $p = \pm(\eta - i\ell) - \pi$. These poles being simple, the expression $(1 + e^{-i(p+\eta-i\ell)})(1 + e^{-i(p-\eta+i\ell)})\hat{X}(p - \pi\omega)$ has no singularities at $p = \pm(\eta - i\ell) - \pi$. This implies the desired. $\square$

Remark 4.1. Using equation (4.6), one can directly check that Υ solves (2.13). By means of (4.7), one proves that $\Upsilon(\bar{z}) = -\overline{\Upsilon(z)}$.

4.2. **Real z .** Here, we prove

Proposition 4.2. *The solution Υ can be continued to a meromorphic function that may have poles only at $z = \pm(\omega k + m)$, $k, m \in \mathbb{N}$.*

Proof. If $\text{Im } z \neq 0$, (4.9) implies that

$$\begin{aligned} \Upsilon(z) &= -\frac{1}{4} \int_{\gamma(z)} (e^{\frac{i(p+\pi\omega+\pi)z}{\omega}} - e^{\frac{i(p+\pi\omega-\pi)z}{\omega}} - e^{\frac{i(p-\pi\omega+\pi)z}{\omega}} + e^{\frac{i(p-\pi\omega-\pi)z}{\omega}}) \hat{X}(p) dp = \\ &= -\frac{1}{4} \sum_{s_1, s_2 = \pm 1} s_1 s_2 \int_{\gamma(z) + s_1\pi\omega + s_2\pi} e^{\frac{ipz}{\omega}} \hat{X}(p - s_1\pi\omega - s_2\pi) dp \\ &= -\frac{1}{4} \sum_{s_1, s_2 = \pm 1} s_1 s_2 \int_{\gamma(z)} e^{\frac{ipz}{\omega}} \hat{X}(p - s_1\pi\omega - s_2\pi) dp + 2\pi i R(z), \end{aligned}$$

where $e^{ipz/\omega} \hat{X}$ is the integrand in (4.9), and $R(z)$ denotes the sum of the residues appeared when deforming the integration contour. The function R is entire. To analyse it, we note that, in (4.9), both the integrand and the part of the integration contour situated in $\{|\text{Im } p| \leq 2l\}$ are independent of z . Furthermore, the poles of the integrand are located on the lines $\text{Im } p = \pm l$. This implies that R is given by one and the same formula both for $\text{Im } z > 0$ and for $\text{Im } z < 0$. By means of (4.1) and (4.6), the last representation for Υ can be transformed to the form

$$\Upsilon(z) = \int_{\gamma(z)} e^{\frac{ipz}{\omega}} \frac{A \hat{X}(p - \pi - \pi\omega)}{(\cos p - \cos(\eta + i\ell))(\cos(p/\omega) - \cos((\eta + i\ell)/\omega))} dp + 2\pi i R(z),$$

where $A = \text{sh } l \text{ sh } (l/\omega) \sin \eta \sin(\eta/\omega)$.

Using (4.2) and (4.3), one can easily see that, for any fixed $C > 0$, in K_C , the integrand admits the estimates $O(e^{i(z-1-\omega)p/\omega})$ for $p \rightarrow -i\infty$ and $O(e^{i(z+1+\omega)p/\omega})$ for $p \rightarrow +i\infty$.

Assume that $|\text{Re } z| < 1 + \omega$. Thanks to the last two estimates, we can deform the integration contour to the imaginary axis (both for $\text{Im } z > 0$ and $\text{Im } z < 0$). In the strip $|\text{Re } z| < 1 + \omega$, the obtained contour integral converges for all z and defines an analytic function. Therefore, Υ is analytic in the strip $\{z \in \mathbb{C} : |\text{Re } z| < 1 + \omega\}$. It can be continued to a meromorphic one directly via equation (1.8). This equation also implies the statement on the poles of Υ . $\square$

4.3. **Behavior of Υ for $\text{Im } z \rightarrow \pm\infty$.** Here, first, we get the asymptotics of Υ for $\text{Im } z \rightarrow \pm\infty$, and then, we check that Υ is a minimal meromorphic solution of (1.8).

Proposition 4.3. *Fix $C > 0$. If $|\eta| < \pi(1 + \omega)$, then, in K_C ,*

$$\Upsilon(z) = e^{(l-i\eta)z/\omega} (a_+ + o(1)) + e^{-(l-i\eta)z/\omega} (a_- + o(1)), \quad \text{Im } z \rightarrow +\infty, \quad (4.12)$$

$$\Upsilon(z) = e^{(l+i\eta)z/\omega} (b_+ + o(1)) + e^{-(l+i\eta)z/\omega} (b_- + o(1)), \quad \text{Im } z \rightarrow -\infty, \quad (4.13)$$

where

$$a_{\pm} = \frac{\pi i}{2} \frac{\sigma(\pi(1+\omega) \mp 2\eta) \sigma(\pi(1+\omega) \mp 2il)}{\sigma(\pi(1+\omega) \mp 2(\eta + il))} \operatorname{res}_{p=\pi(1+\omega)} \frac{1}{\sigma(p)}, \quad (4.14)$$

$$b_{\pm}(\eta, l) = -\overline{a_{\pm}(\eta, l)}. \quad (4.15)$$

Remark 4.2. Formulas (4.14) and the description of the zeros of the σ -function imply that $a_- = b_- = 0$ at $\eta = 0, \omega, 2\omega \dots$

Proof. Assume that $z \in \mathbb{C}_+ \cap K_C$. As $z \notin \mathbb{R}$, we use (4.9). For sufficiently small $\delta > 0$, for all $z \in \mathbb{C}_+ \cap K_C$, $D(z)$ contains rays parallel to the vectors $e^{\pm i\delta}$. Therefore, for all $z \in \mathbb{C}_+ \cap K_C$, in (4.9), we can choose one and the same integration contour γ .

When being translated to the right along the real line, the integration contour can cross poles of the integrand. These are zeros of the denominator in (4.9) located at the points $\pm(il + \eta) + \pi(1 + \omega) + 2\pi(n + \omega m)$, $n, m = 0, 1, 2 \dots$. Let us translate the contour γ to $\gamma + c$, where $c > 0$ is chosen so that, in the course of translation, the contour crosses the poles at $\pm(il + \eta) + \pi(1 + \omega)$ and that, after the translation, it does not contain any pole of the integrand.

The function Υ equals the sum of the term $I(z)$ containing the integral along $\gamma + c$ and $S(z)$, the sum of (a finite number of) the contributions of the residues appeared when deforming γ to $\gamma + c$. One has

$$S(z) = e^{(l-i\eta)z/\omega} (a_+ + o(1)) + e^{-(l-i\eta)z/\omega} (a_- + o(1)), \quad \operatorname{Im} z \rightarrow +\infty, \quad (4.16)$$

with $a_{\pm}$ given by (4.14). When deriving the last formula, we take into account the fact that the zero of σ -function at $p = \pi(1 + \omega)$ is simple. Let us estimate $I(z)$. Along $\gamma + c$, $|\hat{X}(p)|$ is bounded. Therefore,

$$\begin{aligned} |I(z)| &\leq \operatorname{Const} e^{\pi(1+\omega)|\operatorname{Im} z|/\omega} \int_{\gamma+c} |e^{ipz/\omega} dp| \\ &= \operatorname{Const} e^{\pi(1+\omega)|\operatorname{Im} z|/\omega - c \operatorname{Im} z} \int_{\gamma} |e^{ipz/\omega} dp|. \end{aligned}$$

The last integral converges. When $\operatorname{Im} z$ increases, it increases exponentially. Therefore, if c is sufficiently large, then, as $\operatorname{Im} z \rightarrow +\infty$, the I integral becomes small with respect to both exponentials in (4.16). This implies (4.12).

The proof of (4.13) is similar to the proof of (4.12), but one translates γ to the left. Omitting the details, we note only that, to get (4.15), one has to use (4.7). $\square$

Now, one easily checks the main statement of the section:

Theorem 4.1. *The solution Υ is a minimal meromorphic solution to (1.8); its asymptotic coefficients are given in (4.14) and (4.15).*

Theorem 2.2 is an immediate corollary of this theorem.

Proof. By Proposition 4.2, Υ is analytic in $|\operatorname{Re} z| \leq \pi\omega$. Consider the coefficients $A_{\pm}$ in formula (2.10) representing $\psi = \Upsilon$ as a linear combination of the canonical basis solutions $u_{\pm}$. In view of Section 2.1, $A_{\pm}(z) = \pm w(\Upsilon(z), u_{\mp}(z)) / w(u_+(z), u_-(z))$. Assume that $z \in \mathbb{C}_+$ is in the ω -neighborhood of the line $i(l + i\eta)\mathbb{R}$. Then $e^{\pm(l-i\eta)z/\omega}$ are of order of one, and using (4.12), (2.5) and (2.6), we get $A_{\pm}(z) = a_{\pm} + o(1)$ as $\operatorname{Im} z \rightarrow +\infty$. This representation is uniform in $\operatorname{Re} z$. As $A_{\pm}$ are ω -periodic, these representations remain valid and uniform in $\mathbb{C}_+$. One studies the coefficients $B_{\pm}$

in the representation (2.11) for $\psi = \Upsilon$ similarly. This leads to the statement of the theorem. $\square$

4.4. Construction of the canonical Bloch solutions. Here, using techniques developed in [3] and [7], we prove Theorem 2.1. The proof is carried out in several steps:

1. Consider equation (3.8) equivalent to (1.8). As $\text{Im } z \rightarrow +\infty$, in this equation, the matrix takes the form $\mathcal{F}(z, \eta, l) = \begin{pmatrix} 2 \cos(\eta + il) & -1 \\ 1 & 0 \end{pmatrix} + O(e^{-2\pi \text{Im } z})$. The eigenvalues of the leading term equal $\nu^{\pm 1}$, $\nu = e^{-i(\eta + il)}$. Put $\phi = V^{-1}\psi$, where $V = \begin{pmatrix} 1 & 1 \\ 1/\nu & \nu \end{pmatrix}$, and ψ is a vector solution of (3.8). In a neighborhood of $+i\infty$, ϕ solves the equation

$$\phi(z + h) = (D + m(z))\phi(z), \quad D = \begin{pmatrix} \nu & 0 \\ 0 & 1/\nu \end{pmatrix}, \quad m(z) = O(e^{-2\pi \text{Im } z}). \quad (4.17)$$

2. Let $\phi_1(z)$ and $\phi_2(z)$ be the first and the second components of the vector $\phi(z)$. Put $\Phi(z) = \phi_2(z)/\phi_1(z)$. Then

$$\Phi(z + \omega) = \frac{(1/\nu + m_{22}(z))\Phi(z) + m_{21}(z)}{\nu + m_{11}(z) + m_{12}(z)\Phi(z)}. \quad (4.18)$$

We construct a solution of this equation by means of the technique described in Section 4.1.1 from [7]. Consider the sequence of functions defined by the formulas

$$\Phi_{n+1}(z + \omega) = \frac{(1/\nu + m_{22}(z))\Phi_n(z) + m_{21}(z)}{\nu + m_{11}(z) + m_{12}(z)\Phi_n(z)}, \quad n \geq 0, \quad \Phi_0(z) \equiv 0.$$

Let $D \subset \mathbb{C}$ be a domain. Repeating the proof of Proposition 4.1 from [7], we show that if $|\nu| > 1$, and $\sup_{z \in D} |m(z)|$ is sufficiently small, then, for all $n \in \mathbb{N}$ and $z \in D$, $|\Phi_n(z)| \leq 1$, the sequence $\{\Phi_n\}$ converges uniformly in $z \in D$, and the limit Φ solves (4.18). As, in a neighborhood of $+i\infty$, m is analytic and 1-periodic and satisfies the estimate in (4.17), we conclude that, in a neighborhood of $+i\infty$, there exists an analytic 1-periodic bounded solution Φ of equation (4.18). As Φ is 1-periodic and bounded, it can be represented by the Fourier series of the form $\Phi(z) = \sum_{m=0}^{\infty} q_n e^{2\pi i m z}$. Substituting it into (4.18), one checks that $\Phi(z) = O(e^{-2\pi \text{Im } z})$ as $\text{Im } z \rightarrow +\infty$.

3. If Φ solves (4.18) and ϕ_1 satisfies the equation

$$\phi_1(z + \omega) = (\nu + m_{11}(z) + m_{12}(z)\Phi(z))\phi_1(z), \quad (4.19)$$

then the vector with the components $\phi_1(z)$ and $\phi_2 = \phi_1(z)\Phi(z)$ solves (4.17).

4. Let Φ be the function constructed in the second step. To construct a solution of (4.19), we use Lemma 2.3 from [3]. It can be formulated in the following way:

Lemma 4.1. *Let g be a 1-periodic function analytic in a neighborhood of $+i\infty$ such that $g(i\infty) = 0$. Then equation $f(z + \omega) - f(z) = g(z)$ with a fixed $0 < \omega < 1$ has a solution analytic in a neighborhood of $+i\infty$ and decreasing as $\text{Im } z \rightarrow +\infty$ uniformly in $\{z \in \mathbb{C} : |\text{Re } z| \leq C\}$, where $C > 0$ is an arbitrary fixed constant.*

Define $A(z) = (\nu + m_{11}(z) + m_{12}(z)\Phi(z))$. The estimates for Φ and m for $\text{Im } z \rightarrow +\infty$ imply that $A(z) = \nu(1 + o(1))$. Choose the branch of $B = \ln A$ so that $B = -i(\eta + il) + g$ and $g(z) = o(1)$ as $\text{Im } z \rightarrow +\infty$. Let f be the function constructed by means of Lemma 4.1 in terms of g . Then $\phi_1(z) = e^{-i(\eta + il)z/\omega + f(z)}$

solves (4.19). Using the observation made at the third step, one constructs in terms of ϕ_1 a solution of (4.17) analytic in a neighborhood of $+i\infty$ and such that, as $\text{Im } z \rightarrow +\infty$, $\phi(z) = e^{-i(\eta+il)z/\omega} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \right)$ uniformly in $\{z \in \mathbb{C} : |\text{Re } z| \leq C\}$, $C > 0$ being a fixed constant.

5. The function ϕ is a Bloch solution, i.e., $\phi(z+1) = \alpha(z)\phi(z)$, where α is an ω -periodic. Indeed, a direct calculation shows that, for any solution f of the equation $f(z+\omega) - f(z) = g(z)$ with a given 1-periodic function g , $f(z+1) - f(z)$ is ω -periodic. This implies that $\alpha(z) = \phi_1(z+1)/\phi_1(z)$ is ω -periodic. As Φ is 1-periodic, one has $\phi_2(z+1)/\phi_2(z) = \phi_1(z+1)/\phi_1(z)$. This implies the needed.

6. Fix $C_1 > 0$. Let us show that the asymptotics of ϕ is uniform in K_{C_1} . Consider the coefficient α from the definition of the Bloch solution ϕ . As it is ω -periodic, the asymptotics for ϕ in $\{z \in \mathbb{C} : |\text{Im } z| \leq C\}$ implies that, as $\text{Im } z \rightarrow +\infty$, $\alpha(z) = e^{-i(\eta+il)/\omega + O(e^{-2\pi \text{Im } z/\omega})}$ uniformly in $\text{Re } z$. Assume that $z \in K_C$. Let N be the integer part of $\text{Re } z$. One has

$$\phi(z) = \left(\prod_{n=1}^N \alpha(z-n) \right) \phi(z-N) = e^{-i(\eta+il)N/\omega + O(\text{Im } z e^{-2\pi \text{Im } z/\omega})} \phi(z-N).$$

Substituting in this formula the asymptotics for ϕ justified for bounded $|\text{Re } z|$, we obtain the needed.

7. One can easily see that the first component ψ_1 of a vector solution ψ of (3.8) satisfies (1.8). Let $\psi = V\phi$, where ϕ the solution of (4.18) constructed in the previous steps. By the result of the first step, ψ solves (3.8). We construct solutions $u_{\pm}$ of (1.8) by the formulas $u_+(z) = \psi_1(z)$ and $u_-(z) = \overline{u_+(-\bar{z})}$. One can easily check that these solutions have all the properties listed in Theorem 2.1. We omit the elementary calculations.

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