Representations of affine superalgebras and mock theta functions

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To Evgenii Borisovich Dynkin on his 90th birthday

Abstract

We show that the normalized supercharacters of principal admissible modules over the affine Lie superalgebra $\widehat{s\ell}_{2|1}$ (resp. $\widehat{ps\ell}_{2|2}$) can be modified, using Zwegers' real analytic corrections, to form a modular (resp. S-) invariant family of functions. Applying the quantum Hamiltonian reduction, this leads to a new family of positive energy modules over the $N=2$ (resp. $N=4$) superconformal algebras with central charge $3\left(1-\frac{2m+2}{M}\right)$, where $m\in\mathbb{Z}_{\geq 0}$, $M\in\mathbb{Z}_{\geq 2}$, $\gcd(2m+2,M)=1$ if $m>0$ (resp. $6\left(\frac{m}{M}-1\right)$, where $m\in\mathbb{Z}_{\geq 1}, M\in \mathbb{Z}_{\geq 2}$, $\gcd(2m,M)=1$ if $m>1$), whose modified characters and supercharacters form a modular invariant family.

0 Introduction

Modular invariance of characters of affine Lie algebras have been playing an important role in their representation theory and applications to physics (see [K2] and references there).

Recall that an *affine Lie algebra* $\widehat{\mathfrak{g}}$, associated to a simple finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ endowed with a suitably normalized invariant symmetric bilinear form $(\,.\,|\,.\,)$, is the infinite-dimensional Lie algebra over $\mathbb{C}$:

$$\widehat{\mathfrak{g}}=\mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K\oplus\mathbb{C}d\,, \tag{0.1}$$

with the following commutation relations ($a,b\in\mathfrak{g},\ m,n\in\mathbb{Z}$):

$$[at^m,bt^n]=[a,b]t^{m+n}+m\delta_{m,-n}(a|b)K\,,\ [d,at^m]=m\,at^m\,,\ [K,\widehat{\mathfrak{g}}]=0\,. \tag{0.2}$$

We identify $\mathfrak{g}$ with the subalgebra $1\otimes\mathfrak{g}$. The bilinear form $(\,.\,|\,.\,)$ extends from $\mathfrak{g}$ to a non-degenerate symmetric invariant bilinear form on $\widehat{\mathfrak{g}}$ by:

$$(at^m|bt^n)=\delta_{m,-n}(a|b)\,,\ (\mathfrak{g}[t,t^{-1}]|\mathbb{C}K+\mathbb{C}d)=0\,,\ (K|K)=(d|d)=0\,,\ (K|d)=1\,. \tag{0.3}$$

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Choosing a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, one defines the corresponding *Cartan subalgebra* of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{h}} = \mathbb{C}d \oplus \mathfrak{h} \oplus \mathbb{C}K\,. \tag{0.4}$$

The restriction of the bilinear form $(\,.\,|\,.\,)$ from $\widehat{\mathfrak{g}}$ to $\widehat{\mathfrak{h}}$ is non-degenerate, hence we shall identify $\widehat{\mathfrak{h}}$ with its dual $\widehat{\mathfrak{h}}^*$ via this form.

One uses the following coordinates on $\widehat{\mathfrak{h}}$:

$$\widehat{\mathfrak{h}} \ni h = 2\pi i(-\tau d + z + tK)\,, \text{ where } \tau, t \in \mathbb{C}\,,\; z \in \mathfrak{h}\,. \tag{0.5}$$

Choosing a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$ containing $\mathfrak{h}$, where $\mathfrak{n}_+$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, we define the corresponding Borel subalgebra of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus \mathfrak{n}_+ \oplus (\oplus_{n>0}\mathfrak{g}t^n)\,.$$

Given $\Lambda \in \widehat{\mathfrak{h}}^*$, one extends it to a linear function on $\widehat{\mathfrak{b}}$ by zero on all other summands, and defines the *highest weight module* $L(\Lambda)$ over $\widehat{\mathfrak{g}}$ as the irreducible module, which admits an eigenvector of $\widehat{\mathfrak{b}}$ with weight Λ. Since K is a central element of $\widehat{\mathfrak{g}}$, it is represented on $L(\Lambda)$ by a scalar $\Lambda(K)$, called the *level* of $L(\Lambda)$ (and of Λ).

A $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ is called *integrable* if any nilpotent element of $\widehat{\mathfrak{g}}$ is represented by a locally nilpotent operator (hence this module can be "integrated" to a representation of the group, associated to $\widehat{\mathfrak{g}}$). It is well known [K2] that a $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ is integrable iff for all simple roots $\alpha_1, \ldots, \alpha_\ell$ and the highest root θ the numbers

$$2(\Lambda|\alpha_i)/(\alpha_i|\alpha_i)\,,\; i = 1, \ldots, \ell, \text{ and } 2(\Lambda|K-\theta)/(\theta|\theta)$$

are non-negative integers. It is easy to deduce that if the bilinear form on $\mathfrak{g}$ is normalized by the condition $(\theta|\theta) = 2$, then the level $(\Lambda|K)$ is a non-negative integer and $(\Lambda|\theta) \le (\Lambda|K)$.

The *character* of $L(\Lambda)$ is defined as the following series, corresponding to the weight space decomposition with respect to $\widehat{\mathfrak{h}}$, cf. (0.5):

$$\mathrm{ch}_{L(\Lambda)}(\tau, z, t) = \mathrm{tr}_{\,L(\Lambda)} e^{2\pi i(-\tau d + z + tK)}\,.$$

It is known [KP], [K2] that for an integrable $L(\Lambda)$ this series converges in the domain

$$X = \{h \in \widehat{\mathfrak{h}} |\; \mathrm{Re}\,(h|K) > 0\} = \{(\tau, z, t) |\; \mathrm{Im}\,\tau > 0\} \tag{0.6}$$

to a holomorphic function.

Note that, as a $\widehat{\mathfrak{g}}' = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$-module, $L(\Lambda)$ remains irreducible, and it is unchanged if we replace Λ by $\Lambda + aK$, $a \in \mathbb{C}$, and the character of the $\widehat{\mathfrak{g}}$-module gets multiplied by q^a. Here and further $q = e^{2\pi i\tau}$. Note also that the set of highest weights Λ of level K of integrable $\widehat{\mathfrak{g}}$-modules $L(\Lambda)$ is finite $\mod \mathbb{C}K$. We denote this finite set by P_+^k.

An important property of integrable $\widehat{\mathfrak{g}}$-modules is modular invariance of its normalized characters, discovered in [KP]. Recall that the *normalized character* ch_Λ is defined as

$$\mathrm{ch}_\Lambda(\tau, z, t) = q^{m_\Lambda}\mathrm{ch}_{L(\Lambda)}(\tau, z, t)\,,$$

where $m_\Lambda \in \mathbb{Q}$ is the "modular anomaly" (see formula (3.15)). Note that $\mathrm{ch}_{\Lambda+aK} = \mathrm{ch}_\Lambda\,,\; a \in \mathbb{C}$. Recall the action of $SL_2(\mathbb{R})$ in the domain X in coordinates (0.5):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z, t) = \left(\frac{a\tau + b}{c\tau + d}\,, \frac{z}{c\tau + d}\,, t - \frac{c(z|z)}{2(c\tau + d)}\right)\,. \tag{0.7}$$

The modular invariance of normalized characters of integrable $\widehat{\mathfrak{g}}$-modules means that the $\mathbb{C}$-span of the finite set $\{\mathrm{ch}_\Lambda |\ \Lambda \in P_+^k\}$ is $SL_2(\mathbb{Z})$-invariant (for the action (0.7)).

The proof of modular invariance of normalized characters of integrable modules $L(\Lambda)$ relies on the Weyl–Kac character formula

$$\widehat{R}\mathrm{ch}_{L(\Lambda)} = \sum_{w \in \widehat{W}} (\det w)\, w(e^{\Lambda+\widehat{\rho}})\,, \tag{0.8}$$

where $\widehat{R}$ is the affine Weyl denominator, $\widehat{W}$ is the affine Weyl group, $\widehat{\rho}$ is the affine Weyl vector (see [K2] for details). One has [K2]:

$$\widehat{W} = W \ltimes \{t_\alpha |\ \alpha \in L\}\,,$$

where W is the Weyl group of $\mathfrak{g}$, $L \subset \mathfrak{h}$ is the coroot lattice, and $t_\alpha \in \mathrm{End}\,\widehat{\mathfrak{h}}$ is defined by (recall that $k = \Lambda(K)$ is the level):

$$t_\alpha(\Lambda) = \Lambda + k\alpha - ((\Lambda|\alpha) + \frac{k}{2}(\alpha|\alpha))K\,. \tag{0.9}$$

Using this, (0.8) can be rewritten, after multiplying both sides by a suitable power of q, as

$$q^{\frac{\dim \mathfrak{g}}{24}}\widehat{R}\mathrm{ch}_\Lambda = \sum_{w \in W} (\det w)\, w(\Theta_{\Lambda+\widehat{\rho}})\,. \tag{0.10}$$

Here, for $\lambda \in \widehat{\mathfrak{h}}$, such that $n = (\lambda|K)$ is a positive integer, the theta function (= Jacobi form) Θ_λ of degree n is defined by

$$\Theta_\lambda = q^{\frac{(\lambda|\lambda)}{2n}} \sum_{\alpha \in L} t_\alpha(e^\lambda)\,. \tag{0.11}$$

This series converges on X to a holomorphic function, which in coordinates (0.5) takes the usual form, going back to Jacobi:

$$\Theta_\lambda(\tau, z, t) = e^{2\pi i n t} \sum_{\gamma \in \frac{\bar{\lambda}}{n} + L} q^{n\frac{(\gamma|\gamma)}{2}} e^{2\pi i n(\gamma|z)}\,, \tag{0.12}$$

where $\bar{\lambda}$ denotes the orthogonal projection of λ on $\mathfrak{h}$. Now modular invariance of Jacobi forms (which we recall in the Appendix) easily implies the modular invariance of the numerators of normalized characters of integrable modules, and modular invariance of the normalized denominator $q^{\frac{\dim \mathfrak{g}}{24}}\widehat{R}$ easily follows from the Jacobi triple product identity.

As we have discovered out in [KW1], [KW2], modular invariance of normalized characters holds for a much larger class of irreducible highest weight modules $L(\Lambda)$ over $\widehat{\mathfrak{g}}$, which we called *admissible* modules (and we conjectured that these are all $L(\Lambda)$ with modular invariance property, which we were able to verify only for $\mathfrak{g} = s\ell_2$). Roughly speaking, a $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ is called admissible, if the $\mathbb{Q}$-span of coroots of $\widehat{\mathfrak{g}}$ coincides with that of Λ-integral coroots, and with respect to the corresponding affine Lie algebra $\widehat{\mathfrak{g}}_\Lambda$ the weight Λ becomes integrable after a shift by the Weyl vectors.

We showed in [KW1] that a formula similar to (0.8) holds for $\mathrm{ch}_{L(\Lambda)}$ if Λ is an admissible weight: one just has to replace $\widehat{W}$ by the subgroup, generated by reflections with respect to non-isotropic Λ-integral coroots. It follows that the numerators of normalized admissible characters are again expressed as linear combinations of Jacobi forms, which again implies modular invariance of normalized characters of admissible $\widehat{\mathfrak{g}}$-modules.

Furthermore, using the quantum Hamiltonian reduction, one can transfer the modular invariance property from the admissible modules over affine Lie algebras to the "minimal models" of W-algebras [FKW], [KRW], [A2], the simplest example being the Virasoro algebra. (Note that the integrable modules are "erased" by the reduction!)

A natural question arises whether the theory of integrable and admissible modules over affine Lie algebras $\widehat{\mathfrak{g}}$ extends to the case when $\mathfrak{g}$ is a finite-dimensional simple Lie superalgebra. Of course, we need to assume that $\mathfrak{g}$ carries a non-degenerate supersymmetric invariant bilinear form, and also that its even part $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. According to the classification of [K1], a complete list of such Lie superalgebras consists of the classical series $s\ell_{m|n} (m > n \geq 1)$, $ps\ell_{n|n} (n \geq 2)$, $osp_{m|n} (m \geq 1, n \geq 2 \ \text{ even})$ and three exceptional superalgebras.

Of all these Lie superalgebras, the above mentioned results extend without difficulty only for $\mathfrak{g} = osp_{1|n}$, in particular, modular invariance property of normalized characters and supercharacters of integrable and admissible modules still holds [KW1], and the quantum Hamiltonian reduction in the case of $\mathfrak{g} = osp_{1|2}$ leads to modular invariance of characters of the Neveu–Schwarz and Ramond superalgebras ($N = 1$ superconformal algebras).

In general, for a Lie superalgebra $\mathfrak{g}$ in question, a $\widehat{\mathfrak{g}}$-module is integrable iff it is integrable with respect to $\widehat{\mathfrak{g}}_{\bar{0}}$. However, such non-trivial $\widehat{\mathfrak{g}}$-modules $L(\Lambda)$ exist iff $\mathfrak{g}_{\bar{0}}$ has only one simple component (which is the case only when $\mathfrak{g} = s\ell_{n|1}$, $osp_{1|2n}$, or $osp_{2|2n}$). In all other cases one considers *partially integrable* modules, namely those, for which integrability holds for the affine subalgebra, associated to one of the simple components of $\mathfrak{g}_{\bar{0}}$.

Partially integrable $\widehat{\mathfrak{g}}$-modules $L(\Lambda)$ in the "super" case have been classified in [KW4], but the computation of their characters is a very difficult problem in general. However, in the special case of "tame" modules (see Definition 3.5) we found a conjectural formula for the characters (see formula (3.10)). This formula has been proved in all cases that are considered in the present paper (see [KW4], [S], [GK]). Note also that in the "super" case one has to study supercharacters along with the characters (when the trace is replaced by the supertrace), but one can pass from one to the other without difficulty.

The formula for the supercharacter of a tame partially integrable module $L(\Lambda)$ over an affine Lie superalgebra $\widehat{\mathfrak{g}}$ differs little from formula (0.10) for the character of an integrable module $L(\Lambda)$ over an affine Lie algebra. One just has to replace the theta function $\Theta_{\Lambda+\widehat{\rho}}$, defined in (0.11) (or (0.12)) by the mock theta function! Given a finite subset $T \subset \mathfrak{h}$ of pairwise orthogonal vectors, which are also orthogonal to λ, the following series converges to a meromorphic function on X, called a *mock theta function* of degree n:

$$\Theta_{\lambda,T}(\tau, z, t) = e^{2\pi int} \sum_{\gamma \in \frac{\bar{\lambda}}{n} + L} \frac{q^{n\frac{(\gamma|\gamma)}{2}} e^{2\pi in(\gamma|z)}}{\prod_{\beta \in T} \left(1 - q^{-(\gamma|\beta)} e^{-2\pi i(\beta|z)}\right)} . \tag{0.13}$$

These kind of functions (when $\#T = 1$ and $\mathrm{rank}\,\mathrm{L} = 1$) first appeared in the work of Appell [Ap] in the 1880s in his study of elliptic functions "of the third kind", and also, a few years later, in the work of Lerch [L]. More than 100 years later these functions made their way to the representation theory of affine Lie superalgebras [KW4].

One of the simplest results of [KW4] is the following formula for the normalized supercharacter of the integrable $\widehat{s\ell}_{2|1}$-module $L(d)$ (of level 1), obtained via the super boson-fermion correspondence:

$$ch_d^-(\tau, z_1, z_2) = \eta(\tau)^{-3}\vartheta_{11}(\tau, z_1)\vartheta_{11}(\tau, z_2)\mu(\tau, z_1, z_2)\,, \tag{0.14}$$

where $\eta(\tau)$ is the Dedekind eta-function, $\vartheta_{11}(\tau, z)$ is one of the standard Jacobi forms (see the Appendix), and

$$\mu(\tau, z_1, z_2) = \frac{e^{\pi i z_1}}{\vartheta_{11}(\tau, z_2)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{1}{2}(n^2+n)} e^{2\pi i n z_2}}{1 - e^{2\pi i z_1} q^n}\,. \tag{0.15}$$

The function $\mu(\tau, z_1, z_2)$, up to the factor $\vartheta_{11}(\tau, z_1 + z_2)$, is a difference of two simplest mock theta functions (see (5.16)), which we denote by $\Phi^{[1]}(\tau, z_1, z_2, 0)$ (see (5.3), (5.4)).

It is the function $\mu(\tau, z_1, z_2)$ that plays a central role in the work of Zwegers on mock theta functions [Z], which has been a major advance in the understanding of Ramanujan's mock ϑ-functions. Ramanujan defined a mock ϑ-function as a function f of the complex variable q, defined by a q-series of a particular type, which converges for $|q| < 1$ and satisfies the following conditions (see [Z]):

(i) infinitely many roots of unity are exponential singularities,

(ii) for every root of unity ξ there is a ϑ-function $\vartheta_\xi(q)$, such that $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially,

(iii) there is no ϑ-function that works for all ξ.

It turns out that the function $\mu(\tau, z_1, z_2)$ is the prototype for a mock theta function in the sense that, specializing the complex variables z_1 and z_2 to torsion points (i.e. elements of $\mathbb{Q} + \mathbb{Q}\tau$), one gets mock ϑ-functions [Z1].

An important discovery of Zwegers is the real analytic function $R(\tau, u)$, $\tau, u \in \mathbb{C}$, $\mathrm{Im}\ \tau > 0$, such that the modified function

$$\widetilde{\mu}(\tau, z_1, z_2) = \mu(\tau, z_1, z_2) + \frac{i}{2} R(\tau, z_1 - z_2)$$

is a modular invariant function with nice elliptic transformation properties ([Z], Theorem 1.11). Furthermore, Zwegers introduces real analytic functions $R_{m;\ell}(\tau, u)$, similar to $R(\tau, u)$ (they are related by (5.15)), such that, adding to a rank 1 mock theta function of arbitrary degree $m > 0$ (and $\#T = 1$ in our terminology) a suitable linear combination of rank 1 Jacobi forms $\Theta_{m,\ell}$ as coefficients, he obtains a modular invariant real analytic function ([Z], Proposition 3.5). The latter functions are used in the study of Ramanujan's mock theta functions ([Z], Chapter 4).

In our paper (Section 5) we use the functions $R_{m+1;\ell}$ of Zwegers in order to modify the normalized supercharacter of the $\widehat{s\ell}_{2|1}$-module $L(md)$, where m is a positive integer. The normalized supercharacter is given in this case by the following formula:

$$\widehat{R}^- \mathrm{ch}_{md}^-(\tau, z_1, z_2, t) = \Phi^{[m]}(\tau, z_1, z_2, t)\,, \tag{0.16}$$

where $\widehat{R}^-$ is the superdenominator (see (4.2) for its expression in terms of the Jacobi theta function $\vartheta_{11}(\tau,z)$), and $\Phi^{[m]}$ is the following mock theta function

$$\Phi^{[m]}(\tau,z_1,z_2,t)=e^{2\pi i(m+1)t}\sum_{j\in\mathbb{Z}}\left(\frac{e^{2\pi ij(m+1)(z_1+z_2)}q^{j^2(m+1)}}{1-e^{2\pi iz_1}q^j}-\frac{e^{-2\pi ij(m+1)(z_1+z_2)}q^{j^2(m+1)}}{1-e^{-2\pi iz_2}q^j}\right).$$

Following Zwegers' ideas, we introduce the real analytic modified numerator

$$\widetilde{\Phi}^{[m]}(\tau,z_1,z_2,t)=\Phi^{[m]}(\tau,z_1,z_2,t)+\Phi^{[m]}_{\text{add}}(\tau,z_1,z_2,t)\,,$$

where $\Phi^{[m]}_{\text{add}}$ is a real analytic function, similar to Zwegers' correction in higher degree, and prove the following modular transformation properties:
(0.17)

$$\widetilde{\Phi}^{[m]}\left(-\frac{1}{\tau},\frac{z_1}{\tau},\frac{z_2}{\tau},t-\frac{z_1z_2}{\tau}\right)=\tau\bar{\Phi}^{[m]}(\tau,z_1,z_2,t)\,,\ \Phi^{[m]}(\tau+1,z_1,z_2,t)=\Phi^{[m]}(\tau,z_1,z_2,t)\,,$$

along with certain elliptic transformation properties (Theorem 5.10 and Corollary 5.11). This establishes modular invariance of the modified normalized supercharacter $\widetilde{\text{ch}}^-_{md}=\widetilde{\Phi}^{[m]}/\widehat{R}^-$.

Next, in Sections 6 and 7 we discuss modular invariance properties of the modified normalized principal admissible characters of $\widehat{sl}_{2|1}$-modules, associated to a compatible homomorphism of degree M (see Section 3 for the definition of these modules). The main result of Section 6 is the following modular transformation formula (Theorem 6.5(a)):

$$\begin{aligned}(0.18)\quad &\widetilde{\Phi}^{[m]}\left(-\frac{M}{\tau},\frac{z_1}{\tau},\frac{z_2}{\tau},t-\frac{z_1z_2}{\tau M}\right)\\ &=\frac{\tau}{M}\sum_{j,k\in\mathbb{Z}/M\mathbb{Z}}q^{\frac{m+1}{M}jk}e^{\frac{2\pi i(m+1)}{M}(kz_1+jz_2)}\widetilde{\Phi}^{[m]}(M\tau\,,\,z_1+j\tau\,,\,z_2+k\tau\,,\,t)\,,\end{aligned}$$

provided that $\gcd(M,2m+2)=1$ if $m>0$. We deduce that the modified normalized principal admissible characters, supercharacters, and their Ramond twisted analogues, form a modular invariant family under the above conditions on M and m (Theorem 7.3).

In Section 8 we study the behavior under the modular transformation $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ of modified normalized characters of $\widehat{A}_{1|1}$-modules $L(md)$, where m is a non-zero integer. This is related to the $\widehat{s\ell}_{2|1}$ case, using a simple connection between the numerators of $\widehat{A}_{1|1}$ and $\widehat{s\ell}_{2|1}$ normalized supercharacters (for $m\geq 1$):

$$(0.19)\qquad \Phi^{A_{1|1}[m]}(\tau,z_1,z_2,t)=\frac{1}{2\pi i}\left(\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right)\Phi^{[m-1]}(\tau,z_1,z_2,t)\,.$$

The S-transformation of modified normalized characters and supercharacters, and their Ramond twisted analogues is given by Theorem 8.5 for $m>0$ and Theorem 8.7 for $m<0$. Note that in either case we don't have $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$-invariance, and that the process of modification is more complicated than in the $\widehat{s\ell}_{2|1}$ case.

In Section 9 (resp. 10) we study modular invariance of the modified characters of modules, obtained from the principal admissible $\widehat{s\ell}_{2|1}$ (resp. $\widehat{A}_{1|1}$)-modules by the quantum Hamiltonian reduction (developed in [KRW], [KW5], [KW6]).

As a result we obtain in Section 9 a modular invariant family of $N = 2$ modified characters, supercharacters, and their Ramond twisted analogs, of irreducible positive energy modules with central charge $3\left(1 - \frac{2m+2}{M}\right)$, where $m \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 2}$ and $\gcd(2m+2, M) = 1$ if $m > 0$. (Theorem 9.4). If $m = 0$ we obtain the famous $N = 2$ unitary discrete series, for which modular invariance holds without modification, but for $m \geq 1$ we obtain some very interesting new positive energy $N = 2$ modules, which should be of great interest for the conformal field theory. For example, as shown in [W], the fusion coefficients for the $N = 2$ unitary discrete series are equal to 0 or 1. Remarkably, using the same method one can show that the same property holds for arbitrary $m > 0$, such that $\gcd(M, 2m+2) = 1$.

The quantum Hamiltonian reduction of principal admissible $\widehat{A}_{1|1}$-modules, studied in Section 8, produces $N = 4$ irreducible positive energy modules. However we obtain modular invariant families of modified characters only for negative level, the central charge being $6\left(\frac{m}{M} - 1\right)$, where $m \in \mathbb{Z}_{\geq 1}$, $M \in \mathbb{Z}_{\geq 2}$ and $\gcd(2m, M) = 1$ if $m > 1$ (Theorem 10.6).

For the convenience of the reader, we provide in Sections 1–4 some necessary material on Lie superalgebras and their highest weight modules, and in the Appendix we give a brief review of some basic facts about Jacobi theta functions, used throughout the paper.

Note that a general definition of a mock modular form was given by Don Zagier (see [DMZ] for an introduction to the subject), and that connections of the theory of mock modular forms to level 1 integrable $\widehat{s\ell}_{n|1}$-characters, computed in [KW4], have been established in [BF], [BO], [F].

In our subsequent paper we will consider the remaining case of a rank 2 simple finite-dimensional Lie superalgebra, $\mathfrak{g} = osp_{3|2}$, and the corresponding $N = 3$ quantum Hamiltonian reduction.

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1 Kac–Moody superalgebras and their highest weight modules

Let I be a finite index set and let $A = (a_{ij})_{i,j \in I}$ be a symmetric matrix over $\mathbb{R}$. A *realization* of the matrix A is a vector space $\mathfrak{h}_{\mathbb{R}}$ of dimension $|I|+\text{corank } A$ over $\mathbb{R}$ with a linearly independent set of vectors $\{h_i\}_{i \in I}$, and a linearly independent set of linear functions $\Pi = \{\alpha_i\}_{i \in I}$, satisfying

$$\alpha_i(h_j) = a_{ij}\,, \qquad j \in I\,. \tag{1.1}$$

The elements $\alpha_i \in \mathfrak{h}^*$, $i \in I$, are called *simple roots*.

Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$. Given a subset $I_{\bar{1}} \subset I$, one defines the Kac–Moody superalgebra $\mathfrak{g}(A, I_{\bar{1}})$ as follows [K1]. First, denote by $\widetilde{\mathfrak{g}}(A, I_{\bar{1}})$ the Lie superalgebra on generators e_i, f_i, $i \in I$, and $\mathfrak{h}$, the generators e_i, f_i for $i \in I_{\bar{1}}$ being odd and all the other generators being even, and the following *Chevalley relations* $(i, j \in I\,,\ h \in \mathfrak{h})$:

$$[\mathfrak{h}, \mathfrak{h}] = 0, [e_i, f_j] = \delta_{ij} h_i\,,\ [h, e_i] = \alpha_i(h) e_i\,,\ [h, f_i] = -\alpha_i(h) f_i\,.$$

The Lie superalgebra $\widetilde{\mathfrak{g}}(A, I_{\bar{1}})$ has a unique maximal ideal $\widetilde{J}$ among those intersecting the subspace $\mathfrak{h}$ trivially, and we let

$$\mathfrak{g}(A, I_{\bar{1}}) = \widetilde{\mathfrak{g}}(A, I_{\bar{1}})/\widetilde{J}\,.$$

Fix a non-degenerate symmetric bilinear form $(\,.\,|\,.\,)$ on $\mathfrak{h}_{\mathbb{R}}$, such that $(h_i|h_j) = a_{ij}$, $i, j \in I$, and extend to $\mathfrak{h}$ by bilinearity.

Of course, $\mathfrak{g}(A, I_{\bar{1}})$ is a Lie algebra iff $I_{\bar{1}} = \emptyset$. In this case it is isomorphic to a simple Lie algebra $\mathfrak{g}$ if A is the Cartan matrix of $\mathfrak{g}$, and to the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$ if A is the extended Cartan matrix of $\mathfrak{g}$.

Proposition 1.1. *(cf. [K2], Chapter 2). The Lie superalgebra $\mathfrak{g}(A, I_{\bar{1}})$ carries a unique bilinear form $(\,.\,|\,.\,)$, extending that on $\mathfrak{h}$, which is supersymmetric (i.e. $(a|b) = (-1)^{p(a)p(b)}(b|a)$) and invariant (i.e. $([a, b]|c) = (a|[b, c])$). This bilinear form is non-degenerate.*

The abelian subalgebra $\mathfrak{h}$ is called the Cartan subalgebra of the Kac–Moody algebra $\mathfrak{g}(A, I_{\bar{1}})$. As usual, we have the *root space decomposition*:

$$\mathfrak{g}(A, I_{\bar{1}}) = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)\,, \tag{1.2}$$

where $\mathfrak{g}_\alpha = \{a \in \mathfrak{g}(A, I_{\bar{1}})|\ [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{h}\}$ and $\Delta = \{\alpha \in \mathfrak{h}^*|\ \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}$ is the *set of roots*.

Denoting by $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the subalgebra of $\mathfrak{g}(A, I_{\bar{1}})$, generated by all the e_i (resp. f_i), we have the *triangular decomposition*:

$$\mathfrak{g}(A, I_{\bar{1}}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+\,. \tag{1.3}$$

Let $Q = \mathbb{Z}\Pi \subset \mathfrak{h}^*$ be the *root lattice*, and let $Q_+ = \mathbb{Z}_{\geq 0}\Pi$. Let $\Delta_+ = \Delta \cap Q_+$ be the set of *positive roots*. Then $\mathfrak{n}_\pm = \oplus_{\alpha\in\Delta_+}\mathfrak{g}_{\pm\alpha}$.

Since $\mathfrak{g}(A, I_{\bar{1}})$ has the anti-involution which exchanges e_i and f_i and fixes $\mathfrak{h}$ pointwise, we conclude that $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$.

Since the bilinear form $(\,.\,|\,.\,)$ to $\mathfrak{h}$ is nondegenerate, we may (and will) identify $\mathfrak{h}$ with $\mathfrak{h}^*$. For a non-isotropic root $\alpha \in \Delta$, we let

$$\alpha^\vee = 2\alpha/(\alpha|\alpha)\,, \tag{1.4}$$

and define the *reflection* $r_\alpha \in GL(\mathfrak{h}^*)$ by

$$r_\alpha(\lambda) = \lambda - (\lambda|\alpha^\vee)\,\alpha\,, \quad \lambda \in \mathfrak{h}^*\,. \tag{1.5}$$

If $\alpha \in \Delta_+$, then $\alpha = \sum_{i\in I} k_i\alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$, and its parity is $p(\alpha) = \sum_i k_i p(\alpha_i) \mod 2$, which is the same as the parity of $\mathfrak{g}_\alpha$. Denote by $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ the subsets of Δ, consisting of even and odd roots respectively.

Proposition 1.2. *(cf. [K2], Chapter 3). Let $\alpha \in \Delta$ be an even non-isotropic root, and suppose that* ad $\mathfrak{g}_\alpha$ *is locally nilpotent on $\mathfrak{g}(A, I_{\bar{1}})$. Then $r_\alpha(\Delta) \subset \Delta$.*

A root α, satisfying conditions of Proposition 1.2, is called *integrable*. The group, generated by all reflections r_α, where α is integrable, is called the *Weyl group* of the Kac–Moody superalgebra $\mathfrak{g}(A, I_{\bar{1}})$, and is denoted by $W (\subset GL(\mathfrak{h}^*))$. By Proposition 1.2,

$$W\Delta = \Delta\,. \tag{1.6}$$

For $w \in W$ let $w = r_{\gamma_1} ... r_{\gamma_s}$ be a decomposition of w in a product of s reflections with respect to integrable roots, and let s_- be the number of those of them, for which the half is not a root. Define $\varepsilon_+(w) = (-1)^s$ and $\varepsilon_-(w) = (-1)^{s_-}$. (For $\mathfrak{g} = s\ell_{m|n}$, $m > n$, and $g\ell_{m|m}$ one has: $\varepsilon_-(w) = \varepsilon_+(w)$.)

Let $\rho_{\bar{0}}$ (resp. $\rho_{\bar{1}}$) be the half of the sum of positive even (resp. odd) roots. The element $\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$ is called the Weyl vector. One has [K1]:

$$(1.7) \qquad (\rho|\alpha_i) = \frac{1}{2}(\alpha_i|\alpha_i)(= \frac{1}{2}a_{ii}), \quad i \in I.$$

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace (i.e. a vector space, decomposed in a direct sum of subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$, called even and odd respectively). The associative algebra $\operatorname{End} V$ has the corresponding $\mathbb{Z}/2\mathbb{Z}$-grading:

$$\operatorname{End} V = (\operatorname{End} V)_{\bar{0}} \oplus (\operatorname{End} V)_{\bar{1}}, \ \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\},$$

where

$$(\operatorname{End} V)_\alpha = \{a \in \operatorname{End} V |\, aV_\beta \subset V_{\alpha+\beta}\}, \ \alpha, \beta \in \mathbb{Z}/2\mathbb{Z}.$$

One denotes by $g\ell_V$ the vector superspace $\operatorname{End} V$ with the bracket

$$[a, b] = ab - (-1)^{p(a)p(b)} ba.$$

A *module* V over a Lie superalgebra $\mathfrak{g}$ is a homomorphism of $\mathfrak{g}$ to the Lie superalgebra $g\ell_V$. If $\dim V < \infty$, one defines the *supertrace* on $\operatorname{End} V$ by

$$(1.8) \qquad \operatorname{str} a = \operatorname{tr} \mathrm{Fa}, \quad \text{where } \ \mathrm{F}|_{\mathrm{V}_\alpha} = (-1)^\alpha, \quad \alpha \in \mathbb{Z}/2\mathbb{Z},$$

and the superdimension of V by $\operatorname{sdim} V = \operatorname{str} I_V = \dim V_{\bar{0}} - \dim V_{\bar{1}}$. In this case $g\ell_V$ contains the subalgebra $s\ell_V = \{a \in g\ell_V |\ \operatorname{str} a = 0\}$.

For each $\Lambda \in \mathfrak{h}^*$ one defines an *irreducible highest weight module* $L(\Lambda)$ over the Kac–Moody superalgebra $\mathfrak{g}(A, I_{\bar{1}})$ as the (unique) irreducible $\mathfrak{g}(A, I_{\bar{1}})$-module for which there exists an even non-zero vector v_Λ, such that

$$h\mathrm{v}_\Lambda = \Lambda(h)\mathrm{v}_\Lambda \ \text{ for all } \ h \in \mathfrak{h}, \quad \mathfrak{n}_+\mathrm{v}_\Lambda = 0.$$

One has the *weight space decomposition* with respect to $\mathfrak{h}$:

$$(1.9) \qquad L(\Lambda) = \oplus_{\lambda \in \mathfrak{h}^*} L(\Lambda)_\lambda, \qquad \text{where } \ L(\Lambda)_\lambda = \{\mathrm{v} \in L(\Lambda) |\, h\mathrm{v} = \lambda(h)\mathrm{v}, \quad h \in \mathfrak{h}\}.$$

Since $L(\Lambda) = U(\mathfrak{n}_-)\mathrm{v}_\Lambda$, it follows that $\dim L(\Lambda)_\Lambda = 1$ and $\dim L(\Lambda)_\lambda < \infty$.

One then defines the *character*

$$\operatorname{ch}^+_{L(\Lambda)} = \sum_{\lambda \in \mathfrak{h}^*} (\dim L(\Lambda)_\lambda) e^\lambda$$

and the *supercharacter*

$$\operatorname{ch}^-_{L(\Lambda)} = \sum_{\lambda \in \mathfrak{h}^*} (\operatorname{sdim}\ L(\Lambda)_\lambda) e^\lambda.$$

Note that for $h \in \mathfrak{h}$ one has:

$$\operatorname{ch}^+_{L(\Lambda)}(h) = \operatorname{tr}_{\mathrm{L}(\Lambda)} \mathrm{e}^{\mathrm{h}}, \ \ \operatorname{ch}^-_{\mathrm{L}(\Lambda)}(\mathrm{h}) = \operatorname{str}_{\mathrm{L}(\Lambda)} \mathrm{e}^{\mathrm{h}}.$$

An integrable root α is called Λ-*integrable* if $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are locally nilpotent on $L(\Lambda)$ (note that $\mathfrak{g}_\alpha$ with $\alpha \in \Delta_+$ is always locally nilpotent).

Proposition 1.3. *(cf. [K2], Chapter 3). Let α be a Λ-integrable root. Then* $\dim L(\Lambda)_\lambda = \dim L(\Lambda)_{r_\alpha(\lambda)}$*, and the same holds for* sdim*. Equivalently,* $\mathrm{ch}^\pm_{L(\Lambda)}$ *are r_α-invariant.*

Proposition 1.4. *(cf. [K2], Chapter 10). The series* $\mathrm{ch}^\pm_{L(\Lambda)}$ *converge to a holomorphic function in a convex domain, containing the domain $Y := \{h \in \mathfrak{h} |\ \mathrm{Re}\ \alpha_i(h) > 0\,,\ i \in I\}$.*

Remark 1.5. Let $\mathfrak{g}'(A, I_{\bar{1}})$ denote the derived Lie superalgebra of $\mathfrak{g}(A, I_{\bar{1}})$. Then we have:

$$\mathfrak{g}(A, I_{\bar{1}}) = \mathfrak{h} + \mathfrak{g}'(A, I_{\bar{1}})\,, \quad \mathfrak{g}'(A, I_{\bar{1}}) \cap \mathfrak{h} = \mathfrak{h}' := \mathrm{span}\{h_i |\, i \in I\}\,.$$

It follows that $L(\Lambda)$ remains irreducible when restricted to $\mathfrak{g}'(A, I_{\bar{1}})$, and that $\mathrm{ch}^\pm_{L(\Lambda)}$ depend essentially only on $\Lambda|_{\mathfrak{h}'}$, namely:

$$\mathrm{ch}^\pm_{L(\Lambda)} = e^{\Lambda-\Lambda'} \mathrm{ch}^\pm_{L(\Lambda')} \ \text{ if } \ \Lambda|_{\mathfrak{h}'} = \Lambda'|_{\mathfrak{h}'}\,.$$

Definition 1.6. *The $\mathfrak{g}(A, I_{\bar{1}})$-module $L(\Lambda)$ is called integrable if any integrable root α is Λ-integrable.*

If A is the Cartan matrix of a Kac–Moody algebra $\mathfrak{g}(A)$, then the integrable $\mathfrak{g}(A)$-modules $L(\Lambda)$ are precisely the ones that can be "integrated" to the corresponding Kac–Moody group.

If $\mathfrak{g}(A, I_{\bar{1}})$ is a finite-dimensional Kac–Moody superalgebra, then it is easy to show that integrable modules $L(\Lambda)$ are precisely all finite-dimensional irreducible $\mathfrak{g}(A, I_{\bar{1}})$-modules.

If $\mathfrak{g}(A, I_{\bar{1}})$ is an affine Kac–Moody superalgebra with $I_{\bar{1}} \neq \emptyset$, then non 1-dimensional integrable modules $L(\Lambda)$ exist in only a few cases, so that it is natural to consider "partially" integrable modules instead. They are classified in [KW4], and will be discussed in §3.

Proposition 1.7. *Suppose that a Lie superalgebra carries two structures of a Kac–Moody superalgebra with the same set of even positive roots. Then*

(a) One of these Kac–Moody superalgebra structures can be obtained from the other one by a sequence of odd reflections.

(b) If $L(\Lambda)$ is an irreducible highest weight module with highest weight vector v_Λ with respect to the first structure, then it is an irreducible highest weight module with respect to the second structure. Explicitly, if the second structure is obtained from the first one by an odd reflection r_i, then the new highest weight vector is $f_i v_\Lambda$ if $(\Lambda|\alpha_i) \neq 0$, and is v_Λ if $(\Lambda|\alpha_i) = 0$.

(c) If $L(\Lambda)$ is integrable with respect to the first Kac–Moody superalgebra structure, then it is also true for the second one.

Proof. (a) is proved in the same way as Proposition 5.9 from [K2], by making use of odd reflections (described e.g. in [KW4]). (b) is Lemma 1.4 from [KW4] and (c) follows from (a) and (b). ☐

2 Examples of finite-dimensional and affine Kac–Moody superalgebras

Let$V = V_{\bar 0} \oplus V_{\bar 1}$ be a finite-dimensional superspace over $\mathbb{C}$, let $m = \dim V_{\bar 0}$, $n = \dim V_{\bar 1}$. Then one denotes $g\ell_{m|n} = g\ell_V$, $s\ell_{m|n} = s\ell_V$. Since $s\ell_{m|n} \simeq s\ell_{n|m}$, we shall always assume that $m \geq n$.

Let $\mathfrak{g} = s\ell_{m|n}$ if $m > n$, and $\mathfrak{g} = g\ell_{n|n}$ if $m = n$. The Lie superalgebra $\mathfrak{g}$ carries several structures of a Kac–Moody superalgebra, described below (it is easy to show that there are no others).

Fix a basis $\mathrm{v}_1, \ldots, \mathrm{v}_{m+n}$ of V, such that each vector v_i lies either in $V_{\bar 0}$ or in $V_{\bar 1}$; we write $p(\mathrm{v}_i) = \bar 0$ or $\bar 1$ respectively. Let $I = \{1, 2, \ldots, m+n-1\}$ and let $I_{\bar 1} = \{i \in I |\ p(\mathrm{v}_i) \neq p(\mathrm{v}_{i+1})\}$.

The choice of basis of V identifies $\operatorname{End} V$ with the space of $(m+n) \times (m+n)$ matrices. Let $\{E_{ij}\}$ denote the standard basis of this space, and let ε_i denote the linear function on the space of diagonal matrices, which picks out the i^{th} diagonal entry.

Let $\mathfrak{h}$ be the space of all diagonal matrices in $\mathfrak{g}$, and let

(2.1) $$h_{ij} = (-1)^{p(\mathrm{v}_i)} E_{ii} - (-1)^{p(\mathrm{v}_j)} E_{jj}\,.$$

The elements h_{ij} lie in $\mathfrak{h}$, and the elements $h_i := h_{i,i+1}$, $i \in I$, are linearly independent. Let

(2.2) $$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathfrak{h}^*\,.$$

Then the set $\Pi = \{\alpha_i\}_{i \in I}$ is linearly independent.

Let $a_{ij} = \alpha_i(h_j)$. It is easy to see that $A = (a_{ij})_{i,j \in I}$ is a symmetric matrix. Moreover, this matrix is 3-diagonal, with diagonal entries

$$a_{ii} = (-1)^{p(\mathrm{v}_i)} + (-1)^{p(\mathrm{v}_{i+1})}\,,$$

which are equal to 0 if $i \in I_{\bar 1}$, and to ± 2 otherwise, and the non-zero offdiagonal entries are $a_{i,i+1} = a_{i+1,i} = -(-1)^{p(\mathrm{v}_{i+1})}$.

This matrix is depicted by the Dynkin diagram $\bullet - \bullet - \bullet - \cdots - \bullet - \bullet$, consisting of $|I|$ nodes, where the i^{th} node is $\otimes$, called grey, if $i \in I_{\bar 1}$, and is $\bigcirc$, called white, otherwise.

Since the h_i are supertraceless and Π, restricted to the supertraceless diagonal matrices, is linearly independent (resp. dependent) if $m > n$ (resp. $m = n$), we conclude that $\operatorname{corank} A = 0$ if $m > n$, and $\operatorname{corank} A = 1$ if $m = n$. Thus, we have constructed a realization of the matrix A.

The structure of a Kac–Moody superalgebra $\mathfrak{g}(A, I_{\bar 1})$ in $\mathfrak{g}$ is introduced by letting

(2.3) $$e_i = E_{i,i+1}\,,\ f_i = (-1)^{p(\mathrm{v}_i)} E_{i+1,i}\,, \quad i \in I\,.$$

Indeed, it is clear that e_i, f_i and $\mathfrak{h}$ generate $\mathfrak{g}$, and it is easy to check that they satisfy the Chevalley relations. Also $\mathfrak{g}$ has no ideals intersecting $\mathfrak{h}$ trivially.

Furthermore, the supertrace form:

$$(a|b) = \operatorname{str} ab\,.$$

is an invariant bilinear form on $\mathfrak{g}$. The induced bilinear form on $\mathfrak{h}^*$ is given by:

(2.4) $$(\varepsilon_i|\varepsilon_j) = \delta_{ij} (-1)^{p(\mathrm{v}_i)},\ i, j = 1, 2, \ldots, m+n\,.$$

The set of roots is

$$\Delta = \{\varepsilon_i - \varepsilon_j |\ i, j = 1, 2, \ldots, m+n;\ i \neq j\}\,, \tag{2.5}$$

a root $\varepsilon_i - \varepsilon_j$ being even if $p(\mathrm{v}_i) = p(\mathrm{v}_j)$, and odd otherwise. The set of positive roots is

$$\Delta_+ = \{\varepsilon_i - \varepsilon_j |\ i, j = 1, 2, \ldots, m+n\,;\ i < j\}\,. \tag{2.6}$$

The root spaces are $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$, and the triangular decomposition is the usual one: $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) consists of strictly upper (resp. lower) triangular matrices.

Note that, in view of (2.4), a root $\varepsilon_i - \varepsilon_j$ is odd if and only if it is isotropic (this is not the case for other finite-dimensional Kac–Moody superalgebras). The orthogonal reflection with respect to an even root $\varepsilon_i - \varepsilon_j$ (which is integrable since $\dim \mathfrak{g} < \infty$) is the transposition of i and j. Hence the Weyl group W of $\mathfrak{g}$ is $S_m \times S_n$, where S_m (resp. S_n) is the group of permutations of all even (resp. odd) vectors v_i's.

Example 2.1. The constructed above Kac–Moody superalgebra structures on $\mathfrak{g} = s\ell_{2|1}$ correspond to three Dynkin diagrams:

$$\otimes - \otimes\,, \quad \bigcirc - \otimes\,, \quad \otimes - \bigcirc\,,$$

and those on $\mathfrak{g} = g\ell_{2|2}$ to the Dynkin diagrams:

$$\otimes - \bigcirc - \otimes, \quad \bigcirc - \otimes - \bigcirc\,, \quad \otimes - \otimes - \otimes\,.$$

The corresponding Cartan matrices A are respectively:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\,, \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix},$$

and, up to an overall sign (which does not change the Kac–Moody superalgebra structure):

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Recall that a Kac–Moody superalgebra structure exists on an almost simple finite-dimensional Lie superalgebra iff either $\mathfrak{g}$ is a Lie algebra (then it is unique up to conjugacy), or $\mathfrak{g} \simeq s\ell_{m|n}$ with $m > n$, $g\ell_{n|n}$, $osp_{m|n}$, $D(2|1;a)$, $F(4)$ or $G(3)$ [K1].

Next, we proceed to construct examples of Kac–Moody superalgebra structures in the affine superalgebra

$$\widehat{\mathfrak{g}} = \widetilde{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}d\,,$$

associated to a finite-dimensional Kac–Moody superalgebra $\mathfrak{g}$. Here $\widetilde{\mathfrak{g}} = \mathbb{C}F + s\ell_{n|n}[t, t^{-1}]$ if $\mathfrak{g} = g\ell_{n|n}$, and $\widetilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ in all other cases, and K is a non-zero central element of $\widehat{\mathfrak{g}}$. The brackets of all other elements of $\widehat{\mathfrak{g}}$ are as follows ($a, b \in \mathfrak{g}\,, m, n \in \mathbb{Z}$):

$$[at^m, bt^n] = [a, b]t^{m+n} + m(a|b)\delta_{m,-n}K,\ \ [d, at^m] = mat^m\,.$$

In particular, $\mathfrak{g}$ is a subalgebra of $\widehat{\mathfrak{g}}$.

Let $\mathfrak{g} = \mathfrak{g}(A, I_{\bar{1}})$ be a Kac–Moody subalgebra structure on $\mathfrak{g}$ with $\{h_i\}_{i\in I} \subset \mathfrak{h}$, $A = (a_{ij})_{i,j\in I}$ a symmetric matrix, and $\Pi = \{\alpha_i\}_{i\in I}$ defined by (1.1). Let $\theta = \sum_{i\in I} k_i\alpha_i \in \Delta_+$ be the highest root (i.e. $\sum_i k_i$ is maximal).

Let $\widehat{I} = \{0\} \bigcup I$, and let $\widehat{I}_{\bar{1}} = I_{\bar{1}}$ if θ is an even root, and $\widehat{I}_{\bar{1}} = \{0\} \bigcup I_{\bar{1}}$ otherwise. Let $\widehat{A}$ be the $\widehat{I} \times \widehat{I}$-matrix obtained from A by adding 0^{th} row and column, where

$$a_{00} = (\theta|\theta)\,,\; a_{0i} = a_{i0} = -(\theta|\alpha_i) \;\text{ for }\; i \in I\,.$$

Note that $\det \widehat{A} = 0$.

Next, we construct a realization of the matrix $\widehat{A}$. Let

$$\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}d\,, \tag{2.7}$$

and let δ be the linear function on $\widehat{\mathfrak{h}}$, defined by

$$\delta|_{\mathfrak{h}+\mathbb{C}K} = 0\,,\; \delta(d) = 1\,. \tag{2.8}$$

Let $h_0 = K - \theta$, $\alpha_0 = \delta - \theta$. Then $\widehat{\mathfrak{h}}$ and $\{h_i\}_{i\in\widehat{I}}$ define a realization of $\widehat{A}$, with the set of simple roots $\widehat{\Pi} = \{\alpha_0\} \bigcup \Pi$.

We now consider in more detail an important for this paper example $\mathfrak{g} = s\ell_{m|n}$ with $m > n$ or $g\ell_{n|n}$. We introduce a structure of a Kac–Moody superalgebra in $\widehat{\mathfrak{g}}$, associated to the basis $\mathrm{v}_1, \dots, \mathrm{v}_{m+n}$ of V as follows. Let $\widehat{I} = \{0\} \cup I$, and let $\widehat{I}_{\bar{1}} = I_{\bar{1}}$ if v_1 and v_{m+n} have the same parity, and $\widehat{I}_{\bar{1}} = \{0\} \cup I_{\bar{1}}$ otherwise. Let $\widehat{A}$ be the $\widehat{I} \times \widehat{I}$-symmetric matrix, obtained from A by adding 0^{th} row and column, where $a_{00} = (-1)^{p(\mathrm{v}_1)} + (-1)^{p(\mathrm{v}_{m+n})}$, $a_{01} = -(-1)^{p(\mathrm{v}_1)}$, $a_{0,m+n-1} = (-1)^{p(\mathrm{v}_{m+n})}$. Note that the sum of entries of each row of $\widehat{A}$ is zero. Such a matrix is depicted by the extended Dynkin diagram, which is a cycle, where the additional, 0^{th} mode, is grey if $0 \in \widehat{I}_{\bar{1}}$ and is white otherwise.

Clearly, corank $\widehat{A} \leq 1$ if $m > n$ and corank $\widehat{A} \leq 2$ if $m = n$. In fact, we have equalities since $\sum_{i\in\widehat{I}} \alpha_i = \delta$, and, if $m = n$, we have a linear dependence of the α_i , restricted to $\mathfrak{h}'$ (= span of the h_{ij}). Hence $\dim \widehat{\mathfrak{h}} = |\widehat{I}| + \text{corank } \widehat{A}$, and we indeed have constructed a realization of the matrix $\widehat{A}$.

The structure of a Kac–Moody superalgebra $\mathfrak{g}(\widehat{A}, \widehat{I}_{\bar{1}})$ in $\widehat{\mathfrak{g}}$ is introduced by letting e_i, f_i for $i \in I$ being the same as for $\mathfrak{g}$, and

$$e_0 = e_{-\theta}t\,,\; f_0 = e_\theta t^{-1}\,,$$

where $e_{\pm\theta} \in \mathfrak{g}_{\pm\theta}$ are chosen such that $(e_\theta|e_{-\theta}) = 1$. (For $\mathfrak{g} = s\ell_{m|n}$ with $m > n$ and $g\ell_{n|n}$ we take $e_0 = E_{1,m+n}t$, $f_0 = (-1)^{p(v_{m+n})}E_{m+n,1}t^{-1}$.)

The invariant bilinear form $(\,.\,|\,.\,)$ extends from $\mathfrak{g}$ to the invariant bilinear form on $\widehat{\mathfrak{g}}$ by the following formulae ($a, b \in \mathfrak{g},\; i, j \in \mathbb{Z}$):

$$(at^i|bt^j) = \delta_{i,-j}(a|b),\; (at^i|\mathbb{C}K + \mathbb{C}d) = (K|K) = 0,\; (d|d) = 0,\; (K|d) = 1.$$

We identify $\widehat{\mathfrak{h}}^*$ with the space

$$\mathfrak{h}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0\,,$$

where δ is defined by (2.8), and

$$\Lambda_0|_{\mathfrak{h}+\mathbb{C}d} = 0\,,\ \Lambda_0(K) = 1\,. \tag{2.9}$$

Then the induced bilinear form on $\widehat{\mathfrak{h}}^*$ is given by (2.4) and

$$(\mathfrak{h}^*|\mathbb{C}\delta + \mathbb{C}\Lambda_0) = 0\,,\ (\delta|\delta) = 0\,,\ (\Lambda_0|\Lambda_0) = 0\,,\ (\Lambda_0|\delta) = 1\,. \tag{2.10}$$

The set of roots $\widehat{\Delta}$ of $\widehat{\mathfrak{g}}$ is the union of the sets of real roots $\widehat{\Delta}^{re}$ and imaginary roots $\widehat{\Delta}^{im}$, where (cf. (2.5)):

$$\widehat{\Delta}^{re} = \{\alpha + s\delta|\, \alpha\in\Delta\,,\ s\in\mathbb{Z}\}\,,\quad \widehat{\Delta}^{im} = \{s\delta|\, s\in\mathbb{Z}\backslash\{0\}\}\,,$$

the parity of a root $\alpha + s\delta$ equals that of α, and all $s\delta$ are even roots. The set of positive roots is the union of

$$\widehat{\Delta}^{re}_+ = \{\alpha + s\delta|\, \alpha\in\Delta\,,\ s>0\}\cup\Delta_+\,,\quad \widehat{\Delta}^{im}_+ = \{s\delta|\, s>0\}\,.$$

The root spaces are $\widehat{\mathfrak{g}}_{\alpha+s\delta} = \mathfrak{g}_\alpha t^s$, and $\widehat{\mathfrak{g}}_{s\delta} = \mathfrak{h}t^s$, except for $\mathfrak{g} = g\ell_{n|n}$, where $\widehat{\mathfrak{g}}_{s\delta} = t^s\{h\in\mathfrak{h}|\,\mathrm{str}\, h = 0\}$.

The triangular decomposition is:

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+\,,$$

where $\widehat{\mathfrak{n}}_\pm = \mathfrak{n}_\pm + \sum_{s>0} t^{\pm s}\mathfrak{g}$, For example, if $\mathfrak{g} = s\ell_{m|n}$, $m>n$ or $g\ell_{n|n}$, then $\widehat{\mathfrak{n}}_+$ consists of all supertraceless matrices over $\mathbb{C}[t]$, which are strictly upper triangular at $t=0$.

Note that all imaginary roots are isotropic, and the even real roots are not. It follows from the description of root spaces that all even real roots are integrable. Hence, by definition, the Weyl group $\widehat{W}$ of $\widehat{\mathfrak{g}}$ is generated by reflections r_α, $\alpha\in\Delta^{re}_{\bar{0}}$.

An important alternative description of the group $\widehat{W}$ is as follows. Given $\alpha\in\mathfrak{h}^*$, define the following automorphism t_α of the vector space $\widehat{\mathfrak{h}}^*$:

$$t_\alpha(\lambda) = \lambda + \lambda(K)\alpha - ((\lambda|\alpha) + \frac{(\alpha|\alpha)}{2}\lambda(K))\delta\,. \tag{2.11}$$

It is easy to check that t_α preserves the bilinear form $(\,.\,|\,.\,)$ on $\widehat{\mathfrak{h}}^*$ and that $t_\alpha t_\beta = t_{\alpha+\beta}$, $\alpha,\beta\in\mathfrak{h}^*$. Given an additive subgroup $L\subset\mathfrak{h}^*$, let $t_L = \{t_\alpha|\alpha\in L\}$.

Let $L = \mathbb{Z}\{\alpha^\vee|\alpha\in\Delta_{\bar{0}}\}$. Then we have ([K2], Chapter 6):

$$\widehat{W} = W \ltimes t_L. \tag{2.12}$$

Remark 2.2. Note that $\varepsilon_+(t_\alpha) = 1$ for $\alpha\in L$, but $\varepsilon_-(t_\alpha)$ is sometimes -1. However for the affine superalgebra, associated to $s\ell_{m|n}$, $m>n$, or to $g\ell_{n|n}$, it is always 1.

Example 2.3. The extended (symmetric) Cartan matrices $\widehat{A}$ for Example 2.1 are obtained from the matrices A using the property that the sum of entries in each row and each column is zero.

3 Partially integrable and admissible highest weight modules over affine superalgebras

Let $\mathfrak{g}$ be either a simple finite-dimensional Lie algebra or one of the simple finite-dimensional Lie superalgebras $s\ell_{m|n}, m > n$, $osp_{m|n}$, $D(2|1;\alpha)$, $F(4)$, $G(3)$, or $g\ell_{n|n}$, endowed with a structure of a Kac–Moody superalgebra $\mathfrak{g}(A, I_{\bar{1}})$, and let $\widehat{\mathfrak{g}}$ be the corresponding affine superalgebra with the structure of the Kac–Moody superalgebra $\mathfrak{g}(\widehat{A}, \widehat{I}_{\bar{1}})$ (see Section 2).

Let $h^\vee$ be the half of the eigenvalue of the Casimir operator, associated to the bilinear form $(\,.\,|\,.\,)$ on $\mathfrak{g}$. It is given by the usual formulae [K1], p. 85:

$$h^\vee = (\rho|\theta) + \frac{1}{2}(\theta|\theta)\,;\ h^\vee(a|b) = \sum_{\alpha\in\Delta_+} (-1)^{p(\alpha)}\alpha(a)\alpha(b), \quad a, b \in \mathfrak{h}\,. \tag{3.1}$$

Recall that $\theta \in \Delta_+$ is the highest root and ρ is a Weyl vector, defined in Section 1. Recall also that we have the following "strange" formula (cf. [KW3]):

$$(\rho|\rho) = h^\vee(\mathrm{sdim}\ \mathfrak{g})/12\,. \tag{3.2}$$

Now we introduce the important subset $\Delta_{\bar{0}}^{\#}$ of the set of even roots of $\mathfrak{g}$ [KW3]. If $h^\vee \neq 0$ (which happens iff $\mathfrak{g} \neq g\ell_{n|n}$, $osp_{2n+2|2n}$ or $D(2|1;\alpha)$ [K1]), let

$$\Delta_{\bar{0}}^{\#} = \{\alpha \in \Delta_{\bar{0}} |\ h^\vee(\alpha|\alpha) > 0\}. \tag{3.3}$$

In the case $\mathfrak{g} = g\ell_{n|n}$ there are two choices of $\Delta_{\bar{0}}^{\#}$, described in the example below.

Example 3.1. Let $\mathfrak{g} = s\ell_{m|n}$, $m > n$, or $g\ell_{n|n}$ with one of the structures of a Kac–Moody algebra, described in Section 2. Then $h^\vee = m - n$ for the invariant bilinear form $(a|b) = \mathrm{str}\ ab$ on $\mathfrak{g}$. We have: $\Delta_{\bar{0}}^{\#} = \{\varepsilon_i - \varepsilon_j |\, p(v_i) = p(v_j) = \bar{0},\, i \neq j\}$. If $m = n$, there is another choice: $\Delta_{\bar{0}}^{\#} = \{\varepsilon_i - \varepsilon_j |\ p(v_i) = p(v_j) = \bar{1},\, i \neq j\}$. For all $\alpha \in \Delta_{\bar{0}}^{\#}$ we have $(\alpha|\alpha) = 2$, except for the second choice of $\Delta_{\bar{0}}^{\#}$ in the case $m = n$, when $(\alpha|\alpha) = -2$.

Note that in all cases the set $\Delta_{\bar{0}}^{\#}$ is W-invariant. Let $L^{\#}$ be the $\mathbb{Z}$-span of the set $\{\alpha^\vee |\, \alpha \in \Delta_{\bar{0}}^{\#}\}$, and let's introduce the following subgroup of the Weyl group $\widehat{W}$:

$$\widehat{W}^{\#} = W \ltimes t_{L^{\#}}\,. \tag{3.4}$$

Let $\Lambda \in \widehat{\mathfrak{h}}^*$ and let $L(\Lambda)$ be the corresponding $\widehat{\mathfrak{g}} = \mathfrak{g}(\widehat{A}, \widehat{I}_{\bar{1}})$-module. Then the central element K acts as a scalar $\Lambda(K)$, called the *level* of Λ, which we denote by the same letter K, unless where confusion may arise.

Definition 3.2. *The* $\widehat{\mathfrak{g}}$*-module* $L(\Lambda)$ *is called* partially integrable *if*

(i) any root $\alpha + n\delta$*, where* $\alpha \in \Delta_{\bar{0}}^{\#}$*,* $n \in \mathbb{Z}$*, is* Λ*-integrable;*

(ii) any root $\alpha \in \Delta_{\bar{0}}$ *is* Λ*-integrable.*

Condition (i) means that $L(\Lambda)$ is integrable with respect to the subalgebra $\widehat{\mathfrak{g}}_{\bar{0}}^{\#}$ of $\widehat{\mathfrak{g}}$, where $\widehat{\mathfrak{g}}_{\bar{0}}^{\#}$ is the affine subalgebra with the set of real roots $\{\alpha + n\delta |\, \alpha \in \Delta_{\bar{0}}^{\#}, n \in \mathbb{Z}\}$, and condition (ii) means that $L(\Lambda)$ is integrable with respect to $\mathfrak{g}_{\bar{0}}$ (= locally finite with respect to $\mathfrak{g}$).

We let $\alpha_j^\vee = \alpha_j$ if α_j is a simple isotropic root, and define it as in (1.4) otherwise. Define the fundamental weights $\Lambda_i \in \widehat{\mathfrak{h}}^*$, $i \in \widehat{I}$, by

$$(\Lambda_i|\alpha_j^\vee) = \delta_{ij}\,, \quad j \in \widehat{I}\,, \quad \Lambda_j(d) = 0\,, \tag{3.5}$$

and also $\Lambda_j(F) = 0$ for $\mathfrak{g} = g\ell_{n|n}$ (which is consistent with (2.9)).

As has been explained in Section 1, we may assume that $\Lambda(d) = 0$, and $\Lambda(F) = 0$ if $\mathfrak{g} = g\ell_{n|n}$ for a highest weight Λ. Then we can write:

$$\Lambda = \sum_{i\in\widehat{I}} m_i\Lambda_i \ \text{ for some } \ m_i \in \mathbb{C}\,. \tag{3.6}$$

The numbers m_i are called the *labels* of Λ .

Define $\widehat{\rho} \in \widehat{\mathfrak{h}}^*$ by

$$(\widehat{\rho}|\alpha_j) = \frac{1}{2}a_{jj}\,, \quad j \in \widehat{I}\,, \quad (\widehat{\rho}|d) = 0\,, \tag{3.7}$$

and, in addition, $\widehat{\rho}(F) = 0$ if $\mathfrak{g} = g\ell_{n|n}$. It is easy to see that (cf. [K2], Chapter 12)

$$\widehat{\rho} = \rho + h^\vee\Lambda_0\,. \tag{3.8}$$

In [KW4] we gave a classification of partially integrable modules $L(\Lambda)$ in terms of the labels m_i of Λ. Here we give the answer only for the relevant to this paper two examples. Unlike in the present paper, we used in [KW4] Dynkin diagrams with minimal number of grey nodes. Here we use diagrams with a white 0^{th} node and recalculate the results of [KW4] using Proposition 1.7(b).

Example 3.3. $\mathfrak{g} = s\ell_{2|1}$; we use its first Dynkin diagram in Example 2.1 and the corresponding extended Dynkin diagram. The level of a $\widehat{\mathfrak{g}}$-module $L(\Lambda)$, where Λ has labels m_0, m_1, m_2, is:

$$K = m_0 + m_1 + m_2\,.$$

We have $\Delta_{\bar{0}}^{\#} = \{\pm\theta\}$, where $\theta = \alpha_1 + \alpha_2$. Let $K' = m_1 + m_2$. Then $L(\Lambda)$ is partially integrable iff $m_0, K' \in \mathbb{Z}_{\geq 0}$ (hence $K \in \mathbb{Z}_{\geq 0}$), and $m_1 = m_2 = 0$ if $K' = 0$ [KW4] . (Note that in these cases $L(\Lambda)$ is integrable. In fact, any partially integrable $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ is integrable iff $\mathfrak{g}$ is either a Lie algebra, or $\mathfrak{g} \simeq s\ell_{m|1}$ with $m > 1$, or $\mathfrak{g} \simeq osp_{m|n}$ with $m = 1$ or 2 and $n \geq 2$ [KW4]).

Example 3.4. $\mathfrak{g} = g\ell_{2|2}$; we use its first Dynkin diagram in Example 2.1 and the corresponding extended Dynkin diagram. The level of a $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ is

$$K = m_0 + m_1 - m_2 + m_3\,.$$

The first choice of $\Delta_{\bar{0}}^{\#}$ is $\{\pm\theta\}$, where $\theta = \alpha_1 + \alpha_2 + \alpha_3$. Let $K' = m_1 - m_2 + m_3$. Then $L(\Lambda)$ is partially integrable iff: $m_0, K' \in \mathbb{Z}_{\geq 0}$ (hence $K \in \mathbb{Z}_{\geq 0}$), and $m_1 = m_3 = 0$ if $K' = 0$ and $m_1 m_3 = 0$ if $K' = 1$. For this choice, θ, $\delta - \theta$ and α_2 are Λ-integrable. The second choice is $\Delta_{\bar{0}}^{\#} = \{\pm\alpha_2\}$, i.e. $\alpha_2, \delta - \alpha_2$ and θ are Λ-integrable. Let then $K'' = m_0 + m_1 + m_3$. The partial integrability conditions in this case are: $m_2\,, -K'' \in \mathbb{Z}_{\geq 0}$ (hence $-K \in \mathbb{Z}_{\geq 0}$), and $m_1 = m_3 = 0$ if $K'' = 0$, $m_1 m_3 = 0$ if $K'' = -1$. Thus, a $\widehat{g\ell}_{2|2}$-module $L(\Lambda)$ is integrable iff $\dim L(\Lambda) = 1$.

Definition 3.5. *(a) Let $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. A* λ-maximal isotropic *subset of Δ is a subset T_λ, consisting of the maximal number of positive roots β_i, such that $(\lambda|\beta_i) = 0$ and $(\beta_i|\beta_j) = 0$ for all $\beta_i, \beta_j \in T_\lambda$.*

(b) A $\mathfrak{g}$-module $L(\Lambda)$ is called tame *if there exists a $\Lambda + \rho$-maximal isotropic subset of Δ, contained in the set of simple roots Π.*

(c) A $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ of level K is called tame *if the $\mathfrak{g}$-module $L(\bar{\Lambda})$ is tame, where $\bar{\Lambda} = \Lambda|_{\mathfrak{h}}$, and $K + h^\vee \neq 0$.*

As usual, we introduce the Weyl denominator R^+ and superdenominator R^- by:

$$R^{\pm} = \frac{e^{\rho} \prod_{\alpha \in \Delta_{\bar{0}+}} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_{\bar{1}+}} (1 \pm e^{-\alpha})}. \tag{3.9}$$

Conjecture 3.6. *[KW3]. Let $L(\lambda)$ be a tame finite-dimensional $\mathfrak{g}$-module. Then there exists a non-zero integer j_λ, such that*

$$j_\lambda R^+ \mathrm{ch}^+_{L(\lambda)} = \sum_{w \in W} \varepsilon_+(w) w \frac{e^{\lambda+\rho}}{\prod_{\beta \in T_{\lambda+\rho}} (1 + e^{-\beta})}, \tag{3.10}$$

where $T_{\lambda+\rho} \subset \Pi$ is a $\lambda + \rho$-maximal subset of Δ, and $\varepsilon_+(w) = \det_{\mathfrak{h}} w$.

It is not difficult to show, using (1.8), that (3.10) implies the following formula for the supercharacter:

$$j_\lambda R^- \mathrm{ch}^-_{L(\lambda)} = \sum_{w \in W} \varepsilon_-(w) w \frac{e^{\lambda+\rho}}{\prod_{\beta \in T_{\lambda+\rho}} (1 - e^{-\beta})}. \tag{3.11}$$

Remark 3.7. (a) Due to Remark 1.5, the $g\ell_{n|n}$-module $L(\lambda)$ remains irreducible when restricted to $s\ell_{n|n}$. Furthermore, if $\lambda(I_{2n}) = 0$, then $L(\lambda)$ is actually a module over $ps\ell_{n|n} := s\ell_{n|n}/\mathbb{C}I_{2n}$.

(b) Due to Remark 1.5, the $\widehat{g\ell}_{n|n}$-module $L(\Lambda)$ remains irreducible when restricted to $\widehat{s\ell}_{n|n} = s\ell_{n|n}[t, t^{-1}] + \mathbb{C}K + \mathbb{C}d$. Furthermore, if $\Lambda(I_{2n}) = 0$, then $L(\Lambda)$ is actually a module over $\widehat{ps\ell}_{n|n} = ps\ell_{n|n}[t, t^{-1}] + \mathbb{C}K + \mathbb{C}d$.

Conjecture 3.8. *(cf. [KW4]). Let $L(\Lambda)$ be a partially integrable tame $\widehat{\mathfrak{g}}$-module. Then*

$$\widehat{R}^+ \mathrm{ch}^+_{L(\Lambda)} = \sum_{\alpha \in L^\#} t_\alpha (e^{\widehat{\rho} + \Lambda - \rho - \bar{\Lambda}} R^+ \mathrm{ch}^+_{L(\bar{\Lambda})}) \tag{3.12}$$

(and, consequently, the same formula holds for $\mathrm{ch}^-_{L(\Lambda)}$ if we replace $\widehat{R}^+$ by $\widehat{R}^-$ and R^+ by R^-, and insert $\varepsilon_-(t_\alpha)$ in front of t_α). Here in the LHS we have the affine Weyl denominator $\widehat{R}^+$ and superdenominator $\widehat{R}^-$, defined by:

$$\widehat{R}^{\pm} = e^{\widehat{\rho} - \rho} R^{\pm} \prod_{n=1}^{\infty} \left((1 - e^{-n\delta})^{\ell} \frac{\prod_{\alpha \in \Delta_{\bar{0}}} (1 - e^{\alpha - n\delta})}{\prod_{\alpha \in \Delta_{\bar{1}}} (1 \pm e^{\alpha - n\delta})} \right), \tag{3.13}$$

where ℓ is the multiplicity of the root δ.

Note that $\ell = \dim \mathfrak{h}$ if $\mathfrak{g} \neq g\ell_{m|m}$, $\ell = \dim \mathfrak{h} - 1 = 2m-1$ (resp. $= 2m-2$) if $\mathfrak{g} = g\ell_{m|m}$ and $\Lambda(I_{2m}) \neq 0$ (resp. $= 0$).

Note that, using (3.10), formula (3.12) can be rewritten as follows:

$$j_{\bar{\Lambda}} \widehat{R}^{\pm} \mathrm{ch}^{\pm}_{L(\Lambda)} = \sum_{w \in \widehat{W}^{\#}} \varepsilon_{\pm}(w) w \frac{e^{\Lambda + \widehat{\rho}}}{\prod_{\beta \in T_{\bar{\Lambda}+\rho}} (1 \pm e^{-\beta})}. \tag{3.14}$$

Conjectures 3.6 and 3.8 have been verified in many cases ([KW4], [S], [GK]).

Given $\Lambda \in \widehat{\mathfrak{h}}^*$ of level $K \neq -h^\vee$, by analogy with the affine Lie algebra case, we introduce the following number ([K2], Chapter 12):

$$m_\Lambda = \frac{(\Lambda + \widehat{\rho} | \Lambda + \widehat{\rho})}{2(K + h^\vee)} - \frac{\mathrm{sdim}\ \mathfrak{g}}{24}, \tag{3.15}$$

called the *modular anomaly* of Λ. As in the affine algebra case, using the "strange" formula (3.2), we obtain another important expression for m_Λ:

$$m_\Lambda = h(\Lambda) - \frac{c(K)}{24}, \tag{3.16}$$

where

$$h(\Lambda) = \frac{(\Lambda + 2\widehat{\rho} | \Lambda)}{2(K + h^\vee)}, \tag{3.17}$$

$$c(K) = \frac{K\, \mathrm{sdim}\ \mathfrak{g}}{K + h^\vee}. \tag{3.18}$$

As in the affine Lie algebra case, $c(K)$ is the central charge of the Sugawara's Virasoro field $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, and h_Λ is the minimal eigenvalue of L_0 in $L(\Lambda)$ (cf. [K2], Chapter 12).

As in the affine Lie algebra case, in order to "improve" modular invariance properties of characters, one introduces the *normalized character and supercharacter* of the $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ by the formula

$$\mathrm{ch}^{\pm}_{\Lambda} = e^{-m_\Lambda \delta} \mathrm{ch}^{\pm}_{L(\Lambda)}. \tag{3.19}$$

Note that $\mathrm{ch}^{\pm}_{\Lambda}$ depends only on $\Lambda \mod \mathbb{C}\delta$.

Definition 3.9. *An injective homomorphism $\varphi : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ is called* compatible *if it respects the triangular decomposition and preserves the invariant bilinear form.*

It is clear that for a compatible homomorphism φ, the map $\varphi : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}$ is an isomorphism, that $\widehat{\Delta}_+ \subset \varphi^*(\widehat{\Delta}_+)$, and that $\varphi(K) = MK$, where M is a positive integer, called the *degree* of φ. Furthermore, the subset $S = (\varphi^*)^{-1}(\widehat{\Pi}) \subset \widehat{\Delta}_+$ clearly satisfies the following two properties:

$$\alpha - \beta \notin \widehat{\Delta} \quad \text{for} \quad \alpha, \beta \in S; \tag{3.20}$$

$$\mathbb{Q}S = \mathbb{Q}\widehat{\Pi}. \tag{3.21}$$

Such an S is called a *simple subset* of $\widehat{\Delta}_+$. As in [KW2], it is not difficult to classify all simple subsets S in $\widehat{\Delta}_+$. (In fact, for $\mathfrak{g} = s\ell_{m|n}$, $m > n$, and $g\ell_{n|n}$ the answer obviously is the same as for the Lie algebra $s\ell_{m+n}$.) We give here the answer for $\mathfrak{g} = s\ell_{2|1}$ and $g\ell_{2|2}$.

Example 3.10. $\otimes_1 - \otimes_2$ from Example 2.1. There are two types of simple subsets $S \subset \widehat{\Delta}_+$:

$$\begin{aligned} S_1 &= \{\alpha_0 + k_0\delta\,,\, \alpha_1 + k_1\delta\,,\, \alpha_2 + k_2\delta\}\,; \\ S_2 &= \{-\alpha_0 + k_0\delta\,,\, -\alpha_1 + k_1\delta\,,\, -\alpha_2 + k_2\delta\}\,. \end{aligned}$$

Here $k_i \in \mathbb{Z}$ are such that S_1 and S_2 lie in $\widehat{\Delta}_+$ (i.e, $k_i \in \mathbb{Z}_{\geq 0}$ for S_1, and $k_i \in \mathbb{N}$ for S_2).

Example 3.11. $\otimes_1 - \bigcirc_2 - \otimes_3$ from Example 2.1. There are four types of simple subsets $S \subset \widehat{\Delta}_+$:

$$\begin{aligned} S_1 &= \{\alpha_0 + k_0\delta\,,\, \alpha_1 + k_1\delta\,,\, \alpha_2 + k_2\delta\,,\, \alpha_3 + k_3\delta\}; \\ S_2 &= \{-\alpha_0 + k_0\delta\,,\, -\alpha_1 + k_1\delta\,,\, -\alpha_2 + k_2\delta\,,\, -\alpha_3 + k_3\delta\}; \\ S_3 &= \{\alpha_0 + k_0\delta\,,\, \alpha_1 + \alpha_2 + k_1\delta\,,\, -\alpha_2 + k_2\delta\,,\, \alpha_2 + \alpha_3 + k_3\delta\}; \\ S_4 &= \{-\alpha_0 + k_0\delta\,,\, -\alpha_1 - \alpha_2 + k_1\delta\,,\, \alpha_2 + k_2\delta,\, -\alpha_2 - \alpha_3 + k_3\delta\}\,. \end{aligned}$$

Here k_i are all non-negative integers, such that $S \subset \widehat{\Delta}_+$.

Definition 3.12. *Let $\varphi : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ be a compatible homomorphism, and let $S = \varphi^{*-1}(\widehat{\Pi})$. Then $\Lambda \in \widehat{\mathfrak{h}}^*$ is called a (corresponding to φ)* principal admissible *weight (with respect to S) if the following two properties hold:*

(i) $(\Lambda + \widehat{\rho}\,|\,\alpha^\vee) \in \mathbb{Z}$ for $\alpha \in \widehat{\Delta}_{\bar{0}}$ implies that $\alpha \in \mathbb{Z}S \cap \widehat{\Delta}$;

(ii) the $\widehat{\mathfrak{g}}$-module $L(\Lambda^0)$ is partially integrable, where

$$\Lambda^0 = \varphi^*(\Lambda + \widehat{\rho}) - \widehat{\rho}\,. \tag{3.22}$$

Note that the level of Λ^0 is expressed via the level $\Lambda(K)$ of Λ by

$$\Lambda^0(K) = M(\Lambda(K) + h^\vee) - h^\vee\,, \tag{3.23}$$

where M is the degree of φ.

Conjecture 3.13. *Let $\varphi : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ be a compatible homomorphism and let $\Lambda \in \mathfrak{h}^*$ be a corresponding principal admissible weight. Then we have the following formula for normalized characters:*

$$(\widehat{R}^\pm \mathrm{ch}^\pm_\Lambda)(h) = (\widehat{R}^\pm \mathrm{ch}^\pm_{\Lambda^0})(\varphi^{-1}(h)),\ h \in \widehat{\mathfrak{h}}\,. \tag{3.24}$$

This formula is proved, in the case when $\widehat{\mathfrak{g}}$ is an affine Lie algebra, for more general, admissible modules, in [KW1]. We conjecture that a similar result holds also in the Lie superalgebra case. (However, all admissible modules are principal admissible for Lie superalgebras, considered in this paper.) Note that Conjecture 3.8, in the form, given by equation (3.14) is proved in [GK] for all cases, considered in the present paper.

As shown in [KW2], formula (3.24) can be written in a more explicit form as follows. Let M be a positive integer and let $S_{(M)} = \{(M-1)\delta + \alpha_0,\, \alpha_1, \ldots, \alpha_\ell\}$ be a simple set of degree M. Let $\beta \in \mathfrak{h}^*$ and $y \in W$ be such that $S := t_\beta y(S_{(M)}) \subset \widehat{\Delta}_+$. Then S is a simple set. All principal admissible weights with respect to S and of level K are of the form (up to adding a multiple of δ):

$$\Lambda = (t_\beta y)(\Lambda^0 - (M-1)(K + h^\vee)\Lambda_0 + \widehat{\rho}) - \widehat{\rho}\,, \tag{3.25}$$

where Λ^0 is a partially integrable weight, and, by (3.23), we have:

$$K = \frac{m+h^\vee}{M} - h^\vee\,, \quad \text{where } m \text{ is the level of } \Lambda^0\,. \tag{3.26}$$

It is convenient to write formula (3.24) in the following coordinates on $\widehat{\mathfrak{h}}$, which we will be using throughout the paper:

$$h = 2\pi i(-\tau d + z + tK)\,, \quad \text{where } z \in \mathfrak{h}\,,\ t, \tau \in \mathbb{C}\,. \tag{3.27}$$

We will always assume that $\mathrm{Im}\,\tau > 0$ in order to have all our series convergent. Note that $e^{-\delta}(h) = q := e^{2\pi i\tau}$, so that $|q| < 1$.

In these coordinates formula (3.24) becomes (cf. [KW2]):

$$(\widehat{R}^{\pm} ch^{\pm}_{\Lambda})(\tau, z, t) = (\widehat{R}^{\pm} ch^{\pm}_{\Lambda^0})\left(M\tau\,,\ y^{-1}(z+\tau\beta)\,,\ \frac{1}{M}\left(t + (z|\beta) + \frac{\tau(\beta|\beta)}{2}\right)\right)\,,$$

or, a little more explicitly:

$$(\widehat{R}^{\pm}\mathrm{ch}^{\pm}_{\Lambda})(\tau, z, t) = q^{\frac{m+h^\vee}{M}(\beta|\beta)} e^{\frac{2\pi i(m+h^\vee)}{M}(z|\beta)} (\widehat{R}^{\pm}\mathrm{ch}^{\pm}_{\Lambda^0})\left(M\tau\,,\ y^{-1}(z+\tau\beta)\,,\ \frac{t}{M}\right)\,. \tag{3.28}$$

Next, we describe the principal admissible weights in the cases considered in this paper.

Proposition 3.14. *Let $\mathfrak{g} = s\ell_{2|1}$, let $\varphi : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ be a compatible homomorphism of degree M, and let $\Lambda \in \widehat{\mathfrak{h}}^*$ be a principal admissible weight with respect to a simple subset $S \subset \widehat{\Delta}_+$ such that $\Lambda^0 = m\Lambda_0$, $m \in \mathbb{Z}_{\geq 0}$ (see (3.22)), and $gcd(M, m+1) = 1$. Then we have for the level K of Λ:*

$$K = \frac{m+1}{M} - 1\,. \tag{3.29}$$

Furthermore, the value of M and the description of all such principal admissible weights Λ with respect to $S = S_i$, $i = 1, 2$, from Example 3.10, is as follows ($k_0, k_1, k_2 \in \mathbb{Z}_{\geq 0}$):

$$\begin{aligned}
S &= S_1 :\ M = k_0 + k_1 + k_2 + 1,\ k_1, k_2 \geq 0,\ k_1 + k_2 \leq M - 1,\\
&\quad\ \Lambda^{(1)}_{k_1,k_2} = (m - k_0(K+1))\Lambda_0 - k_1(K+1)\Lambda_1 - k_2(K+1)\Lambda_2\,;\\
S &= S_2 :\ M = k_0 + k_1 + k_2 - 1,\ 1 \leq k_1, k_2 \leq M - 1,\ k_1 + k_2 \leq M,\\
&\quad\ \Lambda^{(2)}_{k_1,k_2} = (k_0(K+1) - m - 2)\Lambda_0 + k_1(K+1)\Lambda_1 + k_2(K+1)\Lambda_2\,.
\end{aligned}$$

Proof. Formula (3.29) follows from (3.23) since $h^\vee = 1$. In the case of Λ, admissible with respect to S_1, we have:

$$\begin{aligned}
(\Lambda^0 + \rho|\alpha_0) = m+1 &= (\Lambda + \rho|k_0\delta + \alpha_0) = k_0(K+1) + (\Lambda|\alpha_0) + 1\,;\\
(\Lambda^0 + \rho|\alpha_1) = 0 &= (\Lambda + \rho|k_1\delta + \alpha_1) = k_1(K+1) + (\Lambda|\alpha_1)\,;\\
(\Lambda^0 + \rho|\alpha_2) = 0 &= (\Lambda + \rho|k_2\delta + \alpha_2) = k_2(K+1) + (\Lambda|\alpha_2),
\end{aligned}$$

hence $(\Lambda|\alpha_0) = m - k_0(K+1)$, $(\Lambda|\alpha_1) = -k_1(K+1), (\Lambda|\alpha_2) = -k_2(K+1)$, and similarly for S_2.

The computation of M immediately follows from the fact that $M\delta$ equals to the sum of the elements of S_i. Condition (i) on principal admissible weight is equivalent to (3.29).

□

Proposition 3.15. *Let $\mathfrak{g} = g\ell_{2|2}$, let $\varphi : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ be a compatible homomorphism at degree M, and let $\Lambda \in \widehat{\mathfrak{h}}^*$ be a principal admissible weight with respect to a simple subset $S \subset \widehat{\Delta}_+$, such that $\Lambda^0 = m\Lambda_0$, where m is a non-zero integer and $gcd(m, M) = 1$. Then we have for the level K of Λ:*

$$K = \frac{m}{M}\,. \tag{3.30}$$

Furthermore, the value of M and the description of all such principal admissible weights Λ with respect to $S = S_i$, $i = 1, 2, 3, 4$, from Example 3.11, is as follows ($k_0, ..., k_3 \in \mathbb{Z}_{\geq 0}$ are such that $S \subset \widehat{\Delta}_+$):

$$\begin{aligned}
S = S_1 \quad &: \quad M = \sum_{i=0}^{3} k_i + 1\,, \\
\Lambda^{(1)} \quad &= \quad (m - k_0K)\Lambda_0 - k_1K\Lambda_1 + k_2K\Lambda_2 - k_3K\Lambda_3; \\
S = S_2 \quad &: \quad M = \sum_{i=0}^{3} k_i - 1\,, \\
\Lambda^{(2)} \quad &= \quad (k_0K - m - 2)\Lambda_0 + k_1K\Lambda_1 - (k_2K + 2)\Lambda_2 + k_3K\Lambda_3; \\
S = S_3 \quad &: \quad M = \sum_{i=0}^{3} k_i + 1\,, \\
\Lambda^{(3)} \quad &= \quad (m - k_0K)\Lambda_0 - (1 + (k_1 + k_2)K)\Lambda_1 - (2 + k_2K)\Lambda_2 - (1 + (k_2 + k_3)K)\Lambda_3; \\
S = S_4 \quad &: \quad M = \sum_{i=0}^{3} k_i - 1\,, \\
\Lambda^{(4)} \quad &= \quad (k_0K - m - 2)\Lambda_0 + ((k_1 + k_2)K + 1)\Lambda_1 + k_2K\Lambda_2 + ((k_2 + k_3)K + 1)\Lambda_3\,.
\end{aligned}$$

Proof. It is the same as that of Proposition 3.14 □

4 Characters of partially integrable modules and mock theta functions

In this section we rewrite the characters of partially integrable modules $L(\Lambda)$ over an affine superalgebra $\widehat{\mathfrak{g}}$ in terms of "mock" theta functions. Throughout the section, $\mathfrak{g}$ is a finite-dimensional Lie superalgebra as in Section 3 and ℓ is its rank ($=$ rank $\mathfrak{g}_{\bar{0}}$).

Let m be a positive integer and let $j \in \mathbb{Z}/2m\mathbb{Z}$. In the course of the paper we shall often use the well-known *theta functions* of *degree* m (rather Jacobi forms) $\Theta_{j,m}(\tau, z) = \Theta_{j,m}(\tau, z, 0)$, where $\Theta_{j,m}(\tau, z, t)$ are defined by formula (A.1) in the Appendix. Especially important are the four Jacobi theta functions of degree two [M]:

$$\vartheta_{00} = \Theta_{2,2} + \Theta_{0,2}\,,\ \vartheta_{01} = -\Theta_{2,2} + \Theta_{0,2}\,,\ \vartheta_{10} = \Theta_{1,2} + \Theta_{-1,2}\,,\ \vartheta_{11} = i\Theta_{1,2} - i\Theta_{-1,2}\,,$$

discussed in detail in the Appendix.

Since $\widehat{\rho}(K) = h^\vee$ by (3.8), we obtain the following formula for the affine Weyl denominator and and superdenominator (3.13) in the coordinates (3.27):

$$\widehat{R}^{\pm}(h) = e^{2\pi i h^\vee t}\frac{\prod_{n=1}^{\infty}((1-q^n)^\ell \prod_{\alpha\in\Delta_{\bar{0},+}}(1-e^{-2\pi i\alpha(z)}q^{n-1})(1-e^{2\pi i\alpha(z)}q^n))}{\prod_{n=1}^{\infty}\prod_{\alpha\in\Delta_{\bar{1},+}}(1\pm e^{-2\pi i\alpha(z)}q^{n-1})(1\pm e^{2\pi i\alpha(z)}q^n)}\,.$$

Using formulae (A.7) from the Appendix, we can rewrite these expressions (or rather their normalization by a power of q) in terms of the four Jacobi functions of degree two:

$$(4.1)\qquad q^{\frac{1}{24}\,\mathrm{sdim}\,\mathfrak{g}}\widehat{R}^{+}(h)=e^{2\pi i h^{\vee} t} i^{d_{\bar 0}}\eta(\tau)^{\ell-d_{\bar 0}+d_{\bar 1}}\frac{\prod_{\alpha\in\Delta_{\bar 0,+}}\vartheta_{11}(\tau,\alpha(z))}{\prod_{\alpha\in\Delta_{\bar 1,+}}\vartheta_{10}(\tau,\alpha(z))}\,;$$

$$(4.2)\qquad q^{\frac{1}{24}\,\mathrm{sdim}\,\mathfrak{g}}\widehat{R}^{-}(h)=e^{2\pi i h^{\vee} t} i^{d_{\bar 0}-d_{\bar 1}}\eta(\tau)^{\ell-d_{\bar 0}+d_{\bar 1}}\frac{\prod_{\alpha\in\Delta_{\bar 0,+}}\vartheta_{11}(\tau,\alpha(z))}{\prod_{\alpha\in\Delta_{\bar 1,+}}\vartheta_{11}(\tau,\alpha(z))}\,.$$

Here and further, $d_\alpha=|\Delta_{\alpha,+}|$, $\alpha\in\mathbb{Z}/2\mathbb{Z}$.

In order to obtain a modular invariant family, we need to consider also twisted affine Weyl denominators and superdenominators. For this purpose, fix an element $\xi\in\mathfrak{h}^*$, satisfying

$$(4.3)\qquad (\xi|\alpha)\in\frac{1}{2}p(\alpha)+\mathbb{Z}\,,\quad \alpha\in\Delta\,,$$

and let

$$\widehat{R}^{\mathrm{tw}\,,\pm}=t_\xi(\widehat{R}^{\pm})\,.$$

Recalling (2.11), we have:

$$t_\xi(\delta)=\delta,\ t_\xi(\alpha)=\alpha-(\xi|\alpha)\delta\ \text{ if }\ \alpha\in\Delta\,,\ \widehat{\rho}^{\mathrm{tw}}:=t_\xi(\widehat{\rho})=\widehat{\rho}+h^\vee\xi-(\frac{1}{2}h^\vee(\xi|\xi)+(\rho|\xi))\delta\,.$$

Hence we obtain from (3.13):

$$\widehat{R}^{\mathrm{tw}\,,\pm}=e^{\widehat{\rho}^{\mathrm{tw}}}\prod_{n=1}^{\infty}\frac{(1-q^n)^\ell\prod_{\alpha\in\Delta_{\bar 0,+}}(1-e^{-\alpha}q^{n-1-(\xi|\alpha)})(1-e^{\alpha}q^{n+(\xi|\alpha)})}{\prod_{\alpha\in\Delta_{\bar 1,+}}(1\pm e^{-\alpha}q^{n-1-(\xi|\alpha)})(1\pm e^{\alpha}q^{n+(\xi|\alpha)})}\,.$$

This can be rewritten in terms of the four Jacobi functions, as in the non-twisted case (cf. (4.1), (4.2)). For this we need to use Proposition A.6 from the Appendix. By a straightforward (but a bit lengthy calculation) we derive formulae, similar to (4.1) and (4.2):

$$(4.4)\qquad q^{\frac{1}{24}\,\mathrm{sdim}\,\mathfrak{g}}\widehat{R}^{\mathrm{tw}\,,+}(h)\ =\ e^{2\pi i h^{\vee} t}(-1)^{2(\rho_{\bar 0}|\xi)} i^{d_{\bar 0}}\eta(\tau)^{\ell-d_{\bar 0}+d_{\bar 1}}\frac{\prod_{\alpha\in\Delta_{\bar 0,+}}\vartheta_{11}(\tau,\alpha(z))}{\prod_{\alpha\in\Delta_{\bar 1,+}}\vartheta_{00}(\tau,\alpha(z))}\,,$$

$$(4.5)\qquad q^{\frac{1}{24}\,\mathrm{sdim}\,\mathfrak{g}}\widehat{R}^{\mathrm{tw}\,,-}(h)\ =\ e^{2\pi i h^{\vee} t}(-1)^{2(\rho|\xi)-\frac{1}{2}d_{\bar 1}} i^{d_{\bar 0}}\eta(\tau)^{\ell-d_{\bar 0}+d_{\bar 1}}\frac{\prod_{\alpha\in\Delta_{\bar 0,+}}\vartheta_{11}(\tau,\alpha(z))}{\prod_{\alpha\in\Delta_{\bar 1,+}}\vartheta_{01}(\tau,\alpha(z))}\,.$$

It turns out that modular transformation formulae can be written in a beautiful unified form if we use the following notations for (4.1), (4.2), (4.4), (4.5):

$$(4.1)=\widehat{R}_0^{(\frac{1}{2})}(\tau,z,t),\quad (4.2)=\widehat{R}_0^{(0)}(\tau,z,t),\quad (4.4)=\widehat{R}_{\frac{1}{2}}^{(\frac{1}{2})}(\tau,z,t),\quad (4.5)=\widehat{R}_{\frac{1}{2}}^{(0)}(\tau,z,t),$$

i.e., the superscripts $(\frac{1}{2})$ and (0) refer to the denominator and superdenominator respectively, and the subscripts 0 and $\frac{1}{2}$ refer to the non-twisted and twisted cases respectively.

Theorem 4.1. *Let* $\varepsilon, \varepsilon' = 0$ *or* $\frac{1}{2}$. *Then*

(a) $\widehat{R}^{(\varepsilon)}_{\varepsilon}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{(z|z)}{2\tau}\right) = c_{\varepsilon}(-i\tau)^{\frac{1}{2}\ell} R^{(\varepsilon)}_{\varepsilon}(\tau, z, t)$,
where $c_0 = (-i)^{d_{\bar{0}} - d_{\bar{1}}}$ *and* $c_{\frac{1}{2}} = (-i)^{d_{\bar{0}}}$*;*

$\widehat{R}^{(\varepsilon')}_{\varepsilon}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{(z|z)}{2\tau}\right) = c_{\varepsilon,\varepsilon'}(-i\tau)^{\frac{1}{2}\ell} R^{(\varepsilon)}_{\varepsilon'}(\tau, z, t)$ *if* $\varepsilon \neq \varepsilon'$,
where $c_{0,\frac{1}{2}} = c_{\frac{1}{2},0} = (-1)^{2(\rho|\xi) - \frac{1}{2}d_{\bar{1}} + d_{\bar{0}}} i^{d_{\bar{0}}}$.

(b) $\widehat{R}^{(\varepsilon)}_{0}(\tau + 1, z, t) = e^{\frac{\pi i}{12}\,\mathrm{sdim}\,\mathfrak{g}}\widehat{R}^{(\varepsilon)}_{0}(\tau, z, t)$*;*
$\widehat{R}^{(\varepsilon)}_{\frac{1}{2}}(\tau + 1, z, t) = (-1)^{2(\rho_{\bar{1}}|\xi) - \frac{1}{2}d_{\bar{1}}} e^{\frac{\pi i}{12}(2d_{\bar{0}} + d_{\bar{1}} + \ell)}\widehat{R}^{(\frac{1}{2} - \varepsilon)}_{\frac{1}{2}}(\tau, z, t)$.

Proof. It is immediate by the modular transformation formulae for the four Jacobi theta functions of degree two, given by Proposition A.7 in the Appendix and the modular transformation formulae for the η-function:

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\eta(\tau)\,, \quad \eta(\tau + 1) = e^{\frac{\pi i}{12}}\eta(\tau)\,.$$

□

The representation theoretical meaning of twisted denominators and superdenominators is as follows. Let

$$\widehat{\mathfrak{g}}^{\mathrm{tw}} = \mathfrak{g}_{\bar{0}}[t, t^{-1}] \oplus \mathfrak{g}_{\bar{1}}[t, t^{-1}]t^{\frac{1}{2}} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with the same commutation relations and the same $\widehat{\mathfrak{h}}$ as in Section 2. The action of t_{ξ} on $\widehat{\mathfrak{h}}^*$ induces the action of $t_{-\xi}$ on $\widehat{\mathfrak{h}}$, which extends to the following isomorphism $t_{-\xi} : \widehat{\mathfrak{g}}^{\mathrm{tw}} \xrightarrow{\sim} \widehat{\mathfrak{g}}$:

$$\begin{aligned} t_{-\xi}(h\,t^n) &= ht^n + \xi(h)K\delta_{n,0}\,,\ h \in \mathfrak{h}\,,\ \ t_{-\xi}(K) = K, \\ t_{-\xi}(d) &= d - \xi - \frac{1}{2}(\xi|\xi)K\,,\ t_{-\xi}(e_{\alpha}t^n) = e_{\alpha}t^{n + (\xi|\alpha)}\,,\ e_{\alpha} \in \mathfrak{g}_{\alpha}\,. \end{aligned}$$

Via this isomorphism, the $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ becomes a $\widehat{\mathfrak{g}}^{\mathrm{tw}}$ -module, which we denote by $L^{\mathrm{tw}}(\Lambda)$. The $\widehat{\mathfrak{g}}^{\mathrm{tw}}$-module $L^{\mathrm{tw}}(\Lambda)$ is a highest weight module with respect to the triangular decomposition of $\widehat{\mathfrak{g}}^{\mathrm{tw}}$ induced from that of $\widehat{\mathfrak{g}}$ via the isomorphism $t_{-\xi}$. Its highest weight is

$$\Lambda^{\mathrm{tw}} = t_{\xi}(\Lambda), \tag{4.6}$$

and its character and supercharacter are:

$$\mathrm{ch}^{\pm}_{L^{\mathrm{tw}}(\Lambda)} = t_{\xi}(\mathrm{ch}^{\pm}_{L(\Lambda)}), \tag{4.7}$$

their denominators being $\widehat{R}^{\mathrm{tw},\pm}$. The corresponding normalized twisted character and supercharacter are given by

$$\mathrm{ch}^{\mathrm{tw},\pm}_{\Lambda} = e^{-m^{\mathrm{tw}}_{\Lambda}\delta}\mathrm{ch}^{\pm}_{L^{\mathrm{tw}}(\Lambda)}\,, \tag{4.8}$$

where

$$m^{\mathrm{tw}}_{\Lambda} = \frac{(\Lambda^{\mathrm{tw}} + \widehat{\rho}^{\mathrm{tw}}|\Lambda^{\mathrm{tw}} + \widehat{\rho}^{\mathrm{tw}})}{2(K + h^{\vee})} - \frac{\mathrm{sdim}\,\mathfrak{g}}{24}. \tag{4.9}$$

Remark 4.2. (a) As in the affine Lie algebra case (see [K2], Chapter 12) we have, in view of (3.16) - (3.19), the following formula for the normalized character ch_Λ in coordinates (3.27):

$$\mathrm{ch}^+_\Lambda(h) = \mathrm{ch}_\Lambda(\tau, z, t) = e^{2\pi i K t}\mathrm{tr}_{\,\mathrm{L}(\Lambda)}\mathrm{q}^{\mathrm{L}_0 - \frac{1}{24}\mathrm{c(K)}}\mathrm{e}^{2\pi \mathrm{iz}}, \tag{4.10}$$

where K is the level of $L(\Lambda)$, L_0 is the 0^{th} mode of the Sugawara's Virasoro field $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, and $c(K)$, given by (3.18), is its central charge. A similar formula holds for ch^-_Λ by replacing tr by str.

(b) We have the twisted Sugawara's Virasoro field $L^{\mathrm{tw}}(z) = \sum_{n\in\mathbb{Z}} L^{\mathrm{tw}}_n z^{-n-2}$, for which we take $s_\alpha = -(\xi|\alpha)$, $\alpha \in \Delta_+$ (see [KW6], Section 1). Then $m^{\mathrm{tw}}_\Lambda = h^{\mathrm{tw}}(\Lambda) - \frac{c(K)}{24}$, where $h^{\mathrm{tw}}(\Lambda) = \frac{(\Lambda^{\mathrm{tw}} + 2\widehat{\rho}^{\mathrm{tw}}|\Lambda^{\mathrm{tw}})}{2(K+h^\vee)}$ is the minimal eigenvalue of L^{tw}_0 in $L^{\mathrm{tw}}(\Lambda)$, and we have:

$$\mathrm{ch}^{\mathrm{tw},+}_\Lambda(\tau, z, t) = e^{2\pi i K t}\mathrm{tr}_{\,\mathrm{L}^{\mathrm{tw}}(\Lambda)}\mathrm{q}^{\mathrm{L}^{\mathrm{tw}}_0 - \frac{1}{24}\mathrm{c(K)}}\mathrm{e}^{2\pi \mathrm{iz}}, \tag{4.11}$$

and similarly for $\mathrm{ch}^{\mathrm{tw},-}_\Lambda$, and $\widehat{R}^{\mathrm{tw},\pm}$ are the denominators of $\mathrm{ch}^{\mathrm{tw},\pm}_\Lambda$.

Next, we introduce mock theta functions. Having in mind applications to affine Lie superalgebras, we use notation similar to that above.

Let $\mathfrak{h}_\mathbb{R}$ be an ℓ-dimensional vector space over $\mathbb{R}$ with a non-degenerate symmetric bilinear form $(\,.\,|\,.\,)$ (not necessarily positive definite). We shall identify $\mathfrak{h}_\mathbb{R}$ with $\mathfrak{h}^*_\mathbb{R}$ using this bilinear form. Let $L^\#$ be a positive definite integral sublattice of $\mathfrak{h}^*_\mathbb{R}$. Let $\widehat{\mathfrak{h}}_\mathbb{R} = \mathfrak{h}_\mathbb{R} \oplus (\mathbb{R}K \oplus \mathbb{R}d)$ be the $\ell + 2$-dimensional vector space over $\mathbb{R}$ with a symmetric bilinear form $(\,.\,|\,.\,)$, which coincides with that on $\mathfrak{h}_\mathbb{R}$, and such that $\mathfrak{h}_\mathbb{R} \perp (\mathbb{R}K \oplus \mathbb{R}d)$ and $\mathbb{R}K \oplus \mathbb{R}d$ is the 2-dimensional hyperbolic space, i.e. $(K|K) = (d|d) = 0$, $(K|d) = 1$. We identify $\widehat{\mathfrak{h}}_\mathbb{R}$ with $\widehat{\mathfrak{h}}^*_\mathbb{R} = \mathfrak{h}^*_\mathbb{R} \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0$, using this bilinear form, so that $\mathfrak{h}_\mathbb{R}$ gets identified with $\mathfrak{h}^*_\mathbb{R}$, and K (resp. d) with δ (resp. Λ_0). Given $\alpha \in \mathfrak{h}^*_\mathbb{R}$, define the automorphism t_α of the vector space $\widehat{\mathfrak{h}}^*_\mathbb{R}$ by formula (2.11).

Let $\Lambda \in \widehat{\mathfrak{h}}^*_\mathbb{R}$ be such that $\Lambda(K)$ is a positive integer, which, as before, we denote by K, and $(\Lambda|L^\#) \subset \mathbb{Z}$. Let $T \subset \mathfrak{h}^*_\mathbb{R}$ be a finite set of vectors that spans an isotropic subspace of $\mathfrak{h}^*_\mathbb{R}$, and such that $(T|L^\#) \subset \mathbb{Z}$ and $\bar{\Lambda} \perp T$, where, as before, $\bar{\Lambda} \in \mathfrak{h}^*_\mathbb{R}$ denotes the restriction of Λ to $\mathfrak{h}_\mathbb{R}$.

Definition 4.3. *A* mock theta function *of degree K is defined by the following series:*

$$\Theta^\pm_{\Lambda,T} = e^{-\frac{(\Lambda|\Lambda)}{2K}\delta} \sum_{\alpha\in L^\#} \varepsilon_\pm(t_\alpha) t_\alpha \frac{e^\Lambda}{\prod_{\beta\in T}(1 \pm e^{-\beta})}\,. \tag{4.12}$$

It is not difficult to deduce from (2.11) that the series $\Theta^+_{\Lambda,T}$ (resp. $\Theta^-_{\Lambda,T}$) converges in the domain

$$X := \{h \in \widehat{\mathfrak{h}}|\ \mathrm{Re}\ \delta(h) > 0\}$$

to a meromorphic function with poles at the hyperplanes

$$\{h \in \widehat{\mathfrak{h}}|\ \beta(h) - (\alpha|\beta)\delta(h) = (2n+1)\pi i \,(\text{resp.} = 2n\pi i)\},$$

where $\alpha \in L^\#, \beta \in T, n \in \mathbb{Z}$. It is also clear that $\Theta^\pm_{\Lambda,T}$ depends only on $\Lambda \mod \mathbb{C}\delta$. We shall call also a mock theta function an arbitrary linear combination of functions of the form (4.12).

By (2.11), in coordinates (3.27) the mock theta function (4.12) looks as follows:

$$(4.13)\qquad \Theta^{\pm}_{\Lambda,T}(\tau,z,t)=e^{2\pi iKt}\sum_{\gamma\in L^{\#}+K^{-1}\bar{\Lambda}}\varepsilon_{\pm}(t_\gamma)\frac{q^{\frac{K(\gamma|\gamma)}{2}}e^{2\pi iK\gamma(z)}}{\prod_{\beta\in T}(1\pm q^{-(\gamma|\beta)}e^{-2\pi i\beta(z)})}.$$

Recall that $\varepsilon_+(t_\gamma)=1$ in all cases, and $\varepsilon_-(t_\gamma)=1$ in all cases, considered in this paper.

Remark 4.4. Let D be the Laplace operator, associated with the bilinear form $(\,.\,|\,.\,)$. Using that $D(e^\lambda)=(\lambda|\lambda)e^\lambda$, we immediately see that $D(\Theta^{\pm}_{\Lambda,T})=0$. Also, obviously, these functions are t_α-invariant (rather anti-invariant) for all $\alpha\in L^{\#}$, invariant under the translations $z\mapsto z+2\pi i\alpha$ for $\alpha\in L^{\#}$, and satisfy the degree property: $\Theta(h+aK)=e^{Ka}\Theta(h)$, $a\in\mathbb{C}$. It is known ([K2], Chapter 13) that these properties characterize classical theta functions of degree K. It is an interesting problem to find out what are the properties which, along with the above, characterize mock theta functions.

In the same way as the normalized characters of integrable modules over affine Lie algebras are rewritten in terms of the theta functions (see [K2], Chapter 13), by Conjecture 3.8, the normalized characters and supercharacters of partially integrable tame modules over affine Lie superalgebras can be rewritten in terms of mock theta functions:

$$(4.14)\qquad j_{\bar{\Lambda}}\widehat{R}^{\pm}\mathrm{ch}^{\pm}_{\Lambda}=\sum_{w\in W}\varepsilon_{\pm}(w)\Theta^{\pm}_{w(\Lambda+\widehat{\rho}),w(T_{\bar{\Lambda}+\rho})}\,,$$

where $T_{\bar{\Lambda}+\rho}$ is a maximal subset in $\Delta_{\bar{1},+}$, consisting of linearly independent pairwise orthogonal isotropic roots, which are orthogonal to $\bar{\Lambda}+\rho$.

Remark 4.5. Since $\mathrm{ch}^{\mathrm{tw},\pm}_{\Lambda}=t_\xi(\mathrm{ch}^{\pm}_{\Lambda})$, we obtain in coordinates (3.27):

$$\mathrm{ch}^{\mathrm{tw},\pm}_{\Lambda}(\tau,z,t)=\mathrm{ch}^{\pm}_{\Lambda}(\tau,z+\tau\xi,t+\frac{(z+\tau\xi|z+\tau\xi)-(z|z)}{2\tau}).$$

The main objective of our discussion in the next Section will be modular transformation properties of the *numerator* (cf. the RHS of (4.14)):

$$(4.15)\qquad \Phi^{\pm}_{\Lambda,T}=\sum_{w\in W}\varepsilon_{\pm}(w)\Theta^{\pm}_{w(\Lambda),w(T)}\,,$$

where $\Lambda\in\widehat{\mathfrak{h}}^*_{\mathbb{R}}$ is such that $\Lambda(K)$ is a positive integer (denoted as before by K), and $T\subset\widehat{\mathfrak{h}}^*_{\mathbb{R}}$ consists of pairwise orthogonal linearly independent isotropic roots, which are orthogonal to Λ, and $(T|L^{\#})\subset\mathbb{Z}$.

Recall the action of the group $SL_2(\mathbb{R})$ in the domain

$$X=\{(\tau,z,t)|\,\mathrm{Im}\,\tau>0\}\subset\widehat{\mathfrak{h}}^*$$

in coordinates (3.27):

$$(4.16)\qquad \begin{pmatrix}a\,b\\c\,d\end{pmatrix}\cdot(\tau,z,t)=\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d},t-\frac{c(z|z)}{2(c\tau+d)}\right),$$

and the action of the corresponding metaplectic group $Mp_2(\mathbb{R})$ on the space of meromorphic functions on X:

$$F|_A(\tau,z,t) = (c\tau+d)^{-\frac{\ell}{2}} F(A\cdot(\tau,z,t)) \tag{4.17}$$

(see [K2], Chapter 13 for details). Everywhere in the paper the square root of a complex number $a = re^{i\theta}$, where $r \geq 0$ and $-\pi < \theta < \pi$ is, as usual, chosen to be $a^{1/2} = r^{1/2}e^{i\theta/2}$.

Let's consider the mock theta functions of the form $\Theta^-_{\Lambda,\beta}$, for which the set T consists of one isotropic vector β. Choose a basis $\mathrm{v}_1, \mathrm{v}_2, \dots$ of $\mathfrak{h}$, such that $(\beta|\mathrm{v}_j) = \delta_{1j}$. Then coordinates (3.27) on $\widehat{\mathfrak{h}}$ become $h = 2\pi i(-\tau d + \sum_j z_j \mathrm{v}_j + tK)$, and we have:

$$\Theta^-_{\Lambda,\beta}(h) = e^{2\pi i K t} \sum_{\alpha\in L^{\#}} \frac{e^{2\pi i\left(\sum_{j\geq 1} z_j(\Lambda+K\alpha|\mathrm{v}_j)+\tau(\frac{K}{2}(\alpha|\alpha)+(\Lambda|\alpha))\right)}}{1-e^{-2\pi i(z_1+(\alpha|\beta)\tau)}}.$$

Hence for each fixed τ, $\mathrm{Im}\,\tau > 0$, this function has only simple poles and they occur at the hyperplanes $z_1 = n - (\gamma|\beta)\tau$ ($n \in \mathbb{Z}$, $\gamma \in L^{\#}$). For $\gamma \in L^{\#}$ let $L_\gamma = \{\alpha \in L^{\#} \mid (\alpha|\beta) = (\gamma|\beta)\}$. It is immediate to compute the residue:

$$\begin{aligned}\operatorname*{Res}_{z_1=n-(\gamma|\beta)\tau} \Theta^-_{\Lambda,\beta}(h) = \frac{e^{2\pi i K t}}{2\pi i} \sum_{\alpha\in L_\gamma} & e^{2\pi i\left(n(\Lambda+K\alpha|\mathrm{v}_1)+\sum_{j\geq 2} z_j(\Lambda+K\alpha|\mathrm{v}_j)\right)} \\ & \times e^{2\pi i\tau\left(\frac{K}{2}(\alpha|\alpha)+(\Lambda|\alpha)-(\gamma|\beta)(\Lambda+K\alpha|\mathrm{v}_1)\right)}.\end{aligned} \tag{4.18}$$

On the other hand, for each fixed τ, $\mathrm{Im}\,\tau > 0$, the function, transformed by $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, looks as follows:

$$\begin{aligned}(\Theta^-_{\Lambda,\beta}|_S)(h) = \tau^{-\frac{\ell}{2}}\Theta^-_{\Lambda,\beta}(S\cdot h) &= \tau^{-\frac{\ell}{2}} e^{2\pi i K\left(t-\frac{\|\sum_{j\geq 1} z_j \mathrm{v}_j\|^2}{2\tau}\right)} \\ & \times \sum_{\alpha\in L^{\#}} \frac{e^{\frac{2\pi i}{\tau}\left(\sum_{j\geq 1} z_j(\Lambda+K\alpha|\mathrm{v}_j)-\frac{K}{2}(\alpha|\alpha)-(\Lambda|\alpha)\right)}}{1-e^{-\frac{2\pi i}{\tau}(z_1-(\alpha|\beta))}}.\end{aligned}$$

For each fixed τ, this function has only simple poles as well, and they occur at the hyperplanes $z_1 = (\gamma|\beta) + n\tau$ ($n \in \mathbb{Z}, \gamma \in L^{\#}$). The corresponding residue is

$$\begin{aligned}\operatorname*{Res}_{z_1=(\gamma|\beta)+n\tau} (\Theta^-_{\Lambda,j}|_S)(h) = \frac{\tau^{1-\frac{\ell}{2}} e^{2\pi i K t}}{2\pi i} e^{-\frac{\pi K i}{\tau}\|((\gamma|\beta)+n\tau)\mathrm{v}_1+\sum_{j\geq 2} z_j \mathrm{v}_j\|^2} \\ \times \sum_{\alpha\in L_\gamma} e^{2\pi i n(\Lambda+K\alpha|\mathrm{v}_1)} e^{\frac{2\pi i}{\tau}\left((\gamma|\beta)(\Lambda+K\alpha|\mathrm{v}_1)+\sum_{j\geq 2} z_j(\Lambda+K\alpha|\mathrm{v}_j)-(\frac{K}{2}(\alpha|\alpha)+(\Lambda|\alpha))\right)}.\end{aligned} \tag{4.19}$$

Proposition 4.6. *Let* $\dim\mathfrak{h} = 2$ *and let* $\Lambda \in \widehat{\mathfrak{h}}^*$, $\alpha, \beta \in \mathfrak{h}^*$ *be such that* $\Lambda(K)$ *is a positive integer,* $(\Lambda|\alpha) \in \mathbb{Z}$, $(\Lambda|\beta) = 0$, $\Lambda(K)(\alpha|\alpha)$ *is a positive integer,* $(\alpha|\beta) = 1$, $(\beta|\beta) = 0$. *Let* $L^{\#} = \mathbb{Z}\alpha$. *Then the function* $G = \Theta^-_{\Lambda,\beta} - \Theta^-_{\Lambda,\beta}|_S$ *is holomorphic in the domain* X.

Proof. The function G is holomorphic in the domain X if and only if $\Theta^-_{\Lambda,\beta}$ and $\Theta^-_{\Lambda,\beta}|_S$ have the same poles and equal residues at each pole. For $\gamma \in L^\#$ we have: $L_\gamma = \{\gamma\}$, hence formulae (4.18) and (4.19) become:

$$\operatorname*{Res}_{z_1=n-(\gamma|\beta)\tau} \Theta^-_{\Lambda,\beta}(h) = \frac{e^{2\pi iKt}}{2\pi i} e^{2\pi i(n(\Lambda+K\gamma|\mathrm{v}_1)+z_2(\Lambda+K\gamma|\mathrm{v}_2))} q^{\frac{K}{2}(\gamma|\gamma)+(\Lambda|\gamma)-(\gamma|\beta)(\Lambda+K\gamma|\mathrm{v}_1)},$$

and

$$\operatorname*{Res}_{z_1=(\gamma|\beta)+n\tau} (\Theta^-_{\Lambda,j}|_S)(h) = \frac{e^{2\pi iKt}}{2\pi i} e^{-\frac{\pi Ki}{\tau}|((\gamma|\beta)+n\tau)\mathrm{v}_1+z_2\mathrm{v}_2|^2} e^{2\pi in(\Lambda+K\gamma|\mathrm{v}_1)}$$
$$\times e^{-\frac{\pi i}{\tau}((\gamma|\beta)(\Lambda+K\gamma|\mathrm{v}_1)+z_2(\Lambda+K\gamma|\mathrm{v}_2)-\frac{K}{2}(\gamma|\gamma)-(\Lambda|\gamma))}.$$

Let $\mathrm{v}_1 = \alpha$, $\mathrm{v}_2 = \beta$, $\gamma = a\alpha$, $a \in \mathbb{Z}$. Then the above formulae become:

$$\operatorname*{Res}_{z_1=n-a\tau} \Theta^-_{\Lambda,\beta}(h) = \frac{e^{2\pi iKt}}{2\pi i} e^{2\pi iKaz_2} q^{-\frac{Ka^2}{2}(\alpha|\alpha)}; \quad \operatorname*{Res}_{z_1=a+n\tau} \Theta^-_{\Lambda,\beta}|_S(h) = \frac{e^{2\pi iKt}}{2\pi i} e^{-2\pi iKnz_2} q^{-\frac{Kn^2}{2}(\alpha|\alpha)}.$$

Hence the residues at all poles of the function G are zero. □

5 Transformation properties of the mock theta functions $\Phi^{[m]}$ and their modifications $\widetilde{\Phi}^{[m]}$

Let $\mathfrak{g} = s\ell_{2|1}$ with a structure of a Kac–Moody algebra as in Example 3.3, and let $\widehat{\mathfrak{g}}$ be the corresponding affine superalgebra. We have

$$\mathrm{sdim}\,\mathfrak{g} = 0\,,\ \ell = 2\,,\ \Pi = \Delta_{\bar{1},+} = \{\alpha_1, \alpha_2\},\ \Delta_{\bar{0},+} = \{\theta = \alpha_1 + \alpha_2\},$$
$$(\alpha_i|\alpha_i) = 0,\ (\alpha_1|\alpha_2) = 1\,,\ (\theta|\theta) = 2\,,\ \rho_{\bar{0}} = \rho_{\bar{1}} = \frac{1}{2}\theta\,,\ \rho = 0\,, h^\vee = 1.$$

We introduce the following coordinates in the Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ (cf. (3.27)):

$$h = 2\pi i(-\tau\Lambda_0 - z_1\alpha_2 - z_2\alpha_1 + t\delta) = 2\pi i(-\tau\Lambda_0 + u(\alpha_1+\alpha_2) + v(\alpha_1-\alpha_2) + t\delta)\,. \tag{5.1}$$

In this section we shall study transformation properties of the numerator of the normalized supercharacter of the integrable $\widehat{\mathfrak{g}}$-module $L(m\Lambda_0)$, where m is a non-negative integer (see Example 3.3), using formula (3.14) for $\mathrm{ch}^-_{L(m\Lambda_0)}$. We have: $\widehat{\rho} = \Lambda_0$, $m_{m\Lambda_0} = 0$, hence $\mathrm{ch}^-_{L(m\Lambda_0)} = \mathrm{ch}^-_{m\Lambda_0}$. We choose in this formula $T_0 = \{\alpha_1\}$. Then $j_0 = 1$ and (3.14) gives:

$$\widehat{R}^-\mathrm{ch}^-_{m\Lambda_0} = \sum_{w\in\widehat{W}^\#} \varepsilon(w) w \frac{e^{(m+1)\Lambda_0}}{1-e^{-\alpha_1}}\,. \tag{5.2}$$

Since $L^\# = \mathbb{Z}\theta$ and $W = \{1, r_{\alpha_1+\alpha_2}\}$, due to (3.4) and (2.11), formula (5.2) in coordinates (5.1) looks as follows:

$$(\widehat{R}^-\mathrm{ch}^-_{m\Lambda_0})(\tau, z_1, z_2, t) = e^{2\pi i(m+1)t} \sum_{j\in\mathbb{Z}} \left(\frac{e^{2\pi ij(m+1)(z_1+z_2)} q^{j^2(m+1)}}{1-e^{2\pi iz_1}q^j} - \frac{e^{-2\pi ij(m+1)(z_1+z_2)} q^{j^2(m+1)}}{1-e^{-2\pi iz_2}q^j} \right). \tag{5.3}$$

We denote the right-hand side of this formula by

$$\Phi^{[m]}(\tau, z_1, z_2, t) = \varphi^{[m]}(\tau, u, v, t)\,, \tag{5.4}$$

where $u = -\frac{1}{2}(z_1 + z_2)$, $v = \frac{1}{2}(z_1 - z_2)$. This is the *numerator* of $\mathrm{ch}^-_{m\Lambda_0}$.

The main properties of the functions $\Phi^{[m]}$ are described by the following lemma.

Lemma 5.1. *(a)* $\Phi^{[m]}(\tau, z_1 + a, z_2 + b, t) = \Phi^{[m]}(\tau, z_1, z_2, t)$ *for all* $a, b \in \mathbb{Z}$.

(b) $\Phi^{[m]}(\tau, -z_1, -z_2, t) = -\Phi^{[m]}(\tau, z_1, z_2, t)$.

(c) $\Phi^{[m]}(\tau, z_2, z_1, t) = \Phi^{[m]}(\tau, z_1, z_2, t)$.

(d) $\Phi^{[m]}(\tau, z_1 + \tau, z_2 + \tau, t) = q^{-(m+1)} e^{-2\pi i(m+1)(z_1+z_2)} \Phi^{[m]}(\tau, z_1, z_2, t)$.

(e) $\Phi^{[m]}(\tau, z_1, z_2, t) - e^{2\pi i(m+1)z_1} \Phi^{[m]}(\tau, z_1, z_2 + \tau, t)$
$= \sum_{j=0}^{m-1} e^{\pi i(j+1)(z_1-z_2)} q^{-\frac{(j+1)^2}{4(m+1)}} \big(\Theta_{j+1,m+1}(\tau, z_1 + z_2, t) - \Theta_{-(j+1),m+1}(\tau, z_1 + z_2, t)\big)$,
where $\Theta_{j,m}(\tau, z, t) = e^{2\pi imt}\Theta_{j,m}(\tau, z)$, *and* $\Theta_{j,m}(\tau, z)$ *is given by (A.3).*

Proof. Without loss of generality we may assume that $t = 0$. Property (a) is obvious, while property (b) follows easily from (c). Property (c) follows from the expansion of $\Phi^{[m]}$ from formula (8.3) in [KW3]:

$$\Phi^{[m]}(\tau, z_1, z_2, 0) = \left(\sum_{j,k\geq 0,\, \min(j,k)|m+1} - \sum_{j,k<0,\, \max(j,k)|m+1} \right) e^{2\pi i(jz_1+kz_2)} q^{\frac{jk}{m+1}}.$$

Next, we prove (e). We have:

$$\Phi^{[m]}(\tau, z_1, z_2, 0) = \sum_{j\in\mathbb{Z}} \frac{e^{2\pi ij(m+1)(z_1+z_2)} q^{j^2(m+1)}}{1 - e^{2\pi iz_1} q^j} - \sum_{j\in\mathbb{Z}} \frac{e^{-2\pi ij(m+1)(z_1+z_2)} q^{j^2(m+1)}}{1 - e^{-2\pi iz_2} q^j}\,,$$

and

$$\Phi^{[m]}(\tau, z_1, z_2 + \tau, 0) = e^{-2\pi i(m+1)z_1} \Bigg(\sum_{j\in\mathbb{Z}} \frac{e^{2\pi ij(m+1)(z_1+z_2)} q^{j^2(m+1)} (e^{2\pi iz_1} q^j)^{m+1}}{1 - e^{2\pi iz_1} q^j} - \sum_{j\in\mathbb{Z}} \frac{e^{-2\pi ij(m+1)(z_1+z_2)} q^{j^2(m+1)} (e^{-2\pi iz_2} q^j)^{m+1}}{1 - e^{-2\pi iz_2} q^j} \Bigg).$$

Hence we have:

$$
\begin{aligned}
&\Phi^{[m]}(\tau, z_1, z_2, 0) - e^{2\pi i(m+1)z_1}\Phi^{[m]}(\tau, z_1, z_2+\tau, 0)\\
&= \sum_{j\in\mathbb{Z}} e^{2\pi i j(m+1)(z_1+z_2)} q^{j^2(m+1)} \frac{1-(e^{2\pi i z_1}q^j)^{m+1}}{1-e^{2\pi i z_1}q^j}\\
&\quad - \sum_{j\in\mathbb{Z}} e^{-2\pi i j(m+1)(z_1+z_2)} q^{j^2(m+1)} \frac{1-(e^{-2\pi i z_2}q^j)^{m+1}}{1-e^{-2\pi i z_2}q^j}\\
&= \sum_{k=0}^{m}\sum_{j\in\mathbb{Z}} e^{2\pi i j(m+1)} q^{j^2(m+1)} (e^{2\pi i z_1}q^j)^k - \sum_{k=0}^{m}\sum_{j\in\mathbb{Z}} e^{-2\pi i j(m+1)(z_1+z_2)} q^{j^2(m+1)} (e^{-2\pi i z_2}q^j)^k\\
&= \sum_{k=0}^{m} e^{\pi i k(z_1-z_2)} q^{-\frac{k^2}{4(m+1)}}\\
&\quad\times\left(\sum_{j\in\mathbb{Z}} e^{2\pi i(m+1)(z_1+z_2)} q^{(m+1)\left(j+\frac{k}{2(m+1)}\right)^2} - \sum_{j\in\mathbb{Z}} e^{2\pi i(m+1)(z_1+z_2)(j-\frac{k}{2(m+1)})} q^{(m+1)\left(j-\frac{k}{2(m+1)}\right)^2}\right)\\
&= \sum_{k=1}^{m} e^{\pi i k(z_1-z_2)} q^{-\frac{k^2}{4(m+1)}} \left(\Theta_{k,m+1}(\tau, z_1+z_2) - \Theta_{-k,m+1}(\tau, z_1+z_2)\right).
\end{aligned}
$$

Finally, we prove (d). Exchanging z_1 and z_2 in (e) and using (c), we obtain:

$$
\begin{aligned}
&\Phi^{[m]}(\tau, z_1, z_2, 0) - e^{2\pi i(m+2)}\Phi^{[m]}(\tau, z_1+\tau, z_2, 0)\\
&= \sum_{j=1}^{m} e^{\pi i j(z_2-z_1)} q^{-\frac{j}{4(m+1)}} \left(\Theta_{j,m+1} - \Theta_{-j,m+1}\right)(\tau, z_1+z_2).
\end{aligned}
$$

Replacing z_2 by $z_2+\tau$, we deduce:

$$
\begin{aligned}
&\Phi^{[m]}(\tau, z_1, z_2+\tau, 0) - e^{2\pi i(m+1)z_2} q^{m+1}\Phi^{[m]}(\tau, z_1+\tau, z_2+\tau, 0)\\
&= \sum_{j=1}^{m} e^{\pi i j(z_2-z_1)} q^{\frac{j}{2}-\frac{j^2}{4(m+1)}} \left(\Theta_{j,m+1} - \Theta_{-j,m+1}\right)(\tau, z_1+z_2+\tau)\\
&= \sum_{j=1}^{m} e^{\pi i j(z_2-z_1)} e^{-\pi i(m+1)(z_1+z_2)} q^{-\frac{(m+1-j)^2}{4(m+1)}} \left(\Theta_{-(m+1-j),m+1} - \Theta_{m+1-j,m+1}\right)(\tau, z_1+z_2).
\end{aligned}
$$

Replacing in the summation j by $m+1-j$, and multiplying both sides by $e^{2\pi i(m+1)z_1}$, we obtain:

$$
\begin{aligned}
&e^{2\pi i(m+1)z_1}\Phi^{[m]}(\tau, z_1, z_2+\tau, 0) - e^{2\pi i(m+1)(z_1+z_2)} q^{m+1}\Phi^{[m]}(\tau, z_1+\tau, z_2+\tau, 0)\\
&= -\sum_{j=1}^{m} e^{\pi i j(z_1-z_2)} q^{-\frac{j^2}{4(m+1)}} \left(\Theta_{j,m+1} - \Theta_{-j,m+1}\right)(\tau, z_1+z_2).
\end{aligned}
$$

Adding this equality to (e), we obtain (d).

□

Remark 5.2. Properties (c) and (e) of $\Phi^{[m]}$ have a simple representation theoretical meaning (and proof). Property (c) means that $\mathrm{ch}^-_{L(m\Lambda_0)}$ is unchanged under the flip of the Dynkin diagram of $\widehat{\mathfrak{g}}$. Property (e) follows from the fact that the odd reflection with respect α_1 maps $m\Lambda_0$ to $m\Lambda_2 = m(\Lambda_0+\alpha_1)$ (and doesn't change the supercharacter), hence we have:

$$\sum_{w\in\widehat{W}} \varepsilon(w) w \frac{e^{m\Lambda_0}}{1-e^{-\alpha_1}} - \sum_{w\in\widehat{W}} \varepsilon(w) w \frac{e^{m(\Lambda_0+\alpha_1)}}{1-e^{-\alpha_1}} = -\sum_{w\in\widehat{W}} \varepsilon(w) w (e^{m(\Lambda_0+\alpha_1)} \sum_{j=0}^{m-1} e^{-j\alpha_1}) \,.$$

Lemma 5.1 on properties of the functions $\Phi^{[m]}$ immediately implies the following lemma on properties of the functions $\varphi^{[m]}$.

Lemma 5.3. *(a)* $\varphi^{[m]}(\tau, u+a, v+b, t) = \varphi^{[m]}(\tau,u,v,t)$ *if* $a,b\in\frac{1}{2}\mathbb{Z}$, $a+b\in\mathbb{Z}$.

(b) $\varphi^{[m]}(\tau,-u,v,t) = -\varphi^{[m]}(\tau,u,v,t)$.

(c) $\varphi^{[m]}(\tau,u,-v,t) = \varphi^{[m]}(\tau,u,v,t)$.

(d) $\varphi^{[m]}(\tau,u+\tau,v,t) = q^{-(m+1)} e^{-4\pi i(m+1)u} \varphi^{[m]}(\tau,u,v,t)$,
$\varphi^{[m]}(\tau,u-\tau,v,t) = q^{-(m+1)} e^{4\pi i(m+1)u} \varphi^{[m]}(\tau,u,v,t)$.

(e) $\varphi^{[m]}(\tau,u,v,t) - e^{2\pi i(m+1)(v-u)} \varphi^{[m]}(\tau, u-\frac{\tau}{2}, v-\frac{\tau}{2}, t)$
$= -e^{2\pi i(m+1)t} \sum_{j=1}^{m} e^{2\pi i j v} q^{-\frac{j^2}{4m+4}} (\Theta_{j,m+1} - \Theta_{-j,m+1})(\tau, 2u)$.

Proof. Properties (a), (b), (c), (d), and (e) of $\Phi^{[m]}$ immediately translate into properties (a), (b), (c), the second formula in (d), and (e) of $\varphi^{[m]}$. The first formula in (d) is obtained from the second one by replacing u by $u+\tau$.

□

Note that formulae (d) of Lemma 5.3 imply the following version of property (e):

(5.5)
$$\varphi^{[m]}(\tau,u,v,t) - e^{2\pi i(m+1)(2v-\tau)} \varphi^{[m]}(\tau,u,v-\tau,t) = -e^{2\pi i(m+1)t} \sum_{j=1}^{2m+1} e^{2\pi i j v} q^{-\frac{j^2}{4m+4}} (\Theta_{j,m+1} - \Theta_{-j,m+1})(\tau,2u)\,.$$

Recall that in coordinates τ,u,v,t we have:

$$\varphi^{[m]} = \widehat{R}^- \mathrm{ch}^-_{m\Lambda_0} = \Theta^-_{m\Lambda_0,\alpha_1} - r_\theta \Theta^-_{m\Lambda_0,\alpha_1}\,,$$

and note that the function $\Theta^-_{m\Lambda_0,\alpha_1}$ satisfies all conditions of Proposition 4.6 (with $\alpha=\theta$). Hence the function $\Theta^-_{m\Lambda_0,\alpha_1} - \Theta^-_{m\Lambda_0,\alpha_1}|_S$ is holomorphic in the domain X. Since the action of the Weyl group on X commutes with the action of $SL_2(\mathbb{R})$, the same holds for $r_\theta\Theta^-_{m\Lambda_0,\alpha_1} - r_\theta\Theta^-_{m\Lambda_0,\alpha_1}|_S$. Consequently, the function

(5.6)
$$G(\tau,u,v,t) = \varphi^{[m]}(\tau,u,v,t) - \tau^{-1}\varphi^{[m]}\left(-\frac{1}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, t - \frac{u^2-v^2}{\tau}\right)$$

is holomorphic in the domain X.

Proposition 5.4. *(a)* $G(\tau,u,v+1,t)-G(\tau,u,v,t)$
$= e^{2\pi i(m+1)t}\sqrt{\frac{2}{m+1}}(-i\tau)^{-\frac{1}{2}}\sum_{j=1}^{2m+1}\sum_{k\in\mathbb{Z}/(2m+2)\mathbb{Z}} e^{\frac{2\pi i(m+1)}{\tau}(v+\frac{j}{2m+2})^2}\sin\frac{\pi jk}{m+1}\Theta_{k,m+1}(\tau,2u).$

(b) $G(\tau,u,v,t)-e^{2\pi i(m+1)(2v-\tau)}G(\tau,u,v-\tau,t)$
$= -e^{2\pi i(m+1)t}\sum_{j=1}^{2m+1} e^{2\pi ijv}e^{-\frac{\pi ij^2\tau}{2m+2}}[\Theta_{j,m+1}-\Theta_{-j,m+1}](\tau,2u)$

(c) The holomorphic function G is determined uniquely by the above properties (a) and (b).

Proof. Without loss of generality we may put $t=0$. In order to prove (a), first recall the well-known transformation formula for theta functions (cf. formula (A.5) in the Appendix):

$$\Theta_{j,m}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{\frac{\pi imz^2}{2\tau}}\left(-\frac{i\tau}{2m}\right)^{\frac{1}{2}}\sum_{n\in\mathbb{Z}\bmod 2m\mathbb{Z}} e^{-\frac{\pi inj}{m}}\Theta_{n,m}(\tau,z)\,. \tag{5.7}$$

Using (5.5) and (5.7) we obtain:

$$\varphi^{[m]}\left(-\frac{1}{\tau},\frac{u}{\tau},\frac{v}{\tau},0\right) - e^{\frac{2\pi i}{\tau}(2v+1)}\varphi^{[m]}\left(-\frac{1}{\tau},\frac{u}{\tau},\frac{v+1}{\tau},0\right)$$
$$= i\left(\frac{-2i\tau}{m+1}\right)^{\frac{1}{2}} e^{\frac{2\pi i(m+1)u^2}{\tau}}\sum_{j=1}^{2m+1}\sum_{k\in\mathbb{Z}/(2m+2)\mathbb{Z}} e^{\frac{2\pi ijv}{\tau}}e^{\frac{2\pi ij^2}{(4m+4)\tau}}\sin\frac{\pi jk}{m+1}\Theta_{k,m+1}(\tau,2u)\,.$$

We deduce (a) from this by a straightforward calculation.

In order to prove (b), by a straightforward calculation we get:

$$G(\tau,u,v,0)-e^{2\pi i(m+1)}G(\tau,u,v-\tau,0) = \varphi^{[m]}(\tau,u,v,0)-e^{2\pi i(m+1)(2v-\tau)}\varphi^{[m]}(\tau,u,v-\tau,0)\,.$$

By (5.5), the RHS of this equation is equal to the RHS of (b).

In order to prove (c), note that the difference, say, $F(\tau,u,v,t)$ of two holomorphic functions, satisfying (a) and (b), satisfies the following two equations:

$$F(\tau,u,v+1,t)=F(\tau,u,v,t)\,,\ \ F(\tau,u,v-\tau,t)=e^{-4\pi i(m+1)}F(\tau,u,v,t)\,.$$

Consider the function

$$P(\tau,u,v,t)=F(\tau,u,v,t)\vartheta_{11}((m+1)\tau,(m+1)v)^2\,.$$

Since, by Proposition A.6 from the Appendix,

$$\vartheta_{11}((m+1)\tau,(m+1)(v-\tau)) = -e^{2\pi i(m+1)v}q^{-\frac{m+1}{2}}\vartheta_{11}((m+1)\tau,(m+1)v)\,,$$

we deduce that

$$P(\tau,u,v+1,t)=P(\tau,u,v,t)\,,\ \ P(\tau,u,v-\tau,t)=P(\tau,u,v,t)\,.$$

Since P is a holomorphic function in v, which is doubly periodic (for each fixed value of τ, u and t), we conclude that P is constant in v. Since $\vartheta_{1,1}(\tau,0)=0$ (see formula (A.7)), we conclude that P is identically zero.

□

Now we relate the function G to the functions h_ℓ, introduced by Zwegers in [Z], page 51; we will denote h_ℓ by $h_{m;\ell}$ to emphasize its dependence on m. Replacing x by $x+i$ in Zwegers' formula, it is straightforward to obtain a slightly different formula:

$$h_{m;j}(\tau,v) = ie^{-\frac{\pi i\tau}{2m}(2m-j)^2+2\pi i(2m-j)v}\int_{\mathbb{R}+is}\frac{e^{2\pi im\tau x^2+2\pi(2m-j)\tau x-4\pi m\sqrt{x}}}{1-e^{2\pi x}}\,dx\,, \tag{5.8}$$

where $s\in\mathbb{R}$, $0<s<1$.

Theorem 5.5. $G(\tau,u,v,t) = -e^{2\pi i(m+1)t}\sum_{j=1}^{2m+1} h_{m+1;2m+2-j}(\tau,v)(\Theta_{j,m+1}-\Theta_{-j,m+1})(\tau,2u)$.

Proof. Due to uniqueness of a holomorphic function on X, satisfying properties (a) and (b) of Proposition 5.4, it suffices to show that the RHS satisfies these two properties.

Let $a_j(\tau,v) = h_{m+1;2m+2-j}(\tau,v)$ to simplify notation, $1\le j\le 2m+1$. Replacing x by $x+i\frac{j}{2m+2}$ in (5.8) and taking $s=\frac{j}{2m+2}$, we obtain a simpler expression:

$$a_j(t,v) = i\int_{\mathbb{R}}\frac{e^{2\pi i(m+1)x^2\tau-4\pi(m+1)xv}}{1-e^{2\pi(x+\frac{ij}{2m+2})}}\,dx\,. \tag{5.9}$$

Property (a) of Proposition 5.4 of the function

$$G(\tau,u,v,t) = e^{2\pi it}\sum_{j=1}^{2m-1} a_j(\tau,v)(\Theta_{j,m+1}-\Theta_{-j,m+1})(\tau,2u)$$

is equivalent to the following property of the functions a_j:

$$a_j(\tau,v+1)-a_j(\tau,v) = \frac{i}{\sqrt{2m+2}}(-i\tau)^{-\frac{1}{2}}\sum_{k=1}^{2m-1} e^{-\frac{\pi ijk}{m+1}}e^{\frac{2\pi i(m+1)}{\tau}(v+\frac{k}{2m+2})^2}\,. \tag{5.10}$$

Equation (5.10) is established as follows. Using the expression (5.9) for $a_j(\tau,v)$, we obtain:

$$a_j(\tau,v+1)-a_j(\tau,v) = i\int_{\mathbb{R}} e^{2\pi i(m+1)x^2\tau-4\pi(m+1)xv}R_j(x)dx\,,$$

where

$$\begin{aligned} R_j(x) &= \frac{e^{-4\pi(m+1)x}-1}{1-e^{2\pi i(x+\frac{ij}{2m+2})}} = e^{-4\pi(m+1)x}\frac{1-e^{4\pi(m+1)(x+\frac{ij}{2m+2})}}{1-e^{2\pi(x+\frac{ij}{2m+2})}} \\ &= e^{-4\pi(m+1)x}\sum_{k=0}^{2m+1} e^{2\pi k(x+\frac{ij}{2m+2})}\,, \end{aligned}$$

which, replacing k by $2m+2-k$, is equal to $\sum_{k=1}^{2m+2} e^{-2\pi kx}e^{-\frac{\pi ijk}{m+1}}$. Thus,

$$\begin{aligned} a_j(\tau,v+1)-a_j(\tau,v) &= i\sum_{k=1}^{2m+2} e^{-\frac{\pi ijk}{m+1}}\int_{\mathbb{R}} e^{2\pi i(m+1)x^2\tau-4\pi(m+1)xv-2\pi kx}\,dx \\ &= i\sum_{k=1}^{2m+2} e^{-\frac{\pi ijk}{m+1}}e^{\frac{2\pi i(m+1)}{\tau}(v+\frac{k}{2m+2})}\int_{\mathbb{R}} e^{2\pi i(m+1)\tau\left(x+i\frac{(2m+2)v+k}{(2m+2)\tau}\right)^2}\,dx\,. \end{aligned}$$

We compute the integral in the above expression using the formula

$$\int_{\mathbb{R}+ib} e^{-ax^2}\, dx = \left(\frac{\pi}{a}\right)^{1/2} \quad \text{if} \quad \operatorname{Re} a > 0\,, \quad b \in \mathbb{R}\,,$$

to obtain (5.10).

In order to establish property (b) of Proposition 5.4, we need to prove

$$a_j(\tau,v) - e^{2\pi i j v(m+1)(2v-\tau)} a_j(\tau,v) = -e^{2\pi i j v} e^{-\frac{\pi i j^2}{2m+2}}\,. \tag{5.11}$$

By (5.8) we have:

$$a_j(\tau,v) = i \int_{\mathbb{R}+is} \frac{P(\tau,v,x)}{1-e^{2\pi x}}\, dx \quad (0 < s < 1)\,, \tag{5.12}$$

where

$$P(\tau,v,x) = e^{\frac{-\pi i j^2}{2m+2}\tau + 2\pi i j v} e^{2\pi i (m+1)\tau x^2 - 4\pi(m+1)vx + 2\pi j \tau x}\,.$$

The function P satisfies the identity

$$e^{2\pi i (m+1)(2\tau - v)} P(\tau, v-\tau, x) = P(\tau, v, x-i)\,.$$

Hence

$$e^{2\pi i(m+1)(2v-\tau)} a_j(\tau, v-\tau) = i \int_{\mathbb{R}+is} \frac{P(\tau,v,x-i)}{1-e^{2\pi x}} = i \int_{\mathbb{R}+i(s-1)} \frac{P(\tau,v,x)}{1-e^{2\pi x}}\, dx\,.$$

Using this and (5.12), we obtain:

$$\begin{aligned} a_j(\tau,v) - e^{2\pi i(m+1)(2v-\tau)} a_j(\tau,v-\tau) &= i \left(\int_{\mathbb{R}+is} - \int_{\mathbb{R}+i(s-1)} \right) \frac{P(\tau,v,x)}{1-e^{2\pi x}}\, dx \\ &= 2\pi \operatorname*{Res}_{x=0} \frac{P(\tau,v,x)\, dx}{1-e^{2\pi x}} = P(\tau,v,0)\,, \end{aligned}$$

proving (b), and the theorem. □

Remark 5.6. Let $\phi^{[m]} = \varphi^{[m]} - \frac{1}{2}G = \frac{1}{2}(\varphi^{[m]} + \varphi^{[m]}|_S)$. This function has a good modular transformation property: $\phi^{[m]}|_S = \phi^{[m]}$, but not so good elliptic transformation property:

$$\begin{aligned} &\phi^{[m]}(\tau,u,v-\tau,t) = e^{2\pi i(m+1)(2v-\tau)} \phi^{[m]}(\tau,u,v,t) \\ &-\tfrac{1}{2} \sum_{j=1}^{2m+1} e^{2\pi i j v} q^{-\frac{j^2}{4m+4}} (\Theta_{j,m+1} - \Theta_{-j,m+1})(\tau, 2u)\,. \end{aligned}$$

Following the idea of Zwegers [Z], we introduce below a non-holomorphic modification $\widetilde{\varphi}^{[m]}$ of $\varphi^{[m]}$, which has both good modular and elliptic transformation properties. For that we shall be making use of the functions $R_{m;j}(\tau,v)$, where m is a positive integer and $j \in \mathbb{Z}$, introduced in [Z], p. 51 ($\tau, v \in \mathbb{C}$, $\operatorname{Im}\tau > 0$):

$$R_{m;j}(\tau,v) = \sum_{\substack{n\in\mathbb{Z} \\ n \equiv j \bmod 2m}} \left(\operatorname{sign}\left(n+\tfrac{1}{2}\right) - E\left(\left(n + 2m\frac{\operatorname{Im} v}{\operatorname{Im}\tau}\right) \left(\frac{\operatorname{Im}\tau}{m}\right)^{\frac{1}{2}} \right) \right) e^{-\frac{\pi i n^2}{2m}\tau - 2\pi i n v}\,,$$

where $E(x)$ is the odd entire function $2\int_0^x e^{-\pi u^2}\,du$. The explicit expression of the holomorphic function $E(x)$ is used only in the proof of formula (5.13) below, given in [Z]. The key property of these functions relates them to the functions $h_{m;j}$ as follows (see [Z], Remark 3.6):

$$R_{m;j}(\tau,v)+\frac{i}{(-2mi\tau)^{\frac{1}{2}}}e^{\frac{2\pi imv^2}{\tau}}\sum_{k\in\mathbb{Z}/2m\mathbb{Z}}e^{-\frac{\pi ijk}{m}}R_{m;k}\left(-\frac{1}{\tau}\,,\,\frac{v}{\tau}\right)=2h_{m;j}(\tau,v)\,, \tag{5.13}$$

provided that $0\leq j\leq 2m-1$.

Remark 5.7. Note that the functions $R_{m;j}$ depend on j only $\mod 2m$, while this is not the case for the functions $h_{m;j}$, namely $h_{m;j}(\tau,u)-h_{m;j+2m}(\tau,u)=q^{-\frac{j^2}{4m}}e^{-2\pi iju}$.

The functions $R_{m;j}$ have also the following elliptic transformation properties, which is straightforward to check.

Lemma 5.8. *(a)* $R_{m;j}(\tau,v+\frac{1}{2})=(-1)^jR_{m;j}(\tau,v)$.

(b) For $0\leq j\leq 2m-1$ *one has:*

$$R_{m;j}(\tau,v)-e^{2\pi im(2v-\tau)}R_{m;j}(\tau,v-\tau)=2e^{-\frac{\pi i}{2m}(2m-j)^2\tau+2\pi i(2m-j)v}\,.$$

Now let

$$\begin{aligned}\varphi^{[m]}_{\text{add}}(\tau,u,v,t) &= \tfrac{1}{2}e^{2\pi i(m+1)t}\sum_{j\in\mathbb{Z}/(2m+2)\mathbb{Z}}R_{m+1;j}(\tau,v)(\Theta_{j,m+1}-\Theta_{-j,m+1})(\tau,2u)\,,\\ \widetilde{\varphi}^{[m]}(\tau,u,v,t) &= \varphi^{[m]}(\tau,u,v,t)+\varphi^{[m]}_{\text{add}}(\tau,u,v,t)\,.\end{aligned}$$

The correction function $\varphi^{[m]}_{\text{add}}$ has the following properties.

Lemma 5.9. *(a)* $\varphi^{[m]}_{\text{add}}(\tau,u,v,t)-\varphi^{[m]}_{\text{add}}|_S(\tau,u,v,t)=-G(\tau,u,v,t)$.

(b) $\varphi^{[m]}_{\text{add}}(\tau,u+a,v+b,t)=\varphi^{[m]}_{\text{add}}(\tau,u,v,t)$ *if* $a,b\in\frac{1}{2}\mathbb{Z}$ *are such that* $a+b\in\mathbb{Z}$.

(c) $\varphi^{[m]}_{\text{add}}(\tau,u+\tau,v,t)=q^{-(m+1)}e^{-4\pi i(m+1)u}\varphi^{[m]}_{\text{add}}(\tau,u,v,t)$.

(d) $\varphi^{[m]}_{\text{add}}(\tau,u,v,t)-e^{2\pi i(2v-\tau)}\varphi^{[m]}_{\text{add}}(\tau,u,v-\tau,t)$
$=e^{2\pi i(m+1)t}\sum_{j=1}^{2m+1}e^{2\pi ijv}q^{-\frac{j^2}{4m+4}}(\Theta_{j,m+1}-\Theta_{-j,m+1})(\tau,2u)$.

Proof. We have by definition of $\varphi^{[m]}_{\text{add}}$:

$$\begin{aligned}\varphi^{[m]}_{\text{add}}|_S(\tau,u,v,t) &= \tau^{-1}\varphi^{[m]}_{\text{add}}\left(-\frac{1}{\tau}\,,\,\frac{u}{\tau}\,,\,\frac{v}{\tau}\,,\,t-\frac{u^2-v^2}{\tau}\right)\\ &= \frac{1}{2\tau}e^{2\pi i(m+1)t}e^{-\frac{2\pi i(m+1)}{\tau}(u^2-v^2)}\sum_{j\in\mathbb{Z}/(2m+2)\mathbb{Z}}R_{m+1;j}\left(-\frac{1}{\tau}\,,\,\frac{v}{\tau}\right)(\Theta_{j,m+1}-\Theta_{-j,m+1})\left(-\frac{1}{\tau}\,,\,\frac{2u}{\tau}\right)\,.\end{aligned}$$

Now we apply to the RHS the following modular transformation formula, which is immediate by (5.7):

$$(\Theta_{j,m+1}-\Theta_{-j,m+1})(-\frac{1}{\tau}\,,\,\frac{2u}{\tau})$$
$$=e^{\frac{2\pi i(m+1)}{\tau}}u^2\left(-\frac{i\tau}{2m+2}\right)^{\frac{1}{2}}\sum_{k\in\mathbb{Z}/(2m+2)\mathbb{Z}}e^{-\frac{\pi ijk}{m+1}}(\Theta_{k,m+1}-\Theta_{-k,m+1})(\tau,2u)\,.$$

Substituting this in the previous formula, we obtain (after exchanging j and k):

$$\varphi^{[m]}_{\mathrm{add}}|_S(\tau,u,v,t)=\frac{-i}{2\sqrt{2m+2}}(-i\tau)^{-\frac{1}{2}}e^{2\pi i(m+1)t}$$
$$\times\sum_{j,k\in\mathbb{Z}/(2m+2)\mathbb{Z}}e^{\frac{2\pi i(m+1)}{\tau}v^2}e^{-\frac{\pi ijk}{m+1}}R_{m+1;k}\left(-\frac{1}{\tau}\,,\,\frac{v}{\tau}\right)(\Theta_{j,m+1}-\Theta_{-j,m+1})(\tau,2u)\,.$$

Using this and the key formula (5.13), we have:

$$\varphi^{[m]}_{\mathrm{add}}(\tau,u,v,t)-\varphi^{[m]}_{\mathrm{add}}|_S(\tau,u,v,t)$$
$$=e^{2\pi i(m+1)t}\sum_{j=1}^{2m+1}h_{m+1;j}(\tau,v)(\Theta_{j,m+1}-\Theta_{-j,m+1})(\tau,2u)\,.$$

The RHS of this equation equals to $-G(\tau,u,v,t)$ by Theorem 5.5. This proves claim (a). The remaining claims (b), (c) and (d) follow from Lemma 5.8.

□

Now we can prove the following important theorem.

Theorem 5.10. *(a)* $\widetilde{\varphi}^{[m]}(-\frac{1}{\tau},\frac{u}{\tau},\frac{v}{\tau},t-\frac{u^2-v^2}{\tau})=\tau\widetilde{\varphi}^{[m]}(\tau,u,v,t)$.

(b) $\widetilde{\varphi}^{[m]}(\tau,u+a,v+b,t)=\widetilde{\varphi}^{[m]}(\tau,u,v,t)$ *if* $a,b\in\frac{1}{2}\mathbb{Z}$ *are such that* $a+b\in\mathbb{Z}$.

(c)$\widetilde{\varphi}^{[m]}(\tau,u,-v,t)=\widetilde{\varphi}^{[m]}(\tau,u,v,t),\ \ \widetilde{\varphi}^{[m]}(\tau,-u,v,t)=-\widetilde{\varphi}^{[m]}(\tau,u,v,t)$.

(d) $\widetilde{\varphi}^{[m]}(\tau+1,u,v,t)=\widetilde{\varphi}^{[m]}(\tau,u,v,t,)$.

(e) $\widetilde{\varphi}^{[m]}(\tau,u+a\tau,v+b\tau,t)=q^{(m+1)(b^2-a^2)}e^{4\pi i(m+1)(-au+bv)}\widetilde{\varphi}^{[m]}(\tau,u,v,t)$ *if* $a,b\in\frac{1}{2}\mathbb{Z}$ *are such that* $a+b\in\mathbb{Z}$.

Proof. Adding to the equation of Lemma 5.9(a) formula (5.5), we get $\widetilde{\varphi}^{[m]}-\widetilde{\varphi}^{[m]}|_S=0$, proving claim (a). Adding equations of Lemmas 5.3(a) and 5.9(b), we get claim (b).

Claim (c) is derived as follows. It is straightforward to check that

$$R_{m;j}(\tau,-v)+R_{m;-j}(\tau,v)=2\delta_{0,j},\ \ j\in\mathbb{Z}/2m\mathbb{Z}. \tag{5.14}$$

It follows easily that $\varphi^{[m]}_{\mathrm{add}}(\tau,u,v,t)$ is unchanged if we change the sign of v. It is also immediate to see that $\varphi^{[m]}_{\mathrm{add}}(\tau,u,v,t)$ changes sign if we change the sign of u. This proves (c). Claim (d) is obvious.

Finally, we derive claim (e) from claims (a) and (b) as follows. Replacing u by $u+a\tau$ and v by $v+b\tau$ in (a), we obtain

$$\widetilde{\varphi}^{[m]}\left(-\frac{1}{\tau}\,,\,\frac{u}{\tau}+a\,,\,\frac{v}{\tau}+b\,,\,t\right)$$
$$=e^{\frac{2\pi i(m+1)}{\tau}(u^2-v^2)}e^{4\pi i(m+1)(au-bv)}q^{(m+1)(a^2-b^2)}\widetilde{\varphi}^{[m]}(\tau,u+a\tau,v+b\tau,t)\,.$$

But an equivalent form of (a) is

$$\widetilde{\varphi}^{[m]}\left(-\frac{1}{\tau}\,,\,\frac{u}{\tau}\,,\,\frac{v}{\tau}\,,\,t\right)=\tau e^{\frac{2\pi i(m+1)}{\tau}(u^2-v^2)}\widetilde{\varphi}^{[m]}(\tau,u,v,t)\,.$$

Since, by (b), the LHS of the last two formulae are equal, we conclude that the RHS are equal as well, which gives (e).

□

Translating the discussion on $\varphi^{[m]}_{\text{add}}$ and $\widetilde{\varphi}^{[m]}$ into the language of Φ's, we obtain the following modification of the function $\Phi^{[m]}(\tau,z_1,z_2,t)$:

$$\widetilde{\Phi}^{[m]}(\tau,z_1,z_2,t)=\Phi^{[m]}(\tau,z_1,z_2,t)+\Phi^{[m]}_{\text{add}}(\tau,z_1,z_2,t),$$

where

$$\widetilde{\Phi}^{[m]}_{\text{add}}(\tau,z_1,z_2,t)=-\tfrac{1}{2}e^{2\pi i(m+1)t}\sum_{j\in\mathbb{Z}/(2m+2)\mathbb{Z}}R_{m+1;j}(\tau,\frac{z_1-z_2}{2})(\Theta_{j,m+1}-\Theta_{-j,m+1})(\tau,z_1+z_2)\,.$$

Then we obtain the following corollary of Theorem 5.10.

Corollary 5.11. *(a)* $\widetilde{\Phi}^{[m]}(-\frac{1}{\tau}\,,\,\frac{z_1}{\tau}\,,\,\frac{z_2}{\tau}\,,\,t-\frac{z_1z_2}{\tau})=\tau\widetilde{\Phi}^{[m]}(\tau,z_1,z_2,t)$.

(b) $\widetilde{\Phi}^{[m]}(\tau,z_1+a,z_2+b,t)=\widetilde{\Phi}^{[m]}(\tau,z_1,z_2,t)$ *if* $a,b\in\mathbb{Z}$.

(c) $\widetilde{\Phi}^{[m]}(\tau,-z_1,-z_2,t)=-\widetilde{\Phi}^{[m]}(\tau,z_1,z_2,t)$; $\widetilde{\Phi}^{[m]}(\tau,z_2,z_1,t)=\widetilde{\Phi}^{[m]}(\tau,z_1,z_2,t)$.

(d) $\widetilde{\Phi}^{[m]}(\tau+1,z_1,z_2,t)=\widetilde{\Phi}^{[m]}(\tau,z_1,z_2,t)$.

(e) $\widetilde{\Phi}^{[m]}(\tau,z_1+j\tau,z_2+k\tau,t)=q^{-(m+1)jk}e^{-2\pi i(m+1)(kz_1+jz_2)}\widetilde{\Phi}^{[m]}(\tau,z_1,z_2,t)$ *if* $j,k\in\mathbb{Z}$.

Remark 5.12. The following Appell–Lerch sum plays a key role in Zwegers' paper [Z]:

$$\mu(\tau,z_1,z_2)=\frac{e^{\pi iz_1}}{\vartheta_{11}(\tau,z_2)}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{\frac{1}{2}(n^2+n)}e^{2\pi inz_2}}{1-e^{2\pi iz_1}q^n}\,,$$

along with its non-meromorphic modification

$$\widetilde{\mu}(\tau,z_1,z_2)=\mu(\tau,z_1,z_2)+\tfrac{i}{2}R(\tau,z_1-z_2),$$

where R is defined in [Z], p. 11. Let us compare these functions with the functions $\Phi^{[m]}(\tau, z_1, z_2) := \Phi^{[m]}(\tau, z_1, z_2, 0)$ and $R_{m;j}(\tau, z)$, which play a key role in the present paper. First it is easy to see that

$$R(\tau, z) = R_{2;1}(\tau, \tfrac{z}{2}) - R_{2;-1}(\tau, \tfrac{z}{2})\,. \tag{5.15}$$

Furthermore, in [KW4] (formula (3.22) for $s = 0, m = 2$) we proved the following character formula:

$$\mathrm{ch}^{-}_{\Lambda_0}(\tau, z_1, z_2, 0) = \vartheta_{11}(\tau, z_1)\vartheta_{11}(\tau, z_2)\mu(\tau, z_1, z_2)/\eta(\tau)^3\,.$$

Comparing this formula with (5.3) for $m = 1$, we obtain the following identity:

$$\Phi^{[1]}(\tau, z_1, z_2, 0) = \vartheta_{11}(\tau, z_1 + z_2)\mu(\tau, z_1, z_2)\,. \tag{5.16}$$

On the other hand, using (5.15), it is easy to see that

$$\Phi^{[1]}_{\mathrm{add}}(\tau, z_1, z_2, 0) = \tfrac{i}{2}R(\tau, z_1 - z_2)\vartheta_{11}(\tau, z_1 + z_2)\,. \tag{5.17}$$

Comparing (5.16) and (5.17), we get the "modified" identity

$$\widetilde{\Phi}^{[1]}(\tau, z_1, z_2, 0) = \vartheta_{11}(\tau, z_1 + z_2)\widetilde{\mu}(\tau, z_1, z_2)\,. \tag{5.18}$$

6 Modular transformation formula for the function $\widetilde{\Phi}^{[m]}\left(\frac{\tau}{M}, \frac{u}{M}, \frac{v}{M}, t\right)$

Given an additive subgroup A of $\mathbb{Q}$, consider the following abelian group:

$$\widetilde{\Omega}_A = \left\{(a, b) \in \frac{1}{2}A \times \frac{1}{2}A \mid a + b \in A\right\}\,.$$

For a positive integer M consider the abelian group

$$\Omega_M = \widetilde{\Omega}_{\mathbb{Z}}/\widetilde{\Omega}_{M\mathbb{Z}}\,.$$

The first result of this section is the following theorem.

Theorem 6.1. *Let M be a positive integer and let m be a non-negative integer, such that $gcd(M, 2m + 2) = 1$ if $m > 0$. Then*

(a) $\widetilde{\varphi}^{[m]}\left(\frac{\tau}{M}, \frac{u}{M}, \frac{v}{M}, t\right) = \frac{M}{\tau}\widetilde{\varphi}^{[m]}\left(-\frac{M}{\tau}, \frac{u}{\tau}, \frac{v}{\tau}, t - \frac{u^2-v^2}{\tau M}\right)$.

(b) $\widetilde{\varphi}^{[m]}\left(\frac{\tau}{M}, \frac{u}{M}, \frac{v}{M}, t\right) = \sum\limits_{a,b\in\Omega_M} q^{\frac{m+1}{M}(a^2-b^2)} e^{\frac{4\pi i(m+1)}{M}(au-bv)}\widetilde{\varphi}^{[m]}(M\tau, u + a\tau, v + b\tau, t)$.

Replacing (τ, u, v, t) by $\left(\frac{\tau}{M}, \frac{u}{M}, \frac{v}{M}, t\right)$ in Theorem 5.10(a), we obtain (a). The proof of (b) is based on several lemmas, proven below.

Given coprime positive integers p and q, for each integer $n \in [0, q-1]$ there exist unique integers $n' \in [0, p-1]$ and b_n, such that

$$n = n'q + b_n p\,. \tag{6.1}$$

Furthermore, the set

$$I_{q,p} := \{b_n \mid n = 0, 1, \ldots, q-1\}$$

consists of q distinct integers. Any $n \in \mathbb{Z}$ can be uniquely represented in the form (6.1), where $n' \in \mathbb{Z}$ and $b_n \in I_{q,p}$ and this decomposition has the following properties:

(i) $n \geq 0$ iff $n' \geq 0$;

(ii) if $j, j_0 \in \mathbb{Z}/p\mathbb{Z}$ are such that $j \equiv qj_0 \mod p$, then $n \equiv j \mod p$ iff $n' \equiv j_0 \mod p$.

We shall apply this setup to $p = 2m+2$, $q = M$, and let $I = I_{M,2m-2}$ for short.

Lemma 6.2. *(a)* $\Phi^{[m]}\left(\frac{\tau}{M}\,,\,\frac{z_1}{M}\,,\,\frac{z_2}{M}\,,\,t\right)$

$$= \sum_{\substack{0 \leq a < M \\ b \in I}} e^{\frac{2\pi i(m+1)}{M}((a+2b)z_1 + az_2)} q^{\frac{m+1}{M}(a^2+2ab)} \Phi^{[m]}(M\tau\,,\, z_1 + a\tau\,,\, z_2 + (a+2b)\tau,\, t).$$

(b) $\varphi^{[m]}\left(\frac{\tau}{M}\,,\,\frac{u}{M}\,,\,\frac{v}{M}\,,\,t\right)$

$$= \sum_{\substack{0 \leq a < M \\ b \in I}} e^{-\frac{4\pi i(m+1)}{M}((a+b)u + bv)} q^{\frac{m+1}{M}(a^2+2ab)} \varphi^{[m]}\left(M\tau,\, u - (a+b)\tau,\, v + b\tau,\, t\right).$$

Proof. We prove (a); (b) follows immediately from (a). Recall that

$$\Phi^{[m]}(\tau, z_1, z_2, t) \;=\; e^{2\pi i(m+1)t}(\Phi_1^{[m]}(\tau, z_1, z_2) - \Phi_1^{[m]}(\tau, -z_2, -z_1)),$$

where

$$\Phi_1^{[m]}(\tau, z_1, z_2) \;=\; \sum_{j \in \mathbb{Z}} \frac{e^{2\pi i j(m+1)(z_1+z_2)} q^{j^2(m+1)}}{1 - e^{2\pi i z_1} q^j}\,.$$

We have, by expanding each term in this series in a geometric series, and replacing (τ, z_1, z_2) by $\left(\frac{\tau}{M}\,,\,\frac{z_1}{M}\,,\,\frac{z_2}{M}\right)$:

$$(6.2) \quad \Phi_1^{[m]}\left(\frac{\tau}{M}\,,\,\frac{z_1}{M}\,,\,\frac{z_2}{M}\right) = \left(\sum_{j,k \geq 0} - \sum_{j,k<0}\right) e^{\frac{2\pi i(m+1)}{M} j(z_1+z_2)} e^{\frac{2\pi i k z_1}{M}} q^{\frac{1}{M}(j^2(m+1)+jk)}\,.$$

Now we divide j by M with the remainder a, $0 \leq a < M$: $j = j'M + a$, and decompose k according to (6.1): $k = k'M + (2m+2)b_k$. Since

$$\frac{1}{M}(j^2(m+1)+jk) = j'(j'(m+1)+k')M + j'(a+2b_k)(m+1) + a(j'(m+1)+k') + \frac{a(a+2b_k)(m+1)}{M}\,,$$

we obtain from (6.2):

$$\Phi_1\left(\frac{\tau}{M}\,,\,\frac{z_1}{M}\,,\,\frac{z_2}{M}\right)$$
$$= \sum_{\substack{0 \leq a < M \\ b \in I}} \left(\sum_{j',k' \geq 0} - \sum_{j',k'<0}\right) A_{j',k'} e^{\frac{2\pi i(m+1)a}{M}(z_1+z_2)} e^{\frac{4\pi i(m+1)}{M} bz} q^{Mj'(j'(m+1)+k')} B_{j',k'} q^{\frac{(m+1)a(a+2b)}{M}}\,,$$

where

$$A_{j',k'} = e^{2\pi i(m+1)j'(z_1+z_2)} e^{2\pi i k' z_1}\,,\quad B_{j',k'} = e^{2\pi i\tau(2j'(m+1)(a+b)+ak')}\,.$$

Therefore, using again (6.2), we obtain:

$$\Phi_1^{[m]}\left(\frac{\tau}{M},\frac{z_1}{M},\frac{z_2}{M}\right) = \sum_{\substack{0\le a<M\\ b\in I}} e^{\frac{2\pi i(m+1)}{M}((a+2b)z_1+az_2)} q^{\frac{m+1}{M}(a^2+2ab)} \Phi_1^{[m]}(M\tau, z_1+a\tau, z_2+(a+2b)\tau)\,,$$

which proves (a).

□

Lemma 6.3. *Given $j\in\mathbb{Z}/(2m+2)\mathbb{Z}$, let $j_0\in\mathbb{Z}/(2m+2)\mathbb{Z}$ be the element, such that $j-Mj_0\in(2m+2)\mathbb{Z}$. Then*

(a) $R_{m+1;j}\left(\frac{\tau}{M},\,\frac{v}{M}\right) = \sum_{b\in I} q^{-\frac{m+1}{M}b^2} e^{-\frac{4\pi i(m+1)}{M}bv} R_{m+1;j_0}(M\tau,\,v+b\tau).$

(b) $\Theta_{j,m+1}\left(\frac{\tau}{M},\,\frac{2u}{M}\right) = \sum_{a\in I} q^{\frac{m+1}{M}a^2} e^{\frac{4\pi i(m+1)}{M}au} \Theta_{j_0,m+1}(M\tau,\,2u+2a\tau).$

Proof. By definition of $R_{m;j}$ we have:

$$R_{m+1;j}\left(\frac{\tau}{M},\frac{v}{M}\right) = \sum_{n\equiv j \bmod 2m+2}\left(\mathrm{sign}\,(\mathrm{n}+\tfrac{1}{2})-\mathrm{E}\left(\left(\mathrm{n}+(2\mathrm{m}+2)\frac{\mathrm{Im}\,v}{\mathrm{Im}\,\tau}\right)\frac{1}{\mathrm{M}}\left(\frac{\mathrm{Im}\,(\mathrm{M}\tau)}{\mathrm{m}+1}\right)^{\frac{1}{2}}\right)\right) e^{-\frac{\pi i n^2}{(2m+2)M}\tau-\frac{2\pi i n}{M}v}\,.$$

We have the decomposition (6.1):

$$n = n'M + b_n(2m+2)\,,$$

and, by its property (ii),

$$n\equiv j \bmod 2m+2 \ \text{ iff } n'\equiv j_0 \bmod 2m+2\,.$$

Using property (i) of the decomposition (6.1), we obtain:

$$\begin{aligned} R_{m+1;j}\left(\frac{\tau}{M},\frac{v}{M}\right) &= \sum_{b\in I}\ \sum_{n'\equiv j_0 \bmod 2m+2}\left(\mathrm{sign}\,(\mathrm{n}'+\tfrac{1}{2})-\mathrm{E}\left(\left(\mathrm{n}'+(2\mathrm{m}+2)\frac{\mathrm{Im}\,(v+b\tau)}{\mathrm{Im}\,(\mathrm{M}\tau)}\right)\left(\frac{\mathrm{Im}\,(\mathrm{M}\tau)}{\mathrm{m}+1}\right)^{\frac{1}{2}}\right)\right) \\ &\quad \times e^{-\frac{\pi i M\tau}{2m+2}n'^2-2\pi i n'(v+b\tau)} q^{-\frac{m+1}{M}b^2} e^{-\frac{4\pi i(m+1)}{M}bv} \\ &= \sum_{b\in I} q^{-\frac{m+1}{M}b^2} e^{-\frac{4\pi i(m+1)}{M}bv} R_{m+1;j_0}(M\tau, v+b\tau)\,, \end{aligned}$$

proving (a).

In order to prove (b), note that we have from (A.3):

$$\Theta_{j,m+1}\left(\frac{\tau}{M}, \frac{z}{M}\right) = \sum_{n\equiv j \bmod 2m+2} e^{\frac{2\pi i\tau}{(4m+4)M}n^2} e^{\frac{\pi i n z}{M}}. \tag{6.3}$$

Using the decomposition (6.1): $n = n'M + (2m+2)a$, where $a \in I$, and its property (ii), we deduce:

$$\begin{aligned}\Theta_{j,m+1}\left(\frac{\tau}{M}, \frac{z}{M}\right) &= \sum_{a\in I} \sum_{n'\equiv j_0 \bmod 2m+2} e^{\frac{2\pi i\tau}{(4m+4)M}(n'M+(2m+2)a)^2} e^{\frac{\pi i}{M}(n'M+(2m+2)a)z} \\ &= \sum_{a\in I} q^{\frac{m+1}{M}a^2} e^{\frac{2\pi i(m+1)}{M}az}\Theta_{j_0,m+1}(M\tau, z+2a\tau),\end{aligned}$$

and (b) follows by replacing z by $2u$.

□

Lemma 6.3 implies that for $\varphi^{[m]}_{\mathrm{add}}(\tau,u,v,t)$, defined in Section 5, we have:

$$\varphi^{[m]}_{\mathrm{add}}\left(\frac{\tau}{M}, \frac{u}{M}, \frac{v}{M}, t\right) = \sum_{a,b\in I} q^{\frac{m+1}{M}(a^2-b^2)} e^{\frac{4\pi i(m+1)}{M}(au-bv)}\varphi^{[m]}_{\mathrm{add}}(M\tau, u+a\tau, v+b\tau, t). \tag{6.4}$$

End of the proof of Theorem 6.1(b). First, by induction on $|j|$ we obtain from Lemma 5.9(c) for $j \in \mathbb{Z}$:

$$\varphi^{[m]}(\tau, u+j\tau, v, t) = q^{-j^2(m+1)}e^{-4\pi i j(m+1)u}\varphi^{[m]}(\tau,u,v,t). \tag{6.5}$$

This equation implies that for $j \in \mathbb{C}$, the expression

$$q^{\frac{m+1}{M}j^2}e^{\frac{4\pi i(m+1)}{M}ju}\varphi^{[m]}(M\tau, u+j\tau, v, t) \tag{6.6}$$

remains unchanged if we replace j by $j' = j + Mn$, $n \in \mathbb{Z}$. Indeed, we have: $\varphi^{[m]}(M\tau, u + j'\tau, v, t) = \varphi^{[m]}(M\tau, u+j\tau+\frac{j'-j}{M}M\tau, v, t)$, and we apply (6.5) with u replaced by $u+j\tau$, τ replaced by $M\tau$ and j replaced by $\frac{j'-j}{M}$.

Replacing v in (6.6) by $v + b\tau$ ($b \in \mathbb{C}$) and multiplying it by $q^{-\frac{m+1}{M}b^2}e^{-\frac{4\pi i(m+1)}{M}bu}$, we deduce that the expression

$$q^{\frac{m+1}{M}(a^2-b^2)}e^{\frac{4\pi i(m+1)}{M}(au-bv)}\varphi^{[m]}(M\tau, u+a\tau, v+b\tau, t) \tag{6.7}$$

remains unchanged if we replace $a \in \mathbb{C}$ by $a + Mn$, $n \in \mathbb{Z}$.

It follows that Lemma 6.2(b) can be rewritten as follows:

$$\varphi^{[m]}\left(\frac{\tau}{M}, \frac{u}{M}, \frac{v}{M}, t\right) = \sum_{a,b\in I} q^{\frac{m+1}{M}(a^2-b^2)} e^{\frac{4\pi i(m+1)}{M}(au-bv)}\varphi^{[m]}(M\tau\,,\, u+a\tau\,,\, v+b\tau\,,\, t), \tag{6.8}$$

by making use of the following lemma.

Lemma 6.4. *Given $b \in \mathbb{Z}$, for each $a \in \mathbb{Z}_{\geq 0}$, such that $a < M$ there exists a unique $a' \in I$, such that $-(a+b) \equiv a' \mod M$. Moreover $\{a' |\ 0 \leq a < M\} = I$.*

Proof. We have, by decomposition (6.1) with $p = 2m + 2$, $q = M$:

$$-2(m+1)(a+b) = n'M + (2m+2)a' \ (n' \in \mathbb{Z}\,,\ a' \in I)\,.$$

Hence $-(2m+2)(a+b) \equiv (2m+2)a' \mod M$, and $-(a+b) \equiv a' \mod M$. ☐

We obtain from (6.8) and (6.4):

$$\text{(6.9)} \qquad \widetilde{\varphi}^{[m]}\left(\frac{\tau}{M}\,,\,\frac{u}{M}\,,\,\frac{v}{M}\,,\,t\right) = \sum_{a,b\in I} q^{\frac{m+1}{M}(a^2-b^2)} e^{\frac{4\pi i(m+1)}{M}(au-bv)} \widetilde{\varphi}^{[m]}(M\tau, u+a\tau, v+b\tau, t)\,.$$

It follows from Theorem 5.10(e) that the expression

$$\text{(6.10)} \qquad q^{\frac{m+1}{M}(a^2-b^2)} e^{\frac{4\pi i(m+1)}{M}(au-bv)} \widetilde{\varphi}^{[m]}(M\tau, u+a\tau, v+b\tau, t)$$

is independent of the choice of $(a,b) \in \widetilde{\Omega}_{\mathbb{Z}} \mod \widetilde{\Omega}_{M\mathbb{Z}}$. Along with (6.9), this completes the proof of Theorem 6.1(b).

Next, we translate the obtained results from $\widetilde{\varphi}^{[m]}$ to $\widetilde{\Phi}^{[m]}$. For that we define the map

$$\text{(6.11)} \qquad \mathbb{Z}^2 \to \left(\tfrac{1}{2}\mathbb{Z}\right)^2\,,\ (j,k) \mapsto \left(a = -\frac{j+k}{2}\,,\, b = \frac{j-k}{2}\right)\,.$$

This map induces a bijective map

$$\text{(6.12)} \qquad (\mathbb{Z}/M\mathbb{Z})^2 \to \Omega_M\,.$$

It follows from Corollary 5.11(e) that the function

$$\text{(6.13)} \qquad q^{\frac{m+1}{M}jk} e^{\frac{2\pi i(m+1)}{M}(kz_1+jz_2)} \widetilde{\Phi}^{[m]}(M\tau, z_1+j\tau, z_2+k\tau, t)$$

is independent of the choice of $j, k \mod M\mathbb{Z}$.

Theorem 6.5. *Let M be a positive integer and let m be a non-negative integer, such that $gcd(M, 2m+2) = 1$. Then*

(a) $\widetilde{\Phi}^{[m]}\left(-\frac{M}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau M}\right)$
$= \frac{\tau}{M} \sum_{j,k\in\mathbb{Z}/M\mathbb{Z}} q^{\frac{m+1}{M}jk} e^{\frac{2\pi i(m+1)}{M}(kz_1+jz_2)} \widetilde{\Phi}^{[m]}(M\tau,\ z_1+j\tau,\ z_2+k\tau,\ t).$

(b) $\widetilde{\Phi}^{[m]}\left(\frac{\tau}{M}\,,\,\frac{z_1}{M}\,,\,\frac{z_2}{M}\,,\,t\right) = \sum_{j,k\in\mathbb{Z}/M\mathbb{Z}} q^{\frac{m+1}{M}jk} e^{\frac{2\pi i(m+1)}{M}(kz_1+jz_2)} \widetilde{\Phi}^{[m]}(M\tau,\ z_1+j\tau,\ z_2+k\tau,\ t).$

Proof. Recall that $\widetilde{\Phi}^{[m]}(\tau, z_1, z_2, t) = \widetilde{\varphi}^{[m]}(\tau, u, v, t)$, where $z_1 = v - u$, $z_2 = -(v+u)$. Hence $\widetilde{\varphi}^{[m]}(M\tau\,,\ u+a\tau\,,\ v+b\tau\,,\ t) = \widetilde{\Phi}^{[m]}(\tau\,,\ z_1+(b-a)\tau\,,\ z_2-(b+a)\tau\,,\ t) = \widetilde{\Phi}^{[m]}(\tau\,,\ z_1+j\tau\,,\ z_2+k\tau\,,\ t)$, where $j = b-a$, $k = -(b+a)$. Hence under the map (6.11), (6.12) formulae from Theorem 6.5 correspond to those of Theorem 6.1. ☐

Remark 6.6. If $m=0$, then Theorem 6.5 holds for an arbitrary positive integer M. Indeed, in this case $gcd(M,m+1)=1$, Lemma 6.2 still holds with I replaced by $I'=I_{M,m+1}$ and b replaced by $b'=\frac{1}{2}b$ (proof is the same). Also $\Theta_{j,1}-\Theta_{-j,1}=0$ ($j\in\mathbb{Z}/2\mathbb{Z}$), hence $\varphi^{[0]}_{\text{add}}=0$, and $\widetilde{\varphi}^{[0]}=\varphi^{[0]}$, $\widetilde{\Phi}^{[0]}=\Phi^{[0]}$. Therefore we have for any $M\geq 1$:

$$\Phi^{[0]}\left(-\frac{M}{\tau},\frac{z_1}{\tau},\frac{z_2}{\tau},t-\frac{z_1z_2}{\tau M}\right)$$
$$=\frac{\tau}{M}\sum_{j,k\in\mathbb{Z}/M\mathbb{Z}}q^{\frac{jk}{M}}e^{\frac{2\pi i}{M}(kz_1+jz_2)}\Phi^{[0]}(M\tau\,,\,z_1+j\tau\,,\,z_2+k\tau\,,\,t)\,.$$

7 Modular transformation formulae for modified normalized characters of admissible $\widehat{s\ell}_{2|1}$-modules

Recall (cf. Section 4) that in order to have $SL_2(\mathbb{Z})$-invariance, we need to take, along with characters and supercharacters, also twisted characters and supercharacters. For the twisted $\widehat{s\ell}_{2|1}$ -modules we choose

$$\xi=-\tfrac{1}{2}(\alpha_1+\alpha_2). \tag{7.1}$$

Throughout this section we shall work, as in Section 5, in the following coordinates of the Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{s\ell}_{2|1}$:

$$h=2\pi i(-\tau\Lambda_0-z_1\alpha_2-z_2\alpha_1+t\delta):=(\tau,z_1,z_2,t). \tag{7.2}$$

In particular, we have:

$$t_{-\xi}(\tau,z_1,z_2,t)=\left(\tau,z_1+\tfrac{\tau}{2},z_2+\tfrac{\tau}{2},t+\tfrac{z_1+z_2}{2}+\tfrac{\tau}{4}\right)\,. \tag{7.3}$$

As in Section 4, throughout this section the superscripts (1/2) and (0) will refer to characters and supercharacters, while the subscripts 0 and 1/2 will refer to non-twisted and twisted sectors respectively.

By (4.1)–(4.5) we have the following formula for the normalized affine denominators ($\varepsilon,\varepsilon'=0$ or $\frac{1}{2}$):

$$\widehat{R}^{(\varepsilon)}_{\varepsilon'}(\tau,z_1,z_2,t)=(-1)^{2\varepsilon(1-2\varepsilon')}ie^{2\pi it}\frac{\eta(\tau)^3\vartheta_{11}(\tau,z_1+z_2)}{\vartheta_{1-2\varepsilon',1-2\varepsilon}(\tau,z_1)\vartheta_{1-2\varepsilon',1-2\varepsilon}(\tau,z_2)}\,. \tag{7.4}$$

By Theorem 4.1 we have the following modular transformation formulae:

$$\widehat{R}^{(\varepsilon)}_{\varepsilon'}\left(-\tfrac{1}{\tau},\tfrac{z_1}{\tau},\tfrac{z_2}{\tau},t\right)=(-1)^{4\varepsilon\varepsilon'}i\tau e^{\frac{2\pi iz_1z_2}{\tau}}\widehat{R}^{(\varepsilon')}_{\varepsilon}(\tau,z_1,z_2,t)\,; \tag{7.5}$$

$$\widehat{R}^{(\varepsilon)}_{\varepsilon'}(\tau+1,z_1,z_2,t)=e^{\pi i\varepsilon'}\widehat{R}^{(|\varepsilon-\varepsilon'|)}_{\varepsilon'}(\tau,z_1,z_2,t)\,. \tag{7.6}$$

Fix a positive integer M and a non-negative integer m, such that $gcd(M,2m+2)=1$ if $m>0$. In connection with the study of the numerators of the normalized characters of admissible $\widehat{s\ell}_{2|1}$-modules introduce the following functions ($\varepsilon,\varepsilon'=0$ or $\frac{1}{2}$, $j,k\in\varepsilon'+\mathbb{Z}$):

$$(7.7)\quad \Psi^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau,z_1,z_2,t)=q^{\frac{(m+1)jk}{M}}e^{\frac{2\pi i(m+1)}{M}(kz_1+jz_2)}\Phi^{[m]}(M\tau,z_1+j\tau+\varepsilon,z_2+k\tau+\varepsilon,\frac{t}{M}),$$

and denote by $\widetilde{\Psi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau,z_1,z_2,t)$ the function given by the same formula, except that Φ is replaced by $\widetilde{\Phi}$. Since the functions (6.13) are independent of the choice of $j,k \mod M\mathbb{Z}$, the same holds for the functions

$$e^{\frac{2\pi i(m+1)\varepsilon}{M}(j+k)}\widetilde{\Psi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau,z_1,z_2,t).$$

The following theorem is immediate by Theorem 6.5(a) and Remark 6.6.

Theorem 7.1. *Let $\varepsilon,\varepsilon'=0$ or $1/2$, and let $j,k\in\varepsilon'+\mathbb{Z}/M\mathbb{Z}$. Then*

$$\widetilde{\Psi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}\left(-\tfrac{1}{\tau},\tfrac{z_1}{\tau},\tfrac{z_2}{\tau},t\right)=\frac{\tau}{M}e^{\frac{2\pi i(m+1)}{M}z_1z_2}\sum_{a,b\in\varepsilon+\mathbb{Z}/M\mathbb{Z}}e^{-\frac{2\pi i(m+1)}{M}(ak+bj)}\widetilde{\Psi}^{[M,m;\varepsilon']}_{a,b;\varepsilon}(\tau,z_1,z_2,t);$$

$$\widetilde{\Psi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau+1,\,z_1,\,z_2,\,t)=e^{2\pi i\frac{(m+1)jk}{M}}\widetilde{\Psi}^{[M,m;\varepsilon+\varepsilon']}_{j,k;\varepsilon'}(\tau,\,z_1,\,z_2,\,t).$$

Now we link the functions $\Psi^{[M,m;\varepsilon]}_{j,k;\varepsilon'}$ to the normalized characters of admissible $\widehat{s\ell}_{2|1}$-modules of level $K=\frac{m+1}{M}-1$. Recall (cf. Proposition 3.14) that we have admissible weights of this level K of two types ($j,k\in\mathbb{Z}_{\geq 0}$):

$$\begin{aligned}\Lambda^{(1)}_{j,k} &= \left((j+k+1)\frac{m+1}{M}-1\right)\Lambda_0-j\frac{m+1}{M}\Lambda_1-k\frac{m+1}{M}\Lambda_2,\ 0\leq j,k\,,\ j+k\leq M-1;\\ \Lambda^{(2)}_{j,k} &= -\left((j+k-1)\frac{m+1}{M}+1\right)\Lambda_0+j\frac{m+1}{M}\Lambda_1+k\frac{m+1}{M}\Lambda_2,\ 1\leq j,k\,,\ j+k\leq M.\end{aligned}$$

Also in coordinates (7.2) we have for the corresponding simple subsets S_1 (resp. S_2) $=\{\widetilde{\alpha}_i|\ i=0,1,2\}$:

$$(7.8)\quad \begin{aligned}\widetilde{\alpha}_1(h) &= -2\pi i(z_1+k_1\tau),\ \ \widetilde{\alpha}_2(h)=-2\pi i(z_2+k_2\tau),\\ (\text{resp. } \widetilde{\alpha}_1(h) &= -2\pi i(-z_1+k_1\tau),\ \ \widetilde{\alpha}_2(h)=-2\pi i(-z_2+k_2\tau)).\end{aligned}$$

As before, it is convenient to introduce the following notation for normalized untwisted and twisted characters and supercharacters:

$$(7.9)\quad \mathrm{ch}^+_\Lambda=\mathrm{ch}^{(1/2)}_{\Lambda;0}\ ,\ \ \mathrm{ch}^-_\Lambda=\mathrm{ch}^{(0)}_{\Lambda;0}\ ,\ \ \mathrm{ch}^{\mathrm{tw},+}_\Lambda=\mathrm{ch}^{(1/2)}_{\Lambda;1/2}\ ,\ \ \mathrm{ch}^{\mathrm{tw},-}_\Lambda=\mathrm{ch}^{(0)}_{\Lambda;1/2}.$$

Proposition 7.2. *(a) If $\Lambda=\Lambda^{(1)}_{j,k}$, where $j,k\in\mathbb{Z},\ 0\leq j,k,j+k\leq M-1$, then*

$$(\widehat{R}^{(\varepsilon)}_{\varepsilon'}\mathrm{ch}^{(\varepsilon)}_{\Lambda;\varepsilon'})(\tau,z_1,z_2,t)=\Psi^{[M,m;\varepsilon]}_{j+\varepsilon',k+\varepsilon';\varepsilon'}(\tau,\,z_1,\,z_2,\,t).$$

(b) If $\Lambda=\Lambda^{(2)}_{j,k}$, where $j,k\in\mathbb{Z},\ 1\leq j,k,j+k\leq M$, then

$$(\widehat{R}^{(\varepsilon)}_{\varepsilon'}\mathrm{ch}^{(\varepsilon)}_{\Lambda;\varepsilon'})(\tau,z_1,z_2,t)=\Psi^{[M,m;\varepsilon]}_{M+\varepsilon'-j,M+\varepsilon'-k;\varepsilon'}(\tau,\,z_1,\,z_2,\,t).$$

Proof. We explain how to derive (a), the proof of (b) being similar.

First, the (super)character of an admissible module is obtained from the (super)character of the corresponding partially integrable module $L(\Lambda^0)$, as described by formula (3.24). This amounts to replacing the simple roots $\alpha_i \in \Pi$ by the $\widetilde{\alpha}_i \in S_1$, $i \in I$, described by (7.8), in the numerator of the (super)character of $L(\Lambda^0)$. Thus, the supercharacter of an admissible $\widehat{s\ell}_{2|1}$-module $L(\Lambda^{(1)}_{j,k})$ is obtained from that of $L(m\Lambda_0)$ by the following substitution in the RHS of (5.3): the z_i are replaced according to (7.8), and t is replaced by $\Lambda^{(1)}_{j,k}(h) = \frac{t+(m+1)(z_1k+z_2j)}{M}$.

In order to deduce the formula for the normalized supercharacter (3.19), we find for $\Lambda = \Lambda^{(1)}_{j,k}$:

$$m_\Lambda = \frac{m+1}{M} jk.$$

Then (a) for $\varepsilon = \varepsilon' = 0$ follows.

To deduce (a) for $\varepsilon' = 0$, $\varepsilon = \frac{1}{2}$, note that the character is obtained from the supercharacter by replacing z_i, $i = 1, 2$, by $z_i + \frac{1}{2}$. Finally, to deduce (a) for $\varepsilon' = \frac{1}{2}$, $\varepsilon = 0$ or $\frac{1}{2}$, we should replace Λ by Λ^{tw}, which, according to (4.6), (4.7), (4.8) and (4.9), amounts to replacing j and k by $j + \frac{1}{2}$ and $k + \frac{1}{2}$, and replacing $\mathrm{ch}^{\pm}$ by $t_\xi(\mathrm{ch}^{\pm})$. ☐

In order to state the modular transformation formula for the modified normalized admissible characters it is convenient to change notation as follows:

$$\mathrm{ch}^{[M,m;\varepsilon]}_{j+\varepsilon',k+\varepsilon';\varepsilon'} := \mathrm{ch}^{(\varepsilon)}_{\Lambda^{(1)}_{j,k;\varepsilon'}}, \quad \mathrm{ch}^{[M,m;\varepsilon]}_{M+\varepsilon'-j,M+\varepsilon'-k;\varepsilon'} := \mathrm{ch}^{(\varepsilon)}_{\Lambda^{(2)}_{j,k;\varepsilon'}}.$$

Then

$$\{\mathrm{ch}^{[M,m;\varepsilon]}_{j+\varepsilon',k+\varepsilon';\varepsilon'} \mid j,k \in \mathbb{Z}\,,\ 0 \le j,k \le M-1,\ \varepsilon,\varepsilon' = 0, 1/2\}$$

is precisely the set of all admissible characters (resp. supercharacters) if $\varepsilon' = 0$ and $\varepsilon = 1/2$ (resp. $\varepsilon = 0$), and it is the set of all twisted admissible characters (resp. supercharacters) if $\varepsilon' = 1/2$ and $\varepsilon = 1/2$ (resp. $\varepsilon = 0$). In view of these observations, introduce the *modified* normalized characters ($\varepsilon = 1/2\,,\ \varepsilon' = 0$), supercharacters ($\varepsilon = 0\,,\ \varepsilon' = 0$), twisted characters ($\varepsilon = 1/2\,,\ \varepsilon' = 1/2$), and twisted supercharacters ($\varepsilon = 0\,,\ \varepsilon' = 1/2$), letting

$$\widetilde{\mathrm{ch}}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t) = \frac{\widetilde{\Psi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t)}{\widehat{R}^{(\varepsilon)}_{\varepsilon'}(\tau, z_1, z_2, t)},\ j,k \in \varepsilon' + \mathbb{Z}\,,\ 0 \le j,k < M\,. \tag{7.10}$$

Then from (7.5), (7.6) and Theorem 7.1 we obtain the following theorem.

Theorem 7.3. *Let M be a positive integer and let m be a non-negative integer, such that $gcd(M, 2m+2) = 1$ if $m > 0$. One has the following modular transformation formulae ($\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$; $j,k \in \varepsilon' + \mathbb{Z}\,,\ 0 \le j,k < M$):*

$$\widetilde{\mathrm{ch}}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}\left(-\frac{1}{\tau}\,,\ \frac{z_1}{\tau}\,,\ \frac{z_2}{\tau}\,,\ t - \frac{z_1z_2}{\tau}\right)$$

$$= -(-1)^{4\varepsilon\varepsilon'} \frac{1}{M} \sum_{\substack{a,b\in\varepsilon+\mathbb{Z} \\ 0\le a,b<M}} e^{-\frac{2\pi i(m+1)}{M}(ak+bj)} \widetilde{\mathrm{ch}}^{[M,m;\varepsilon']}_{a,b;\varepsilon}(\tau, z_1, z_2, t)\,;$$

$$\widetilde{\mathrm{ch}}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau+1, z_1, z_2, t) = e^{2\pi i\frac{(m+1)jk}{M} - \pi i\varepsilon'} \widetilde{\mathrm{ch}}^{[M,m;|\varepsilon-\varepsilon'|]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t)\,.$$

Remark 7.4. If $m = 0$, we have admissible $\widehat{sl}_{2|1}$-modules of *boundary level* $K = \frac{1}{M} - 1$ [KW4]. In this case, by Remark 6.6, $\widetilde{\Phi}^{[0]} = \Phi^{[0]}$, and, since $\mathrm{ch}_0^- = 1$, we have, by (5.3), (5.4):

$$\Phi^{[0]} = \widehat{R}_0^{(0)}.$$

(Note that this is the famous Ramanujan summation formula for the bilateral basic hypergeometric function ${}_1\Psi_1$, cf. [KW3].) Hence in this case (7.10) becomes ($j, k \in \varepsilon' + \mathbb{Z}, 0 \le j, k < M$):

$$\widetilde{\mathrm{ch}}^{[M,0;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t) = q^{\frac{jk}{M}} e^{\frac{2\pi i}{M}(kz_1 + jz_2)} \frac{\widehat{R}^{(\varepsilon)}_{\varepsilon'}(M\tau, z_1 + j\tau + \varepsilon, z_2 + k\tau + \varepsilon, t)}{\widehat{R}^{(\varepsilon)}_{\varepsilon'}(\tau, z_1, z_2, t)}.$$

These are the normalized characters and supercharacters of all admissible untwisted and twisted modules of boundary level $K = \frac{1}{M} - 1$. In this case one needs no modifications, and modular transformation formulae for these characters and supercharacters is given by Theorem 7.3 for $m = 0$ with tildes removed.

8 Modular transformation formulae for modified normalized characters of admissible $\widehat{A}_{1|1}$-modules

Consider the Lie superalgebra $g\ell_{2|2}$, endowed with the structure of a Kac–Moody superalgebra as in Example 3.4, and let $\widehat{g\ell}_{2|2}$ be the corresponding affine Lie superalgebra (see Section 2). On the Lie superalgebra $s\ell_{2|2}$ the supertrace form is degenerate with kernel $\mathbb{C}I_4$, but it induces a non-degenerate invariant bilinear form, which we again denote by $(\,.\,|\,.\,)$, on the Lie superalgebra $A_{1|1} = ps\ell_{2|2}(= s\ell_{2|2}/\mathbb{C}I_4)$. The associated affinization (see Section 2)

$$\widehat{A}_{1|1} = A_{1|1}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

is a Lie superalgebra with a non-degenerate invariant bilinear form $(\,.\,|\,.\,)$ induced from $\widehat{g\ell}_{2|2}$.

Throughout this section, $\mathfrak{g} = A_{1|1}$ and $\widehat{\mathfrak{g}} = \widehat{A}_{1|1}$. The Lie superalgebras $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ inherit from $g\ell_{2|2}$ and $\widehat{g\ell}_{2|2}$ all the basic features of a Kac–Moody superalgebra, discussed in Sections 1 and 2, like the root space decomposition, the triangular decomposition, the Weyl group, etc. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is, by definition, the quotient of the space of diagonal matrices from $s\ell_{2|2}$ by $\mathbb{C}I_4$. (Thus, $\dim \mathfrak{h} = 2$, $|I| = 3$, the corank of the Cartan matrix is 1, and Π is not a linearly independent set, so that $\mathfrak{g}$ is, strictly speaking, not a Kac–Moody superalgebra, but a simple variation of it.) The Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ is, as before, defined by (2.7).

It is easy to see that an irreducible highest weight module $L(\Lambda)$ over $\widehat{g\ell}_{2|2}$ is actually a $\widehat{\mathfrak{g}}$-module, provided that $\Lambda(I_4) = 0$, and it remains irreducible when restricted to $\widehat{\mathfrak{g}}$. The condition $\Lambda(I_4) = 0$ is equivalent to the following condition on labels of Λ (defined by (3.5), (3.6)):

(8.1) $$m_1 = m_3\,.$$

Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ be the set of simple roots of $\mathfrak{g}$ (note that $\alpha_1 = \alpha_3$). We denote $\alpha_{12} = \alpha_1 + \alpha_2$, $\alpha_{23} = \alpha_2 + \alpha_3$, $\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$, etc, for short. We have:

$$\mathrm{sdim}\ \mathfrak{g} = -2,\ \ell = 2,\ \Delta_{\bar 0}^+ = \{\alpha_2,\ \theta = \alpha_{123}\}\,, \Delta_{\bar 1}^+ = \{\alpha_1, \alpha_3, \alpha_{12}, \alpha_{23}\}\,,$$
$$(\alpha_1|\alpha_1) = (\alpha_3|\alpha_3) = (\alpha_1|\alpha_3) = 0\,,\ (\alpha_2|\alpha_2) = -2\,,\ (\alpha_1|\alpha_2) = (\alpha_2|\alpha_3) = 1\,,\ (\theta|\theta) = 2\,,$$
$$2\rho_{\bar 0} = \alpha_{12} + \alpha_{23}\,,\ \rho_{\bar 1} = \theta\,,\ 2\rho = -\alpha_{13}\,,\ h^\vee = 0\,.$$

We choose $\xi \in \mathfrak{h}$ by letting

(8.2) $$(\xi|\alpha_1) = (\xi|\alpha_3) = \tfrac{1}{2}\,,\ (\xi|\alpha_2) = 0\,,$$

so that

$$(\rho_{\bar{0}}|\xi) = \tfrac{1}{2}\,,\ (\rho_{\bar{1}}|\xi) = 1\,,\ (\rho|\xi) = -\tfrac{1}{2}.$$

We choose the following coordinates in $\widehat{\mathfrak{h}}$ (cf. (3.27)):

(8.3) $$h = 2\pi i(-\tau\Lambda_0 - (z_1+z_2)\alpha_1 - z_1\alpha_2 + t\delta) := (\tau, z_1, z_2, t)\,.$$

Note that for $z = -(z_1+z_2)\alpha_1 - z_1\alpha_2$ we have: $(z|z) = 2z_1z_2$.

By (4.1)–(4.5) we have the following formulae for the normalized affine denominators ($\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$):

(8.4) $$\widehat{R}^{(\varepsilon)}_{\varepsilon'}(\tau, z_1, z_2) = (-1)^{2\varepsilon'}\eta(\tau)^4 \frac{\vartheta_{11}(\tau, z_1-z_2)\vartheta_{11}(\tau, z_1+z_2)}{\vartheta_{1-2\varepsilon',1-2\varepsilon}(\tau, z_1)^2\vartheta_{1-2\varepsilon',1-2\varepsilon}(\tau, z_2)^2}.$$

Note that these denominators are independent of t since $h^\vee = 0$.

By Theorem 4.1, we have the following modular transformation formulae:

(8.5) $$\widehat{R}^{(\varepsilon)}_{\varepsilon'}\left(-\tfrac{1}{\tau}\,,\ \tfrac{z_1}{\tau}\,,\ \tfrac{z_2}{\tau}\right) = (-1)^{2(\varepsilon-\varepsilon')} i\tau e^{\frac{\pi i z_1 z_2}{\tau}} R^{(\varepsilon')}_{\varepsilon}(\tau, z_1, z_2)\,,$$

(8.6) $$\widehat{R}^{(\varepsilon)}_{\varepsilon'}(\tau+1, z_1, z_2) = e^{\pi i(2\varepsilon'-\frac{1}{2})}\widehat{R}^{(|\varepsilon-\varepsilon'|)}_{\varepsilon'}(\tau, z_1, z_2)\,.$$

Next, we study modular transformation properties of the numerator of the normalized supercharacter of the partially integrable $\widehat{\mathfrak{g}}$-module $L(m\Lambda_0)$, where m is a non-zero integer (see Example 3.4), using formula (3.14) for $\mathrm{ch}^-_{L(m\Lambda_0)}$.

We have: $W = \{1, r_{\alpha_2}, r_\theta, r_{\alpha_2}r_\theta\}$. We choose $T_\rho = \{\alpha_1, \alpha_3\}$ in (3.14) . Since $r_{\alpha_2}r_\theta$ fixes $e^{m\Lambda_0+\rho}/(1-e^{-\alpha_1})(1-e^{-\alpha_3})$, we deduce that $j_0 = 2$, and the summation in (3.14) is over the semidirect product of the group $\{1, r_{\alpha_2}\}$ and $t_{L^\#}$. Furthermore, $L^\# = \mathbb{Z}\theta$ if $m > 0$, and $L^\# = \mathbb{Z}\alpha_2$ if $m < 0$. Hence formula (3.14) gives, using (2.11), the following supercharacter formulae, where m is a positive integer:

(8.7)
$$e^{-m\Lambda_0-\rho}\widehat{R}^-\mathrm{ch}^-_{L(m\Lambda_0)} = \sum_{j\in\mathbb{Z}}\left(\frac{e^{jm\theta}q^{mj^2+j}}{(1-e^{-\alpha_1}q^j)(1-e^{-\alpha_3}q^j)} - \frac{e^{-\alpha_2-jm\theta}q^{mj^2+j}}{(1-e^{-\alpha_1-\alpha_2}q^j)(1-e^{-\alpha_2-\alpha_3}q^j)}\right)\,,$$

(8.8)
$$e^{m\Lambda_0-\rho}\widehat{R}^-\mathrm{ch}^-_{L(-m\Lambda_0)} = \sum_{j\in\mathbb{Z}}\left(\frac{e^{jm\alpha_2}q^{mj^2+j}}{(1-e^{-\alpha_1}q^j)(1-e^{-\alpha_3}q^j)} - \frac{e^{-\alpha_2-jm\alpha_2}q^{mj^2+j}}{(1-e^{-\alpha_1-\alpha_2}q^j)(1-e^{-\alpha_2-\alpha_3}q^j)}\right)\,.$$

After passing to the normalized supercharacter, equation (8.7) in coordinates (8.3) looks as follows:

(8.9) $$(\widehat{R}^{(0)}_0\mathrm{ch}^-_{m\Lambda_0})(\tau, z_1, z_2, t)$$
$$= e^{2\pi imt}\sum_{j\in\mathbb{Z}}\left(\frac{e^{2\pi ijm(z_1+z_2)}e^{2\pi iz_1}q^{mj^2+j}}{(1-e^{2\pi iz_1}q^j)^2} - \frac{e^{-2\pi ijm(z_1+z_2)}e^{-2\pi iz_2}q^{mj^2+j}}{(1-e^{-2\pi iz_2}q^j)^2}\right)\,.$$

We denote the RHS of (8.9) by $\Phi^{A_{1|1}[m]}(\tau, z_1, z_2, t)$. Recall that m is a positive integer.

Lemma 8.1.
$$\Phi^{A_{1|1}[m]}(\tau, z_1, z_2, t) = D_0 \Phi^{[m-1]}(\tau, z_1, z_2, t),$$
where
$$D_0 = \tfrac{1}{2\pi i}\left(\tfrac{\partial}{\partial z_1} - \tfrac{\partial}{\partial z_2}\right),$$
and $\Phi^{[m-1]}$ *is the RHS of (5.3) with* m *replaced by* $m-1$.

Proof. It is straightforward, using that $D_0 e^{2\pi i j m(z_1+z_2)} = 0$. □

Next, introduce the following differential operator:
$$D_1 = \tfrac{1}{2\pi i}\left(\tfrac{\partial}{\partial z_1} - \tfrac{\partial}{\partial z_2} - \tfrac{z_1 - z_2}{2\tau}\tfrac{\partial}{\partial t}\right).$$
It is immediate to check that

$$(D_1 F)|_S = \tau D_1 (F|_S), \tag{8.10}$$

where (cf. (4.16) and (4.17))
$$(F|_S)(\tau, z_1, z_2, t) = \tau^{-1} F\left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau}\right).$$
Consider the following non-meromorphic *modification* of the *numerator* $\Phi^{A_{1|1}[m]}$:
$$\widetilde{\Phi}^{A_{1|1}[m]}(\tau, z_1, z_2, t) = D_1 \widetilde{\Phi}^{[m-1]}(\tau, z_1, z_2, t)$$
and let (see (7.7))
$$\widetilde{\Psi}^{A_{1|1}[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t) = D_1 \widetilde{\Psi}^{[M,m-1;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t).$$
Theorem 8.2. *(a) If* m *is a positive integer, then*
$$\widetilde{\Phi}^{A_{1|1}[m]}(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau}) = \tau^2 \widetilde{\Phi}^{A_{1|1}[m]}(\tau, z_1, z_2, t).$$
(b) Let M *and* m *be positive integers, such that* $\gcd(M, 2m) = 1$ *if* $m > 1$. *Let* $\varepsilon, \varepsilon' = 0$ *or* $1/2$, *and let* $j, k \in \varepsilon' + \mathbb{Z}/M\mathbb{Z}$. *Then*
$$\widetilde{\Psi}^{A_{1|1}[M,m;\varepsilon]}_{j,k;\varepsilon'}\left(-\tfrac{1}{\tau}, \tfrac{z_1}{\tau}, \tfrac{z_2}{\tau}, t - \frac{z_1 z_2}{\tau}\right) = \frac{\tau^2}{M} \sum_{a,b \in \varepsilon + \mathbb{Z}/M\mathbb{Z}} e^{-\frac{2\pi i m}{M}(ak+bj)} \widetilde{\Psi}^{A_{1|1}[M,m;\varepsilon']}_{a,b;\varepsilon}(\tau, z_1, z_2, t).$$

Proof. It follows immediately from Corollary 5.11 and Theorem 7.1, using (8.10). □

Next, we link the functions $\widetilde{\Psi}^{A_{1|1}[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t)$ to the normalized characters of admissible $\widehat{A}_{1|1}$-modules of level $K = \frac{m}{M}$, where m and M are positive coprime integers, for which the corresponding partially integrable module is $L(m\Lambda_0)$. The highest weights of these modules are admissible with respect to the simple subsets of type S_j, $j = 1, 2, 3, 4$, described in Example 3.11. These highest weights are listed in Proposition 3.15 , and they should satisfy the additional condition (8.1). Below we introduce a more convenient indexing of them, where $k_i \in \mathbb{Z}_{\geq 0}$, $i = 0, 1, 2, 3,\ k_1 = k_3$ (the last condition is equivalent to (8.1)).

Recall that

$$S_1 = \{\widetilde{\alpha}_i = k_i\delta + \alpha_i |\, i = 0, 1, 2, 3,\ \sum_{i=0}^{3} k_i = M - 1\,\}.$$

In coordinates (8.3) we have:

$$\widetilde{\alpha}_1(h) = -2\pi i(z_1 + k_1\tau)\,,\ (\widetilde{\alpha}_1 + \widetilde{\alpha}_2)(h) = -2\pi i(z_2 + (k_1 + k_2)\tau)\,. \tag{8.11}$$

Let $j = k_1,\ k = k_1 + k_2$. Then the pairs (j, k) that determine the corresponding highest weight $\Lambda^{(1)}_{k_1,k_2}$, run over the following set of pairs of integers:

$$k \geq j \geq 0\,,\ j + k \leq M - 1\,. \tag{8.12}$$

We denote the corresponding admissible highest weights by Λ_{jk}, and let $s = 1$.

The simple subsets of the second type are

$$S_2 = \{\widetilde{\alpha}_i = k_i\delta - \alpha_i |\, i = 0, 1, 2, 3,\ \sum_{i=0}^{3} k_i = M + 1,\ k_i > 0\,\}.$$

In coordinates (8.3) we have:

$$\widetilde{\alpha}_1(h) = -2\pi i(-z_1 + k_1\tau)\,,\ (\widetilde{\alpha}_1 + \widetilde{\alpha}_2)(h) = -2\pi i(-z_2 + (k_1 + k_2)\tau)\,. \tag{8.13}$$

Let $j = M - k_1,\ k = M - k_1 - k_2$. Then the pairs (j, k) that determine the corresponding highest weight $\Lambda^{(2)}_{k_1,k_2}$ run over the following set of pairs of integers:

$$M - 1 \geq j > k \geq 1\,,\ j + k \geq M\,. \tag{8.14}$$

We denote the corresponding admissible highest weights by Λ_{jk}, and let $s = 2$.

The simple subsets of the third type are

$$S_3 = \{\widetilde{\alpha}_0 = k_0\delta + \alpha_0, \widetilde{\alpha}_1 = k_1\delta + \alpha_{12}, \widetilde{\alpha}_2 = k_2\delta - \alpha_2, \widetilde{\alpha}_3 = k_3\delta + \alpha_{23} |\ \sum_{i=0}^{3} k_i = M - 1,\ k_2 > 0\}.$$

In coordinates (8.3) we have:

$$\widetilde{\alpha}_1(h) = -2\pi i(z_2 + k_1\tau)\,,\quad (\widetilde{\alpha}_1 + \widetilde{\alpha}_2)(h) = -2\pi i(z_1 + (k_1 + k_2)\tau)\,. \tag{8.15}$$

Let $j = k_1 + k_2,\ k = k_1$. Then the pairs (j, k) that determine the corresponding highest weight $\Lambda^{(3)}_{k_1,k_2}$ run over the following set of pairs of integers:

$$0 \leq k < j\,,\ j + k \leq M - 1\,. \tag{8.16}$$

We denote the corresponding admissible weights by Λ_{jk}, and let $s = 3$.

The simple subsets of the fourth type are

$$S_4 = \{\widetilde{\alpha}_0 = k_0\delta - \alpha_0\,,\ \widetilde{\alpha}_1 = k_1\delta - \alpha_{12},\ \widetilde{\alpha}_2 = k_2\delta + \alpha_2,\ \widetilde{\alpha}_3 = k_3\delta - \alpha_{23} |\ \sum_{i=0}^{3} k_i = M + 1,\ k_0, k_1 > 0\}.$$

In coordinates (8.3) we have:

$$(8.17) \qquad \widetilde{\alpha}_1(h) = -2\pi i(-z_2 + k_1\tau)\,,\ (\widetilde{\alpha}_1 + \widetilde{\alpha}_2)(h) = -2\pi i(-z_1 + (k_1+k_2)\tau)\,.$$

Let $j = M - k_1 - k_2$, $k = M - k_1$. Then the pairs (j,k) that determine the corresponding highest weight $\Lambda^{(4)}_{k_1,k_2}$ run over the following set of pairs of integers:

$$(8.18) \qquad 1 \le j \le k \le M-1\,,\ j+k \ge M\,.$$

We denote the corresponding admissible weights by Λ_{jk}, and let $s = 4$.

Summing up, we observe the following.

Remark 8.3. The sets of pairs (j,k) indexing the admissible highest weights Λ_{jk}, fill up without overlapping the set of pairs of integers in the square $\{(j,k) \in \mathbb{Z}^2 |\ 0 \le j,k \le M-1\}$.

Proposition 8.4. *We have the following formula for the normalized supercharacters of all admissible $\widehat{A}_{1|1}$-modules $L(\Lambda_{jk})$:*

$$(\widehat{R}^{(0)}_0 ch^-_{\Lambda_{jk}})(\tau, z_1, z_2, t) = \varepsilon_s q^{\frac{mjk}{M}} e^{\frac{2\pi i m}{M}(kz_1+jz_2)} \Phi^{A_{1|1}[m]}\left(M\tau, z_1 + j\tau, z_2 + k\tau\,,\ \frac{t}{M}\right),$$

where $\varepsilon_s = (-1)^{\frac{(s-1)(s-2)}{2}}$, and the function $\Phi^{A_{1|1}[m]}$ is given by Lemma 8.1.

Proof. Instead of proving the proposition in the same way as Proposition 7.2, we shall use formula (3.28) for $\Lambda^0 = m\Lambda_0$. For this we need to compute $\beta \in \mathfrak{h}^*$ and $y \in W$, such that $S_s = t_\beta y(S_{(M)})$ for each $s = 1,2,3,4$. A straightforward computation gives:

$$s=1 \quad : \quad y = 1,\ \beta = -(k_1 + \frac{1}{2}k_2)\alpha_{13} - k_1\alpha_2\,;$$

$$s=2 \quad : \quad y = r_{\alpha_{123}} r_{\alpha_2},\ \beta = (k_1 + \frac{1}{2}k_2)\alpha_{13} + k_1\alpha_2\,;$$

$$s=3 \quad : \quad y = r_{\alpha_2},\ \beta = -(k_1 + \frac{1}{2}k_2)\alpha_{13} - (k_1+k_2)\alpha_2\,;$$

$$s=4 \quad : \quad y = r_{\alpha_{12}},\ \beta = (k_1 + \frac{1}{2}k_2)\alpha_{13} + (k_1+k_2)\alpha_2\,.$$

Now, using (3.25), we see that the highest weight coincides with the $\Lambda^{(s)}_{k_1,k_2}$, $s = 1,2,3,4$, up to adding a multiple of δ, which can be ignored since it does not change the normalized character. Applying formula (3.28) and using the correspondence $\Lambda^{(s)}_{k_1,k_2} \mapsto \Lambda_{jk}$ gives the result after a straightforward computation in all four cases. ∎

We use the same notation (7.9) as before for normalized untwisted and twisted characters and supercharacters, and use ξ given by (8.2) for the twisted characters and supercharacters, given by (4.7). Then, as in Section 7, we get from Proposition 8.4, after the change of notation, similar to (7.9), the following unified formula for normalized characters and supercharacters of all admissible twisted and untwisted $\widehat{A}_{1|1}$-modules ($\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$, $j,k \in \varepsilon' + \mathbb{Z}$):

(8.19)

$$(\widehat{R}^{(\varepsilon)}_{\varepsilon'} \mathrm{ch}^{(\varepsilon)}_{\Lambda_{jk};\varepsilon'})(\tau, z_1, z_2, t) = \varepsilon_s q^{\frac{mjk}{M}} e^{\frac{2\pi i m}{M}(kz_1+jz_2)} \Phi^{A_{1|1}[m]}\left(M\tau, z_1 + \varepsilon + j\tau, z_2 + \varepsilon + k\tau\,,\ \frac{t}{M}\right).$$

In view of this formula, we introduce the modified (super)characters of untwisted and twisted $\widehat{A}_{1|1}$-modules by the formula ($\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$; $j, k \in \varepsilon' + \mathbb{Z}$, $0 \le j, k < M$):

$$\widetilde{\mathrm{ch}}^{A_{1|1}[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t) = \frac{\widetilde{\Psi}^{A_{1|1}[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t)}{\widehat{R}^{(\varepsilon)}_{\varepsilon'}(\tau, z_1, z_2)},$$

where the denominators are given by (8.4). Recalling Lemma 8.1, we see that the modification amounts to replacing the meromorphic function $\Phi^{[m-1]}$ by its non-meromorphic modification $\widetilde{\Phi}^{[m-1]}$ (as in the case of $s\ell_{2|1}$), and, furthermore, putting the operator D_0 in front and replacing it by D_1 (and dropping the unessential sign ε_s).

The following theorem follows immediately from (8.5) and Theorem 8.2.

Theorem 8.5. *Let M and m be positive integers, such that $gcd(M, 2m) = 1$ if $m > 1$. Let $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$ and let $j, k \in \varepsilon' + \mathbb{Z}/M\mathbb{Z}$. Then:*

$$\begin{aligned}&\widetilde{\mathrm{ch}}^{A_{1|1}[M,m;\varepsilon]}_{j,k;\varepsilon'}\left(-\tfrac{1}{\tau}, \tfrac{z_1}{\tau}, \tfrac{z_2}{\tau}, t - \tfrac{z_1 z_2}{\tau}\right)\\ &= -(-1)^{2(\varepsilon-\varepsilon')}\tfrac{i\tau}{M}\sum_{a,b\in\varepsilon+\mathbb{Z}/M\mathbb{Z}} e^{-\frac{2\pi i m}{M}(ak+bj)}\widetilde{\mathrm{ch}}^{A_{1|1}[M,m;\varepsilon']}_{a,b;\varepsilon}(\tau, z_1, z_2, t).\end{aligned}$$

Remark 8.6. In the case $m = 1$ we have:

$$\widetilde{\Phi}^{A_{1|1}[1]}(\tau, z_1, z_2, 0) = \Phi^{A_{1|1}[1]}(\tau, z_1, z_2, 0) = D_0\Phi^{(0)}(\tau, z_1, z_2, 0) = -iD_0\frac{\eta(\tau)^3\vartheta_{11}(\tau, z_1 + z_2)}{\vartheta_{11}(\tau, z_1)\vartheta_{11}(\tau, z_2)}.$$

Hence, in view of Lemma 8.1 and formula (8.4), the supercharacter formula (8.9) for $m = 1$ becomes:

$$\widehat{R}^{(0)}_0(\tau, z_1, z_2)\mathrm{ch}^-_{\Lambda_0}(\tau, z_1, z_2, 0) = D_0(\widehat{R}^{(0)}_0(\tau, z_1, z_2)),$$

where

$$\widehat{R}^{(0)}_0(\tau, z_1, z_2) = \eta(\tau)^4\frac{\vartheta_{11}(\tau, z_1 - z_2)\vartheta_{11}(\tau, z_1 + z_2)}{\vartheta_{11}(\tau, z_1)^2\vartheta_{11}(\tau, z_2)^2}.$$

In the remainder of this section we shall study the partially integrable $\widehat{A}_{1|1}$-modules $L(-m\Lambda_0)$ where m is a positive integer, and the corresponding admissible $\widehat{A}_{1|1}$-modules of level $K = -\frac{m}{M}$, where M is a positive integer, coprime with m.

We introduce coordinates, different from (8.3):

$$h = 2\pi i(-\tau\Lambda_0 - (z_1 - z_2)\alpha_1 - z_1\alpha_2 + t\delta)\,. \tag{8.20}$$

In these coordinates the normalized supercharacter of $L(-m\Lambda_0)$ looks exactly like the RHS of (8.9) (except for the sign change of t):

(8.21)

$$(\widehat{R}^{(0)}_0\mathrm{ch}^-_{-m\Lambda_0})(\tau, z_1, z_2, t) = e^{-2\pi i m t}\sum_{j\in\mathbb{Z}}\left(\frac{e^{2\pi i j m(z_1+z_2)}e^{2\pi i z_1}q^{mj^2+j}}{(1 - e^{2\pi i z_1}q^j)^2} - \frac{e^{-2\pi i m(z_1+z_2)}e^{-2\pi i z_2}q^{mj^2+j}}{(1 - e^{-2\pi i z_2}q^j)^2}\right).$$

Note that the RHS of this equation is the function $\Phi^{A_{1|1}[m]}(\tau, z_1, z_2, -t)$. Note also that passing from coordinates (8.3) to coordinates (8.20) amounts to the change of sign of z_2, hence the normalized affine denominators (8.4) remain unchanged.

Next, we derive a formula for normalized supercharacters of level $K=-\frac{m}{M}$, where m and M are positive coprime integers, for which the corresponding partially integrable module is $L(-m\Lambda_0)$. As in the case of positive level K, there are four types of such modules, with highest weights $\Lambda^{(s)}_{k_1,k_2}$ $(s=1,2,3,4)$ given by the same formulae as in Proposition 3.15 except that m is replaced by $-m$. We consider the same reparametrization Λ_{jk} of these highest weights (cf. (8.12),(8.14),(8.16),(8.18)), so that, according to Remark 8.3, the set of pairs (j,k), indexing this admissible highest weights, is the set of integer points in the square $0\le x,y\le M-1$.

As in the case of positive level, we use formula (3.28) in order to compute the normalized superhcaracters $ch^-_{\Lambda^{(s)}_{k_1,k_2}}$ of level $-\frac{m}{M}$ in coordinates (8.20). We get:

$$(8.22)\qquad (\widehat{R}^{(0)}_0 \mathrm{ch}^-_{\Lambda^{(s)}_{k_1,k_2}})(\tau,z_1,z_2,t)$$
$$=\varepsilon_s q^{-\frac{m}{M}jk}e^{\frac{2\pi i m}{M}(-kz_1+jz_2)}(D_0\Phi^{[m-1]})(M\tau,z_1+j\tau,z_2-k\tau,-\frac{t}{M})\ \text{ if }\ s=1\ \text{ or }\ 3\,.$$

By Lemma 5.1(b),(c) and by formula (3.28), we have:

$$\begin{aligned}(\widehat{R}^{(0)}_0 \mathrm{ch}^-_{\Lambda^{(2)}_{k_1,k_2}})(\tau,z_1,z_2,t) &= q^{-\frac{m}{M}k_1(k_1+k_2)}e^{\frac{2\pi i m}{M}((k_1+k_2)z_1-k_1z_2)}\\ &\times\ D_0\left(\Phi^{[m-1]}(M\tau,z_1-k_1\tau\,,\,z_2+(k_1+k_2)\tau\,,\,-\frac{t}{M})\right)\\ &= \varepsilon_2 q^{-\frac{m}{M}(j-M)(k-M)}e^{\frac{2\pi i m}{M}(-(k-M)z_1+(j-M)z_2)}\left(D_0\Phi^{[m-1]}\right)\\ &\quad\left(M\tau,z_1+(j-M)\tau,z_2-(k-M)\tau,-\frac{t}{M}\right)\,.\end{aligned}$$

A similar calculation shows that an analogous formula holds also for $\Lambda^{(4)}_{k_1,k_2}$ (for the corresponding values of j and k). Combining this with (8.22), we obtain the following unified formula for $s=1,2,3,4$:

$$(8.23)\qquad (\widehat{R}^{(0)}_0 \mathrm{ch}^-_{\Lambda^{(s)}_{k_1,k_2}})(\tau,z_1,z_2,t)$$
$$=\ \varepsilon_s q^{-\frac{m}{M}j'k'}e^{\frac{2\pi i m}{M}(-k'z_1+j'z_2)}(D_0\Phi^{[m-1]})\left(M\tau,z_1+j'\tau,z_2-k'\tau,-\frac{t}{M}\right)\,,$$

where $j'=j$, $k'=k$ in cases $s=1,3$, and $j'=j-M$, $k'=k-M$ in cases $s=2,4$.

Next, we compute the normalized supercharacters of all twisted admissible $\widehat{A}_{1|1}$-modules of negative level $K=-\frac{m}{M}$. For this, consider the following automorphism of the Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}=\widehat{A}_{1|1}$:

$$w=t_{-\frac{1}{2}\alpha_2}r_{\alpha_2}\,.$$

Choosing a lifting $\widetilde{r}_{\alpha_2}$ of r_{α_2} in the corresponding $SL_2(\mathbb{C})$, we can lift w to an isomorphism $\widetilde{w}=t_{-\frac{1}{2}\alpha_2}\widetilde{r}_{\alpha_2}:\widehat{\mathfrak{g}}^{\mathrm{tw}}\overset{\sim}{\to}\widehat{\mathfrak{g}}$, cf. Section 4. As in Section 4, via this isomorphism, the $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ becomes a $\widehat{\mathfrak{g}}^{\mathrm{tw}}$-module $L^{\mathrm{tw}}(\Lambda)$, with the highest weight $\Lambda^{\mathrm{tw}}=w(\Lambda)$, and the character and supercharacter $\mathrm{ch}^{\pm}_{L^{\mathrm{tw}}(\Lambda)}=w(\mathrm{ch}^{\pm}_{L(\Lambda)})$.

We have in coordinates (8.20):

$$w^{-1}(\tau,z_1,z_2,t)=\left(\tau,-z_2-\frac{\tau}{2},-z_1-\frac{\tau}{2},t-\frac{\tau}{4}-\frac{z_1+z_2}{2}\right),$$

hence the normalized twisted character of an admissible twisted $\widehat{A}_{1|1}$-module $L^{\mathrm{tw}}(\Lambda)$ is computed by the following formula:

$$\begin{aligned}&(\widehat{R}^{(0)}_{\frac{1}{2}}\mathrm{ch}^{-,\mathrm{tw}}_{\Lambda})(\tau,z_1,z_2,t)\\&=q^{\frac{m}{4M}}e^{\frac{\pi im}{M}(z_1+z_2)}(\widehat{R}^{(0)}_0\mathrm{ch}^{-}_{\Lambda})(\tau,-z_2-\frac{\tau}{2},-z_1-\frac{\tau}{2},t)\,.\end{aligned}$$

Using this formula, we derive from (8.23) the following formula for the normalized twisted supercharacter (by making use of the properties of $\Phi^{[m]}$, given by Lemma 5.1(b), (c), (d)):

$$\begin{aligned}(8.24)\qquad (\widehat{R}^{(0)}_{\frac{1}{2}}\mathrm{ch}^{-,\mathrm{tw}}_{\Lambda^{(s)}_{k_1,k_2}})(\tau,z_1,z_2,t)&=\varepsilon_s q^{-\frac{m}{M}(j'-\frac{1}{2})(k'+\frac{1}{2})}e^{\frac{2\pi im}{M}(-(j'-\frac{1}{2})z_1+(k'+\frac{1}{2})z_2)}\\&\times\left(D_0\Phi^{[m-1]}\right)\left(M\tau,z_1+\left(k'+\tfrac{1}{2}\right)\tau,z_2+(j'-\tfrac{1}{2})\tau,-\tfrac{t}{M}\right).\end{aligned}$$

As in the case of positive level $K=\frac{m}{M}$ considered above, we express the normalized characters of non-twisted and twisted admissible $\widehat{A}_{1|1}$-modules of negative level $K=-\frac{m}{M}$ via their normalized supercharacters, given by (8.23) and (8.24), and we use for them a similar notation $\mathrm{ch}^{A_{1|1}[M,-m;\varepsilon]}_{j,k;\varepsilon'}$. Similarly, we define their non-holomorphic modifications ($\varepsilon,\varepsilon'=0$ or $\frac{1}{2}$; $j,k\in\varepsilon'+\mathbb{Z}$) :

$$\begin{aligned}(8.25)\qquad (\widehat{R}^{(\varepsilon)}_0\widetilde{\mathrm{ch}}^{A_{1|1}[M,-m;\varepsilon]}_{j,k;\varepsilon'})(\tau,z_1,z_2,t)&=D_1\widetilde{\Psi}^{[M,m-1;\varepsilon]}_{j,-k;0}(\tau,z_1,z_2,-t)\,;\\(\widehat{R}^{(\varepsilon)}_{\frac{1}{2}}\widetilde{\mathrm{ch}}^{A_{1|1}[M,-m;\varepsilon]}_{j,k;\varepsilon'})(\tau,z_1,z_2,t)&=D_1\widetilde{\Psi}^{[M,m-1;\varepsilon]}_{k+\frac{1}{2},-j+\frac{1}{2};\frac{1}{2}}(\tau,z_1,z_2,-t)\,.\end{aligned}$$

Since the pairs (j,k) fill up all the integral points in the square $0\le j,k<M$, consider the sets ($\varepsilon'=0$ or $\frac{1}{2}$):

$$\Omega^{A_{1|1}}_{\varepsilon'}=\left\{(j,k)|\ \ j,k\in\varepsilon'+\mathbb{Z}\quad 0\le j<M\,;\,-\frac{1}{2}\le k<M-\frac{1}{2}\right\}.$$

This set parametrizes the normalized modified characters and supercharacters in the non-twisted case when $\varepsilon'=0$ and in the twisted case when $\varepsilon'=\frac{1}{2}$. Then the numerators of the modified non-twisted characters and supercharacters are:

$$D_1\widetilde{\Psi}^{[M,m-1;\varepsilon]}_{j,-k;0}(\tau,z_1,z_2,t)\,,\quad (j,k)\in\Omega^{A_{1|1}}_0\,,$$

and that of the twisted ones are:

$$D_1\widetilde{\Psi}^{[M,m-1;\varepsilon]}_{j,-k;\frac{1}{2}}(\tau,z_1,z_2,-t)\,,\quad (j,k)\in\Omega^{A_{1|1}}_{\frac{1}{2}}\,.$$

In the same way as above we obtained in Theorem 8.5 a modular transformation formula in the case of the positive level $K=\frac{m}{M}$, we otain now a similar formula in the case of the negative level $K=-\frac{m}{M}$.

Theorem 8.7. *Let M and m be positive integers, such that $gcd(M, 2m) = 1$ if $m > 1$. Let $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$ and let $j, k \in \Omega_{\varepsilon'}^{A_{1|1}}$. Then*

$$\widetilde{\mathrm{ch}}_{j,k;\varepsilon'}^{A_{1|1}[M,-m;\varepsilon]}\left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t + \frac{z_1 z_2}{\tau}\right)$$
$$= (-1)^{2(\varepsilon-\varepsilon')}\frac{\tau}{iM}\sum_{a,b\in\varepsilon+\mathbb{Z}/M\mathbb{Z}} e^{\frac{2\pi i m}{M}(ak+bj)}\widetilde{\mathrm{ch}}_{a,b;\varepsilon}^{A_{1|1}[M,-m;\varepsilon']}(\tau, z_1, z_2, t)\,.$$

Remark 8.8. In coordinates (8.3) we have, for $z = -(z_1 + z_2)\alpha_1 - z_1\alpha_2$: $(z|z) = 2z_1z_2$, while in coordinates (8.20) we have, for $z = -(z_1 - z_2)\alpha_1 - z_1\alpha_2$: $(z|z) = -2z_1z_2$. Thus, the modular transformations of Theorem 8.5 and 8.7 are consistent with the usual action of $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in SL_z(\mathbb{Z})$, given by formula (4.16).

Remark 8.9. Each summand of the RHS of the transformation formulae for the modified characters in Theorems 8.5 and 8.7 remains unchanged after adding to a or b an integer multiple of M, but this is not the case for the modified characters in these summands.

9 Modular transformation formulae for modified characters of admissible $N = 2$ modules

Let $\mathfrak{g} = s\ell_{2|1}$ or $A_{1|1}$ $(= ps\ell_{2|2})$, and let $\mathfrak{h}$ be its Cartan subalgebra; recall that $\ell := \dim\mathfrak{h} = 2$ in both cases. Let $x = \frac{1}{2}\theta \in \mathfrak{h}^* = \mathfrak{h}$, where θ is the highest root, which we assume to be even. With respect to ad x we have the following eigenspace decomposition:

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_{-\frac{1}{2}} + \mathfrak{g}_0 + \mathfrak{g}_{\frac{1}{2}} + \mathfrak{g}_1\,,$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}e_{\pm\theta}$, $\mathfrak{g}_{\pm\frac{1}{2}}$ are purely odd, and $\mathfrak{g}_0 = \mathbb{C}x + \mathfrak{g}^{\#}$, where $\mathfrak{g}^{\#}$ is the orthogonal complement to x in $\mathfrak{g}_0$ with respect to $(\,.\,|\,.\,)$. The subalgebra $\mathfrak{g}^{\#}$ is spanned by the element $J_0 = \alpha_2 - \alpha_1$ in the case of $\mathfrak{g} = s\ell_{2|1}$. The subalgebra $\mathfrak{g}^{\#}$ is isomorphic to $s\ell_2$ in the case $\mathfrak{g} = A_{1|1}$, and $\mathfrak{g}^{\#} \cap \mathfrak{h}$ is spanned by the element $J_0 = -\alpha_2$.

Recall that the quantum Hamiltonian reduction associates to a $\widehat{\mathfrak{g}}$-module $L(\Lambda)$, such that $K + h^\vee \neq 0$, a module $H(\Lambda)$ over the corresponding superconformal algebra, which is $N = 2$ (resp. $N = 4$) algebra if $\mathfrak{g} = s\ell_{2|1}$ (resp. $\mathfrak{g} = A_{1|1}$), for which the following properties hold, [KRW], [KW5], [A1]:

(i) the module $H(\Lambda)$ is either 0 or an irreducible positive energy module over the superconformal algebra;

(ii) $H(\Lambda) = 0$ iff $(\Lambda|\alpha_0) \in \mathbb{Z}_{\geq 0}$;

(iii) the irreducible module $H(\Lambda)$ is characterized by three numbers:

(α) the central charge

(9.1) $$c_K = c(K) - 6K + h^\vee - 4, \text{ where } c(K) = \frac{K\,\mathrm{sdim}\,\mathfrak{g}}{K + h^\vee},$$

(β) the lowest energy (i.e. the minimal eigenvalue of L_0)

$$h_\Lambda = h(\Lambda) - (x + d|\Lambda), \text{ where } h(\Lambda) = \frac{(\Lambda + 2\widehat{\rho}|\Lambda)}{2(K + h^\vee)}, \tag{9.2}$$

(γ) the spin

$$s_\Lambda = \Lambda(J_0); \tag{9.3}$$

(iv) the (super)character of the module $H(\Lambda)$ is given by the following formula:

$$\mathrm{ch}^{\pm}_{H(\Lambda)}(\tau,\, z) := \mathrm{tr}^{\pm}_{\mathrm{H}(\Lambda)} \mathrm{q}^{\mathrm{L}_0 - \frac{\mathrm{c}_\mathrm{K}}{24}} \mathrm{e}^{2\pi \mathrm{i} \mathrm{z} \mathrm{J}_0} = (\widehat{\mathrm{R}}^{\pm} \mathrm{ch}^{\pm}_{\Lambda})(\tau, -\tau \mathrm{x} + \mathrm{J}_0 \mathrm{z}, 0) \overset{\mathrm{N}}{\mathrm{R}}{}^{\pm}(\tau, \mathrm{z})^{-1}, \tag{9.4}$$

where

$$\overset{N}{R}{}^{\pm}(\tau, z) = \eta(\tau)^{\ell} \prod_{n=1}^{\infty} \frac{\prod_{\substack{\alpha \in \Delta_+ \\ \alpha(x)=0}} (1 - q^{n-1} e^{-2\pi i z \alpha(J_0)})(1 - q^n e^{2\pi i z \alpha(J_0)})}{\prod_{\substack{\alpha \in \Delta_+ \\ \alpha(x)=1/2}} (1 \pm q^{n-\frac{1}{2}} e^{2\pi i z \alpha(J_0)})}$$

is the N superconformal algebra denominator and superdenominator.

It is easy to rewrite these denominators in terms of the four Jacobi theta functions, using (A.7). We have:

$$\begin{aligned} & q^{\frac{2d_{\bar{0}} + d_{\bar{1}}/2}{24}} e^{2\pi i z \rho(J_0)} i^{d_{\bar{1}} - 1} \overset{N}{R}{}^{\pm}(\tau, z) \\ & = \eta(\tau)^{\ell - d_{\bar{0}} + d_{\bar{1}}/2 + 1} \frac{\prod_{\substack{\alpha \in \Delta_+ \\ \alpha(x)=0}} \vartheta_{11}(\tau, z\alpha(J_0))}{\prod_{\substack{\alpha \in \Delta_+ \\ \alpha(x)=1/2,\, \alpha(J_0)>0}} \vartheta_{0a}(\tau, z\alpha(J_0))}, \end{aligned} \tag{9.5}$$

where $a = 0$ (resp. 1) in the case of $+$ (resp. $-$). Here and further we assume that $\alpha(J_0) \neq 0$ for all $\alpha \in \Delta_{\bar{1}}$, which is the case for $N = 2$ or 4 (but not for $N = 1$ and 3).

Now we turn to the Ramond twisted sector. For each $\alpha \in \Delta_+$ choose an integer if α is even (resp. $\frac{1}{2}+$ integer if α is odd), denoted by s_α, cf. (4.3), such that $s_\theta = 0$ and $s_\alpha + s_{\theta - \alpha} = 0$ if $\alpha, \theta - \alpha \in \Delta_{\bar{1},+}$. Recall [KW6], [A1] that, given such a suitable choice of s_α's, the twisted Hamiltonian reduction associates to a $\widehat{\mathfrak{g}}^{\mathrm{tw}}$-module $L^{\mathrm{tw}}(\Lambda)$ of non-critical level (i. e. $\Lambda(K) + h^\vee \neq 0$), a positive energy module $H^{\mathrm{tw}}(\Lambda)$ over the corresponding Ramond twisted superconformal algebra, for which the properties (i) and (ii) hold with H replaced by H^{tw}. The irreducible module $H^{\mathrm{tw}}(\Lambda)$ is again characterized by three numbers: the same central charge c_K, given by (9.1), the lowest energy (i.e. the minimal eigenvalue of L_0^{tw})

$$h_\Lambda^{\mathrm{tw}} = h(\Lambda^{\mathrm{tw}}) - (x + d|\Lambda^{\mathrm{tw}}) + \frac{1}{16} \,\mathrm{sdim}\, \mathfrak{g}_{\frac{1}{2}}, \tag{9.6}$$

and the spin

$$s_\Lambda^{\mathrm{tw}} = \Lambda^{\mathrm{tw}}(J_0) + \frac{1}{2} \sum_{\substack{\alpha \in \Delta \\ \alpha(x)=1/2}} s_\alpha \alpha(J_0). \tag{9.7}$$

Furthermore, the (super)character of the module $H^{\mathrm{tw}}(\Lambda)$ is given by the following formula:
(9.8)

$$\mathrm{ch}^{\pm}_{H^{\mathrm{tw}}(\Lambda)}(\tau,z) := \mathrm{tr}^{\pm}_{\mathrm{H^{tw}}(\Lambda)} \mathrm{q}^{\mathrm{L}_0^{\mathrm{tw}}-\frac{\mathrm{c_K}}{24}} \mathrm{e}^{2\pi \mathrm{i z J}_0^{\mathrm{tw}}} = (\widehat{\mathrm{R}}^{\mathrm{tw},\pm}\mathrm{ch}^{\mathrm{tw},\pm}_{\Lambda})(\tau, -\tau \mathrm{x} + \mathrm{J}_0 \mathrm{z}\,,\,0) \overset{\mathrm{N}}{\mathrm{R}}{}^{\mathrm{tw}\,,\pm}(\tau,\mathrm{z})^{-1},$$

where

$$\overset{\mathrm{N}}{R}{}^{\mathrm{tw}\,,\pm}(\tau,z) = \eta(\tau)^{\ell} \prod_{n=1}^{\infty} \frac{\prod_{\substack{\alpha\in\Delta_+\\ \alpha(x)=0}} (1-q^{n-1+s_\alpha} e^{-2\pi i z \alpha(J_0)})(1-q^{n-s_\alpha}e^{2\pi i z\alpha(J_0)})}{\prod_{\substack{\alpha\in\Delta_+\\ \alpha(x)=1/2}} (1\pm q^{n-\frac{1}{2}+s_\alpha} e^{2\pi i z\alpha(J_0)})}$$

is the N superconformal algebra twisted denominator and superdenominator. It is easy to rewrite these denominators in terms of the four Jacobi theta functions, using (A.7). We have:

$$\text{(9.9)} \qquad i^{d_{\bar{1}}-1} e^{\pi i z (\sum_{\alpha\in\Delta_+,\alpha(x)=0}(2s_\alpha-1)\alpha(J_0)+\sum_{\alpha\in\Delta_{\bar{1},+},s_\alpha>0}\alpha(J_0))} q^{\frac{d_{\bar{0}}-1+d_{\bar{1}}/2}{12}} \overset{\mathrm{N}}{R}{}^{\mathrm{tw}\,,\pm}(\tau,z)$$
$$= \eta(\tau)^{\ell-d_{\bar{0}}+d_{\bar{1}}/2+1} \frac{\prod_{\substack{\alpha\in\Delta_+\\ \alpha(x)=0}} \vartheta_{11}(\tau, z\alpha(J_0))}{\prod_{\substack{\alpha\in\Delta_+\\ \alpha(x)=1/2,\, s_\alpha>0}} \vartheta_{1a}(\tau, z\alpha(J_0))},$$

where $a=0$ (resp. 1) in the case of $+$ (resp. $-$). The right hand sides of fomulas (9.5) (resp. (9.9)) are called the normalized untwisted (resp. twisted) denominators (for $+$) and superdenominators (for -).

In order to write down modular transformation formulae for these normalized denominators, we denote them by $\overset{\mathrm{N}(\varepsilon)}{R_{\varepsilon'}}(\tau,z)$, where $\varepsilon,\varepsilon'=0$ or $\frac{1}{2}$, and, as before, the superscript refers to the normalized denominator (resp. superdenominator) if $\varepsilon=\frac{1}{2}$ (resp. $\varepsilon=0$), and the subscript refers to the untwisted case, also called the Neveu–Schwarz sector (resp. to the twisted case, also called the Ramond sector), if, not as before, $\varepsilon'=\frac{1}{2}$ (resp. $\varepsilon'=0$.) Then we have

$$\text{(9.10)} \qquad \overset{\mathrm{N}(\varepsilon)}{R_{\varepsilon'}}(\tau,z) = \eta(\tau)^{\ell+1-d_{\bar{0}}+d_{\bar{1}}/2} \frac{\prod_{\substack{\alpha\in\Delta_{\bar{0},+}\\ \alpha(x)=0}} \vartheta_{11}(\tau,z\alpha(J_0))}{\prod_{\substack{\alpha\in\Delta_{\bar{1},+}\\ s_\alpha>0}} \vartheta_{1-2\varepsilon',1-2\varepsilon}(\tau,z\alpha(J_0))}.$$

Using (9.10) and Propositon A.7 we deduce the following modular transformation formulae for these denominators:

Proposition 9.1.

(a) $\overset{N(\varepsilon')}{R_{\varepsilon}}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = (-i)^{d_{\bar{0}}-1-(1-2\varepsilon)(1-2\varepsilon')d_{\bar{1}}/2}(-i\tau)^{\ell/2}\, e^{\frac{\pi i z^2}{2\tau}(h^\vee(J_0|J_0)+\sum_{\alpha\in\Delta_{\bar{0},+}}\alpha(J_0)^2)} \overset{N(\varepsilon)}{R_{\varepsilon'}}(\tau,z)$.

(b) $\overset{N(\varepsilon)}{R_0}(\tau+1,z) = e^{\frac{\pi i}{12}(\dim\mathfrak{g}_0-\dim\mathfrak{g}_{\frac{1}{2}})} \overset{N(\varepsilon)}{R_0}, \quad \overset{N(\varepsilon)}{R_{\frac{1}{2}}}(\tau+1,z) = e^{\frac{\pi i}{12}(\dim\mathfrak{g}_0+\frac{1}{2}\dim\mathfrak{g}_{\frac{1}{2}})} \overset{N(\frac{1}{2}-\varepsilon)}{R_{\frac{1}{2}}}(\tau,z)$.

For the rest of this section we shall consider the case $\mathfrak{g}=s\ell_{2|1}$. Let $L(\Lambda)$ be an admissible $\widehat{\mathfrak{g}}$-module of level $K=\frac{m+1}{M}-1$, where M is a positive integer, m is a non-negative integer, and $gcd(M,2m+2)=1$ if $m>0$ (see Section 7). As a result of a quantum Hamiltonian reduction of the $\widehat{\mathfrak{g}}$-module $L(\Lambda)$, where $\Lambda=m_0\Lambda_0+m_1\Lambda_1+m_2\Lambda_2$, we obtain a positive energy module

$H(\Lambda)$ over the Neveu–Schwarz type $N=2$ superconformal algebra. Recall that $H(\Lambda)=0$ iff $m_0 \in \mathbb{Z}_{\geq 0}$, and that $H(\Lambda)$ is irreducible otherwise.

If $M=1$, then the $\widehat{\mathfrak{g}}$-module is partially integrable, hence $m_0 \in \mathbb{Z}_{\geq 0}$, and therefore $H(\Lambda)=0$. If $M>1$, then it follows from Proposition 3.14 that for an admissible Λ we have $m_0 \in \mathbb{Z}_{\geq 0}$ iff $k_0=0$. Thus, in what follows we may assume that $M \geq 2$ and $k_0 \neq 0$. Then $H(\Lambda)$ is an irreducible module over the NS type $N=2$ superconformal algebra. The corresponding three characteristic numbers are easy to compute:

$$c_K = -6K-3 = 3\left(1-\frac{2m+2}{M}\right), \tag{9.11}$$

$$h_\Lambda = \frac{Mm_1m_2}{m+1} - \frac{m_1+m_2}{2}, \tag{9.12}$$

$$s_\Lambda = m_1 - m_2\,. \tag{9.13}$$

As has been pointed out in Section 4, in order to get modular invariant family of characters and supercharacters, we need to introduce the Ramond twisted sector, for which we need a choice of $\xi \in \mathfrak{h}^*$, satisfying (4.3). However, in order to apply the twisted quantum Hamiltonian reduction we need a more special choice of ξ (cf. [KW6]). In the case $\mathfrak{g}=s\ell_{2|1}$ we made the choice (7.1) of ξ which gave nice formulae for twisted characters and supercharacters, but unfortunately, it is not compatible with the twisted quantum Hamiltonian reduction. Instead we make the choice

$$\xi' = \frac{1}{2}(\alpha_1 - \alpha_2) \tag{9.14}$$

and let $s_\alpha = -(\xi'|\alpha)$, $\alpha \in \Delta_+$. Then we have in coordinates (7.2):

$$t_{-\xi'}(\tau, z_1, z_2, t) = (\tau, z_1+\frac{\tau}{2}, z_2-\frac{\tau}{2}, t+\frac{z_2-z_1}{2}-\frac{\tau}{4})\,. \tag{9.15}$$

While the non-twisted normalized denominators and super-denominators remain as given in formula (7.4) for $\varepsilon'=0$, the twisted ones change, namely, we let

$$\widehat{R}^{(\varepsilon)}_{\frac{1}{2}}(h) := \widehat{R}^{(\varepsilon)}_0(t_{-\xi'}(h))\,.$$

The obtained denominators differ only by a sign from (7.4), and only when $\varepsilon=\varepsilon'=\frac{1}{2}$, so we keep for them the same notation in the hope that no confusion may arise. The modular transformation formulae differ from (7.5) and (7.6) also by a sign:

$$\widehat{R}^{(\varepsilon)}_{\varepsilon'}\left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t\right) = (-1)^{4\varepsilon\varepsilon'}\tau e^{\frac{2\pi i z_1 z_2}{\tau}} \widehat{R}^{(\varepsilon')}_{\varepsilon}(\tau, z_1, z_2, t)\,; \tag{9.16}$$

$$\widehat{R}^{(\varepsilon)}_{\varepsilon'}(\tau+1, z_1, z_2, t) = e^{-\pi i \varepsilon'} \widehat{R}^{(|\varepsilon-\varepsilon'|)}_{\varepsilon}(\tau, z_1, z_2, t)\,. \tag{9.17}$$

Using the above notation for the denominators, the same notation as (7.9) for the characters, and the same proof as that of Proposition 7.2, we obtain the following analogue of it for the choice of ξ' instead of ξ.

Proposition 9.2. *(a) If* $\Lambda = \Lambda^{(1)}_{j,k}$*, where* $j,k \in \mathbb{Z}$*,* $0 \le j,k,j+k \le M-1$*, then*

$$\left(\widehat{R}^{(\varepsilon)}_{\varepsilon'}\mathrm{ch}^{(\varepsilon)}_{\Lambda;\varepsilon'}\right)(\tau,z_1,z_2,t) = \Psi^{[M,m:\varepsilon]}_{j+\varepsilon',k-\varepsilon';\varepsilon'}(\tau,z_1,z_2,t)\,.$$

(b) If $\Lambda = \Lambda^{(2)}_{j,k}$*, where* $j,k \in \mathbb{Z}$*,* $1 \le j,k,j+k \le M$*, then*

$$\left(\widehat{R}^{(\varepsilon)}_{\varepsilon'}\mathrm{ch}^{(\varepsilon)}_{\Lambda;\varepsilon'}\right)(\tau,z_1,z_2,t) = -\Psi^{[M,m;\varepsilon]}_{M+\varepsilon'-j,M-\varepsilon';\varepsilon'}(\tau,z_1,z_2,t)\,.$$

For the twisted $\widehat{\mathfrak{g}}$-module $L^{\mathrm{tw}}(\Lambda)$ the highest weight is $\Lambda^{\mathrm{tw}} = t_{\xi'}(\Lambda)$, the quantum Hamiltonian reduction produces a module $H^{\mathrm{tw}}(\Lambda)$ over the Ramond type $N=2$ superconformal algebra. As in the non-twisted case, $H^{\mathrm{tw}}(\Lambda) = 0$ if $M=1$, and if $M>1$ and $m_0 = 0$. Otherwise $H^{\mathrm{tw}}(\Lambda)$ is an irreducible positive energy module with central charge (9.1), the remaining two characteristic numbers being

$$h^{\mathrm{tw}}_{\Lambda} = \frac{Mm_1m_2}{m+1} - m_2 - \frac{m+1}{4M} - \frac{1}{8}\,, \tag{9.18}$$

$$s^{\mathrm{tw}}_{\Lambda} = m_1 - m_2 - \frac{m+1}{M} - \frac{1}{2}\,. \tag{9.19}$$

It follows from the description of admissible weights of level K given in Proposition 3.14 that the list of Λ, for which $H(\Lambda) \neq 0$ (resp. $H^{\mathrm{tw}}(\Lambda) \neq 0$), consists of two sets:

$$\begin{aligned} \mathcal{A}^{(1)} &= \{\Lambda^{(1)}_{k_1,k_2} \,|\, k_1,k_2 \in \mathbb{Z}_{\ge 0}\,,\ k_1+k_2 \le M-2\}\,, \\ \mathcal{A}^{(2)} &= \{\Lambda^{(2)}_{k_1,k_2} \,|\, k_1,k_2 \in 1+\mathbb{Z}_{\ge 0}\,,\ k_1+k_2 \le M\}\,. \end{aligned} \tag{9.20}$$

Note that we have the following bijective map:

$$\nu : \mathcal{A}^{(1)} \to \mathcal{A}^{(2)}\,,\ \nu(\Lambda^{(1)}_{k_1,k_2}) = \Lambda^{(2)}_{k_2+1,k_1+1}.$$

It is immediate to see that

$$h_{\Lambda^{(1)}_{k_1,k_2}} = h_{\nu(\Lambda^{(1)}_{k_1,k_2})},\ s_{\Lambda^{(1)}_{k_1,k_2}} = s_{\nu(\Lambda^{(1)}_{k_1,k_2})}$$

for $\Lambda^{(1)}_{k_1,k_2} \in \mathcal{A}^{(1)}$, and the same holds for h^{tw} and s^{tw}. Hence for the quantum Hamiltonian reduction it suffices to consider only the highest weights $\Lambda^{(1)}_{k_1,k_2} \in \mathcal{A}^{(1)}$.

In order to compute the characters and supercharacters of the corresponding $N=2$ modules $H(\Lambda)$ and $H^{\mathrm{tw}}(\Lambda)$ we use formulae (9.4) and (9.8).

First, we have from (9.10)

$$\overset{2}{R}{}^{(\varepsilon)}_{\varepsilon'}(\tau,z) = \frac{\eta(\tau)^3(-1)^{(1-2\varepsilon)(1-2\varepsilon')}}{\vartheta_{1-2\varepsilon',1-2\varepsilon}(\tau,z)}\,. \tag{9.21}$$

Using (9.21) and Proposition A.7 (or Proposition 9.2) we deduce the following modular transformation formulae of the $N=2$ normalized denominators.

Lemma 9.3. *(a)* $\overset{2}{R}{}^{(\varepsilon)}_{\varepsilon'}\left(-\frac{1}{\tau}\,,\,\frac{z}{\tau}\right) = i^{4\varepsilon\varepsilon'-2\varepsilon-2\varepsilon'}\tau e^{-\frac{\pi i z^2}{\tau}}\overset{2}{R}{}^{(\varepsilon')}_{\varepsilon}(\tau,z).$

(b) $\overset{2}{R}{}^{(\varepsilon)}_0(\tau+1,z)=\overset{2}{R}{}^{(\varepsilon)}_0(\tau,z),\ \ \overset{2}{R}{}^{(\varepsilon)}_{\frac12}(\tau+1,z)=e^{\frac{\pi i}{4}}\overset{2}{R}{}^{(\frac12-\varepsilon)}_{\frac12}(\tau,z).$

Let M be an integer ≥ 2, and let $m\in\mathbb{Z}_{\geq 0}$ be such that $gcd(M,2m+2)=1$ if $m>0$. Recall that we have irreducible positive energy $N=2$ modules $H(\Lambda^{(1)}_{k_1,k_2})$ (resp. $H^{\rm tw}(\Lambda^{(1)}_{k_1,k_2})$) in the Neveu-Schwarz (resp. Ramond) sector with central charge $c_K=3\left(1-\frac{2m+2}{M}\right)$, obtained by the quantum Hamiltonian reduction from the $\widehat{s\ell}_{2|1}$-modules $L(\Lambda^{(1)}_{k_1,k_2})$ (resp. $L^{\rm tw}(\Lambda^{(1)}_{k_1,k_2})$) of level $K=\frac{m+1}{M}-1$, where $\Lambda^{(1)}_{k_1,k_2}\in\mathcal{A}^{(1)}$.

It will be convenient to introduce the following two reindexings of the set of weights $\mathcal{A}^{(1)}$:

$$\begin{aligned}\mathcal{A}_{NS}&=\{\Lambda_{jk}=\Lambda^{(1)}_{j-\frac12,k-\frac12}\,|\,j,k\in\tfrac12+\mathbb{Z}_{\geq0}\,,\ j+k\leq M-1\}\,,\\ \mathcal{A}_R&=\{\Lambda_{jk}=\Lambda^{(1)}_{j-1,k}|\,j,k\in\mathbb{Z}_{\geq0}\,,\ j>0\,,\ j+k\leq M-1\}.\end{aligned}$$

We let

$$H_{NS}(\Lambda_{jk})=H(\Lambda^{(1)}_{j-\frac12,k-\frac12})\,,\ j,k\in\mathcal{A}_{NS};\ \ H_R(\Lambda_{jk})=H(\Lambda^{(1)}_{j-1,k}),\ j,k\in\mathcal{A}_R\,.$$

It follows from (9.12)–(9.19) that the lowest energy and the spin of these $N=2$ modules with central charge $c_K=3\left(1-\frac{2m+2}{M}\right)$ are as follows:

$$h^{NS}_{jk}=\frac{m+1}{M}jk-\frac{m+1}{4M}\,,\ \ s^{NS}_{jk}=\frac{m+1}{M}(k-j)\,; \tag{9.22}$$

$$h^{R}_{jk}=\frac{m+1}{M}jk-\frac{m+1}{4M}-\frac18\,,\ \ s^{R}_{jk}=\frac{m+1}{M}(k-j)+\frac12\,. \tag{9.23}$$

Introduce the following notation for the normalized characters and supercharacters of these $N=2$ modules:

$$\begin{aligned}ch^{N=2[M,m;\varepsilon]}_{j,k;\frac12}(\tau,z)&=ch^{\pm}_{H_{NS}(\Lambda_{jk})}(\tau,z)\,,\ \Lambda_{jk}\in\mathcal{A}_{NS}\,;\\ ch^{N=2[M,m;\varepsilon]}_{j,k;0}(\tau,z)&=ch^{\pm}_{H_R(\Lambda_{jk})}(\tau,z)\,,\ \Lambda_{jk}\in\mathcal{A}_R\,.\end{aligned}$$

Formulae (9.4) and (9.8) imply the following expressions for these characters:

$$(\overset{2}{R}{}^{(\varepsilon)}_{\varepsilon'}ch^{N=2[M,m;\varepsilon]}_{j,k;\varepsilon'})(\tau,z)=\Psi^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau,-z,z,0), \tag{9.24}$$

where the functions $\Psi^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau,z_1,z_2,t)$ are defined by (7.7) and $j,k\in\varepsilon'+\mathbb{Z}_{\geq0}$, subject to restrictions $j+k\leq M-1$, $j>0$.

Introduce the *modified* normalized $N=2$ characters and supercharacters, letting

$$(\overset{2}{R}{}^{(\varepsilon)}_{\varepsilon'}\widetilde{ch}^{N=2[M,m;\varepsilon]}_{j,k;\varepsilon'})(\tau,z)=\widetilde{\Psi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau,-z,z,0),$$

where the modification $\widetilde{\Psi}$ of Ψ was introduced in Section 7 (after (7.7)). Theorem 7.1 along with Lemma 9.3 give the following modular transformation properties of the modified normalized $N=2$ characters and supercharacters.

Theorem 9.4. *Let M be an integer ≥ 2 and let $m \in \mathbb{Z}_{\geq 0}$ be such that $gcd(M, 2m+2) = 1$ if $m > 0$. Let $c_{M,m} = 3\left(1 - \frac{2m+2}{M}\right)$. Let $\varepsilon, \varepsilon' = 0$ or $\frac{1}{2}$, and let $\Omega_\varepsilon^{(M)} = \{(j,k) \in (\varepsilon + \mathbb{Z}_{\geq 0})^2 | \, j+k \leq M-1, \, j > 0\}$. Then we have the following modular transformation formulae for $\widetilde{ch}^{N=2[M,m;\varepsilon]}_{j,k;\varepsilon'}$, $j, k \in \Omega_{\varepsilon'}^{(M)}$:*

$$\widetilde{ch}^{N=2[M,m;\varepsilon]}_{j,k;\varepsilon'}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{\frac{\pi i c_{M,m}}{6\tau} z^2} \sum_{(a,b) \in \Omega_\varepsilon^{(M)}} S^{[M,m,\varepsilon,\varepsilon']}_{(j,k),(a,b)} \, \widetilde{ch}^{N=2[M,m;\varepsilon']}_{a,b;\varepsilon}(\tau, z)\,,$$

where

$$S^{[M,m,\varepsilon,\varepsilon']}_{(j,k),(a,b)} = (-i)^{(1-2\varepsilon)(1-2\varepsilon')} \frac{2}{M} e^{\frac{\pi i (m+1)}{M}(j-k)(a-b)} \sin \frac{m+1}{M}(j+k)(a+b)\pi\,;$$

$$\widetilde{ch}^{N=2[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau+1, z) = e^{\frac{2\pi i(m+1)}{M} jk - \frac{\pi i \varepsilon'}{2}} \widetilde{ch}^{N=2[M,m;\varepsilon+\varepsilon']}_{j,k;\varepsilon'}(\tau, z)\,.$$

Remark 9.5. (a) Letting $\widetilde{ch}^{N=2[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z, t) = e^{2\pi i t c_{M,m}} \, \widetilde{ch}^{N=2[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z)$, we can rewrite the transformation formula in Theorem 9.4 in a more suggestive form:

$$\widetilde{ch}^{N=2[M,m;\varepsilon]}_{j,k;\varepsilon'}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{z^2}{6\tau}\right) = \sum_{(a,b) \in \Omega_\varepsilon^{(M)}} S^{[M,m,\varepsilon,\varepsilon']}_{(j,k)(a,b)} \, \widetilde{ch}^{N=2[M,m;\varepsilon']}_{a,b;\varepsilon}(\tau, z, t)\,.$$

(b) If $m = 0$, we have the well-known $N = 2$ unitary discrete series modules with central charge $3\left(1 - \frac{2}{M}\right)$ and their well-known modular transformation properties (with tilde in Theorem 9.4 removed), see e.g. [KW3].

10 Modular transformation formulae for modified characters of admissible $N = 4$ modules

In this section we study quantum Hamiltonian reduction of admissible $\widehat{A}_{1|1}$-modules of negative level $K = -\frac{m}{M}$, where m and M are coprime positive integers. As we have seen in Section 8, S-invariance of modified characters holds for arbitrary non-zero level (Theorems 8.5 and 8.7), however the $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$-invariance fails due to the fact that the operator D_1 is not translation ($\tau \to \tau+1$) invariant). Moreover, for $K > 0$ translation invariance still fails after the reduction, but, as we shall see, for $K < 0$, after the quantum Hamiltonian reduction the translation invariance gets restored.

Consider an $\widehat{A}_{1|1}$-module $L(\Lambda)$ of negative level $K = -\frac{m}{M}$ with the highest weight

$$\Lambda = (K - 2m_1 + m_2)\Lambda_0 + m_1(\Lambda_1 + \Lambda_3) + m_2\Lambda_2\,.$$

It follows from the remarks in the beginning of Section 9 that the module $H(\Lambda)$ over the $N = 4$ superconformal algebra of Neveu–Schwarz type, obtained from $L(\Lambda)$ by the quantum Hamiltonian reduction, is either 0 (which happens off $K - 2m_1 + m_2 \in \mathbb{Z}_{\geq 0}$), or is irreducible.

In the latter case the characteristic numbers of $H(\Lambda)$ are obtained from formulae (9.1)–(9.3):

$$c_K = 6\left(\frac{m}{M}-1\right), \tag{10.1}$$

$$h_\Lambda = -\frac{Mm_1(m_1-m_2-1)}{m} - m_1 + \frac{m_2}{2}, \tag{10.2}$$

$$s_\Lambda = m_2. \tag{10.3}$$

The same holds in the Ramond type case, also the central charge c_K is the same, and the remaining two characteristic numbers are obtained from (9.6) and (9.7):

$$h^{\rm tw}_\Lambda = -\frac{Mm_1(m_1-m_2-1)}{m} - m_1 - \frac{M-m}{4M}, \tag{10.4}$$

$$s^{\rm tw}_\Lambda = -m_2 - \frac{M-m}{M}. \tag{10.5}$$

In order to compute the characters and supercharacters of the corresponding $N=4$ modules $H(\Lambda)$ and $H^{\rm tw}(\Lambda)$, we use formulas (9.4) and (9.8) in the case $\mathfrak{g}=A_{1|1}$.

First, we have from (9.10):

$$\overset{4}{R}{}^{(\varepsilon)}_{\varepsilon'}(\tau,z) = \eta(\tau)^3 \frac{\vartheta_{11}(\tau,2z)}{\vartheta_{1-2\varepsilon',1-2\varepsilon}(\tau,z)^2}. \tag{10.6}$$

From (10.6) and Proposition A.7 (or from Proposition 9.1) we deduce the following transformation formulae for $N=4$ normalized denominators, where we use the notation

$$\overset{4}{R}{}^{(\varepsilon)}_{\varepsilon'}(\tau,z,t) = e^{-2\pi it}\overset{4}{R}{}^{(\varepsilon)}_{\varepsilon'}(\tau,z).$$

Lemma 10.1. *(a)* $\overset{4}{R}{}^{(\varepsilon)}_{\varepsilon'}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t+\frac{z^2}{\tau}\right) = -(-1)^{(1-2\varepsilon)(1-2\varepsilon')}\tau\overset{4}{R}{}^{(\varepsilon)}_{\varepsilon'}(\tau,z,t).$

(b) $\overset{4}{R}{}^{(\varepsilon)}_{\varepsilon'}(\tau+1,z,t) = e^{\pi i\varepsilon'}\overset{4}{R}{}^{(|\varepsilon-\varepsilon'|)}_{\varepsilon'}(\tau,z,t).$

Next, formulae (9.4) and (9.8) for characters and supercharacters of $N=4$ non-twisted and twisted modules are respectively:

$$(\overset{4}{R}{}^{\pm}\mathrm{ch}^{\pm}_{H(\Lambda)})(\tau,z,t) = (\widehat{R}^{A_{1|1}\pm}\mathrm{ch}^{\pm}_\Lambda)\left(\tau, z+\frac{\tau}{2}, z-\frac{\tau}{2}, t\right); \tag{10.7}$$

$$(\overset{4}{R}{}^{\rm tw\ ,\pm}\mathrm{ch}^{\pm}_{H^{\rm tw}(\Lambda)})(\tau,z,t) = \left(\widehat{R}^{A_{1|1}\rm tw\,,\pm}\mathrm{ch}^{\rm tw\,,\pm}_\Lambda\right)\left(\tau, z+\frac{\tau}{2}, z-\frac{\tau}{2}, t\right). \tag{10.8}$$

As before, we change now notation by letting $\varepsilon=\frac12$ (resp. $=0$) in the case of $+$ (resp. $-$), and $\varepsilon'=\frac12$ (resp. $=0$) in the non-twisted (resp. twisted) case. For example, $\overset{4}{R}{}^{\rm tw+} = \overset{4}{R}{}^{\frac12}_0$.

Applying formulae (10.7) and (10.8) to $\Lambda=\Lambda^{(s)}_{k_1,k_2}$ $(s=1,2,3,4)$ and using the modification (8.25) of the numerators of the admissible characters of $\widehat{A}_{1|1}$, we obtain the following *partially modified* $N=4$ characters:

$$\overset{4}{R}{}^{(\varepsilon)}_{\frac12}\mathrm{ch}^{(\varepsilon)\prime}_{H(\Lambda^{(s)}_{k_1,k_2})}(\tau,z,t) = D_1\widetilde{\Psi}^{[M,m-1;\varepsilon]}_{j,-k;0}\left(\tau, z+\frac{\tau}{2}, z-\frac{\tau}{2}, -t\right);$$

$$\left(\overset{4}{R}{}^{(\varepsilon)}_0\mathrm{ch}^{(\varepsilon)\prime}_{H^{\rm tw}(\Lambda^{(s)}_{k_1,k_2})}\right)(\tau,z,t) = D_1\widetilde{\Psi}^{[M,m-1;\varepsilon]}_{k+\frac12,-j+\frac12;\frac12}(\tau, z+\frac{\tau}{2}, z-\frac{\tau}{2}, -t).$$

The *modified* $N=4$ characters are obtained from the partially modified ones by the shifts $j \mapsto j+\frac{1}{2}$, $k \mapsto k+\frac{1}{2}$, and the corresponding shifts $z \mapsto z \pm \frac{\tau}{2}$. Namely, we let

$$\left(\overset{4}{R}{}^{(\varepsilon)}_{\frac{1}{2}} \widetilde{\mathrm{ch}}^{(\varepsilon)}_{H\left(\Lambda^{(s)}_{k_1,k_2}\right)}\right)(\tau,z,t) = G^{[M,m;\varepsilon]}_{j+\frac{1}{2},-k-\frac{1}{2};\frac{1}{2}}(\tau,z,t)\,; \tag{10.9}$$

$$\left(\overset{4}{R}{}^{(\varepsilon)}_{0} \widetilde{\mathrm{ch}}^{(\varepsilon)}_{H^{\mathrm{tw}}\left(\Lambda^{(s)}_{k_1,k_2}\right)}\right)(\tau,z,t) = G^{[M,m;\varepsilon]}_{k+1,-j;0}(\tau,z,t)\,, \tag{10.10}$$

where

$$G^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau,z,t) = D_1 \widetilde{\Psi}^{[M,m-1;\varepsilon]}_{j,k;\varepsilon'}(\tau,z,z,-t)\,. \tag{10.11}$$

Remark 10.2. (a) One can replace D_1 by D_0 in (10.11).

(b) We have:

$$\left(D_1 \Psi^{[M,m;\varepsilon]}_{j,-k;\varepsilon'}\right)\left(\tau, z+\frac{\tau}{2}, z-\frac{\tau}{2}, -t\right) = q^{\frac{m+1}{4M}} \left(D_1 \Psi^{[M,m;\varepsilon]}_{j+\frac{1}{2},-k-\frac{1}{2};\frac{1}{2}-\varepsilon'}\right)(\tau,z,z,-t)\,,$$

and the same identity holds for $\widetilde{\Psi}$. Hence passing from the partial modification to the modification of the $N=4$ characters amounts to removing the factor $q^{\frac{m+1}{4M}}$.

Since the action of $SL_2(\mathbb{Z})$ commutes with D_1, we deduce from Theorem 7.1 modular transformation formulae for the numerators of the modified $N=4$ characters:

$$G^{[M,m;\varepsilon]}_{j,k;\varepsilon'}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t+\frac{z^2}{\tau}\right) = \frac{\tau^2}{M} \sum_{a,b\in\varepsilon+\mathbb{Z}/M\mathbb{Z}} e^{-\frac{2\pi i m}{M}(ak+bj)} G^{[M,m;\varepsilon']}_{a,b;\varepsilon}(\tau,z,t)\,, \tag{10.12}$$

$$G^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau+1,z,t) = e^{\frac{2\pi i m jk}{M}} G^{[M,m;|\varepsilon-\varepsilon'|]}_{j,k;\varepsilon'}(\tau,z,t)\,. \tag{10.13}$$

We rewrite formula (10.12), using that $G^{[M,m;\varepsilon]}_{k,j;\varepsilon'}$ changes sign if we permute k and j. Permuting a and b in the RHS of (10.12) and adding the obtained equation to (10.12) we obtain, after some simple manipulations:

$$\begin{aligned} & G^{[M,m;\varepsilon]}_{j,-k;\varepsilon'}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t+\frac{z^2}{\tau}\right) \\ &= \frac{i\tau^2}{M} \sum_{a,b\in\varepsilon+\mathbb{Z}/M\mathbb{Z}} e^{-\frac{\pi i m}{M}(a-b)(j-k)} \sin\frac{\pi m}{M}(a+b)(j+k) G^{[m,M;\varepsilon']}_{a,-b;\varepsilon}(\tau,z,t)\,. \end{aligned} \tag{10.14}$$

We have used above the parametrization of the admissible weights $\Lambda^{(s)}_{k_1,k_2}$ by the pairs (j,k), given by (8.12), (8.14), (8.16) and (8.18). In order to write down a unified formula for modified $N=4$ characters, it is convenient to introduce new parameters:

$$\begin{aligned} \widetilde{j} &= j+\frac{1}{2}\,, \widetilde{k} = k+\frac{1}{2} \text{ in the non-twisted case,} \\ \widetilde{j} &= k+1\,, \widetilde{k} = j \text{ in the twisted case.} \end{aligned} \tag{10.15}$$

Then (10.9) and (10.10) can be written in a unified way:

$$\left(R^{4(\varepsilon)}_{\varepsilon'}\,\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},\widetilde{k};\varepsilon'}\right)(\tau,z,t)=G^{[M,m;\varepsilon]}_{\widetilde{j},-\widetilde{k};\varepsilon'}(\tau,z,t)\,. \tag{10.16}$$

We describe below the precise parametrization of these modified $N=4$ characters. First, the following lemma is obtained by a direct calculation from (10.2)–(10.5).

Lemma 10.3. *Let $\Lambda=\Lambda^{(s)}_{k_1,k_2}$ be an admissible weight of level $K=-\frac{m}{M}$ for $\widehat{A}_{1|1}$. Assume that $H(\Lambda)\neq 0$, so that the $N=4$ superconformal algebra modules $H(\Lambda)$ and $H^{\mathrm{tw}}(\Lambda)$ are irreducible. Then their lowest energies and spins are described by the following formulae in the parameters $(\widetilde{j},\widetilde{k})$ given by (10.15):*

$$\begin{aligned}
s=1: h_\Lambda &= K\widetilde{j}\widetilde{k}+\widetilde{j}-\frac{K+2}{4}\,,\ s_\Lambda=K(\widetilde{k}-\widetilde{j})\,,\\
h^{\mathrm{tw}}_\Lambda &= K\widetilde{j}\widetilde{k}+\widetilde{k}-\frac{K+1}{4}\,,\ s^{\mathrm{tw}}_\Lambda=K(\widetilde{k}-\widetilde{j})-1\,;\\
s=2: h_\Lambda &= K(M-\widetilde{j})(M-\widetilde{k})+M-\widetilde{j}-\frac{K+2}{4}\,,\ s_\Lambda=K(\widetilde{k}-\widetilde{j})-2\,,\\
h^{\mathrm{tw}}_\Lambda &= K(M-\widetilde{j})(M-\widetilde{k})+M-\widetilde{k}-\frac{k+1}{4}\,,\ s^{\mathrm{tw}}_\Lambda=K(\widetilde{k}-\widetilde{j})+1\,;\\
s=3: h_\Lambda &= K\widetilde{j}\widetilde{k}+\widetilde{k}-\frac{K+2}{4}\,,\ s_\Lambda=K(\widetilde{k}-\widetilde{j})-2\,,\\
h^{\mathrm{tw}}_\Lambda &= K\widetilde{j}\widetilde{k}+\widetilde{j}-\frac{K+1}{4}\,,\ s^{\mathrm{tw}}_\Lambda=K(\widetilde{k}-\widetilde{j})+1\,;\\
s=4: h_\Lambda &= K(M-\widetilde{j})(M-\widetilde{k})+M-\widetilde{k}-\frac{K+2}{4}\,,\ s_\Lambda=K(\widetilde{k}-\widetilde{j})\,,\\
h^{\mathrm{tw}}_\Lambda &= K(M-\widetilde{j})(M-\widetilde{k})+M-\widetilde{j}-\frac{K+1}{4}\,,\ s^{\mathrm{tw}}_\Lambda=K(\widetilde{k}-\widetilde{j})-1\,.
\end{aligned}$$

Since the positive energy irreducible modules over the $N=4$ superconformal algebras are determined by their characteristic numbers, we obtain the following corollary.

Corollary 10.4. *We have the following isomorphisms of modules over the $N=4$ superconformal algebra of Neveu–Schwarz type:*

$$H(\Lambda^{(1)}_{k_1,k_2})\simeq H(\Lambda^{(4)}_{k_1+1,k_2})\,,\ H(\Lambda^{(3)}_{k_1,k_2})\simeq H(\Lambda^{(2)}_{k_1+1,k_2})\,,$$

and similar isomorphims for the $N=4$ superconformal algebra of Ramond type, replacing H by H^{tw} .

Now we turn to modular transformation formulae for the modified $N=4$ characters, defined by (10.16). Using Lemma 10.1(a) and (10.14), we obtain:

$$\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},\widetilde{k};\varepsilon'}\left(-\frac{1}{\tau},\frac{z}{\tau},t+\frac{z^2}{\tau}\right) \tag{10.17}$$
$$=-(-1)^{(1-2\varepsilon)(1-2\varepsilon')}\frac{i\tau}{M}\sum_{\widetilde{a},\widetilde{b}\in\varepsilon+\mathbb{Z}/M\mathbb{Z}}e^{-\frac{\pi im}{M}(\widetilde{a}-\widetilde{b})(\widetilde{j}-\widetilde{k})}\sin\frac{\pi m}{M}(\widetilde{a}+\widetilde{b})(\widetilde{j}+\widetilde{k})\ \widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon']}_{\widetilde{a},\widetilde{b};\varepsilon}(\tau,z,t)\,.$$

In order to obtain the final modular transformation formulae we need the following remarks.

Remark 10.5. (a) $\widetilde{\mathrm{ch}}^{N=4[M,m;0]}_{\widetilde{j},\widetilde{k};\varepsilon'}$ and $e^{\frac{\pi i m(\widetilde{j}-\widetilde{k})}{M}}\widetilde{\mathrm{ch}}^{N=4[M,m;\frac{1}{2}]}_{\widetilde{j},\widetilde{k};\varepsilon'}$ remain unchanged if we add to $\widetilde{j}$ and to $\widetilde{k}$ some integer multiples of M.

(b) $\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},\widetilde{k};\varepsilon'} = -\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{-\widetilde{k},-\widetilde{j};\varepsilon'}$.

(c) $\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},\widetilde{k};\varepsilon'} = 0$ if $\widetilde{j}+\widetilde{k}\in M\mathbb{Z}$.

(d) The $(\widetilde{a},\widetilde{b})$ coefficient and the $(M-\widetilde{b},M-\widetilde{a})$ coefficient in (10.17) are equal.

(e) $\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{0,\widetilde{k};0} = -e^{2\pi i m\varepsilon}\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{M-\widetilde{k},0;0}$ hence $\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{0,\widetilde{k};0}$ for $0<\widetilde{k}<M$ can be replaced in (10.17) by $\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},0;0}$, $0<\widetilde{j}<M$.

Claims (a) and (b) follow, by (10.11), from the corresponding properties of the functions $\widetilde{\Psi}^{[M,m-1;\varepsilon]}_{j,k;\varepsilon'}(\tau,z,z,t)$. For (c), letting $\widetilde{k}=-\widetilde{j}$ in (b), we obtain that $\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},-\widetilde{j};\varepsilon'}=0$. Hence, if $\widetilde{k}+\widetilde{j}=nM$ with $n\in\mathbb{Z}$, we have, by (a): $\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},\widetilde{k};\varepsilon'} = \widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},nM-\widetilde{j};\varepsilon'} = \widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},-\widetilde{j};\varepsilon'} = 0$. The proof of (d) is straightforward. Claim (e) is obtained by letting $\widetilde{j}=0$ in (b) and using (a).

Letting, as before, $\varepsilon=\frac{1}{2}$ in the Neveu–Schwarz case and $\varepsilon=0$ in the Ramond case, introduce the following subsets in the $\widetilde{j},\widetilde{k}$-plane:

$$\Omega^{N=4(M)}_{\varepsilon} = \{(\widetilde{j},\widetilde{k})\in\varepsilon+\mathbb{Z}\,|\ 0<\widetilde{j},\widetilde{j}+\widetilde{k}<M\,,\,0\le\widetilde{k}<M\}\,.$$

It follows from Remark 10.5 and Corollary 10.4 that these subsets parametrize the non-zero modified characters of irreducible modules over the Neveu–Schwarz type and the Ramond type $N=4$ superconformal algebras, obtained by the quantum Hamiltonian reduction from all the admissible $\widehat{A}_{1|1}$-modules $L(\Lambda^{(s)}_{k_1,k_2})$ and $L^{\mathrm{tw}}(\Lambda^{(s)}_{k_1,k_2})$. As a result, we can rewrite (10.17) as in the following theorem. The T-transformation formula in this theorem follows from Lemma 10.1(b) and (10.13).

Theorem 10.6. *Let M and m be positive coprime integers, such that $M\ge 2$ and $\gcd(M,2m)=1$ if $m>1$, and let $\varepsilon,\varepsilon'=0$ or $\frac{1}{2}$. Let $\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},\widetilde{k};\varepsilon'}(\tau,z,t)$, $(\widetilde{j},\widetilde{k})\in\Omega^{N=4(M)}_{\varepsilon'}$, be the modified characters and supercharacters of modules over the $N=4$ Neveu–Schwarz (resp. Ramond) type superconformal algebras if $\varepsilon=\frac{1}{2}$ (resp. $\varepsilon=0$), obtained by the quantum Hamiltonian reduction from level $K=-\frac{m}{M}$ admissible $\widehat{A}_{1|1}$- modules (resp. twisted modules). Then*

$$\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},\widetilde{k};\varepsilon'}\left(-\frac{1}{\tau},\frac{z}{\tau},t+\frac{z^2}{\tau}\right) = -(-1)^{(1-2\varepsilon)(1-2\varepsilon')}\frac{2i\tau}{M}$$

$$\times\sum_{(\widetilde{a},\widetilde{b})\in\Omega^{N=4(M)}_{\varepsilon}} e^{-\frac{\pi i m}{M}(\widetilde{a}-\widetilde{b})(\widetilde{j}-\widetilde{k})}\sin\frac{\pi m}{M}(\widetilde{a}+\widetilde{b})(\widetilde{j}+\widetilde{k})\ \widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon']}_{\widetilde{a},\widetilde{b};\varepsilon}(\tau,z,t)\,;$$

$$\widetilde{\mathrm{ch}}^{N=4[M,m;\varepsilon]}_{\widetilde{j},\widetilde{k};\varepsilon'}(\tau+1,z,t) = e^{-\frac{2\pi i m\widetilde{j}\widetilde{k}}{M}-\pi i\varepsilon'}\widetilde{\mathrm{ch}}^{N=4[M,m;|\varepsilon-\varepsilon'|]}_{\widetilde{j},\widetilde{k};\varepsilon'}(\tau,z,t)\,.$$

Remark 10.7. Note that $H(-m\Lambda_0)$, where $m\in\mathbb{Z}_{\ge1}$, is a non-zero $N=4$ module with $M=1$, whose modified character is zero. For example, the numerator of $\mathrm{ch}^{-}_{-\Lambda_0}$ is equal to D_1(normalization factor$\times\Phi^{[0]}$), which gives 0 when we apply the quantum Hamiltonian reduction. (It is easy to show that such a situation may occur only if $s=1$ or 3 and $\widetilde{j}+\widetilde{k}=M$.)

Remark 10.8. If $m = 1$, we obtain a family of positive energy $N = 4$ modules with central charge $6(\frac{1}{M} - 1)$, where M is a positive integer. It is easy to deduce from our calculations in Section 10 the following formulae for their characters:

$$\left(\overset{4}{R}{}^{(\varepsilon)}_{\varepsilon'} \widetilde{\mathrm{ch}}^{N=4[M,1;\varepsilon]}_{\widetilde{j},\widetilde{k};\varepsilon'} \right)(\tau, z, t) = -i(-1)^{2\varepsilon} e^{-\frac{2\pi i t}{M}} q^{-\frac{\widetilde{j}\widetilde{k}}{M}} e^{\frac{2\pi i}{M}(\widetilde{j}-\widetilde{k})z}$$
$$\times D_0 \left(\frac{\eta(M\tau)^3 \vartheta_{11}(M\tau, z_1 + z_2 + (\widetilde{j} - \widetilde{k})\tau)}{\vartheta_{11}(M\tau, z_1 + \widetilde{j}\tau + \varepsilon)\vartheta_{11}(M\tau, z_2 - \widetilde{k}\tau + \varepsilon)} \right) |_{z_1=z_2=z}.$$

In order to obtain the corresponding modified characters (hence a modular invariant family), one has to add to the RHS the expression:

$$i(-1)^{2\varepsilon} \frac{\widetilde{j} + \widetilde{k}}{M} e^{-\frac{2\pi i t}{M}} q^{-\frac{\widetilde{j}\widetilde{k}}{M}} e^{\frac{2\pi i}{M}(\widetilde{j}-\widetilde{k})z} \frac{\eta(M\tau)^3 \vartheta_{11}(M\tau, 2z + (\widetilde{j} - \widetilde{k})\tau)}{\vartheta_{11}(M\tau, z + \widetilde{j}\tau + \varepsilon)\vartheta_{11}(M\tau, z - \widetilde{k}\tau + \varepsilon)}.$$

A Appendix. A brief review of theta functions.

In this appendix we review some basic facts about theta functions (rather Jacobi forms), following the exposition in [K2], Chapter 13.

Let L be a positive definite integral lattice of rank ℓ with a positive definite symmetric bilinear form $(\,.\,|\,.\,)$. Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ be the complexification of L with the bilinear form $(\,.\,|\,.\,)$, extended from L by bilinearity. Let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ be an $\ell + 2$-dimensional vector space over $\mathbb{C}$ with the (non-degenerate) symmetric bilinear form $(\,.\,|\,.\,)$, extended from $\mathfrak{h}$ by letting $\mathfrak{h} \perp (\mathbb{C}K + \mathbb{C}d)$, $(K|K) = 0$, $(d|d) = 0$, $(K|d) = 1$. We shall identify $\widehat{\mathfrak{h}}$ with $\widehat{\mathfrak{h}}^*$, using this bilinear form, so that any $h \in \widehat{\mathfrak{h}}$ defines a linear function l_h on $\widehat{\mathfrak{h}}$ via $l_h(h_1) = (h|h_1)$.

Let $X = \{h \in \widehat{\mathfrak{h}} | \operatorname{Re}(K|h) > 0\}$. Define the following action of the additive group of the vector space $\mathfrak{h}$ on $\widehat{\mathfrak{h}}^*$ (cf. (2.11)):

$$t_\alpha(h) = h + (K|h)\alpha - ((\alpha|h) + \frac{(\alpha|\alpha)}{2}(K|h))K\,,\ \alpha \in \mathfrak{h}.$$

This action leaves the bilinear form $(\,.\,|\,.\,)$ on $\widehat{\mathfrak{h}}$ invariant and fixes K, hence leaves the domain X invariant.

A theta function (rather Jacobi form) of degree $k \in \mathbb{Z}_{\geq 0}$ is a holomorphic function F in the domain X, satisfying the following four properties:

(i) $F(t_\alpha(h)) = F(h)$;

(ii) $F(h + 2\pi i\alpha) = F(h)$ for $\alpha \in L$;

(iii) $F(h + aK) = e^{ka}F(v)$ for all $a \in \mathbb{C}$;

(iv) $DF = 0$, where D is the Laplace operator on $\widehat{\mathfrak{h}}$, associated to $(\,.\,|\,.\,)$.

Denote by Th_k, $k \in \mathbb{Z}_{\geq 0}$, the vector space over $\mathbb{C}$ of all theta functions of degree k.

Let $P_k = \{\lambda \in \widehat{\mathfrak{h}} | (\lambda|K) = k \text{ and } \bar{\lambda} \in L^*\}$, where $\bar{\lambda}$ stands for the projection of λ on $\mathfrak{h}$ and $L^* \subset \mathfrak{h}$ is the dual lattice of the lattice L. Given $\lambda \in P_k$, where k is a positive integer, let

$$\Theta_\lambda = e^{-\frac{(\lambda|\lambda)}{2k}K} \sum_{\alpha \in L} e^{t_\alpha(\lambda)}\,.$$

This series converges to a holomorphic function in the domain X, which is an example of a theta function of degree $k > 0$ (properties (i)–(iii) are obvious, and property (iv) holds since $De^\lambda = (\lambda|\lambda)e^\lambda$). Note that

$$\Theta_{\lambda + k\alpha + aK} = \Theta_\lambda \quad \text{for} \quad \alpha \in L,\ a \in \mathbb{C}\,.$$

Proposition A.1. *The set* $\{\Theta_\lambda | \lambda \in P_k \mod (kL + \mathbb{C}K)\}$ *is a* $\mathbb{C}$*-basis of* Th_k *if* $k > 0$*, and* $Th_0 = \mathbb{C}$*.*

Proof. See the proof of Proposition 13.3 and Lemma 13.2 in [K2]. □

Introduce coordinates (τ, z, t) on $\widehat{\mathfrak{h}}$ by (3.27), so that $X = \{(\tau, z, t) |\ \mathrm{Im}\ \tau > 0\}$ and $q := e^{2\pi i\tau} = e^{-K}$. In these coordinates we have the usual formula for a Jacobi form Θ_λ, $\lambda \in P_k$, of degree $k > 0$:

$$\Theta_\lambda(\tau, z, t) = e^{2\pi ikt} \sum_{\gamma \in L + \frac{\bar{\lambda}}{k}} q^{k\frac{(\gamma|\gamma)}{2}} e^{2\pi ik(\gamma|z)}\,. \tag{A.1}$$

Proposition A.2. *One has the following elliptic transformation formula of a Jacobi form* Θ_λ *of degree* k *for* $\alpha \in L^*$*:*

$$\Theta_\lambda(\tau, z + \alpha\tau, t) = q^{-\frac{k}{2}(\alpha|\alpha)} e^{-2\pi ik(\alpha|z)} \Theta_{\lambda + k\alpha}(\tau, z, t)\,.$$

Proof. It is straightforward.

□

Recall the action of the group $SL_2(\mathbb{R})$ in the domain X and the action of the corresponding metaplectic group on the space of meromorphic functions on X, given in Section 4.

Proposition A.3. *One has the following modular transformation formulae of a Jacobi form* Θ_λ *of degree* $k > 0$*:*

(a) $\Theta_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{(z|z)}{2\tau}\right) = (-i\tau)^{\frac{\ell}{2}} |L^*/kL|^{-\frac{1}{2}} \sum_{\mu \in P_k \bmod (kL + \mathbb{C}K)} e^{-\frac{2\pi i}{k}(\bar{\lambda}|\bar{\mu})} \Theta_\mu(\tau, z, t).$

(b) $\Theta_\lambda(\tau + 1, z, t) = e^{\frac{\pi i(\lambda|\lambda)}{k}} \Theta_\lambda(\tau, z, t),$
provided that $k(\alpha|\alpha) \in 2\mathbb{Z}$ *for* $\alpha \in L$ *(in particular, provided that the lattice* L *is even).*

(c) The space Th_k *is invariant with respect to the (right) action of the group* $SL_2(\mathbb{Z})$*, provided that* $k(\alpha|\alpha) \in 2\mathbb{Z}$ *for all* $\alpha \in L$*.*

Proof. The proof of (a) is based on the formula (see (4.16) and (4.17) for notation):

$$(DF)|_A = (c\tau + d)^2 D(F|_A), \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \tag{A.2}$$

The rest is straightforward. See [K2], Theorem 13.5. □

Remark A.4. Note that $P_k = \{kd + \bar\lambda + aK |\ \bar\lambda \in L^*,\ a \in \mathbb{C}\}$. Hence we may use a slightly different notation:

$$\Theta_\lambda = \Theta_{\bar\lambda,k}\,,$$

so that the basis $\{\Theta_\lambda |\ \lambda \in P_k \operatorname{mod} kL + \mathbb{C}K\}$ of Th_k is identified with the basis $\{\Theta_{\bar\lambda,k} |\ \bar\lambda \in L^*/kL\}$.

Example A.5. Let $L = \mathbb{Z}$ with the bilinear form $(a|b) = 2ab$, so that $L^* = \frac{1}{2}\mathbb{Z}$. Then for a positive integer k we have the following basis of Th_k $(\tau, z, t \in \mathbb{C}\,, \operatorname{Im}\tau > 0)$:

$$\Theta_{j,k}(\tau, z, t) = e^{2\pi i k t} \sum_{n \in \mathbb{Z} + \frac{j}{2k}} q^{kn^2} e^{2\pi i k n z}\,, \quad j \in \mathbb{Z}/2k\mathbb{Z}\,. \tag{A.3}$$

The elliptic transformation formula is as follows $(n \in \mathbb{Z})$:

$$\Theta_{j,k}(\tau, z + n\tau, t) = q^{-\frac{kn^2}{4}} e^{-\pi i k n z} \Theta_{j+kn,k}(\tau, z, t)\,. \tag{A.4}$$

The modular transformation formulae are:

$$\Theta_{j,k}\left(-\frac{1}{\tau}\,, \frac{z}{\tau}\,, t - \frac{z^2}{2\tau}\right) = \left(\frac{-i\tau}{2k}\right)^{\frac{1}{2}} \sum_{j' \in \mathbb{Z}/2k\mathbb{Z}} e^{-\frac{\pi i j j'}{k}} \Theta_{j',k}(\tau, z, t)\,, \tag{A.5}$$

$$\Theta_{j,k}(\tau + 1, z, t) = e^{\frac{\pi i j^2}{2k}} \Theta_{j,k}(\tau, z, t). \tag{A.6}$$

Especially important are the celebrated four Jacobi theta functions of degree two (we put $t = 0$ here) [M]:

$$\vartheta_{00} = \Theta_{2,2} + \Theta_{0,2}\,,\ \vartheta_{01} = -\Theta_{2,2} + \Theta_{0,2}\,,\ \vartheta_{10} = \Theta_{1,2} + \Theta_{-1,2}\,,\ \vartheta_{11} = i\Theta_{1,2} - i\Theta_{-1,2}\,.$$

Due to the Jacobi triple product identity, the following infinite products can be expressed in terms of the four Jacobi theta functions and the η-function $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^\infty (1 - q^n)$:

$$\prod_{n=1}^{\infty} (1 + e^{2\pi i z} q^{n-\frac{1}{2}})(1 + e^{-2\pi i z} q^{n-\frac{1}{2}}) = q^{\frac{1}{24}} \frac{\vartheta_{00}(\tau, z)}{\eta(\tau)},$$

$$\prod_{n=1}^{\infty} (1 - e^{2\pi i z} q^{n-\frac{1}{2}})(1 - e^{-2\pi i z} q^{n-\frac{1}{2}}) = q^{\frac{1}{24}} \frac{\vartheta_{01}(\tau, z)}{\eta(\tau)},$$

$$\prod_{n=1}^{\infty} (1 + e^{-2\pi i z} q^{n})(1 + e^{2\pi i z} q^{n-1}) = q^{-\frac{1}{12}} e^{\pi i z} \frac{\vartheta_{10}(\tau, z)}{\eta(\tau)}, \tag{A.7}$$

$$\prod_{n=1}^{\infty} (1 - e^{-2\pi i z} q^{n})(1 - e^{2\pi i z} q^{n-1}) = i q^{-\frac{1}{12}} e^{\pi i z} \frac{\vartheta_{11}(\tau, z)}{\eta(\tau)}\,.$$

Proposition A.6. *[M] For $a, b = 0$ or 1 and $n \in \mathbb{Z}$ one has:*

$$\vartheta_{ab}(\tau, z + n\tau) = (-1)^{bn} q^{-\frac{n^2}{2}} e^{-2\pi i n z} \vartheta_{ab}(\tau, z)\,.$$

Proof. It follows from (A.4). □

Proposition A.7. *([M], p. 36) For $a, b = 0$ or 1 one has:*

$$\begin{aligned}
\vartheta_{ab}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= (-i)^{ab}(-i\tau)^{\frac{1}{2}} e^{\frac{\pi i z^2}{\tau}} \vartheta_{ba}(\tau, z)\,; \\
\vartheta_{0a}(\tau + 1, z) &= \vartheta_{0b}(\tau, z)\,, \text{ where } a \neq b\,; \\
\vartheta_{1a}(\tau + 1, z) &= e^{\frac{\pi i}{4}} \vartheta_{1a}(\tau, z)\,.
\end{aligned}$$

Proof. It follows from (A.5), (A.6). □

References

[Ap] M.P. Appell, Sur le fonctions doublement periodique de troisieme espese, *Annals Sci. l'Ecole norm. Sup., 3e serie* **1**, p. 135, **2**, p.9, **3**, p.9 (1884-1886).

[A1] T. Arakawa, Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture, *Duke Math. J.* **130** (2005), 435–478.

[A2] T. Arakawa, Representation theory of W-algebras, *Invent. Math.* **169** (2007), 219–320.

[BF] K. Bringmann, A.Folsom, Almost harmonic Maass forms and Kac-Wakimoto characters, arXiv:1112.4726.

[BO] K. Bringmann, K. Ono, Some characters of Kac and Wakimoto and modular functions, *Math. Annalen* **345** (2009), 547–558.

[DMZ] A. Dabholkar, S.Murthy, D. Zagier, Quantum black holes, wall crossing, and mock modular forms, arXiv:1208.4074.

[F] A.Folsom, Kac-Wakimoto characters and universal mock theta-functions, *Trans. Amer. Math. Soc.* **363** (2011), 439-455.

[FKW] E.V. Frenkel, V.G. Kac, M. Wakimoto, Characters and fusion rules for W-algebras via quantized Drinfeld-Sokolov reduction, *Comm. Math. Phys.* **147** (1992), 295-328.

[GK] M. Gorelik, V.G. Kac, On admissible modules over affine Lie superalgebras (in preparation).

[K1] V.G. Kac, Lie superalgebras, *Advances in Math.* **26**, No. 1 (1977), 8–96.

[K2] V.G. Kac, Infinite-dimensional Lie algebras, Third edition, Cambridge University Press, 1990.

[KP] V.G. Kac, D.H. Peterson, Infinite-dimensional Lie algebras, theta functions and modular forms, *Adv. Math.* **53** (1984), 125–264.

[KRW] V.G. Kac, S.-S. Roan, M. Wakimoto, Quantum reduction for affine superalgebras, *Comm. Math. Phys.* **241** (2003), 307–342.

[KW1] V.G. Kac, M. Wakimoto, Modular invariant representations of infinite-dimensional Lie algebras and superalgebras, *Proc. Nat'l. Acad. Sci. USA* **85** (1988), 4956–4960.

[KW2] V.G. Kac, M. Wakimoto, Classification of modular invariant representations of affine algebras, Advanced Ser. Math. Phys. **7**, World Sci., 1989, pp 138-177.

[KW3] V.G. Kac, M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory, Progress in Math. **123**, Birkhäuser, 1994, 415-456.

[KW4] V.G. Kac, M. Wakimoto, Integrable highest weight modules over affine superalgebras and Appell's function, *Commun. Math. Phys.* **215** (2001), 631-682.

[KW5] V.G. Kac, M. Wakimoto, Quantum reduction and representation theory of superconformal algebras. *Adv. Math.* **185** (2004), 400-458. Corrigendum *Adv. Math.* **193** (2005), 453-455.

[KW6] V.G. Kac, M. Wakimoto, Quantum reduction in the twisted case, Progress in Math. **237**, Birkhäuser, 2005, 85-126.

[L] M. Lerch, Bemerkungen zur Theorie der elliptichen fonctionen, *Jahrbuch uber die fortschritte der mathematik* **24**(1892), 442-445.

[M] D. Mumford, Tata lectures on theta I, Progress in Mathematics **28**, Birkhäuser, 1983.

[S] V. Serganova, Kac-Moody superalgebras and integrability, in Developments and trends in infinite-dimensional Lie theory, Progress in Math. **288**, Birkhäuser, 2011, 169-218.

[W] M. Wakimoto, Fusion rules for $N = 2$ superconformal modules, arXiv:hep-th/9807144

[Z] S.P. Zwegers, Mock theta functions. arXiv:0807.4834

[Z1] S.P. Zwegers, private communication.