LONG TIME BEHAVIOUR OF SOLUTIONS TO THE MKDV

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ABSTRACT. In this paper we consider the long time behaviour of solutions to the modified Korteweg-de Vries equation on $\mathbb{R}$. For sufficiently small, smooth, decaying data we prove global existence and derive modified asymptotics without relying on complete integrability. We also consider the asymptotic completeness problem. Our result uses the method of testing by wave packets, developed in the work of Ifrim and Tataru on the $1d$ cubic nonlinear Schrödinger and $2d$ water wave equations.

1. Introduction

In this article we consider the long-time behaviour of solutions to the mKdV equation

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = \sigma(u^3)_x, \quad u\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\sigma = \pm 1$ and u_0 is sufficiently small, smooth and decaying data.

The Cauchy problem for (1.1) has been studied extensively. For a summary of known results we refer the reader to [28]. In particular, the mKdV is both locally well-posed [22–24] and globally well-posed [3, 11, 26] in H^s for $s \geq \frac{1}{4}$. Below $s = \frac{1}{4}$ the solution map fails to be uniformly continuous [1, 25] although weaker forms of well-posedness hold [2]. Local well-posedness in non-L^2-based spaces closer to the critical scaling has also been obtained [9, 10].

As the mKdV is completely integrable, global existence and the asymptotic behaviour can be studied using inverse scattering techniques such as in Deift and Zhou [5] and references therein. A natural question to ask is whether it is possible to study the asymptotic behaviour of the mKdV without relying on the completely integrable structure. Hayashi and Naumkin [12, 15] were able to prove global existence and derive modified asymptotics in a neighbourhood of a self-similar solution without relying on the complete integrability, with errors bounded in L^p for $4 < p \leq \infty$. Our result presents a significant improvement by proving modified scattering in $L^2 \cap L^\infty$. We also derive the leading asymptotic in the oscillatory region and use slightly weaker assumptions on the initial data.

In the related case of the cubic nonlinear Schrödinger equation on $\mathbb{R}$, modified asymptotics have been proved without inverse scattering techniques using both spatial methods [29] and Fourier methods [13, 21]. In this paper we use the *method of testing by wave packets*, based on the work of Ifrim and Tataru on the $1d$ cubic NLS [18] and $2d$ water wave [19, 20] equations. This method essentially interpolates between the spatial side and Fourier side approaches, by localising in both space and frequency at the scale of the uncertainty principle.

In order to give a more complete picture of the asymptotic behaviour of solutions, we consider the reciprocal problem: given a function with a suitable asymptotic profile, can we construct a solution to (1.1) matching this asymptotic behaviour as $t \to +\infty$? Hayashi-Naumkin [16] showed that under strong conditions on the data, including that it has mean zero, it is possible to find such a solution. Our result holds for a much larger class of data, including those with non-trivial mean. In the case of the gKdV, where solutions scatter to free solutions, asymptotic completeness was established by Cote [4] and refined by Farah-Pastor [8]. Similar results have also been obtained for the cubic NLS, see for example [18] and references therein.

As in the case of the NLS [6, 13, 18, 29] our result is also true for short-range perturbations of the form

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = (\sigma u^3 + F(u))_x, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where $F \in C^2(\mathbb{R})$ satisfies

$$|F(u)| = O(|u|^p), \qquad |u| \to 0, \quad p > 3, \tag{1.3}$$

with some minor modifications if $p \in (3, \frac{7}{2})$. For completeness we briefly outline these modifications in Appendix A.

While preparing this paper we learned that some similar results have been obtained by Germain, Pusateri and Rousset.

For $t > 0$, the solution to the linear KdV equation

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = 0, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

is given by

$$u(t,x) = t^{-\frac{1}{3}} \int \mathrm{Ai}(t^{-\frac{1}{3}}(x-y))u_0(y)\, dy, \tag{1.5}$$

where the Airy function is defined by the oscillatory integral

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int e^{i(\frac{1}{3}\xi^3 + x\xi)}\, d\xi.$$

The linear KdV has Hamiltonian $h(\xi) = -\frac{1}{3}\xi^3$ and hence the Hamiltonian flow associated to the linear KdV operator is

$$(x, \xi) \mapsto (x - t\xi^2, \xi). \tag{1.6}$$

In particular, given a speed $v \geq 0$, we expect wave packets initially localised in phase space near $(x, \xi) = (0, \pm\sqrt{v})$ to travel along the ray $\Gamma_v = \{x + tv = 0\}$. As all wave packets travel towards $x = -\infty$, we expect the solution to (1.4) to decay rapidly as $t^{-\frac{1}{3}}x \to +\infty$ and oscillate as $t^{-\frac{1}{3}}x \to -\infty$.

For Schwartz initial data, we can roughly divide the asymptotic behaviour of solutions to the linear KdV (1.4) into three distinct regions as $t \to +\infty$. In the decaying region $t^{-\frac{1}{3}}x \to +\infty$,

$$u(t,x) = O(t^{-\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{-N}).$$

In the self-similar region $t^{-\frac{1}{3}}|x| \lesssim 1$,

$$u(t,x) = t^{-\frac{1}{3}}\, \mathrm{Ai}(t^{-\frac{1}{3}}x) \int u_0\, dx + O(t^{-\frac{2}{3}}).$$

In the oscillatory region $t^{-\frac{1}{3}}x \to -\infty$,

$$u(t,x) = \pi^{-\frac{1}{2}} t^{-\frac{1}{3}} (t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}} \operatorname{Re}\left(e^{i\phi}\hat{u}_0(t^{-\frac{1}{2}}|x|^{\frac{1}{2}})\right) + O(t^{-\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{-\frac{7}{4}}),$$

where the phase is given by

$$\phi(t,x) = -\frac{2}{3}t^{-\frac{1}{2}}|x|^{\frac{3}{2}} + \frac{\pi}{4}. \tag{1.7}$$

From (1.5), we observe that if our initial data satisfies $\|u_0\|_{H^{0,1}} \leq \epsilon$, the linear solution satisfies the dispersive estimates

$$|u(t,x)| \lesssim \epsilon t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}, \qquad |u_x(t,x)| \lesssim \epsilon t^{-\frac{2}{3}} \langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}, \tag{1.8}$$

and in particular,

$$|uu_x| \lesssim \epsilon^2 t^{-1}.$$

We expect solutions to the nonlinear equation (1.1) to behave like solutions to the linear equation for sufficiently short times. So, if our initial data is of size $\epsilon > 0$ in a suitable norm, we expect it to satisfy (1.8), at least for sufficiently small $T > 1$. In particular, if $\|\cdot\|$ is a Sobolev-type norm in x, then

$$\|u(t)\| \lesssim \|u(1)\| + \epsilon^2 \int_1^t \|u(s)\| \frac{ds}{s}.$$

The integral is bounded by $\sup_t \|u(t)\|$ up to time $T \approx e^{\epsilon^{-2}}$ and hence we only expect linear behaviour up to this time. So while we may still have a global solution, we expect the asymptotic behaviour of the solution to differ from the linear solution by a logarithmic difference in t.

Our first result is that this is indeed the case.

Theorem 1.1. *There exists $\epsilon > 0$ such that for all $u_0 \in H^{1,1}$ satisfying*

$$\|u_0\|_{H^{1,1}} \leq \epsilon, \tag{1.9}$$

there exists a unique global solution u to (1.1), such that $S(-t)u \in C(\mathbb{R}; H^{1,1})$, satisfying the estimates for $t \geq 1$ and a.e. $x \in \mathbb{R}$

$$|u(t,x)| \lesssim \epsilon t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}, \qquad |u_x(t,x)| \lesssim \epsilon t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}. \tag{1.10}$$

Further, we have the following asymptotics as $t \to +\infty$.

In the decaying region $\Omega_\rho^+ = \{x > 0 : t^{-\frac{1}{3}}x \gtrsim t^{2\rho}\}$ we have the estimates

$$\|t^{\frac{1}{3}}(t^{-\frac{1}{3}}x)^{\frac{3}{4}}u\|_{L^\infty(\Omega_\rho^+)} \lesssim \epsilon, \qquad \|t^{\frac{1}{6}}(t^{-\frac{1}{3}}x)u\|_{L^2(\Omega_\rho^+)} \lesssim \epsilon. \tag{1.11}$$

In the self-similar region $\Omega_\rho^0 = \{x \in \mathbb{R} : t^{-\frac{1}{3}}|x| \lesssim t^{2\rho}\}$, where $0 \leq \rho \leq \frac{1}{3}(\frac{1}{6} - C\epsilon^2)$, there exists a solution $Q(y)$ to the Painlevé II equation

$$yQ - Q_{yy} + 3\sigma Q^3 = 0, \tag{1.12}$$

satisfying

$$|Q(y)| \lesssim \epsilon, \qquad \text{p.v.}\int Q = \int u_0, \tag{1.13}$$

and we have the estimates

$$\|u - t^{-\frac{1}{3}}Q(t^{-\frac{1}{3}}x)\|_{L^\infty(\Omega_\rho^0)} \lesssim \epsilon t^{-\frac{1}{2}(\frac{5}{6} - C\epsilon^2)}, \quad \|u - t^{-\frac{1}{3}}Q(t^{-\frac{1}{3}}x)\|_{L^2(\Omega_\rho^0)} \lesssim \epsilon t^{-\frac{2}{3}(\frac{5}{12} - C\epsilon^2)}. \tag{1.14}$$

In the oscillatory region $\Omega_\rho^- = \{x < 0 : t^{-\frac{1}{3}}|x| \gtrsim t^{2\rho}\}$, there exists a unique (complex-valued) function W defined on $(0,\infty)$ such that for $C > 0$ sufficiently large,

$$\|W\|_{H^{1-C\epsilon^2,1}\cap L^\infty(0,\infty)} \lesssim \epsilon, \tag{1.15}$$

and

$$u(t,x) = \pi^{-\frac{1}{2}} t^{-\frac{1}{3}} (t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}} \operatorname{Re}\left(e^{i\phi + \frac{3i\sigma}{4\pi}|W(t^{-\frac{1}{2}}|x|^{\frac{1}{2}})|^2 \log t} W(t^{-\frac{1}{2}}|x|^{\frac{1}{2}})\right) + \mathbf{err}_x, \tag{1.16}$$

where the error satisfies the estimates

$$\|t^{\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{\frac{3}{8}}\mathbf{err}_x\|_{L^\infty(\Omega_\rho^-)} \lesssim \epsilon, \qquad \|t^{\frac{1}{6}}(t^{-\frac{1}{3}}|x|)^{\frac{1}{4}}\mathbf{err}_x\|_{L^2(\Omega_\rho^-)} \lesssim \epsilon. \tag{1.17}$$

In the corresponding frequency region $\widehat{\Omega}_\rho^- = \{\xi > 0 : t^{\frac{1}{3}}\xi \gtrsim t^\rho\}$ we have

$$\hat{u}(t,\xi) = e^{\frac{1}{3}it\xi^3 + \frac{3i\sigma}{4\pi}|W(\xi)|^2 \log t} W(\xi) + \mathbf{err}_\xi, \tag{1.18}$$

where the error satisfies

$$\|(t^{\frac{1}{3}}\xi)^{\frac{1}{4}}\mathbf{err}_\xi\|_{L^\infty(\widehat{\Omega}_\rho^-)} \lesssim \epsilon, \qquad \|t^{\frac{1}{6}}(t^{\frac{1}{3}}\xi)^{\frac{1}{2}}\mathbf{err}_\xi\|_{L^2(\widehat{\Omega}_\rho^-)} \lesssim \epsilon. \tag{1.19}$$

Remark 1.2. As (1.1) has time reversal symmetry given by

$$u(t,x) \mapsto u(-t,-x),$$

we get corresponding asymptotics as $t \to -\infty$.

The loss of regularity of W in Theorem 1.1 can be compared to the similar results [18, 20]. Indeed, as the direct scattering problem for the cubic NLS and mKdV is the same, we expect the correspondence between the W of Theorem 1.1 and u_0 to be the same as in Theorem 1 of [18]. From the inverse scattering theory, see for example [5, 6], we expect this loss of regularity to be logarithmic in nature.

For the asymptotic completeness a key object of study will be the one-parameter family of solutions to the Painlevé II equation (1.12). We first state the following result about the asymptotic behaviour of these solutions.

Theorem 1.3 (Deift-Zhou [7])**.** *Given $W \in \mathbb{R}$ there exists a unique solution $Q(y;W)$ to the Painlevé II equation* (1.12) *such that*

$$\begin{aligned} Q(y;W) &= \pi^{-\frac{1}{2}}|y|^{-\frac{1}{4}} \operatorname{Re}\left(e^{i\phi + \frac{3i\sigma}{4\pi}W^2 \log|y|^{\frac{3}{2}} + i\sigma\theta(W^2)} W\right) \\ &\qquad + O(|y|^{-\frac{5}{4}}\log|y|), \quad y \to -\infty, \\ Q(y;W) &= q_\sigma(W)\operatorname{Ai}(y) + O(|y|^{-\frac{1}{4}} e^{-\frac{4}{3}y^{\frac{3}{2}}}), \quad y \to +\infty, \end{aligned} \tag{1.20}$$

where

$$\theta(W^2) = \tfrac{9\log 2}{4\pi}W^2 - \arg\Gamma\left(\tfrac{3i}{4\pi}W^2\right) - \tfrac{\pi}{2},$$
$$q_\sigma(W) = \operatorname{sgn} W \left(\tfrac{2\sigma}{3}\left(1 - e^{-\frac{3\sigma}{2}W^2}\right)\right)^{\frac{1}{2}}.$$

For a real-valued function W defined on the half-line $(0,\infty)$ we define

$$u_{\text{asymp}}(t,x) = t^{-\frac{1}{3}} Q(t^{-\frac{1}{3}}x; W(t^{-\frac{1}{2}} x_-^{\frac{1}{2}})). \tag{1.21}$$

We observe that from Theorem 1.3, this has an asymptotic profile matching that of Theorem 1.1. We then look for a solution to the problem

$$\begin{cases} u_t + \frac{1}{3}u_{xxx} = \sigma(u^3)_x, \\ \lim_{t\to+\infty} (u(t) - u_{\text{asymp}}(t)) = 0. \end{cases} \tag{1.22}$$

We define the space Y with norm

$$\|W\|_Y = \|W\|_{H^{1+C\epsilon^2,1}(0,\infty)} + \||D|^{C\epsilon^2} W\|_{\dot{H}^{0,1}(0,\infty)} + \|\log\langle z\rangle W\|_{L^2(0,\infty)} \tag{1.23}$$

and then have the following asymptotic completeness result.

Theorem 1.4. *There exist $\epsilon, C > 0$ such that for all $W \in Y$ satisfying*

$$\|W\|_Y \leq \epsilon, \tag{1.24}$$

there exists a unique solution to (1.22) *such that $S(-t)u \in C(\mathbb{R}; H^{1,1})$.*

Remark 1.5. Similar to Theorem 1.1 we have a loss of regularity between W and u. In order to close the argument we require an extra $C\epsilon^2$ derivatives for both W_z and zW. As there is a logarithmic difference between the W of Theorems 1.1 and 1.4 and we also need to assume logarithmic decay of W_z in L^2.

We conclude this section by giving an outline of the proof of Theorems 1.1 and 1.4. In order to control the spatial localisation of solutions we look to control the "vector field"

$$Lu = S(t)xS(-t)u = (x - t\partial_x^2)u. \tag{1.25}$$

However, L does not behave well with respect to the nonlinearity, so as in [12,14,15] we instead work with

$$\Lambda u = \partial_x^{-1}(3t\partial_t + x\partial_x + 1)u. \tag{1.26}$$

We observe that if u is a solution to (1.1) then

$$\Lambda u = Lu + 3t\sigma u^3. \tag{1.27}$$

As $3t\partial_t + x\partial_x + 1$ generates the mKdV scaling symmetry

$$u(t,x) \mapsto \lambda u(\lambda^3 t, \lambda x), \qquad u_0(x) \mapsto \lambda u_0(\lambda x),$$

the function $v = \Lambda u$ satisfies the linearised equation

$$\begin{cases} v_t + \frac{1}{3}v_{xxx} = 3\sigma u^2 v_x, \\ v(0) = xu_0. \end{cases} \tag{1.28}$$

For a large fixed constant $M_0 \geq 2$ we define the space X with norm

$$\|u\|_X = \|u\|_{H^1} + \langle t\rangle^{-\delta} \|\Lambda u\|_{L^2}, \tag{1.29}$$

where

$$\delta = 3M_0^2\epsilon^2. \tag{1.30}$$

We then have the following local well-posedness result that can be proved as in Kenig-Ponce-Vega [23, 24].

Theorem 1.6. *If $u_0 \in H^{1,1}$ satisfies (1.9) then there exists $T = T(\epsilon) \to \infty$ as $\epsilon \to 0$ and a unique solution $u \in C([0,T];X)$ such that*

$$\sup_{t\in[0,T]} \|u(t)\|_X \leq 2\epsilon. \tag{1.31}$$

Further, the solution map $u_0 \mapsto u(t)$ is locally Lipschitz.

Remark 1.7. From Sobolev embedding

$$\|\Lambda u\|_{L^2} - C(t, \|u\|_{H^1}) \lesssim \|Lu\|_{L^2} \lesssim \|\Lambda u\|_{L^2} + C(t, \|u\|_{H^1}). \tag{1.32}$$

In particular, on any bounded subset of X we may define an equivalent norm

$$\|u\|_{\widetilde{X}} = \|u\|_{H^1} + \langle t\rangle^{-\frac{1}{6}}\|Lu\|_{L^2}. \tag{1.33}$$

In Section 3 we prove Theorem 1.1. Using the local well-posedness result Theorem 1.6, for $\epsilon > 0$ sufficiently small we can find $T > 1$ and a unique solution $u \in C([0,T];X)$ to (1.1). We then make the bootstrap assumption that u satisfies the linear pointwise estimate

$$\sup_{t\in[1,T]} \left(\|t^{\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}}u\|_{L^\infty} + \|t^{\frac{2}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}u_x\|_{L^\infty}\right) \leq M_0\epsilon \tag{1.34}$$

and show that under this assumption, for $\epsilon > 0$ sufficiently small, we have the energy estimate

$$\sup_{t\in[0,T]} \|u\|_X \lesssim \epsilon, \tag{1.35}$$

with a constant independent of M_0, T. To complete the proof of global existence we need to close the bootstrap estimate (1.34).

To control the pointwise behaviour of solutions we use the method of testing by wave packets [18–20]. A wave packet is an approximate solution localised in both space and frequency on the scale of the uncertainty principle. We define a wave packet Ψ_v adapted to the ray Γ_v and measure u along Γ_v by considering

$$\gamma(t,v) = \int u(t,x)\overline{\Psi}_v(t,x)\,dx. \tag{1.36}$$

The key innovation of Ifrim-Tataru is to choose the wave packet to be localised at a t-dependent scale. For the KdV, a wave packet adapted to the ray Γ_v will be localised at scale $\lambda = t^{-\frac{1}{3}}\langle t^{\frac{2}{3}}v\rangle^{-\frac{1}{4}}$ in frequency and at scale λ^{-1} in space. However, as we only make use of the wave packets in the region

$$\mathbf{\Omega}_0^- = \{v > 0 : t^{\frac{2}{3}}v \gtrsim 1\} \tag{1.37}$$

corresponding to the region Ω_0^-, we instead define

$$\lambda = t^{-\frac{1}{2}} v^{-\frac{1}{4}}. \tag{1.38}$$

We then reduce closing the bootstrap estimate (1.34) to proving global bounds for γ. To derive these bounds, we show that γ satisfies an ODE of the form

$$\dot\gamma(t,v) = 3i\sigma t^{-1}|\gamma(t,v)|^2\gamma(t,v) + \text{error}.$$

The logarithmic correction to the phase then arises as a consequence of solving this ODE.

In Section 4 we prove Theorem 1.4. The key idea here is to replace u_{asymp} by a regularised version u_{app}, where the regularisation is on the scale of the wave packets. The result then follows by applying a contraction mapping argument to the resulting equation for the difference $v = u - u_{\text{app}}$ in a suitable space.

2. Notation and definitions

We recall that solutions to (1.1) have conserved quantities

$$E_0(t) = \int u\,dx, \tag{2.1}$$

$$E_1(t) = \int u^2\,dx, \tag{2.2}$$

$$E_2(t) = \int u_x^2 + \tfrac{3}{2}\sigma u^4\,dx. \tag{2.3}$$

We note that as (1.1) is completely integrable there are an infinite number of higher order conserved quantities.

We define the Fourier transform of a Schwartz function $f \in \mathcal{S}(\mathbb{R})$ to be

$$\hat f(\xi) = \mathcal{F}f(\xi) = \int f(x)e^{-ix\xi}\,dx, \tag{2.4}$$

with inverse

$$\check f(x) = \mathcal{F}^{-1}f(x) = \frac{1}{2\pi}\int f(\xi)e^{ix\xi}\,d\xi. \tag{2.5}$$

The linear KdV propagator $S(t)$ can then be written as

$$S(t)f = \frac{1}{2\pi}\int \hat f(\xi)e^{i(\frac{1}{3}t\xi^3 + x\xi)}\,d\xi. \tag{2.6}$$

Given a real-valued, even function $\psi \in C_0^\infty$ satisfying $0 \le \psi \le 1$, supported on $(-2,2)$ and identically 1 on $[-1,1]$, we define $\varphi_0(\xi) = \psi(\xi)$ and for $j \ge 1$, $\varphi_j(\xi) = \psi(2^{-j}\xi) - \psi(2^{-(j-1)}\xi)$. We then have the Littlewood-Paley projections

$$P_j u = \varphi_j(D)u. \tag{2.7}$$

We also define the projections to positive and negative frequencies

$$P_\pm u = \mathbf{1}_{(0,\infty)}(\pm D)u. \tag{2.8}$$

We recall the Bernstein inequality, for $1 \le p \le q \le \infty$,

$$\|P_j u\|_{L^q} \lesssim 2^{j(\frac{1}{p}-\frac{1}{q})}\|P_j u\|_{L^p}, \tag{2.9}$$

and the Sobolev estimate

$$\|u\|_{L^\infty} \lesssim \|u\|_{L^2}^{\frac{1}{2}}\|u_x\|_{L^2}^{\frac{1}{2}}. \tag{2.10}$$

We define the weighted Sobolev norms $H^{k,j}$ by

$$\|u\|_{H^{k,j}} = \|\langle\xi\rangle^k \hat{u}\|_{L^2} + \|\langle x\rangle^j u\|_{L^2}, \tag{2.11}$$

and denote the homogeneous norms by $\dot{H}^{k,j}$.

We call a pair of indices (p,q) admissible if

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}, \quad 4 \leq p \leq \infty, \quad 2 \leq q \leq \infty. \tag{2.12}$$

If we define

$$\Phi f = \int_t^\infty S(t-s) f(s)\, ds, \tag{2.13}$$

then for admissible pairs $(p_1,q_1),(p_2,q_2)$ and any decomposition $f = f_1 + f_2$ we have the estimate [23, 24]

$$\|\Phi f\|_{L_t^\infty L_x^2} + \||D|^{1-\frac{5}{p}} \Phi f\|_{L_x^{p_1} L_t^{q_1}} \lesssim \|f_1\|_{L_t^1 L_x^2} + \||D|^{\frac{5}{p_2}-1} f_2\|_{L_x^{p_2'} L_t^{q_2'}}. \tag{2.14}$$

For (p,q) admissible we have the estimate [2]

$$\|u\|_{L_t^\infty L_x^2} + \||D|^{1-\frac{5}{p}} u\|_{L_x^p L_t^q} \lesssim \|u\|_{U_S^2}, \tag{2.15}$$

where the space U_S^2 is defined as in [27]. We also have the embedding (see for example [27]), for $1 < p < q < \infty$

$$V_{rc}^p \subset U^q, \tag{2.16}$$

where V_{rc}^p is the space of right-continuous functions of bounded p-variation. From the definition of V^p we have the embedding

$$\dot{W}_t^{1,1} L_x^2 \subset V^p. \tag{2.17}$$

3. Modified Scattering

3.1. **Energy estimates.** We first derive energy estimates for u under the the bootstrap assumption (1.34). Our argument is similar to Hayashi-Naumkin [12, 14, 15].

Proposition 3.1. *For $\epsilon > 0$ chosen sufficiently small and $t \in [0,T]$ we have the energy estimates*

$$\|u\|_{H^1} \lesssim \epsilon, \tag{3.1}$$

$$\|\Lambda u\|_{L^2} \lesssim \epsilon \langle t\rangle^\delta, \tag{3.2}$$

where δ is defined as in (1.30) and the constants are independent of M_0, T.

Proof. From conservation of mass (2.2), we have

$$\|u\|_{L^2} = \|u_0\|_{L^2} \leq \epsilon.$$

From the Sobolev estimate (2.10), for any $\theta > 0$, we have

$$\int u^4\, dx \lesssim \|u\|_{L^2}^3 \|u_x\|_{L^2} \lesssim \theta^{-1}\|u\|_{L^2}^6 + \theta\|u_x\|_{L^2}^2 \lesssim \theta^{-1}\epsilon^4\|u\|_{L^2}^2 + \theta\|u_x\|_{L^2}^2.$$

Defining $E_j(t)$ for $j = 1, 2$ as in (2.2) and (2.3), for $\theta > 0$ chosen sufficiently small we then have

$$\|u(t)\|_{H^1} \sim E_1(t) + E_2(t) = E_1(0) + E_2(0) \sim \|u_0\|_{H^1} \leq \epsilon, \tag{3.3}$$

where the constants are independent of M_0.

If $v = \Lambda u$, from (1.31) we have

$$\sup_{t\in[0,1]} \|v(t)\|_{L^2_x} \lesssim \epsilon.$$

For $t \geq 1$, as a consequence of (1.31) and (1.34) we have the estimate

$$\|uu_x\|_{L^\infty} \lesssim M_0^2\epsilon^2 t^{-1}, \quad t \geq 1, \tag{3.4}$$

and we may then use the equation (1.28) to get

$$\begin{aligned}\partial_t \|v\|^2_{L^2} &= 6\sigma \int u^2 v_x v\, dx \\ &= -6\sigma \int uu_x v^2\, dx \\ &\leq 6M_0\epsilon^2 \|v\|^2_{L^2}.\end{aligned}$$

Using Gronwall's inequality for $t \geq 1$, we have (3.2).

□

Remark 3.2. We note that from (1.31) and (1.34) we have the estimate

$$\|u\|_{L^p} \lesssim M_0\epsilon \langle t\rangle^{\frac{1}{3p}-\frac{1}{3}}, \quad p \in (4,\infty], \tag{3.5}$$

and hence

$$\|Lu\|_{L^2} \lesssim \|\Lambda u\|_{L^2} + 3t\|u\|^3_{L^6} \lesssim \epsilon\langle t\rangle^\delta + (M_0\epsilon)^3\langle t\rangle^{\frac{1}{6}}.$$

So, provided $\epsilon > 0$ is chosen sufficiently small that $\delta \in (0, \frac{1}{6}]$, we have the estimate

$$\sup_{t\in[0,T]} \|u\|_{\tilde{X}} \lesssim \epsilon, \tag{3.6}$$

with a constant independent of M_0, T.

We also observe that from (1.34), for $t \geq 1$ we have the improvement

$$\|Lu\|_{L^2(\Omega^-_{1/3-2\delta})} \lesssim \epsilon t^\delta. \tag{3.7}$$

3.2. **Construction of wave packets.** Let $\chi \in C_0^\infty(\mathbb{R})$ be a real-valued function, supported on a neighbourhood of the origin of size ~ 1 and localised in frequency near 0 at scale ~ 1 satsifying $\int \chi = 1$. We then define our wave packet

$$\Psi_v(t,x) = \chi(\lambda(x+tv))e^{i\phi}, \tag{3.8}$$

where ϕ, λ are defined as in (1.7), (1.38). We note that if $v \in \mathbf{\Omega}_0^-$, then Ψ_v is supported on Ω_0^-. The following lemma shows that Ψ_v is also a good approximation to a free solution in Fourier space.

Lemma 3.3. *For $t \geq 1$ and $v \in \mathbf{\Omega}_0^-$*

$$\hat{\Psi}_v(t,\xi) = \pi^{\frac{1}{2}}\lambda^{-1}\chi_1(\lambda^{-1}(\xi-\xi_v))e^{\frac{1}{3}it\xi^3}, \tag{3.9}$$

where $\xi_v = \sqrt{v}$, and $\chi_1 \in \mathcal{S}(\mathbb{R})$ is localised at scale 1 in space and frequency satisfying

$$\int \chi_1(\xi) = 1 + O\left((t^{\frac{2}{3}}v)^{-\frac{3}{4}}\right). \tag{3.10}$$

Proof. We consider the Taylor approximation of ϕ at $x = -tv$,

$$\phi(t,x) = \tfrac{1}{3}t\xi_v^3 + x\xi_v + \frac{\pi}{4} - \frac{1}{4}(\lambda(x+tv))^2 + R(\lambda(x+tv), t^{\frac{2}{3}}v),$$

where

$$R(x,y) = -\int_0^1 \frac{y^{-\frac{3}{4}}x^3(1-h)^2}{8|y^{-\frac{3}{4}}xh - 1|^{\frac{3}{2}}}\,dh$$

is well defined for $x \in \operatorname{supp}\Psi_v$ whenever $v \in \mathbf{\Omega}_0^-$. We may then define

$$\chi_1(\xi) = \pi^{-1}e^{-\frac{1}{3}it\lambda^3\xi^3}\int e^{-2i\xi\eta}e^{i\eta^2}\hat{\chi}_2(\eta)\,d\eta,$$

where

$$\chi_2(x) = \chi(x)e^{iR(x,t^{\frac{2}{3}}v)}.$$

As $e^{\frac{1}{3}it\lambda^3\xi^3} = 1 + O((t^{\frac{2}{3}}v)^{-\frac{3}{4}}\xi^3)$ and $\chi_2 \in \mathcal{S}$ we have

$$\int \chi_1 = \hat{\chi}_2(0) + O((t^{\frac{2}{3}}v)^{-\frac{3}{4}}),$$

and similarly, as $e^{iR(x,y)} = 1 + O(y^{-\frac{3}{4}}x^3)$,

$$\hat{\chi}_2(0) = 1 + O((t^{\frac{2}{3}}v)^{-\frac{3}{4}}).$$

□

3.3. **Self-similar coordinates and elliptic estimates.** We now prove a number of estimates for u that will allow us to reduce the closing bootstrap estimate (1.34) to an estimate for γ. Our argument is similar to [17, 19, 20].

To simplify the notation we will work in self-similar coordinates

$$U(s,y) = s^{\frac{1}{3}}u(s, s^{\frac{1}{3}}y), \tag{3.11}$$

and rewrite equation (1.1) as an equation for U

$$\begin{cases} U_s = \frac{1}{3}s^{-1}\partial_y\left(yU - U_{yy} + 3\sigma U^3\right), \\ U(1,y) = u(1,y). \end{cases} \tag{3.12}$$

From (3.2), for $s \in [1,T]$ we have

$$\|yU - U_{yy} + 3\sigma U^3\|_{L^2} \lesssim \epsilon s^{\delta - \frac{1}{6}}. \tag{3.13}$$

So, integrating (3.12), for $s \in [1,T]$ we have

$$\|U\|_{H^{-1}} \lesssim \epsilon. \tag{3.14}$$

We also note that from (3.6) we have

$$\|(y - \partial_y^2)U\|_{L^2_y} = s^{-\frac{1}{6}}\|Lu\|_{L^2_x} \lesssim \epsilon. \tag{3.15}$$

We dyadically decompose

$$U = \sum_{j\geq 0} U_j$$

and for $j \geq 1$ we may further decompose $U_j = U_j^+ + U_j^-$. We observe that

$$\gamma(s,v) = \int U(s,y)\overline{\Psi}_v(s, s^{\frac{1}{3}}y)\,dy$$

and for $v \in \mathbf{\Omega}_0^-$, from the frequency localisation (3.9) and the estimate (3.14),

$$\left|\gamma(s,v) - \sum_{2^{2j} \sim s^{\frac{2}{3}} v} \int U_j^+(s,y)\overline{\Psi}_v(s, s^{\frac{1}{3}} y)\, dy\right| \lesssim_N \epsilon (s^{\frac{2}{3}} v)^{-N}. \tag{3.16}$$

For each $j \geq 1$ the operator $y - \partial_y^2$ is elliptic away from $\{y \sim -2^{2j}\}$. With this in mind, for each $j \geq 1$ we construct a partition of unity

$$1 \equiv \chi_j^{\mathrm{in}} + \chi_j^{\mathrm{out}} + \chi_j^{\mathrm{mid},+} + \chi_j^{\mathrm{mid},-},$$

where $\chi_j^{\mathrm{in}}, \chi_j^{\mathrm{out}}, \chi_j^{\mathrm{mid},+}, \chi_j^{\mathrm{mid},-} \in C^\infty(\mathbb{R})$ are smooth functions localised on the inner $|y| \ll 2^{2j}$, outer $|y| \gg 2^{2j}$, positive middle $y \sim 2^{2j}$ and negative middle $y \sim -2^{2j}$ regions respectively. We assume these functions are real-valued, sharply localised at frequency $\ll 2^j$ and localised in space on the corresponding sets up to rapidly decaying tails at scale $\lesssim 2^{-j}$. We then have the following estimates.

Lemma 3.4. *For $s \in [1, T]$ we have the estimates*

$$\|\langle y \rangle U_0\|_{L^2} \lesssim \epsilon, \tag{3.17}$$

$$\|\sum_{j\geq 1} \chi_j^{\mathrm{in}} 2^{2j} U_j\|_{L^2} \lesssim \epsilon, \tag{3.18}$$

$$\|\sum_{j\geq 1} \chi_j^{\mathrm{out}} y U_j\|_{L^2} \lesssim \epsilon, \tag{3.19}$$

$$\|\sum_{j\geq 1} \chi_j^{\mathrm{mid},+}(|y| + 2^{2j}) U_j\|_{L^2} \lesssim \epsilon, \tag{3.20}$$

$$\|\sum_{j\geq 1} \chi_j^{\mathrm{mid},-}(|y|^{\frac{1}{2}} + 2^j)(|y|^{\frac{1}{2}} + i\partial_y) U_j^+\|_{L^2} \lesssim \epsilon. \tag{3.21}$$

Proof.

Low frequencies (3.17). As a consequence of (3.14) and (3.15) we have

$$\|U_0\|_{L^2} \lesssim \epsilon,$$
$$\|yU_0\|_{L^2} \lesssim \|(y - \partial_y^2)U_0\|_{L^2} + \|\partial_y^2 U_0\|_{L^2} \lesssim \epsilon.$$

Inner region (3.18). From the frequency localisation we have

$$\|\chi_j^{\mathrm{in}} \partial_y^2 U_j\|_{L_y^2} \lesssim \|\chi_j^{\mathrm{in}}(y - \partial_y^2) U_j\|_{L_y^2} + \|\chi_j^{\mathrm{in}} y U_j\|_{L_y^2}$$

and

$$\|\chi_j^{\mathrm{in}}(y - \partial_y^2) U_j\|_{L^2} \lesssim \|P_j (y - \partial_y^2) U\|_{L^2} + 2^{-j} \|\tilde{P}_j U\|_{L^2},$$

where $\tilde{P}_j$ localises to frequencies $\sim 2^j$ and satisfies $\tilde{P}_j P_j = P_j$. Due to the spatial localisation,

$$\|\chi_j^{\mathrm{in}} y U_j\|_{L^2} \lesssim \mu \|2^{2j} \chi_j^{\mathrm{in}} U_j\|_{L^2} + 2^{-j} \|U_j\|_{L^2},$$

where $0 < \mu \ll 1$. Square-summing over $j \geq 1$ and using (3.14), (3.15) and the frequency localisation, we have

$$\sum_{j\geq 1} \|\chi_j^{\mathrm{in}} 2^{2j} U_j\|_{L^2}^2 \lesssim \epsilon^2.$$

Outer region (3.19). Here we proceed similarly and estimate

$$\|\chi_j^{\mathrm{out}} y U_j\|_{L^2} \lesssim \|\chi_j^{\mathrm{out}}(y-\partial_y^2)U_j\|_{L^2} + \|\chi_j^{\mathrm{out}}\partial_y^2 U_j\|_{L^2}.$$

The estimate then follows by square summation, (3.14) and (3.15).

Positive middle region (3.20). Integrating by parts and using the localisation we have

$$\begin{aligned}
&\|\chi_j^{\mathrm{mid},+} y U_j\|_{L^2}^2 + \|\chi_j^{\mathrm{mid},+}\partial_y^2 U_j\|_{L^2}^2 \\
&\lesssim \|(y-\partial_y^2)U_j\|_{L^2}^2 + 2^{-2j}\|U_j\|_{L^2}^2 - 2\int (\chi_j^{\mathrm{mid},+})^2 y(\partial_y U_j)^2\,dy \\
&\lesssim \|(y-\partial_y^2)U_j\|_{L^2}^2 + 2^{-2j}\|U_j\|_{L^2}^2.
\end{aligned}$$

The estimate follows from the localisation, (3.14) and square summation.

Negative middle region (3.21). In this region the operator $y-\partial_y^2$ is no longer elliptic. However, we can approximately factorise $y-\partial_y^2 \approx -(|y|^{\frac12} - i\partial_y)(|y|^{\frac12}+i\partial_y)$ where $|y|^{\frac12} - i\partial_y$ is elliptic on positive frequencies. In particular,

$$\begin{aligned}
&\|\chi_j^{\mathrm{mid},-}|y|^{\frac12}(|y|^{\frac12}+i\partial_y)U_j^+\|_{L^2}^2 + \|\chi_j^{\mathrm{mid},-}\partial_y(|y|^{\frac12}+i\partial_y)U_j^+\|_{L^2}^2 \\
&\lesssim \|(y-\partial_y^2)U_j^+\|_{L^2}^2 + 2^{-2j}\|U_j^+\|_{L^2}^2 \\
&+ 2\Im\int |\chi_j^{\mathrm{mid},-}|^2|y|^{\frac12}(|y|^{\frac12}+i\partial_y)U_j^+\overline{\partial_y(|y|^{\frac12}+i\partial_y)U_j^+}\,dy \\
&\lesssim \|(y-\partial_y^2)U_j^+\|_{L^2}^2 + 2^{-2j}\|U_j^+\|_{L^2}^2.
\end{aligned}$$

□

3.4. **Energy estimates for γ.**

Lemma 3.5. *For $t\in[1,T]$ we have the energy estimates*

$$\|\gamma\|_{H^{0,1}_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \epsilon, \tag{3.22}$$

$$\|\gamma\|_{\dot H^1_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim M_0\epsilon t^{\frac16}, \tag{3.23}$$

as well as the improved estimate in the region $\widehat{\Omega}^-_{1/3-2\delta}$

$$\|\gamma\|_{\dot H^1_{\xi_v}(\widehat{\Omega}^-_{1/3-2\delta})} \lesssim M_0\epsilon t^{\delta}. \tag{3.24}$$

Proof. Making a suitable change of variables valid for $\xi_v\in\widehat{\Omega}_0^-$, we have the estimate

$$\|\gamma\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \|u\|_{L^2(\Omega_0^-)}.$$

We calculate

$$\xi_v\Psi_v = -i\partial_x\Psi_v + \lambda\tilde\Psi_v,$$

where

$$\tilde\Psi_v(t,x) = \left(\lambda^{-1}(\xi_v - t^{-\frac12}|x|^{\frac12})\chi(\lambda(x+tv)) + i\chi'(\lambda(x+tv))\right)e^{i\phi}$$

has similar localisation to Ψ_v. Integrating by parts in the first term we have

$$\|\xi_v\gamma\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \|u\|_{H^1(\Omega_0^-)}.$$

For (3.23) we calculate

$$\partial_{\xi_v}\left(\chi(\lambda(x+tv))\right) = 2\lambda^{-1}\chi'(\lambda(x+tv)) - \frac{1}{2}\xi_v^{-1}\lambda(x+tv)\chi'(\lambda(x+tv)).$$

For the second term we use the bootstrap estimate (1.34) to get

$$\left\| \int u(t,x)\xi_v^{-1}\lambda(x+tv)\chi'(\lambda(x+tv))e^{-i\phi}\,dx \right\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim M_0\epsilon \|\xi_v^{-1}\|_{L^2(\widehat{\Omega}_0^-)} \lesssim M_0\epsilon t^{\frac{1}{6}}.$$

For the first term we define

$$v^+_{\sim\xi_v}(t,x) = e^{-i\phi}t^{-\frac{1}{3}} \sum_{2^j\sim t^{\frac{1}{3}}\xi_v} U_j^+(t, t^{-\frac{1}{3}}x) \tag{3.25}$$

and observe that if $\xi_v \in \widehat{\Omega}_0^-$, then from (3.21) we have

$$\|t^{\frac{5}{6}}\xi_v\partial_x v^+_{\sim\xi_v}\|_{L^2_x(\lambda(x+tv)\lesssim 1)} \lesssim \epsilon. \tag{3.26}$$

Estimating as in (3.16), then integrating by parts and using (3.26) we have

$$\begin{aligned}
&\left\| \int u(t,x)\lambda^{-1}\chi'(\lambda(x+tv))e^{-i\phi}\,dx \right\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \\
&\lesssim \left\| \int v^+_{\sim\xi_v}(t,x)\lambda^{-1}\chi'(\lambda(x+tv))\,dx \right\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} + \epsilon\|(t^{\frac{1}{3}}\xi_v)^{-N}\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \\
&\lesssim \left\| \int \lambda^{-2}\partial_x v^+_{\sim\xi_v}(t,x)\chi(\lambda(x+tv))\,dx \right\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} + \epsilon t^{\frac{1}{6}} \\
&\lesssim \epsilon t^{\frac{1}{6}}.
\end{aligned}$$

For the improved estimate (3.24), estimating as in (3.23) and (3.16), it suffices to show that, for $\xi_v \gtrsim t^{-2\delta}$

$$\|t^{\frac{5}{6}}\xi_v\partial_x v^+_{\sim\xi_v}\|_{L^2(\lambda(x+tv)\lesssim 1)} \lesssim \epsilon t^{\delta-\frac{1}{6}}$$

In order to prove this we observe that if $2^j \sim s^{\frac{1}{3}}\xi_v \gtrsim s^{\frac{1}{3}-2\delta}$ then we may use (1.34) to replace (3.14) by

$$2^{-j}\|\chi_j^{\mathrm{mid},-}U_j\|_{L^2} \lesssim M_0\epsilon 2^{-\frac{j}{2}},$$

and use (3.7) to replace (3.15) by

$$\|\chi_j^{\mathrm{mid},-}(y-\partial_y^2)U\|_{L^2} \lesssim \epsilon s^{\delta-\frac{1}{6}},$$

giving us the improvement to (3.21)

$$\|\sum_{2^j\sim s^{\frac{1}{3}}\xi_v} \chi_j^{\mathrm{mid},-}(|y|^{\frac{1}{2}}+2^j)(|y|^{\frac{1}{2}}+i\partial_y)U_j\|_{L^2} \lesssim \epsilon s^{\delta-\frac{1}{6}}.$$

□

3.5. **Reduction of pointwise estimates to wave packets.** The following lemma allows us to reduce closing the bootstrap estimate (1.34) to proving

$$\|\gamma\|_{L^\infty_v(\mathbf{\Omega}_0^-)} \lesssim \epsilon \tag{3.27}$$

with a constant independent of M_0, T.

Proposition 3.6. *For $t \in [1,T]$ we have the following estimates.*
I) Decaying region.

$$\|t^{\frac{1}{3}}(t^{-\frac{1}{3}}x)^{\frac{3}{4}}u\|_{L^\infty(\Omega_0^+)} \lesssim \epsilon, \qquad \|t^{\frac{2}{3}}(t^{-\frac{1}{3}}x)^{\frac{1}{4}}u_x\|_{L^\infty(\Omega_0^+)} \lesssim \epsilon, \tag{3.28}$$

$$\|t^{\frac{1}{6}}(t^{-\frac{1}{3}}x)u\|_{L^2(\Omega_0^+)} \lesssim \epsilon. \tag{3.29}$$

II) Self-similar region.

$$\|t^{\frac{1}{3}}u\|_{L^\infty(\Omega_0^0)} \lesssim \epsilon, \qquad \|t^{\frac{2}{3}}u_x\|_{L^\infty(\Omega_0^0)} \lesssim \epsilon, \tag{3.30}$$

$$\|t^{\frac{1}{6}}u\|_{L^2(\Omega_0^0)} \lesssim \epsilon. \tag{3.31}$$

III) Oscillating region.

$$\|t^{\frac{1}{3}}u\|_{L^\infty(\Omega_0^-)} \lesssim \epsilon, \tag{3.32}$$

$$\left\|t^{\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{\frac{3}{8}}\left(P_+u(t,x) - t^{-\frac{1}{3}}(t^{-\frac{1}{3}}x)^{-\frac{1}{4}}e^{i\phi}\gamma(t,t^{-1}x)\right)\right\|_{L^\infty(\Omega_0^-)} \lesssim \epsilon, \tag{3.33}$$

$$\left\|t^{\frac{2}{3}}(t^{-\frac{1}{3}}|x|)^{-\frac{1}{8}}\left(P_+u_x(t,x) - it^{-\frac{2}{3}}(t^{-\frac{1}{3}}x)^{\frac{1}{4}}e^{i\phi}\gamma(t,t^{-1}x)\right)\right\|_{L^\infty(\Omega_0^-)} \lesssim \epsilon, \tag{3.34}$$

$$\left\|t^{\frac{1}{6}}(t^{-\frac{1}{3}}|x|)^{\frac{1}{4}}\left(P_+u(t,x) - t^{-\frac{1}{3}}(t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}}e^{i\phi}\gamma(t,t^{-1}x)\right)\right\|_{L^2(\Omega_0^-)} \lesssim \epsilon. \tag{3.35}$$

IV) Fourier-space estimates. Suppose that (3.27) *is true, then provided $t^{\frac{1}{3}}|\xi|$ is sufficiently large whenever $\xi \in \widehat{\Omega}_0^-$ we have*

$$\|(t^{\frac{1}{3}}\xi)^{\frac{1}{4}}(\hat{u}(t,\xi) - \pi^{-\frac{1}{2}}e^{\frac{1}{3}it\xi^3}\gamma(t,\xi^2))\|_{L^\infty_\xi(\widehat{\Omega}_0^-)} \lesssim \epsilon, \tag{3.36}$$

$$\|t^{\frac{1}{6}}(t^{\frac{1}{3}}\xi)^{\frac{1}{2}}(\hat{u}(t,\xi) - \pi^{-\frac{1}{2}}e^{\frac{1}{3}it\xi^3}\gamma(t,\xi^2))\|_{L^2_\xi(\widehat{\Omega}_0^-)} \lesssim \epsilon. \tag{3.37}$$

Proof.
I & II) Decaying and self-similar regions. The L^2-estimates (3.29) and (3.31) follow from the estimates (3.17)–(3.20). The L^∞-estimates (3.28) and (3.30) are similar using Bernstein's inequality (2.9).

III) Oscillating region. The estimate (3.32) follows from applying (2.10) to $e^{-i\phi}u^+$ with (3.17)–(3.19) and (3.21).

For (3.35), using (3.16) and (3.17)–(3.19) it suffices to show that

$$\left\|\lambda^{-1}v^+_{\sim\xi_v}(t,-tv) - \int v^+_{\sim\xi_v}(t,x)\chi(\lambda(x+tv))\,dx\right\|_{L^2_v(\mathbf{\Omega}_0^-)} \lesssim \epsilon t^{-\frac{1}{3}}.$$

As $\int \chi = 1$ we have

$$\begin{aligned}
&\lambda^{-1} v^+_{\sim\xi_v}(t,-tv) - \int v^+_{\sim\xi_v}(t,x)\chi(\lambda(x+tv))\,dx \\
&= \int \left(v^+_{\sim\xi_v}(t,-tv) - v^+_{\sim\xi_v}(t,x)\right)\chi(\lambda(x+tv))\,dx \\
&= -\int\int_0^1 (\partial_x v^+_{\sim\xi_v})(t, x-(x+tv)h)(x+tv)\chi(\lambda(x+tv))\,dhdx.
\end{aligned}$$

The estimate then follows from (3.26).

For (3.33) we estimate similarly, using (3.26) to show that

$$(t^{\frac{1}{3}}\xi_v)^{\frac{1}{4}}|(v^+_{\sim\xi_v}(t,-tv) - v^+_{\sim\xi_v}(t,x))\chi(\lambda(x+tv))| \lesssim \epsilon\lambda^{\frac{3}{2}}|x+tv|^{\frac{1}{2}}|\chi(\lambda(x+tv))|.$$

For (3.34) we first use Bernstein's inequality (2.9) and (3.21) to show that, for $\xi_v \in \widehat{\Omega}_0^-$,

$$\|t^{\frac{5}{6}}\xi_v^{\frac{1}{2}}\partial_x v^+_{\sim\xi_v}\|_{L^\infty_x(\lambda(x+tv)\lesssim 1)} \lesssim \epsilon. \tag{3.38}$$

The estimate then follows from (3.33).

IV) Fourier-space estimates. We use (3.9) to write

$$\begin{aligned}
&e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v) - \pi^{-\frac{1}{2}}\gamma(t,\xi_v) \\
&= \pi^{-\frac{1}{2}}\int \left(e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v) - e^{-\frac{1}{3}it\xi^3}\hat{u}(t,\xi)\right)\lambda^{-1}\chi_1(\lambda^{-1}(\xi-\xi_v))\,d\xi \\
&+ O\left((t^{\frac{1}{3}}\xi_v)^{-\frac{3}{2}}e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v)\right).
\end{aligned}$$

For the difference we have

$$\begin{aligned}
&e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v) - e^{-\frac{1}{3}it\xi^3}\hat{u}(t,\xi) \\
&= -i\int_0^1 e^{-\frac{1}{3}it\eta^3}\widehat{(Lu)}(t, h(\xi_v-\xi)+\xi)\,dh\,(\xi_v-\xi).
\end{aligned}$$

For the error terms we then use that

$$\begin{aligned}
&\|t^{\frac{1}{6}}(t^{\frac{1}{3}}\xi_v)^{-1}e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v)\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \\
&\lesssim \|t^{\frac{1}{6}}(t^{\frac{1}{3}}\xi_v)^{-1}(e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v) - \gamma(t,\xi_v^2))\|_{L^2_{\xi_v},(\widehat{\Omega}_0^-)} + \|\gamma(t,\xi_v^2)\|_{L^\infty_{\xi_v}(\widehat{\Omega}_0^-)}
\end{aligned}$$

which gives us (3.37) provided $t^{\frac{1}{3}}\xi_v \gg 1$ for $\xi_v \in \widehat{\Omega}_0^-$.

For (3.36) we use that

$$|e^{-\frac{1}{3}it\xi_v^3}\hat{u}(t,\xi_v) - e^{-\frac{1}{3}it\xi^3}\hat{u}(t,\xi)| \lesssim \|Lu\|_{L^2}|\xi_v-\xi|^{\frac{1}{2}},$$

and estimate similarly. □

3.6. **Global existence.** In order to prove (3.27), we consider the ODE satisfied by γ

$$\dot{\gamma}(t,v) = \sigma\int (u^3)_x\overline{\Psi}_v\,dx + \int u\overline{(\partial_t + \tfrac{1}{3}\partial_x^3)\Psi_v}\,d\xi = I_0 + I_1 + I_2, \tag{3.39}$$

where we define

$$I_0 = i\sigma \int u^3 \overline{\Psi}_v t^{-\frac{1}{2}} |x|^{\frac{1}{2}}\, dx,$$
$$I_1 = -\sigma \int u^3 \lambda \overline{\chi}_x e^{-i\phi}\, dx,$$
$$I_2 = \int u \overline{(\partial_t + \tfrac{1}{3}\partial_x^3)\Psi_v}\, dx.$$

We then have the following estimates

Lemma 3.7. *For $t \in [1,T]$ and $\epsilon > 0$ sufficiently small, we have estimates*

$$\|t(t^{\frac{2}{3}}v)^{\frac{1}{8}}(I_0 - 3i\sigma t^{-1}|\gamma|^2\gamma - 2i\sigma t^{-1} e^{-i\phi(t,tv)} \operatorname{Re}(e^{3i\phi(t,tv)}\gamma^3))\|_{L^\infty_v(\mathbf{\Omega}_0^-)} \lesssim \epsilon, \tag{3.40}$$

$$\|t^{\frac{7}{6}}(t^{\frac{2}{3}}v)^{\frac{1}{4}}(I_0 - 3i\sigma t^{-1}|\gamma|^2\gamma - 2i\sigma t^{-1} e^{-i\phi(t,tv)} \operatorname{Re}(e^{3i\phi(t,tv)}\gamma^3))\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \epsilon, \tag{3.41}$$

$$\|t(t^{\frac{2}{3}}v)^{\frac{1}{8}}(I_1 + I_2)\|_{L^\infty_v(\mathbf{\Omega}_0^-)} \lesssim \epsilon, \qquad \|t^{\frac{7}{6}}(t^{\frac{2}{3}}v)^{\frac{1}{4}}(I_1 + I_2)\|_{L^2_{\xi_v}(\widehat{\Omega}_0^-)} \lesssim \epsilon. \tag{3.42}$$

Proof. As u is real-valued,

$$u^3 = 2\operatorname{Re}((u^+)^3) + 3|u^+|^2 u.$$

For $x \in \operatorname{supp}\Psi_v$, from (3.33) and (3.35), we have

$$t^{-\frac{1}{2}}|x|^{\frac{1}{2}} u(t,x)^3 = 2\operatorname{Re}(t^{-\frac{4}{3}}(t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}} e^{3i\phi}\gamma(t,t^{-1}x)^3) + 3t^{-1}|\gamma(t,t^{-1}x)|^2 u(t,x) + \mathbf{err},$$

where the error satisfies

$$\begin{aligned} &\|t^{\frac{4}{3}}(t^{-\frac{1}{3}}|x|)^{\frac{3}{8}}\mathbf{err}\|_{L^\infty(\Omega_0^-)} \lesssim M_0^3\epsilon^3,\\ &\|t^{\frac{7}{6}}(t^{-\frac{1}{3}}|x|)^{\frac{1}{4}}\mathbf{err}\|_{L^2(\Omega_0^-)} \lesssim M_0^3\epsilon^3. \end{aligned} \tag{3.43}$$

Further, for $x \in \operatorname{supp}\Psi_v$, using (3.23) we have

$$2\operatorname{Re}(t^{-\frac{4}{3}}(t^{-\frac{1}{3}}|x|)^{-\frac{1}{4}} e^{3i\phi}\gamma(t,t^{-1}x)^3) = 2t^{-1}\lambda e^{i\phi(t,x)} e^{-i\phi(t,tv)} \operatorname{Re}\left(e^{3i\phi(t,tv)}\gamma(t,v)^3\right) + \mathbf{err},$$

$$3t^{-1}|\gamma(t,t^{-1}x)|^2 u(t,x) = 3t^{-1}|\gamma(t,v)|^2 u(t,x) + \mathbf{err},$$

where the errors satisfy the estimate (3.43). The estimates (3.40) and (3.41) then follow from these expressions.

For I_1 we use (1.34) to estimate

$$\|t(t^{-\frac{1}{3}}|x|)^{\frac{1}{8}} u^3\|_{L^\infty(\Omega_0^-)} \lesssim M_0^3\epsilon^3, \qquad \|t^{\frac{5}{6}} u^3\|_{L^2(\Omega_0^-)} \lesssim M_0^3\epsilon^3.$$

For I_2 we start by calculating, for $v \in \mathbf{\Omega}_0^-$

$$(\partial_t + \tfrac{1}{3}\partial_x^3)\Psi_v = t^{-1}\lambda^{-1}\partial_x\tilde{\Psi}_v - \frac{1}{4} i t^{-\frac{1}{2}}|x|^{-\frac{3}{2}}\Psi_v,$$

where

$$\tilde{\Psi}_v = \left(\frac{1}{2}\lambda(x+tv)\chi + i\lambda^2 t^{\frac{1}{2}}|x|^{\frac{1}{2}}\chi' + \frac{1}{3}t\lambda^3\chi''\right)e^{i\phi}$$

has the same localisation as Ψ_v. For the first term we estimate by integrating by parts and using (3.26). For the second term we use the hyperbolic estimate (3.32). □

We now use Lemma 3.7 to solve (3.39) for $t \in [\max\{1, Cv^{-\frac{3}{2}}\}, T]$, where $C > 0$ is chosen such that $v \in \mathbf{\Omega}_0^-$ for $t \geq \max\{1, Cv^{-\frac{3}{2}}\}$. For velocities $v \geq C^{\frac{2}{3}}$, the ray Γ_v lies outside the self-similar region for all $t \geq 1$, so from (1.31) and (3.9) we may take initial data

$$|\gamma(1, v)| \lesssim \|\hat{u}(1)\|_{L^\infty} \lesssim \epsilon. \tag{3.44}$$

For velocities $0 < v < C^{\frac{2}{3}}$, the ray Γ_v lies inside the self-similar region up to time $t = Cv^{-\frac{3}{2}}$, so using (3.16) and (3.17) we have initial data

$$|\gamma(Cv^{-\frac{3}{2}}, v)| \lesssim \|\hat{U}_0(Cv^{-\frac{3}{2}}, y)\|_{L^\infty} + \epsilon \lesssim \epsilon. \tag{3.45}$$

From (3.40) and (3.42), for $v \in \mathbf{\Omega}_0^-$, we have the estimate

$$\dot{\gamma} = 3i\sigma t^{-1}|\gamma|^2\gamma + 2i\sigma t^{-1}e^{-i\phi(t,tv)}\operatorname{Re}(e^{3i\phi(t,tv)}\gamma^3) + O\left(t^{-1}(t^{\frac{2}{3}}v)^{-\frac{1}{8}}\right).$$

We observe that

$$\begin{aligned} 2i\sigma t^{-1}e^{-i\phi}\operatorname{Re}(e^{3i\phi}\gamma^3) = \partial_t\left(-\frac{3\sigma}{4}(t^{\frac{2}{3}}v)^{-\frac{3}{2}}e^{2i\phi}\gamma^3 + \frac{3}{8}\sigma(t^{\frac{2}{3}}v)^{-\frac{3}{2}}e^{-4i\phi}\overline{\gamma}^3\right) \\ + O((t^{\frac{2}{3}}v)^{-\frac{3}{2}}|\gamma|^2|\dot{\gamma}|), \end{aligned}$$

so as $|\dot{\gamma}| \lesssim \epsilon t^{-1}$ this term is integrable. As $3i\sigma t^{-1}|\gamma|^2$ is imaginary we can then solve (3.39) to find a solution satisfying (3.27). This completes the proof of global existence.

3.7. **Asymptotic behaviour.** From Lemma 3.7 there exists a unique function W defined on $(0, \infty)$ such that for $t \geq 1$

$$\|(t^{\frac{2}{3}}v)^{\frac{1}{8}}(\gamma(t, v) - (2\pi)^{-\frac{1}{2}}W(\xi_v)e^{\frac{3i\sigma}{4\pi}|W(\xi_v)|^2\log t})\|_{L_v^\infty(\mathbf{\Omega}_0^-)} \lesssim \epsilon, \tag{3.46}$$

$$\|t^{\frac{1}{6}}(t^{\frac{2}{3}}v)^{\frac{1}{4}}(\gamma(t, v) - (2\pi)^{-\frac{1}{2}}W(\xi_v)e^{\frac{3i\sigma}{4\pi}|W(\xi_v)|^2\log t})\|_{L_{\xi_v}^2(\widehat{\Omega}_0^-)} \lesssim \epsilon. \tag{3.47}$$

As every $\xi_v > 0$ lies in $\widehat{\Omega}_0^-$ for sufficiently large $t > 0$, from (3.22) and (3.27) we have

$$\|W\|_{L_{\xi_v}^\infty \cap H_{\xi_v}^{0,1}(0,\infty)} \lesssim \epsilon. \tag{3.48}$$

From (3.47) and (3.48) we have

$$\left\|e^{-\frac{3i\sigma}{4\pi}|\gamma(t,v)|^2\log t}\gamma(t, v) - W(\xi_v)\right\|_{L_{\xi_v}^2(\widehat{\Omega}_{1/3-2\delta}^-)} \lesssim \epsilon t^{-\frac{1}{3}+\delta}\log t,$$

and from (3.24) we have

$$\left\|\partial_{\xi_v}\left(e^{-\frac{3i\sigma}{4\pi}|\gamma(t,v)|\log t}\gamma(t, v)\right)\right\|_{L_{\xi_v}^2(\widehat{\Omega}_{1/3-2\delta}^-)} \lesssim \epsilon t^\delta \log t.$$

By interpolation, for sufficiently large $C > 0$

$$\|W\|_{\dot{H}_{\xi_v}^{1-C\epsilon^2}(0,\infty)} \lesssim \epsilon. \tag{3.49}$$

To derive the asymptotic behaviour in the self-similar region let $\rho = \frac{1}{3}(\frac{1}{6} - \delta)$ and define $U_{\rm lo}(s, y) = P_{\lesssim s^\rho}U$. From (3.12) and (3.18) have

$$\|\partial_s U_{\rm lo}\|_{L^\infty(\Omega_\rho^0)} \lesssim \epsilon s^{-1-\frac{3}{2}\rho}, \qquad \|\partial_s U_{\rm lo}\|_{L^2(\Omega_\rho^0)} \lesssim \epsilon s^{-1-2\rho}. \tag{3.50}$$

If we take $U_{\rm hi} = U - U_{\rm lo}$ then from (3.18) we have

$$\|U_{\rm hi}\|_{L^2(\Omega_\rho^0)} \lesssim \epsilon s^{-2\rho}, \qquad \|U_{\rm hi}\|_{L^\infty(\Omega_\rho^0)} \lesssim \epsilon s^{-\frac{3}{2}\rho}.$$

In particular there exists a solution Q to (1.12), satsifying (1.13) such that

$$\|U(s,y) - Q(y)\|_{L^\infty(\Omega_\rho^0)} \lesssim \epsilon s^{-\frac{3\rho}{2}}, \qquad \|U(s,y) - Q(y)\|_{L^2(\Omega_\rho^0)} \lesssim \epsilon s^{-2\rho}. \tag{3.51}$$

4. Asymptotic Completeness

In this section we prove Theorem 1.4. We note that from the local theory it suffices to prove existence of a solution $u(t)$ on $[1,\infty)$ satisfying

$$\|u(1)\|_X \lesssim \epsilon. \tag{4.1}$$

We define δ as in (1.30) and assume that W satisfies (1.24) with $C\epsilon^2 = \delta$. In this section we will define Λu by (1.27) instead of (1.26) and note that the definitions agree for solutions to (1.22).

4.1. **Regularisation of W.** Instead of working with u_{asymp}, we work with an approximation u_{app} given by regularising W at the scale corresponding to the wave packets.

First we extend W to $\mathbb{R}$ and dyadically decompose

$$W(z) = \sum_{j\geq 0} W_j(z).$$

Next we take a smooth function $\chi \in C^\infty$ such that $\chi \equiv 1$ for $|z| \geq 2$ and $\chi \equiv 0$ for $|z| \leq 1$. For each $j \geq 1$ we define the function

$$\chi_j(t,z) = P_{\leq j-10}\left(\chi(2^{-j} t^{\frac13} \langle t^{\frac13} z\rangle^{\frac12})\right),$$

where the Littlewood-Paley projection acts on the z-variable. We observe that the χ_j are localised at frequency $\ll 2^j$ and localised in space on the set $A_j = \{t^{\frac13}\langle t^{\frac13} z\rangle^{\frac12} \gtrsim 2^j\}$ up to rapidly decaying tails at scale 2^{-j}. We then define

$$\mathbf{W}(t,z) = W_0(z) + \sum_{j\geq 1} \chi_j(t,z) W_j(z).$$

By construction, the map

$$x \mapsto \mathbf{W}(t, t^{-\frac12} x_-^{\frac12})$$

is smooth on the scale of the wave packets on $\mathbb{R}\backslash\{0\}$. However, to ensure that u_{app} is a good approximation on $\mathbb{R}$ we require additional smoothing at $x=0$. To do this we fix $N_0 \geq 5$ and decompose

$$\mathbf{W}(t,z) = \mathbf{W}_*(t,z) + \sum_{j=1}^{2N_0-1} \frac{1}{j!}\partial_z^j \mathbf{W}(t,0) z^j.$$

Finally, we define

$$\mathcal{W}(t,z) = \mathbf{W}_*(t,z) + \chi(t^{\frac13} z)\sum_{j=1}^{2N_0-1} \frac{1}{j!}\partial_z^j \mathbf{W}(t,0) z^j, \tag{4.2}$$

where $\chi \in C^\infty$ is defined as above. We define the corresponding approximate solution by

$$u_{\mathrm{app}}(t,x) = t^{-\frac13} Q(t^{-\frac13}x; \mathcal{W}(t, t^{-\frac12} x_-^{\frac12})). \tag{4.3}$$

As a straightforward consequence of the localisation of $\mathcal{W}$ we have the following Lemma.

Lemma 4.1. *For $t \geq 1$ we have the following estimates.*

I) Estimates for $\mathcal{W} = \mathcal{W}(t, t^{-\frac{1}{2}} x_-^{\frac{1}{2}})$.

$$\begin{gathered}\|t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{-\frac{1}{4}} e^{-\frac{2}{3} t^{-\frac{1}{2}} x_+^{\frac{3}{2}}} \mathcal{W}\|_{L^2} \lesssim \epsilon, \\ \|t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}} \partial_x \mathcal{W}\|_{L^2} \lesssim \epsilon, \qquad \|t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}} \log\langle t^{-\frac{1}{2}} x_-^{\frac{1}{2}}\rangle \partial_x \mathcal{W}\|_{L^2} \lesssim \epsilon, \\ \|(t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{N+\delta} \partial_x^N \mathcal{W}\|_{L^2} \lesssim \epsilon, \qquad 2 \leq N \leq N_0.\end{gathered} \tag{4.4}$$

$$\begin{gathered}\|t^{-1} (t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}}) e^{-\frac{2}{3} t^{-\frac{1}{2}} x_+^{\frac{3}{2}}} \mathcal{W}\|_{L^2} \lesssim \epsilon, \\ \|t^{-1} (t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{N+1+\delta} \partial_x^N \mathcal{W}\|_{L^2} \lesssim \epsilon, \qquad 1 \leq N \leq N_0.\end{gathered} \tag{4.5}$$

$$\begin{gathered}\|\mathcal{W}\|_{L^\infty} \lesssim \epsilon, \\ \|(t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{N+\frac{1}{2}+\delta} \partial_x^N \mathcal{W}\|_{L^\infty} \lesssim \epsilon, \qquad 1 \leq N \leq N_0.\end{gathered} \tag{4.6}$$

II) Estimates for $\mathcal{W}_t = \mathcal{W}_t(t, t^{-\frac{1}{2}} x_-^{\frac{1}{2}})$.

$$\begin{gathered}\|t (t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{\delta} e^{-\frac{2}{3} t^{-\frac{1}{2}} x_+^{\frac{3}{2}}} \mathcal{W}_t\|_{L^2} \lesssim \epsilon, \\ \|t (t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{N+\delta} \partial_x^N \mathcal{W}_t\|_{L^2} \lesssim \epsilon, \qquad 1 \leq N \leq N_0.\end{gathered} \tag{4.7}$$

$$\begin{gathered}\|(t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{1+\delta} e^{-\frac{2}{3} t^{-\frac{1}{2}} x_+^{\frac{3}{2}}} \mathcal{W}_t\|_{L^2} \lesssim \epsilon, \\ \|(t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{N+1+\delta} \partial_x^N \mathcal{W}_t\|_{L^2} \lesssim \epsilon, \qquad 1 \leq N \leq N_0.\end{gathered} \tag{4.8}$$

III) Estimates for $W - \mathcal{W}$.

$$\|(t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{\delta} e^{-\frac{2}{3} t^{-\frac{1}{2}} x_-^{\frac{3}{2}}} (W - \mathcal{W})\|_{L^2} \lesssim \epsilon, \tag{4.9}$$

$$\|(t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{\frac{1}{2}+\delta} (W - \mathcal{W})\|_{L^\infty} \lesssim \epsilon. \tag{4.10}$$

Proof. We may assume that $\chi(t^{\frac{1}{3}} z) \equiv 1$ on $\widehat{\Omega}_0^-$. We consider the regions Ω_0^-, Ω_0^+ and Ω_0^0 separately.

In for $x \in \Omega_0^-$ we have that $\mathcal{W} = \mathbf{W}$ so

$$\|t^{-\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{-\frac{1}{4}} \mathcal{W}\|_{L^2(\Omega_0^-)} \lesssim \|W\|_{L^2},$$

and differentiating

$$\begin{gathered}\|t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}} \partial_x \mathcal{W}\|_{L^2(\Omega_0^-)} \lesssim \|W_z\|_{L^2}, \\ \|t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}} \log\langle t^{-\frac{1}{2}} x_-^{\frac{1}{2}}\rangle \partial_x \mathcal{W}\|_{L^2(\Omega_0^-)} \lesssim \|\log\langle z\rangle W_z\|_{L^2}.\end{gathered}$$

For $N \geq 2$ we calculate

$$\partial_x^N \mathcal{W} = \sum_{k=1}^{N} c_{k,N} t^{-\frac{N+k}{3}} (t^{-\frac{1}{3}} |x|)^{\frac{k}{2} - N} (\partial_z^k \mathbf{W})(t, t^{-\frac{1}{2}} |x|^{\frac{1}{2}}),$$

so

$$\begin{aligned}&\|(t^{\frac{1}{3}} \langle t^{-\frac{1}{3}} x\rangle^{\frac{1}{4}})^{N+\delta} \partial_x^N \mathcal{W}\|_{L_x^2(\Omega_0^-)} \\ &\lesssim \sum_{k=1}^{N} c_{k,N} \|t^{\frac{1+\delta-k}{3}} (t^{\frac{1}{3}} z)^{\frac{1+\delta+2k-3N}{2}} (\partial_z^k \mathbf{W})(t, z)\|_{L_z^2(\widehat{\Omega}_0^-)}.\end{aligned}$$

For $k = N$ we have

$$\|(t^{\frac{1}{3}}(t^{\frac{1}{3}}z)^12)^{1+\delta-N}\partial_z^k W_0\|_{L^2(\widehat{\Omega}_0^-)} \lesssim t^{\frac{1+\delta-N}{3}}\|W\|_{L^2},$$

and for higher frequencies we use the localisation to estimate

$$\begin{aligned}\|\sum_{j\geq 1}(t^{\frac{1}{3}}(t^{\frac{1}{3}}z)^12)^{1+\delta-N}\partial_z^k(\chi_j W_j)\|^2_{L^2(\widehat{\Omega}_0^-)} &\lesssim \sum_{j\geq 1} 2^{2Nj}\|(t^{\frac{1}{3}}(t^{\frac{1}{3}}z)^12)^{1+\delta-N}\chi_j W_j\|^2_{L^2}\\ &\lesssim \sum_{j\geq 1}\|\chi_j W_j\|^2_{H^{1+\delta}}\\ &\lesssim \|W\|^2_{H^{1+\delta}}.\end{aligned}$$

For $1 \leq k < N$, we estimate

$$\begin{aligned}&\|t^{\frac{1+\delta-k}{3}}(t^{\frac{1}{3}}z)^{\frac{1+\delta+2k-3N}{2}}(\partial_z^k\mathbf{W})(t,z)\|_{L^2_z(\widehat{\Omega}_0^-)}\\ &\lesssim \|t^{\frac{1}{6}}(t^{\frac{1}{3}}z)^{\frac{6k-6N+1}{4}}\|_{L^2(\widehat{\Omega}_0^-)}\|(t^{\frac{1}{3}}\langle t^{\frac{1}{3}}z\rangle^{\frac{1}{2}})^{-(k-\frac{1}{2}-\delta)}\partial_z^k\mathbf{W}\|_{L^\infty_z}\\ &\lesssim \|W\|_{H^{1+\delta}}.\end{aligned}$$

The estimates (4.5), (4.6), (4.7), (4.8) in Ω_0^- are similar.

In Ω_0^+ we observe that $\partial_x\mathcal{W} \equiv 0$ and hence it suffices to consider

$$\|t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}e^{-\frac{2}{3}t^{-\frac{1}{2}}x_+^{\frac{3}{2}}}\mathcal{W}\|_{L^2(\Omega_0^+)} \lesssim \|t^{-\frac{1}{3}}\langle t^{-\frac{1}{3}}x\rangle^{-\frac{1}{4}}e^{-\frac{2}{3}t^{-\frac{1}{2}}x_+^{\frac{3}{2}}}\|_{L^2(\Omega_0^+)}\|\mathcal{W}\|_{L^\infty}.$$

The estimates (4.5), (4.7) are similar.

Finally we consider Ω_0^0. We can write

$$\mathbf{W}_*(t,z) = \mathbf{W}(t,0) + \int_0^1 \frac{(\partial_z^{2N_0}\mathbf{W})(t,hz)}{(2N_0-1)!}(1-h)^{2N_0-1}z^{2N_0}\,dh,$$

so in Ω_0^0 we have, for $1 \leq N \leq N_0$

$$\|\partial_x^N(\mathbf{W}_*(t,t^{-\frac{1}{2}}x_-^{\frac{1}{2}}))\|_{L^\infty(\Omega_0^0)} \lesssim t^{-\frac{1+2N+2\delta}{6}}\|W\|_{H^{1+\delta}}.$$

For $1 \leq N \leq N_0$ we use that $\chi(t^{-\frac{1}{6}}x_-^{\frac{1}{2}})$ is supported away from $x = 0$ to get

$$\left\|\partial_x^N\left(\chi(t^{-\frac{1}{6}}x_-^{\frac{1}{2}})\sum_{j=1}^{2N_0-1}\frac{1}{j!}\partial_z^j\mathbf{W}(t,0)t^{-\frac{j}{2}}x_-^{\frac{j}{2}}\right)\right\|_{L^\infty(\Omega_0^0)} \lesssim t^{-\frac{1+2N+2\delta}{6}}\|W\|_{H^{1+\delta}}.$$

The estimates (4.5), (4.6), (4.7), (4.8) are similar.

For (4.9) and (4.10) we write

$$W - \mathbf{W} = \sum_{j\geq 1}(1-\chi_j)W_j,$$

and estimate similarly, using that $1-\chi_j$ is localised on the complement of A_j. □

4.2. **Estimates for $u_{\rm app}$.** We now look to derive estimates for $u_{\rm app}$. We first state the following lemma giving estimates for solutions to the Painlevé II equation (1.12) which can be proved using Theorem 1.3 and variation of parameters.

Lemma 4.2. *Let $|W| \ll 1$ and $Q(y;W)$ be the solution to* (1.12) *satisfying* (1.20). *We then have the following estimates.*

$$|\langle y\rangle^{\frac{1-2N}{4}} e^{\frac{2}{3}y_+^{\frac{3}{2}}} \partial_y^N Q(y;W)| \lesssim |W|, \tag{4.11}$$

$$|\langle y\rangle^{\frac{1-2N}{4}} e^{\frac{2}{3}y_+^{\frac{3}{2}}} \partial_y^N \partial_W Q(y;W)| \lesssim 1 + |W|^2 \log\langle y\rangle, \tag{4.12}$$

$$|\langle y\rangle^{\frac{1-2N}{4}} e^{\frac{2}{3}y_+^{\frac{3}{2}}} \partial_y^N \partial_W^2 Q(y;W)| \lesssim |W| \log\langle y\rangle(1 + |W|^2 \log\langle y\rangle), \tag{4.13}$$

$$|\langle y\rangle^{\frac{1-2N}{4}} e^{\frac{2}{3}y_+^{\frac{3}{2}}} \partial_y^N \partial_W^3 Q(y;W)| \lesssim \log\langle y\rangle(1 + |W|^2 \log\langle y\rangle)^2, \tag{4.14}$$

$$|\langle y\rangle^{\frac{1-2N}{4}} e^{\frac{2}{3}y_+^{\frac{3}{2}}} \partial_y^N \partial_W^4 Q(y;W)| \lesssim |W|(\log\langle y\rangle)^2(1 + |W|^2 \log\langle y\rangle)^2. \tag{4.15}$$

Using the estimates of Lemmas 4.1 and 4.2 we can show that $u_{\rm app}$ is a good approximation to $u_{\rm asymp}$.

Lemma 4.3. *For $t \geq 1$ we have estimates for $u_{\rm app}$*

$$\|t^{\frac{1}{3}} \langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}} e^{\frac{2}{3}t^{-\frac{1}{2}}x_+^{\frac{3}{2}}} u_{\rm app}\|_{L^\infty} \lesssim \epsilon, \tag{4.16}$$

$$\|u_{\rm app}\|_{H^1} \lesssim \epsilon, \tag{4.17}$$

$$\|\Lambda u_{\rm app}\|_{L^2} \lesssim \epsilon(1 + \epsilon^2 \log t), \tag{4.18}$$

and estimates for the difference $u_{\rm app} - u_{\rm asymp}$

$$\|t^{\frac{1+\delta}{3}} \langle t^{-\frac{1}{3}}x\rangle^{\frac{1}{4}} (u_{\rm app} - u_{\rm asymp})\|_{L^2_x} \lesssim \epsilon, \tag{4.19}$$

$$\|t^{\frac{3+2\delta}{6}} \langle t^{-\frac{1}{3}}x\rangle^{\frac{3}{8}} e^{\frac{2}{3}t^{-\frac{1}{2}}x_+^{\frac{3}{2}}} (u_{\rm app} - u_{\rm asymp}) \|_{L^\infty_x} \lesssim \epsilon. \tag{4.20}$$

Further, if $T \geq 1$ is a dyadic integer we have the estimates

$$\|u_{\rm app}\|_{L^4_x L^\infty_T} \lesssim \epsilon T^{-\frac{1}{4}}, \tag{4.21}$$

$$\|(\Lambda u_{\rm app})_x\|_{L^\infty_x L^2_T} \lesssim \epsilon(1 + \epsilon^2 \log T), \tag{4.22}$$

where we use the notation $L^p_T = L^p([T, 2T])$.

Proof. The estimate (4.16) simply follows from (4.6) and (4.11).

For (4.17) we first observe that from (4.4) and (4.11) we have

$$\|u_{\rm app}\|_{L^2} \lesssim \epsilon.$$

Differentiating we have

$$\partial_x u_{\rm app} = t^{-\frac{2}{3}} Q_y(t^{-\frac{1}{3}}x; \mathcal{W}) + t^{-\frac{1}{3}} Q_w(t^{-\frac{1}{3}}x; \mathcal{W}) \partial_x \mathcal{W},$$

which we may estimate using (4.4)–(4.6) and (4.11)–(4.12).

For (4.18) we use (1.12) to get

$$\Lambda u_{\mathrm{app}} = -2t^{\frac{1}{3}} Q_{wy} \partial_x \mathcal{W} - t^{\frac{2}{3}} Q_w \partial_x^2 \mathcal{W} - t^{\frac{2}{3}} Q_{ww} (\partial_x \mathcal{W})^2,$$

and estimate similarly.

For (4.19) and (4.20) we write

$$u_{\mathrm{app}} - u_{\mathrm{asymp}} = \int_0^1 t^{-\frac{1}{3}} Q_w(t^{-\frac{1}{3}}x; h\mathcal{W} + (1-h)W)(\mathcal{W} - W)\, dh,$$

and estimate using (4.9), (4.10) and (4.12).

To prove (4.21) we first note that for $x > 0$ we have the estimate

$$\|u_{\mathrm{app}}\|_{L_x^4([0,\infty); L_T^\infty)} \lesssim \|T^{-\frac{1}{3}} \langle T^{-\frac{1}{3}} x\rangle^{-\frac{1}{4}} e^{-\frac{2}{3} T^{-\frac{1}{2}} x^{\frac{3}{2}}} \|_{L_x^4([0,\infty))} \|W\|_{L^\infty}$$

For $x < 0$ we take a dyadic partition of unity

$$1 = \sum_{j \geq 0} \varphi_j$$

and consider

$$\begin{aligned}
\|u_{\mathrm{app}}\|_{L_x^4((-\infty,0); L_T^\infty)} &\lesssim \left(\sum_{j\geq 0} \|u_{\mathrm{app}} \varphi_j (t^{-\frac{1}{2}} x_-^{\frac{1}{2}})\|^2_{L_x^4((-\infty,0); L_T^\infty)} \right)^{\frac{1}{2}} \\
&\lesssim \|T^{-\frac{1}{3}} \langle T^{-\frac{1}{3}} x\rangle^{-\frac{1}{4}} \|_{l^\infty L_x^4} \|W\|_{l^2 L^\infty} \\
&\lesssim T^{-\frac{1}{4}} \|W\|_{H^1},
\end{aligned}$$

where the last line follows from Sobolev embedding (2.10) and Cauchy-Schwarz.

For (4.22) we calculate

$$\begin{aligned}
\partial_x \Lambda u_{\mathrm{app}} = &-2Q_{wyy} \partial_x \mathcal{W} - 3t^{\frac{1}{3}} Q_{wwy} (\partial_x \mathcal{W})^2 - 3t^{\frac{1}{3}} Q_{wy} \partial_x^2 \mathcal{W} \\
&- 3t^{\frac{2}{3}} Q_{ww} \partial_x \mathcal{W} \partial_x^2 \mathcal{W} - t^{\frac{2}{3}} Q_w \partial_x^3 \mathcal{W} - t^{\frac{2}{3}} Q_{www} (\partial_x \mathcal{W})^3.
\end{aligned}$$

We then define $X = Tt^{-1}x$ and observe that

$$\|g(t, t^{-1}x)\|_{L_x^\infty L_T^2} \lesssim \|T^{\frac{1}{2}} X^{-\frac{1}{2}} g(TX^{-1}x, X)\|_{L_x^\infty L_X^2 (X \sim x)}.$$

Making this change of variables, we may then estimate as in Lemma 4.1 to get (4.22).

□

4.3. **An equation for $v = u - u_{\mathrm{app}}$.** We now define the function f such that u_{app} satisfies the equation

$$(\partial_t + \tfrac{1}{3}\partial_x^3) u_{\mathrm{app}} = \sigma (u_{\mathrm{app}}^3)_x + f. \tag{4.23}$$

If we define $v = u - u_{\mathrm{app}}$ then (1.22) becomes

$$\begin{cases} v_t + \frac{1}{3} v_{xxx} = N(u_{\mathrm{app}}, v) - f, \\ \lim_{t\to+\infty} v(t) = 0, \end{cases} \tag{4.24}$$

where

$$N(u_{\mathrm{app}}, v) = \sigma \left((v + u_{\mathrm{app}})^3 - u_{\mathrm{app}}^3 \right)_x.$$

We define the norms

$$\|u\|_Z = \sup_{T\geq 1}\left\{T^{\frac{1}{3}+\frac{\delta}{3}}\|u\|_{L^\infty_T L^2_x} + T^{\frac{1}{4}+\frac{\delta}{3}}\|v\|_{L^4_x L^\infty_T} + T^{\frac{\delta}{3}}\|u_x\|_{L^\infty_T L^2_x}\right\},$$

$$\|u\|_{\tilde{Z}} = \sup_{T\geq 1}\left\{\frac{T^{\frac{\delta}{3}}}{1+\epsilon^2\log T}\left(\|u\|_{L^\infty_T L^2_x} + \|u_x\|_{L^2_x L^\infty_T}\right)\right\},$$

and look to solve (4.24) using a contraction mapping argument in the space

$$Z_\epsilon = \{v : \|v\|_Z + \|\Gamma v\|_{\tilde{Z}} \leq B\epsilon\}, \tag{4.25}$$

where

$$\Gamma v = \Lambda(v + u_{\text{app}}) - \Lambda u_{\text{app}}.$$

If we define Φ as in (2.13) then the solution to (4.24) satisfies

$$v = \Phi N - \Phi f, \qquad \Gamma v = \Phi\tilde{N} - \Phi\tilde{f},$$

where

$$\tilde{N} = 3\sigma(v+u_{\text{app}})^2(\Gamma v)_x + 3\sigma(v^2 + 2vu_{\text{app}})\partial_x\Lambda u_{\text{app}},$$
$$\tilde{f} = Lf + 9\sigma t u_{\text{app}}^2 f.$$

4.4. **Nonlinear estimates.** For the nonlinear term we have the following estimates.

Lemma 4.4. *Let $T \geq 1$ be a dyadic integer and $v_1, v_2 \in Z_\epsilon$ where Z_ϵ is defined as in* (4.25)*. Then, if δ is defined as in* (1.30)*, for $M_0 > 0$ chosen sufficiently large and $\epsilon > 0$ chosen sufficiently small we have the estimates*

$$\|\Phi(N(u_{\text{app}}, v_1) - N(u_{\text{app}}, v_2))\|_Z \ll \|v_1 - v_2\|_Z \tag{4.26}$$

$$\|\Phi(\tilde{N}(u_{\text{app}}, v_1) - \tilde{N}(u_{\text{app}}, v_2))\|_{\tilde{Z}} \ll \|v_1 - v_2\|_Z + \|\Gamma v_1 - \Gamma v_2\|_{\tilde{Z}} \tag{4.27}$$

Proof. It suffices to consider $v_1 = v$, $v_2 = 0$ as the general case follows by applying identical estimates. From (2.14) we have

$$\|\Phi N\|_Z \lesssim \sup_{T_0\geq 1}\left(T_0^{\frac{1+\delta}{3}}\sum_{T\geq T_0}\|D_x^{-1}N\|_{L^1_x L^2_T} + T_0^{\frac{\delta}{3}}\sum_{T\geq T_0}\|N\|_{L^1_x L^2_T}\right),$$

$$\|\Phi\tilde{N}\|_{\tilde{Z}} \lesssim \sup_{T_0\geq 1}\left(\frac{T_0^{\frac{\delta}{3}}}{1+\epsilon^2\log T_0}\sum_{T\geq T_0}T^{\frac{1}{2}}\|\tilde{N}\|_{L^2_x L^2_T}\right),$$

where we assume T, T_0 are dyadic integers.

Using (4.21) we can estimate

$$\begin{aligned}\|(v+u_{\text{app}})^3 - u_{\text{app}}^3\|_{L^1_x L^2_T} &\lesssim (\|v\|_{L^4_x L^\infty_T} + \|u_{\text{app}}\|_{L^4_x L^\infty_T})^2\|v\|_{L^2_T L^2_x}\\ &\lesssim T^{-\frac{1+\delta}{3}}(T^{-\frac{\delta}{3}}\|v\|_Z + \epsilon)^2\|v\|_Z,\end{aligned}$$

and similarly, using (4.17) and (4.21),

$$\|((v+u_{\text{app}})^3 - u_{\text{app}}^3)_x\|_{L^1_x L^2_T} \lesssim T^{-\frac{\delta}{3}}(T^{-\frac{\delta}{3}}\|v\|_Z + \epsilon)^2\|v\|_Z.$$

Summing over $T \geq T_0$ and using that $\delta^{-1}\epsilon^2 \lesssim M_0^{-1} \ll 1$ we have (4.26).

Similarly, for $\Phi\tilde{N}$ we estimate

$$\begin{aligned}\|(v+u_{\mathrm{app}})^2(\Gamma v)_x\|_{L^2_xL^2_T} &\lesssim (\|v\|_{L^4_xL^\infty_T}+\|u_{\mathrm{app}}\|_{L^4_xL^\infty_T})\|(\Gamma v)_x\|_{L^\infty_xL^2_T}\\ &\lesssim T^{-\frac12-\frac{\delta}{3}}(1+\epsilon^2\log T)(T^{-\frac{\delta}{3}}\|v\|_Z+\epsilon)^2\|\Gamma v\|_{\tilde{Z}},\end{aligned}$$

$$\|(v^2+2vu_{\mathrm{app}})\partial_x\Lambda u_{\mathrm{app}}\|_{L^2_xL^2_T} \lesssim T^{-\frac12-\frac{\delta}{3}}\epsilon(1+\epsilon^2\log T)(T^{-\frac{\delta}{3}}\|v\|_Z+\epsilon)\|v\|_Z.$$

□

4.5. **Inhomogeneous estimates.** In order to complete the proof of Theorem 1.4 we prove the following estimates for the inhomogeneous terms $f,\tilde{f}$.

Lemma 4.5. *We have the estimates*

$$\|\Phi f\|_Z \lesssim \epsilon, \tag{4.28}$$

$$\|\Phi\tilde{f}\|_{\tilde{Z}} \lesssim \epsilon. \tag{4.29}$$

Proof. We start by calculating

$$\begin{aligned}f(t,x) &= t^{-\frac13}Q_w\mathcal{W}_t + 6\sigma t^{-1}Q^2Q_w\partial_x\mathcal{W} + t^{-\frac23}Q_{wwy}(\partial_x\mathcal{W})^2\\ &\quad + t^{-\frac23}Q_{wy}\partial_x^2\mathcal{W} + \tfrac13 t^{-\frac13}Q_{www}(\partial_x\mathcal{W})^3 + t^{-\frac13}Q_{ww}\partial_x\mathcal{W}\partial_x^2\mathcal{W} + \tfrac13 t^{-\frac13}Q_w\partial_x^3\mathcal{W},\end{aligned}$$

where we have used that $\partial_x^N\mathcal{W}=0$ for $x\geq 0$, for $x<0$

$$\partial_t(\mathcal{W}(t,t^{-\frac12}x_-^{\frac12})) = -t^{-1}x\partial_x(\mathcal{W}(t,t^{-\frac12}x_-^{\frac12})) + \mathcal{W}_t(t,t^{-\frac12}x_-^{\frac12}),$$

and that Q_w satisfies the equation

$$yQ_w - Q_{yyw} + 9\sigma Q^2Q_w = 0.$$

Estimating Φf. We claim that

$$\|f\|_{L^2} \lesssim \epsilon t^{-\frac{4+\delta}{3}}, \tag{4.30}$$

and hence

$$\|f\|_{L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{1+\delta}{3}}.$$

To prove (4.30) we estimate the first term using (4.6), (4.7) and (4.12). For the second term we use (4.6), (4.11) and (4.12) to get

$$\begin{aligned}\|t^{-1}Q^2Q_w\partial_x\mathcal{W}\|_{L^2} &\lesssim t^{-1}\|\mathcal{W}\|_{L^\infty}^2\|\partial_x\mathcal{W}\|_{L^\infty}\|\langle t^{-\frac13}x\rangle^{-\frac34}(1+\mathcal{W}^2\log\langle t^{-\frac13}x\rangle)\|_{L^2_x}\\ &\lesssim \epsilon^3 t^{-\frac{4+\delta}{3}}.\end{aligned}$$

For the remaining terms we can estimate similarly using (4.4), (4.6) and Lemma 4.2.

Next we decompose $f_x = g_0 + b_0$, where we define the bad part

$$b_0 = t^{-\frac23}Q_{wy}\mathcal{W}_t + t^{-1}Q_{wyy}\partial_x^2\mathcal{W}.$$

From Lemmas 4.1 and 4.2 we may estimate the good part by

$$\|g_0\|_{L^2} \lesssim \epsilon t^{-\frac{5+\delta}{3}} + \epsilon^3 t^{-(1+\frac{\delta}{3})},$$

so using (1.30) we have

$$\|g_0\|_{L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{\delta}{3}}.$$

For the bad part we have the estimate

$$\|b_0\|_{L^2} \lesssim \epsilon t^{-(1+\frac{\delta}{3})},$$

which is not quite sufficient to prove (4.28). As $Q(y;W)$ is odd as a function of W, for $W \in \mathbb{R}$ we have

$$Q_w(y;W) = \mathrm{Ai}(y) + P(y;W),$$

where

$$|P(y;W)| \lesssim \langle y\rangle^{-\frac{1}{4}} W^2 \log\langle y\rangle e^{-\frac{2}{3}y_+^{\frac{3}{2}}}.$$

We then define

$$b_1 = t^{-\frac{2}{3}} \mathrm{Ai}_y \, \mathcal{W}_t + t^{-1} \mathrm{Ai}_{yy} \, \partial_x^2 \mathcal{W},$$

and if $g_1 = b_0 - b_1$, we have

$$\|g_1\|_{L^2_x} \lesssim \epsilon^3 t^{-(1+\frac{\delta}{3})},$$

so we may include g_1 in g.

For the remaining terms we first remove the low frequency component by defining $\tilde{\mathbf{W}} = \mathbf{W} - W_0$ and the corresponding $\tilde{\mathcal{W}}$. We then have

$$g_2 = t^{-\frac{2}{3}} \mathrm{Ai}_y(\mathcal{W}_t - \tilde{\mathcal{W}}_t) + t^{-1} \mathrm{Ai}_{yy} \, \partial_x^2(\mathcal{W} - \tilde{\mathcal{W}}),$$

which satisfies the improved estimate

$$\|g_2\|_{L^2} \lesssim \epsilon t^{-\frac{11}{6}},$$

so it remains to consider $b = b_1 - g_2$.

For each $j \geq 1$ we decompose

$$\chi_j = \sum_{k \geq 0} \chi_j^{(k)},$$

where the $\chi_j^{(k)}$ are localised at frequency 2^j and spatially localised on the set $\{t^{\frac{1}{3}}\langle t^{\frac{1}{3}} z\rangle^{\frac{1}{2}} \gtrsim 2^j\} \cap \{|z| \sim 2^k\}$ up to rapidly decaying tails at scale 2^{-j}. We then define

$$W_{jk}(t,z) = \chi_j^{(k)}(t,z) W_j(z),$$

and observe that from almost orthogonality and (1.24)

$$\sum_{j\geq 1}\sum_{k\geq 0} 2^{2\delta j + 2k} \|W_{jk}\|_{L^2}^2 \lesssim \epsilon^2.$$

Further, due to the localisation, up to rapidly decaying tails, we can assume that on the support of W_{jk} we have $t \gtrsim 2^{2j-k}$.

Defining b_{jk} to be the corresponding part of b, we have the estimate

$$\|b_{jk}\|_{L^2} \lesssim (t^{-2} 2^{2j} + t^{-\frac{11}{6}} 2^{\frac{3j}{2}}) \|W_{jk}\|_{L^2},$$

so using the localisation of W_{jk}, we have

$$\sum_{j\geq 1}\sum_{k\geq 0} \|S(-t) b_{jk}\|_{L^1([T,\infty);L^2)}^2 \lesssim \sum_{j\geq 1}\sum_{k\geq 0} T^{-\frac{2\delta}{3}} 2^{2\delta j + 2k} \|W_{jk}\|_{L^2}^2.$$

In particular we have the estimate

$$\|S(-t) b\|_{l^2 L^1([T,\infty);L^2)} \lesssim \epsilon T^{-\frac{\delta}{3}},$$

where the l^2 summation is in j,k. From (2.17), for any fixed $p \in (1,2)$ we have

$$\|\Phi b\|_{l^2 V_S^p} \lesssim \epsilon T^{-\frac{\delta}{3}}$$

Finally we use the embeddings (2.15), (2.16) and that the U^2 norm commutes with the square summation to complete the proof of (4.28).

Estimating $\Phi\tilde{f}$. Using (4.16) and (4.30) we have

$$\|tu_{\mathrm{app}}^2 f\|_{L^2} \lesssim \epsilon^2 t^{-(1+\frac{\delta}{3})}.$$

We then estimate Lf in a similar manner to f_x, defining a bad part

$$\begin{aligned} b_0 = & -2t^{\frac{1}{3}} Q_{wy}\partial_x \mathcal{W}_t - t^{\frac{2}{3}} Q_w \partial_x^2 \mathcal{W}_t - t^{-\frac{1}{3}} Q_w \partial_x^2 \mathcal{W} - 2Q_{wyy}\partial_x^3 \mathcal{W} \\ & - 3t^{\frac{1}{3}} Q_{wy}\partial_x^4 \mathcal{W} - t^{\frac{2}{3}} Q_w \partial_x^5 \mathcal{W}, \end{aligned}$$

and estimating the good part using Lemmas 4.1 and 4.2 to get

$$\|Lf - b_0\|_{L^2} \lesssim \epsilon^3 t^{-(1+\frac{\delta}{3})}.$$

An identical analysis of the bad part b_0, using that

$$\sum_{j\geq 1}\sum_{k\geq 0} 2^{2(1+\delta)j}\|W_{jk}\|_{L^2}^2 \lesssim \|W\|_{H^{1+\delta}}^2,$$

gives us the estimate (4.29). □

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Appendix A. Short-range perturbations

In this appendix we briefly outline some modifications to Theorems 1.1 and 1.4 in the case of short-range perturbations (1.2).

When $p \in [\frac{7}{2}, \infty)$ the results are essentially unchanged. For $p \in (3, \frac{7}{2})$ the energy estimate (3.2) fails for δ defined as in (1.30). This is due to the fact that $v = \Lambda u$ satisfies the equation

$$\begin{cases} v_t + \frac{1}{3} v_{xxx} = (3\sigma u^2 + F'(u))v_x + 3F(u) - F'(u)u, \\ v(0) = xu_0. \end{cases}$$

So, if $\epsilon > 0$ is sufficiently small, estimating $\|\Lambda v\|_{L^2}$ as in (3.2) we have to re-define

$$\delta = \frac{7-2p}{6} \in (0, \tfrac{1}{6}). \tag{A.1}$$

In particular, for sufficiently small $\epsilon > 0$, the loss of regularity in (1.15) is controlled by p rather than ϵ, giving us the revised estimate

$$\|W\|_{H^{1-C\delta,1}\cap L^\infty(0,\infty)} \lesssim \epsilon.$$

A corresponding modification is needed in Theorem 1.4. Here we define the space Y by replacing $C\epsilon^2$ by $C\delta$ in (1.23) with δ defined as in (A.1).

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