JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Binary Operations

Czesław Byliński
Warsaw University
Białystok

Summary. In this paper we define binary and unary operations on domains. We also define the following predicates concerning the operations: ... is commutative, ... is associative, ... is the unity of ..., and ... is distributive wrt .... A number of schemes useful in justifying the existence of the operations are proved.

MML Identifier: `BINOP_1`.

WWW: `http://mizar.org/JFM/Vol1/binop_1.html`

The articles [4], [3], [5], [6], [1], and [2] provide the notation and terminology for this paper.

Let f be a function and let a, b be sets. The functor $f(a, b)$ yielding a set is defined by:

(Def. 1) $f(a,b) = f(\langle a, b \rangle)$.

In the sequel A is a set.

Let A, B be non empty sets, let C be a set, let f be a function from $[: A, B :]$ into C, let a be an element of A, and let b be an element of B. Then $f(a, b)$ is an element of C.

The following proposition is true

(2)[1] Let A, B, C be non empty sets and f_1, f_2 be functions from $[: A, B :]$ into C. Suppose that for every element a of A and for every element b of B holds $f_1(a, b) = f_2(a, b)$. Then $f_1 = f_2$.

Let A be a set. A unary operation on A is a function from A into A. A binary operation on A is a function from $[: A, A :]$ into A.

We adopt the following convention: u is a unary operation on A, o, o' are binary operations on A, and a, b, c, e, e_1, e_2 are elements of A.

In this article we present several logical schemes. The scheme *BinOpEx* deals with a non empty set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

> There exists a binary operation o on $\mathcal{A}$ such that for all elements a, b of $\mathcal{A}$ holds $\mathcal{P}[a, b, o(a, b)]$

provided the following condition is satisfied:

- For all elements x, y of $\mathcal{A}$ there exists an element z of $\mathcal{A}$ such that $\mathcal{P}[x, y, z]$.

The scheme *BinOpLambda* deals with a non empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

> There exists a binary operation o on $\mathcal{A}$ such that for all elements a, b of $\mathcal{A}$ holds $o(a, b) = \mathcal{F}(a, b)$

for all values of the parameters.

Let us consider A, o. We say that o is commutative if and only if:

(Def. 2) For all a, b holds $o(a, b) = o(b, a)$.

[1] The proposition (1) has been removed.

We say that o is associative if and only if:

(Def. 3) For all a, b, c holds $o(a, o(b, c)) = o(o(a, b), c)$.

We say that o is idempotent if and only if:

(Def. 4) For every a holds $o(a, a) = a$.

Let us mention that every binary operation on $\emptyset$ is empty, associative, and commutative.
Let us consider A, e, o. We say that e is a left unity w.r.t. o if and only if:

(Def. 5) For every a holds $o(e, a) = a$.

We say that e is a right unity w.r.t. o if and only if:

(Def. 6) For every a holds $o(a, e) = a$.

Let us consider A, e, o. We say that e is a unity w.r.t. o if and only if:

(Def. 7) e is a left unity w.r.t. o and a right unity w.r.t. o.

We now state several propositions:

(11)[2] e is a unity w.r.t. o iff for every a holds $o(e, a) = a$ and $o(a, e) = a$.

(12) If o is commutative, then e is a unity w.r.t. o iff for every a holds $o(e, a) = a$.

(13) If o is commutative, then e is a unity w.r.t. o iff for every a holds $o(a, e) = a$.

(14) If o is commutative, then e is a unity w.r.t. o iff e is a left unity w.r.t. o.

(15) If o is commutative, then e is a unity w.r.t. o iff e is a right unity w.r.t. o.

(16) If o is commutative, then e is a left unity w.r.t. o iff e is a right unity w.r.t. o.

(17) If e_1 is a left unity w.r.t. o and e_2 is a right unity w.r.t. o, then $e_1 = e_2$.

(18) If e_1 is a unity w.r.t. o and e_2 is a unity w.r.t. o, then $e_1 = e_2$.

Let us consider A, o. Let us assume that there exists e which is a unity w.r.t. o. The functor $\mathbf{1}_o$ yields an element of A and is defined by:

(Def. 8) $\mathbf{1}_o$ is a unity w.r.t. o.

Let us consider A, o', o. We say that o' is left distributive w.r.t. o if and only if:

(Def. 9) For all a, b, c holds $o'(a, o(b, c)) = o(o'(a, b), o'(a, c))$.

We say that o' is right distributive w.r.t. o if and only if:

(Def. 10) For all a, b, c holds $o'(o(a, b), c) = o(o'(a, c), o'(b, c))$.

Let us consider A, o', o. We say that o' is distributive w.r.t. o if and only if:

(Def. 11) o' is left distributive w.r.t. o and right distributive w.r.t. o.

The following propositions are true:

(23)[3] o' is distributive w.r.t. o iff for all a, b, c holds $o'(a, o(b, c)) = o(o'(a, b), o'(a, c))$ and $o'(o(a, b), c) = o(o'(a, c), o'(b, c))$.

(24) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is distributive w.r.t. o if and only if for all elements a, b, c of A holds $o'(a, o(b, c)) = o(o'(a, b), o'(a, c))$.

[2] The propositions (3)–(10) have been removed.
[3] The propositions (19)–(22) have been removed.

(25) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is distributive w.r.t. o if and only if for all elements a, b, c of A holds $o'(o(a, b), c) = o(o'(a, c), o'(b, c))$.

(26) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is distributive w.r.t. o if and only if o' is left distributive w.r.t. o.

(27) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is distributive w.r.t. o if and only if o' is right distributive w.r.t. o.

(28) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is right distributive w.r.t. o if and only if o' is left distributive w.r.t. o.

Let us consider A, u, o. We say that u is distributive w.r.t. o if and only if:

(Def. 12) For all a, b holds $u(o(a, b)) = o(u(a), u(b))$.

Let A be a non empty set and let o be a binary operation on A. Let us observe that o is commutative if and only if:

(Def. 13) For all elements a, b of A holds $o(a, b) = o(b, a)$.

Let us observe that o is associative if and only if:

(Def. 14) For all elements a, b, c of A holds $o(a, o(b, c)) = o(o(a, b), c)$.

Let us observe that o is idempotent if and only if:

(Def. 15) For every element a of A holds $o(a, a) = a$.

Let A be a non empty set, let e be an element of A, and let o be a binary operation on A. Let us observe that e is a left unity w.r.t. o if and only if:

(Def. 16) For every element a of A holds $o(e, a) = a$.

Let us observe that e is a right unity w.r.t. o if and only if:

(Def. 17) For every element a of A holds $o(a, e) = a$.

Let A be a non empty set and let o', o be binary operations on A. Let us observe that o' is left distributive w.r.t. o if and only if:

(Def. 18) For all elements a, b, c of A holds $o'(a, o(b, c)) = o(o'(a, b), o'(a, c))$.

Let us observe that o' is right distributive w.r.t. o if and only if:

(Def. 19) For all elements a, b, c of A holds $o'(o(a, b), c) = o(o'(a, c), o'(b, c))$.

Let A be a non empty set, let u be a unary operation on A, and let o be a binary operation on A. Let us observe that u is distributive w.r.t. o if and only if:

(Def. 20) For all elements a, b of A holds $u(o(a, b)) = o(u(a), u(b))$.

REFERENCES

[1] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[5] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[6] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received April 14, 1989

Published January 2, 2004