LARGE PROPORTION OF THE ZEROS OF THE RIEMANN ZETA FUNCTION ON THE CRITICAL LINE[1]

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Abstract. Following the method of Levinson and Conrey and applying an analytic identity we prove that large proportion (at least 47%) of the zeros of the Riemann zeta function is on the critical line. We briefly discuss a generalization of the argument that leads to the statement that almost all of the zeta zeros are critical.

Contents

1. Introduction. The Riemann zeta-function $\zeta(s)$ is defined for $\Re s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

and for other s by the analytic continuation. It is a meromorphic function in the whole complex plane with the only singularity $s = 1$, which is a simple pole with residue 1.

The Euler product links the zeta-function and prime numbers: for $\Re s > 1$

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1}.$$

The functional equation for $\zeta(s)$ may be written in the form

$$\xi(s) = \xi(1-s),$$

where $\xi(s)$ is an entire function defined by

$$\xi(s) = H(s)\zeta(s)$$

with

$$H(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

This implies that $\zeta(s)$ has zeros at $s = -2, -4, \ldots$ These zeros are called the "trivial" zeros. It is known that $\zeta(s)$ has infinitely many nontrivial zeros $s = \rho = \beta + i\gamma$, and all of them are in the "critical strip" $0 < \Re s = \sigma < 1$, $-\infty < \Im s = t < \infty$. The pair of nontrivial zeros with the smallest value of $|\gamma|$ is $\frac{1}{2} \pm i(14.134725\ldots)$.

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If $N(T)$ denotes the number of zeros $\rho = \beta + i\gamma$ (β and γ real), for which $0 < \gamma \leqslant T$, then

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

with

$$S(T) = \frac{1}{\pi} \arg\zeta\left(\frac{1}{2} + iT\right)$$

and

$$S(T) = O(\log T).$$

This is the Riemann–von Mangoldt formula for $N(T)$.

Let $N_0(T)$ be the number of zeros of $\zeta\left(\frac{1}{2} + it\right)$ when $0 < t \leqslant T$, each zero counted with multiplicity. The Riemann hypothesis is the conjecture that $N_0(T) = N(T)$. Let

$$\kappa = \liminf_{T\to\infty} \frac{N_0(T)}{N(T)}.$$

Important results about $N_0(T)$ include:

- [H14]: Hardy proved that $N_0(T) \to \infty$ as $T \to \infty$.
- [HL21]: Hardy and Littlewood obtained that $N_0(T) \geqslant AT$ for some $A > 0$ and all sufficiently large T.
- [Sel42]: Selberg proved that $\kappa \geqslant A$ for an effectively computable positive constant A.
- [Lev74]: Levinson proved that $\kappa \geqslant 0.34\ldots$
- [Con89]: Conrey obtained $\kappa \geqslant 0.40\ldots$
- [Fen12]: Feng obtained $\kappa \geqslant 0.41\ldots$

In this article we establish

Theorem 1. *We have*

$$\kappa \geqslant 0.47\ldots$$

The key idea is Lemma 1 which allows to "shift" the sum in Conrey's construction [Con83] by

$$\Delta\sigma = \frac{2\pi - \varepsilon}{\log T}.$$

The rest of the argument is an application of the method of Levinson and Conrey (see Iwaniec' lecture notes [Iw12]).

We then describe how to generalize the argument to get

$$\kappa = 1.$$

2. Main lemma.

Lemma 1. *Let $f(s)$ be an analytic function, $s \in \mathbb{C}$, $\Delta\sigma \in \mathbb{R}$, $\mathcal{K} \geqslant 1$ be an odd integer. Then*

$$f(s+\Delta\sigma) = f(s) + \sum_{\substack{k \text{ odd} \\ k \leqslant \mathcal{K}}} \left(g_k(\Delta\sigma) f^{(k)}(s) + g_k(\Delta\sigma) f^{(k)}(s+\Delta\sigma) \right)$$
$$+ \frac{4(-1)^{(\mathcal{K}+1)/2}(\Delta\sigma)^{\mathcal{K}+1}}{\pi^{\mathcal{K}+2}} \int\limits_{s}^{s+\Delta\sigma} f^{(\mathcal{K}+2)}(w) \left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{\mathcal{K}+2}} \sin\left(\frac{(2n-1)\pi(s+\Delta\sigma-w)}{\Delta\sigma} \right) \right) dw, \tag{1}$$

where

$$g_k(\Delta\sigma) = \frac{4(-1)^{(k-1)/2}(\Delta\sigma)^k}{\pi^{k+1}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{k+1}}.$$

Proof. We use induction on $\mathcal{K}$. First establish induction base $\mathcal{K} = 1$. We have

$$f(s+\Delta\sigma) = f(s) + \int\limits_{s}^{s+\Delta\sigma} f'(w)\, dw.$$

Let $\operatorname{sgn}_{2\Delta\sigma}(x)$ be the $2\Delta\sigma$-periodic real-valued function defined by

$$\operatorname{sgn}_{2\Delta\sigma}(x) = \begin{cases} 1 & \text{if } x \in (0, \Delta\sigma), \\ 0 & \text{if } x = -\Delta\sigma, 0, \Delta\sigma, \\ -1 & \text{if } x \in (-\Delta\sigma, 0). \end{cases}$$

Using the Fourier expansion

$$\operatorname{sgn}_{2\Delta\sigma}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi x}{\Delta\sigma} \right) \tag{2}$$

we obtain

$$f(s+\Delta\sigma) = f(s) + \frac{4}{\pi} \int\limits_{s}^{s+\Delta\sigma} f'(w) \left(\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi(s+\Delta\sigma-w)}{\Delta\sigma} \right) \right) dw,$$
$$f(s+\Delta\sigma) = f(s) + \frac{4}{\pi} \left(\int\limits_{s}^{s+\varepsilon} \cdots + \int\limits_{s+\varepsilon}^{s+\Delta\sigma-\varepsilon} \cdots + \int\limits_{s+\Delta\sigma-\varepsilon}^{s+\Delta\sigma} \cdots \right).$$

The series (2) converges uniformly in $x \in [\varepsilon, \Delta\sigma - \varepsilon]$ so by integrating by parts

$$\begin{aligned}
&\frac{4}{\pi}\int\limits_{s+\varepsilon}^{s+\Delta\sigma-\varepsilon} f'(w)\, d\left(\sum_{n=1}^{\infty}\frac{\Delta\sigma}{(2n-1)^2\pi}\cos\left(\frac{(2n-1)\pi(s+\Delta\sigma-w)}{\Delta\sigma}\right)\right)\\
=&\frac{4\Delta\sigma}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\Big(f'(s+\Delta\sigma-\varepsilon)+f'(s+\varepsilon)\Big)\\
-&\frac{4\Delta\sigma}{\pi^2}\int\limits_{s+\varepsilon}^{s+\Delta\sigma-\varepsilon} f''(w)\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi(s+\Delta\sigma-w)}{\Delta\sigma}\right)\right)dw+\delta_1(\varepsilon)\\
=&g_1(\Delta\sigma)\Big(f'(s)+f'(s+\Delta\sigma)\Big)\\
-&\frac{4\Delta\sigma}{\pi^2}\int\limits_{s}^{s+\Delta\sigma} f''(w)\, d\left(\sum_{n=1}^{\infty}-\frac{\Delta\sigma}{(2n-1)^3\pi}\sin\left(\frac{(2n-1)\pi(s+\Delta\sigma-w)}{\Delta\sigma}\right)\right)+\delta_2(\varepsilon)\\
=&g_1(\Delta\sigma)\Big(f'(s)+f'(s+\Delta\sigma)\Big)\\
-&\frac{4\Delta\sigma}{\pi^2}\left(-\int\limits_{s}^{s+\Delta\sigma}-\frac{\Delta\sigma}{\pi}f'''(w)\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\left(\frac{(2n-1)\pi(s+\Delta\sigma-w)}{\Delta\sigma}\right)\right)dw\right)+\delta_3(\varepsilon),
\end{aligned}$$

and $\delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon) \to 0$ as $\varepsilon \to 0$. This proves the induction base. The induction step is proven by integrating by parts in (1) as above, with the uniform convergence of the series in the integrand when $\mathcal{K} \geqslant 1$.

Remark. We have

$$g_1(\Delta\sigma) = \frac{4\Delta\sigma}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{4\Delta\sigma}{\pi^2}\zeta(2)\left(1-\frac{1}{2^2}\right) = \frac{\Delta\sigma}{2},$$

and in general for k odd

$$g_k(\Delta\sigma) = \frac{4(-1)^{(k-1)/2}(\Delta\sigma)^k}{\pi^{k+1}}\zeta(k+1)\left(1-\frac{1}{2^{k+1}}\right) = -\left(\frac{\Delta\sigma}{2}\right)^k\frac{2^{k+1}-4^{k+1}}{(k+1)!}B_{k+1}, \qquad (3)$$

where B_{k+1} is the Bernoulli number.

The series

$$\sum_{\substack{k\geqslant 1\\ k \text{ odd}}}\left(g_k(\Delta\sigma)f^{(k)}(s)+g_k(\Delta\sigma)f^{(k)}(s+\Delta\sigma)\right)$$

obtained by successive integrations by parts in (1) may be divergent. However, we have the following

Lemma 2. *Suppose that* $0<\varepsilon<2\pi$, $\alpha = 2\pi-\varepsilon$ *and*

$$\Delta\sigma = \frac{\alpha}{\log T} = \frac{2\pi-\varepsilon}{\log T}.$$

Then the series

$$\sum_{\substack{k\geqslant 1\\ k \text{ odd}}}\Big(-g_k(\Delta\sigma)\Big)(\log T)^k\left(\frac{1}{2}-x\right)^k$$

converges on $x \in [0,1]$ *and*

$$\sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \Big(-g_k(\Delta\sigma)\Big)(\log T)^k \left(\frac{1}{2} - x\right)^k = -\tanh\left(\frac{\alpha}{2}\left(\frac{1}{2} - x\right)\right).$$

Proof. From (3) we have

$$\Big(-g_k(\Delta\sigma)\Big)(\log T)^k \left(\frac{1}{2} - x\right)^k = \frac{2\left(\alpha\left(\frac{1}{2} - x\right)\right)^k B_{k+1}}{(k+1)!} - \frac{4(\alpha(1-2x))^k B_{k+1}}{(k+1)!}.$$

By the definition of the Bernoulli numbers,

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m,$$

with $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_3 = B_5 = B_7 = \cdots = 0$, the radius of convergence of the series being 2π. Then

$$\begin{aligned} &2\sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{\left(\alpha\left(\frac{1}{2} - x\right)\right)^k B_{k+1}}{(k+1)!} - 4\sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{(\alpha(1-2x))^k B_{k+1}}{(k+1)!} \\ =&\frac{2}{\alpha(1/2 - x)}\left(\frac{\alpha(1/2 - x)}{e^{\alpha(1/2 - x)} - 1} - 1 + \frac{\alpha(1/2 - x)}{2}\right) - \frac{4}{\alpha(1-2x)}\left(\frac{\alpha(1-2x)}{e^{\alpha(1-2x)} - 1} - 1 + \frac{\alpha(1-2x)}{2}\right), \end{aligned}$$

and the lemma follows.

Lemma 3. *Suppose that* α *is real and*

$$\Delta\sigma = \frac{\alpha}{\log T}.$$

Then in the rectangle

$$s = \sigma + it, \quad \frac{1}{3} \leqslant \sigma \leqslant A, \quad T \leqslant t \leqslant 2T$$

with $A \geqslant 3$ *and* $T \geqslant 2A$ *we have*

$$H(s + \Delta\sigma) = \left(e^{\alpha/2} + O\left(\frac{1}{\log T}\right)\right) H(s).$$

3. Proof of Theorem 1 and a remark on $\kappa = 1$. Suppose that $0 < \varepsilon < 2\pi$, $\alpha = 2\pi - \varepsilon$ and

$$\Delta\sigma = \frac{\alpha}{\log T} = \frac{2\pi - \varepsilon}{\log T}.$$

Define $G(s + \Delta\sigma) = G_{\varepsilon_1, \mathcal{K}, \Delta\sigma}(s + \Delta\sigma)$ by

$$2e^{-\alpha/2} H(s + \Delta\sigma) G(s + \Delta\sigma) = \xi(s) + \varepsilon_1 \xi(s) + \sum_{\substack{k \text{ odd} \\ k \leqslant \mathcal{K}}} g_k(\Delta\sigma) \xi^{(k)}(s),$$

where ε_1 is a real number to be chosen later. For k odd $\xi^{(k)}(s)$ is purely imaginary on the line $\Re s = \frac{1}{2}$, so

$$\Re 2e^{-\alpha/2} H(s + \Delta\sigma) G(s + \Delta\sigma) = (1 + \varepsilon_1)\xi(s) \qquad \text{if } \Re s = \frac{1}{2}.$$

That is, the critical zeros of $\zeta(s)$ are precisely the points on the line $\Re s = \frac{1}{2}$ for which we have either $G(s+\Delta\sigma) = 0$ or

$$G(s+\Delta\sigma) \neq 0 \qquad \text{and} \qquad \arg H(s+\Delta\sigma)G(s+\Delta\sigma) \equiv \frac{\pi}{2} \pmod{\pi}.$$

Next for

$$\alpha = 2\pi - \varepsilon, \qquad \Delta\sigma = \frac{\alpha}{\log T} = \frac{2\pi - \varepsilon}{\log T}$$

we obtain (see [Iw12])

$$G(s+\Delta\sigma) = \sum_{l \leqslant T} Q_{1,\varepsilon_1,\mathcal{K},\alpha}\left(\frac{\log l}{\log T} + \delta_1(s)\right) l^{-s} + O\left(T^{-\frac{1}{4}}\right),$$

where

$$\delta_1(s) \ll \frac{1}{\log T}$$

and $Q_{1,\varepsilon_1,\mathcal{K},\alpha}(x)$ is the polynomial

$$Q_{1,\varepsilon_1,\mathcal{K},\alpha}(x) = \frac{1+\varepsilon_1}{2} + \frac{2}{\pi} \sum_{\substack{k \text{ odd} \\ k \leqslant \mathcal{K}}} \left(1 - \frac{\varepsilon}{2\pi}\right)^k (-1)^{(k-1)/2} \zeta(k+1) \left(1 - \frac{1}{2^{k+1}}\right) (1-2x)^k.$$

Now we choose ε_1 so that the polynomial $Q_{1,\varepsilon_1,\mathcal{K},\alpha}(x)$ satisfies

$$Q_{1,\varepsilon_1,\mathcal{K},\alpha}(0) = 1.$$

Let $N_{01}(T,2T)$ denote the number of zeros of $\zeta(s)$ when

$$s = \rho = \frac{1}{2} + i\gamma, \qquad T \leqslant \gamma \leqslant 2T,$$

counted without multiplicity. By estimating the argument variations we get

$$N_{01}(T,2T) \geqslant N(T,2T) - 2N_G(\mathcal{R}) + O(T),$$

where $N(T,2T)$ is the number of all zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $T \leqslant \gamma \leqslant 2T$ counted with multiplicity, and $N_G(\mathcal{R})$ denotes the number of zeros counted with multiplicity of $G(s+\Delta\sigma)$ inside the closed rectangle $\mathcal{R}$ that has the segment

$$\Re s = \frac{1}{2}, \qquad T \leqslant \Im s \leqslant 2T$$

as its left side, with small circular dents to the right centered at the common critical zeros of $\zeta(s)$ and $G(s+\Delta\sigma)$, and that has the segment

$$\Re s = A, \qquad T \leqslant \Im s \leqslant 2T$$

with A a large enough constant as its right side.

By Lemma 1 we can write

$$2e^{-\alpha/2} H(s+\Delta\sigma)G(s+\Delta\sigma) = \xi(s+\Delta\sigma) + \sum_{\substack{k \text{ odd} \\ k \leqslant \mathcal{K}}} \Big(-g_k(\Delta\sigma)\Big)\xi^{(k)}(s+\Delta\sigma) + \mathcal{R}_1(\mathcal{K}, s, \Delta\sigma, \varepsilon_1),$$

where

$$\mathcal{R}_1(\mathcal{K}, s, \Delta\sigma, \varepsilon_1) = \varepsilon_1 \xi(s)$$
$$+ \frac{4(-1)^{(\mathcal{K}-1)/2}(\Delta\sigma)^{\mathcal{K}+1}}{\pi^{\mathcal{K}+2}} \int_s^{s+\Delta\sigma} \xi^{(\mathcal{K}+2)}(w) \left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{\mathcal{K}+2}} \sin\left(\frac{(2n-1)\pi(s+\Delta\sigma-w)}{\Delta\sigma} \right) \right) dw,$$

and

$$-g_k(\Delta\sigma) = \frac{4(-1)^{(k+1)/2}(\Delta\sigma)^k}{\pi^{k+1}} \zeta(k+1) \left(1 - \frac{1}{2^{k+1}}\right).$$

From this we obtain another representation for $e^{-\alpha/2}G(s+\Delta\sigma) = e^{-\alpha/2}G_{\varepsilon_1,\mathcal{K},\Delta\sigma}(s+\Delta\sigma)$:

$$e^{-\alpha/2}G(s+\Delta\sigma) = \sum_{l \leqslant T} Q_{\mathcal{K},\alpha}\left(\frac{\log l}{\log T} + \delta(s)\right) l^{-(s+\Delta\sigma)} + \frac{\varepsilon_1}{2}(1+\delta_0(s)) \sum_{l \leqslant T} l^{-s}$$
$$+ \sum_{l \leqslant T} Q_{2,\mathcal{K},\Delta\sigma}\left(\frac{\log l}{\log T} + \delta_{2,\mathcal{K},\Delta\sigma}(s)\right) l^{-(s+\Delta\sigma)} + \sum_{l \leqslant T} Q_{3,\mathcal{K},\Delta\sigma}\left(\frac{\log l}{\log T} + \delta_{3,\mathcal{K},\Delta\sigma}(s)\right) l^{-s},$$

where

$$\delta(s), \delta_0(s), \delta_{2,\mathcal{K},\Delta\sigma}(s), \delta_{3,\mathcal{K},\Delta\sigma}(s) \ll \frac{1}{\log T}$$

and $Q_{\mathcal{K},\alpha}(x)$ is the polynomial

$$Q_{\mathcal{K},\alpha}(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{\substack{k \text{ odd} \\ k \leqslant \mathcal{K}}} \left(1 - \frac{\varepsilon}{2\pi}\right)^k (-1)^{(k+1)/2} \zeta(k+1) \left(1 - \frac{1}{2^{k+1}}\right) (1-2x)^k.$$

The polynomials $Q_{2,\mathcal{K},\Delta\sigma}(x)$ and $Q_{3,\mathcal{K},\Delta\sigma}(x)$ are obtained by making the integration in $\mathcal{R}_1(\mathcal{K}, s, \Delta\sigma, \varepsilon_1)$. Note that by Lemma 2 we have

$$Q_1(x) = \lim_{\substack{\mathcal{K}\to\infty \\ \alpha \to 2\pi-}} Q_{1,\varepsilon_1,\mathcal{K},\alpha}(x) = \frac{1}{2} - \frac{1}{2}\tanh\left(\pi\left(x - \frac{1}{2}\right)\right) + \frac{\varepsilon_1}{2}, \tag{4}$$

so that $Q_1(0) = 1$ implies $\frac{\varepsilon_1}{2} = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\pi}{2}\right)$ and

$$Q_1(x) = 1 - \frac{1}{2}\tanh\left(\pi\left(x - \frac{1}{2}\right)\right) - \frac{1}{2}\tanh\left(\frac{\pi}{2}\right).$$

Also by Lemma 2

$$Q(x) = \lim_{\substack{\mathcal{K}\to\infty \\ \alpha \to 2\pi-}} Q_{\mathcal{K},\alpha}(x) = \frac{1}{2} + \frac{1}{2}\tanh\left(\pi\left(x - \frac{1}{2}\right)\right), \tag{5}$$

hence

$$Q(0) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\pi}{2}\right).$$

Consider

$$F_{\varepsilon_1,\mathcal{K},\Delta\sigma,R}(s+\Delta\sigma) = e^{-\alpha/2}G_{\varepsilon_1,\mathcal{K},\Delta\sigma}(s+\Delta\sigma) M_{\varepsilon_1,\mathcal{K},\Delta\sigma,R}(s+\Delta\sigma)$$

with the mollifier

$$M_{\varepsilon_1,\mathcal{K},\Delta\sigma,R}(s+\Delta\sigma) = \sum_{m \leqslant T^\theta} \mu(m) P_{\varepsilon_1,\mathcal{K},\Delta\sigma}\left(\frac{\log m}{\log T}\right) m^{-\left(s+\Delta\sigma - \frac{R}{\log T}\right)},$$

and the real-valued function $P_{\varepsilon_1,\mathcal{K},\Delta\sigma}(x)$ such that

$$P_{\varepsilon_1,\mathcal{K},\Delta\sigma}(0) = \frac{1}{Q_{\mathcal{K},\alpha}(0) + \varepsilon_1/2 + Q_{2,\mathcal{K},\Delta\sigma}(0) + Q_{3,\mathcal{K},\Delta\sigma}(0)}.$$

We then have $N_G(\mathcal{R}) \leqslant N_F(\mathcal{R})$ and applying the Littlewood lemma for

$$a = \frac{1}{2} + \frac{R}{\log T}, \quad 0 \leqslant \frac{R}{\log T} < \Delta\sigma = \frac{\alpha}{\log T} = \frac{2\pi - \varepsilon}{\log T}$$

we arrive at the following analog of the principal inequality of the Levinson–Conrey method:

Theorem 2. *Suppose that ε, R are fixed, $0 < \varepsilon < 2\pi$, $0 \leqslant R < 2\pi - \varepsilon$, $\mathcal{K}$ is a fixed large odd integer, T goes to infinity,*

$$\Delta\sigma = \frac{2\pi - \varepsilon}{\log T}, \quad a = \frac{1}{2} + \frac{R}{\log T}.$$

Let $N_{00}(T, 2T)$ be the number of zeros $s = \rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$ counted without multiplicity which are not zeros of $G_{\varepsilon_1,\mathcal{K},\Delta\sigma}(s + \Delta\sigma)$. Then

$$N_{00}(T, 2T) \geqslant N(T, 2T) \left(1 - \frac{2}{2\pi - \varepsilon - R} \log I(R) + O\left(\frac{1}{\log T}\right)\right),$$

where

$$I(R) = \frac{1}{T} \int_T^{2T} |F_{\varepsilon_1,\mathcal{K},\Delta\sigma,R}(a + it)|\, dt.$$

From this theorem, letting $\mathcal{K} \to \infty$ and $\varepsilon \to 0+$ (so that $\alpha \to 2\pi-$), by (4) and (5) and the estimate for the mean square of the Dirichlet polynomial we obtain

$$\kappa \geqslant 1 - \frac{2}{2\pi - R} \log\left(C(R)^{\frac{1}{2}} + c(R)^{\frac{1}{2}}\right).$$

Here

$$C(R) = P(0)^2 S(0)^2 + \frac{P(0)^2 S(1)^2}{2} - \frac{P(0)^2 S(0)^2}{2} + (AA_1)^{\frac{1}{2}} P(0)^2 \frac{\cosh(\lambda\theta)}{\sinh(\lambda\theta)},$$

with

$$P(0) = \frac{1}{1 - \tanh\left(\frac{\pi}{2}\right)}, \quad S(y) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\pi\left(y - \frac{1}{2}\right)\right)\right) e^{-Ry} + \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\pi}{2}\right)\right)$$

and

$$A = \int_0^1 S(y)^2 dy, \quad A_1 = \int_0^1 (S'(y))^2, \quad \lambda = (A_1/A)^{\frac{1}{2}}, \quad \theta = \frac{4}{7}.$$

The jump from $S(1)$ to 0 accounts for the term $c(R)$. The choice $R = \pi$ gives $S(1) = S(0) = \frac{1}{P(0)}$ and

$$\kappa \geqslant 0.47\ldots$$

Remark on $\kappa = 1$. The obstacle to proving that almost all of the zeta zeros are critical is that $\alpha < 2\pi$. This could be avoided if one proves that for α arbitrarily large the analytic function given for $\Re s > 1$ by

$$g_{\alpha,T}(s) = -\frac{1}{2} \sum_{l=1}^{\infty} \tanh\left(\frac{\alpha}{2}\left(\frac{\log l}{\log T} - \frac{1}{2}\right)\right) l^{-s}$$

obeys two types of symmetries:

1. For $\Delta\sigma = \frac{\alpha}{\log T}$

$$2e^{\alpha/2}\left(\frac{\zeta(s+\Delta\sigma)}{2} - g_{\alpha,T}(s+\Delta\sigma)\right) = 2\left(\frac{\zeta(s)}{2} + g_{\alpha,T}(s)\right). \tag{6}$$

2. For $s = \sigma + it$, $T \leqslant t \leqslant 2T$, the function

$$H(s)g_{\alpha,T}(s) + \text{small perturbation} \tag{7}$$

   is purely imaginary for $\Re s = \frac{1}{2}$.

To prove (6) we note that

$$\tanh\left(\frac{u}{2}\right) = \frac{2}{e^{-u}+1} - 1$$

and substitute this with

$$u = \alpha\left(\frac{\log l}{\log T} - \frac{1}{2}\right)$$

into the left-hand side of (6), obtaining the Dirichlet series

$$\sum_{l=1}^{\infty} \frac{2e^{\alpha/2}}{\left(e^{-\alpha\left(\frac{\log l}{\log T}-\frac{1}{2}\right)}+1\right) l^s e^{\alpha\frac{\log l}{\log T}}}.$$

In the right-hand side of (6) we use

$$\tanh\left(\frac{u}{2}\right) = -\frac{2}{e^{u}+1} + 1$$

obtaining

$$\sum_{l=1}^{\infty} \frac{2}{\left(e^{\alpha\left(\frac{\log l}{\log T}-\frac{1}{2}\right)}+1\right) l^s}.$$

The two Dirichlet series are the same.

To prove (7), we note that

$$\tanh\left(\frac{u}{2}\right)$$

is an odd function of u, so for $u \in [-A, A]$ it can be uniformly approximated by finite sums of the odd powers of u. Thus

$$H(s)g_{\alpha,T}(s)$$

is approximated by odd derivatives of the ξ function with real coefficients, which are purely imaginary on the critical line.

Letting $\alpha \to +\infty$ in the above argument we obtain

$$\kappa = 1.$$

References

[Con83] J. B. Conrey, *Zeros of derivatives of the Riemann's ξ-function on the critical line,* J. Number Theory **16** (1983), 49–74.

[Con89] J. B. Conrey, *More than two fifths of the zeros of the Riemann zeta function are on the critical line,* J. reine angew. Math. **399** (1989), 1–26.

[Fen12] S. Feng, *Zeros of the Riemann zeta function on the critical line,* J. Number Theory **132** (2012), 511–542.

[H14] G. H. Hardy, *Sur les zéros de la fonction $\zeta(s)$ de Riemann,* C. R. **158** (1914), 1012–1014.

[HL21] G. H. Hardy and J. E. Littlewood, *The zeros of Riemann's zeta-function on the critical line,* Math. Z. **10** (1921), 283–317.

[Iw12] H. Iwaniec, Lectures on the Riemann Zeta Function (Rutgers, Fall 2012).

[Lev74] N. Levinson, *More than one third of zeros of Riemann's zeta-function are on $\sigma = \frac{1}{2}$,* Adv. Math. **13** (1974), 383–436.

[Sel42] A. Selberg, *On the zeros of Riemann's zeta-function,* Skr. Norske Vid. Akad. Oslo **10** (1942), 1–59.