Maxima of independent, non-identically distributed Gaussian vectors

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Let $X_{i,n}, n \in \mathbb{N}, 1 \le i \le n$, be a triangular array of independent $\mathbb{R}^d$-valued Gaussian random vectors with covariance matrices $\Sigma_{i,n}$. We give necessary conditions under which the row-wise maxima converge to some max-stable distribution which generalizes the class of Hüsler-Reiss distributions. In the bivariate case the conditions will also be sufficient. Using these results, new models for bivariate extremes are derived explicitly. Moreover, we define a new class of stationary, max-stable processes as limits of suitably normalized and randomly rescaled maxima of n independent Gaussian processes whose finite dimensional margins coincide with the above limit distributions. As an application, we show that these processes realize a large set of extremal correlation functions, a natural dependence measure for max-stable processes. This set includes all functions $\psi(\sqrt{\gamma(h)}), h \in \mathbb{R}^d$, where ψ is a completely monotone function and γ is an arbitrary variogram.

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1. Introduction

It is well known that the standard normal distribution Φ is in the max-domain of attraction of the Gumbel distribution, i.e.

$$\lim_{n\to\infty} \Phi(b_n + x/b_n)^n = \exp(-\exp(-x)), \quad \text{for all } x \in \mathbb{R},$$

where b_n is a sequence of normalizing constants defined by

$$b_n := \sqrt{2\log n} - \frac{(1/2)\log\log n + \log(2\sqrt{\pi})}{\sqrt{2\log n}} + o\left((\log n)^{-1/2}\right), \tag{1}$$

such that $b_n = n\phi(b_n)$, where ϕ is the standard normal density. Sibuya [22] showed that the maxima of i.i.d. bivariate normal distributions with correlation $\rho < 1$ asymptotically always become independent. However, for triangular arrays with i.i.d. entries where the correlation in the different rows approaches 1 with an appropriate speed, Hüsler and Reiss [15] proved that the row-wise maxima converge to a new class of max-stable bivariate distributions, namely

$$F_\lambda(x, y) = \exp\left[-\Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y} - \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x}\right], \quad x, y \in \mathbb{R}. \tag{2}$$

Here $\lambda \in [0, \infty]$ parameterizes the dependence in the limit, 0 and ∞ corresponding to complete dependence and asymptotic independence, respectively. In fact, Kabluchko et al. [18] provide a simple argument that these are also the only possible limit points for such triangular arrays.

More generally, Hüsler and Reiss [15] consider triangular arrays with i.i.d. entries of d-variate zero-mean, unit-variance normal distributions with covariance matrix Σ_n in the n-th row satisfying

$$\lim_{n\to\infty} b_n^2(\mathbf{1}\mathbf{1}^\top - \Sigma_n)/2 = \Lambda \in [0, \infty)^{d\times d}, \tag{3}$$

where $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^d$ and $^\top$ denotes the transpose sign. Under this assumption, the row-wise maxima converge to the d-variate, max-stable Hüsler-Reiss distribution whose dependence structure is fully characterized by the matrix Λ. Note that condition (3) implies that all off-diagonal entries of Σ_n converge to 1 as $n \to \infty$. A slightly more general representation is given in Kabluchko [17] in terms of Poisson point processes and negative definite kernels.

In fact, it turns out that these distributions not only attract Gaussian arrays but also classes of related distributions. For instance, Hashorva [12] shows, that the convergence of maxima holds for triangular arrays of general bivariate elliptical distributions, if the random radius is in the domain of attraction of the Gumbel distribution. The generalization to multivariate elliptical distributions can be found in Hashorva [13]. Moreover, Hashorva et al. [14] prove that also non-elliptical distributions are in the domain of attraction of the Hüsler-Reiss distribution, for instance multivariate χ^2-distributions.

Apart from being one of the few known parametric families of multivariate extreme value distributions, the Hüsler-Reiss distributions play a prominent role in modeling spatial extremes since they are the finite dimensional distributions of Brown-Resnick processes [6, 18].

In this paper we consider independent triangular arrays $\mathbf{X}_{i,n}$, $n \in \mathbb{N}$ and $1 \le i \le n$, where $\mathbf{X}_{i,n}$ is a zero-mean Gaussian random vector with covariance matrix $\Sigma_{i,n}$. Thus, in each row the random variables are independent, but may have different dependence structures. We are interested in the convergence of row-wise maxima

$$\mathbf{M}_n = \max_{i=1,\dots,n} b_n(\mathbf{X}_{i,n} - b_n), \tag{4}$$

as $n \to \infty$, and the respective limit distributions.

In Section 2 we start with bivariate triangular arrays. For this purpose, we introduce a sequence of counting measures which capture the dependence structure in each row and which is used to state necessary and sufficient conditions for the convergence of $\mathbf{M}_n$ in (4). Moreover, the limits turn out to be new max-stable distributions that generalize (2). The results on triangular arrays are used to completely characterize the max-limits of independent sequences of bivariate Gaussian vectors. Explicit examples for the bivariate limit distributions are given at the end of this section. The multivariate case is treated in Section 3, giving rise to a class of d-dimensional max-stable distributions. In Section 4, we show how these distributions arise as the finite dimensional margins of suitably normalized and randomly rescaled maxima of n independent Gaussian processes. In fact, the limit processes are max-stable, stationary random fields which can be seen as max-mixtures of Brown-Resnick processes. The extremal dependence structure of the bivariate mixture distributions is analyzed in Section 5 in terms of the spectral measure. Furthermore, it is shown that the processes from Section 4 offer a large variety of extremal correlation functions which makes them interesting for modeling dependencies in spatial extremes. Finally, Section 6 comprises the proofs of the main theorems.

2. The bivariate case

In order to state the main results in the bivariate case, we need probability measures on the extended positive half-line $[0,\infty]$. To this end, let $([0,\infty],d)$ be a compact metric space such that a function $g:[0,\infty]\to\mathbb{R}$ is continuous iff it is continuous in the usual topology on $[0,\infty)$ and the limit $\lim_{x\to\infty} g(x)$ exists in $\mathbb{R}$.
Consider a triangular array of independent bivariate Gaussian random vectors $\mathbf{X}_{i,n} = (X^{(1)}_{i,n}, X^{(2)}_{i,n})$, $n\in\mathbb{N}$ and $1\le i\le n$, with zero expectation and covariance matrix

$$\mathrm{Cov}(\mathbf{X}_{i,n}) = \begin{pmatrix} \sigma^2_{i,n,1} & \sigma_{i,n,1,2} \\ \sigma_{i,n,1,2} & \sigma^2_{i,n,2} \end{pmatrix}.$$

Further, denote by $\rho_{i,n} = \sigma_{i,n,1,2}/(\sigma_{i,n,1}\sigma_{i,n,2})$ the correlation of $\mathbf{X}_{i,n}$. For $n\in\mathbb{N}$, we define a probability measure η_n on $[0,\infty]\times\mathbb{R}^2$ by

$$\eta_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\left(\sqrt{b_n^2(1-\rho_{i,n})/2},\, b_n^2(1-1/\sigma_{i,n,1}),\, b_n^2(1-1/\sigma_{i,n,2})\right)}$$

which keeps track of the variances and correlations in each row. In this general situation, the next theorem gives a sufficient condition in terms of η_n for the convergence of row-wise maxima of this triangular array.

Theorem 1. *For $n\in\mathbb{N}$ and $1\le i\le n$, let $\mathbf{X}_{i,n}$ and η_n be defined as above. Further suppose that for some $\epsilon>0$ the measures $(\eta_n)_{n\in\mathbb{N}}$ satisfy the uniform integrability condition*

$$\lim_{K\to\infty}\sup_{n\in\mathbb{N}}\int_{[0,\infty]\times(K,\infty)^2} e^{\theta(1+\epsilon)} + e^{\gamma(1+\epsilon)}\,\eta_n(d(\lambda,\theta,\gamma)) = 0. \tag{5}$$

If for $n \to \infty$, η_n converges weakly to some probability measure η on $[0,\infty] \times \mathbb{R}^2$, i.e. $\eta_n \Rightarrow \eta$, then

$$\max_{i=1,\dots,n} b_n(\mathbf{X}_{i,n} - b_n) \tag{6}$$

converges in distribution to a random vector with distribution function F_η given by

$$-\log F_\eta(x,y) = \int\limits_{[0,\infty]\times\mathbb{R}^2} \Phi\left(\lambda + \frac{y-x+\theta-\gamma}{2\lambda}\right) e^{-(x-\theta)} + \Phi\left(\lambda - \frac{y-x+\theta-\gamma}{2\lambda}\right) e^{-(y-\gamma)}\, \eta(d(\lambda,\theta,\gamma)),$$

for $x, y \in \mathbb{R}$.

Remark 1. *An equivalent condition for the uniform integrability in (5) is that for some $\tau > 0$*

$$\sup_{n\in\mathbb{N}} \int_{[0,\infty]\times\mathbb{R}^2} e^{\theta(1+\tau)} + e^{\gamma(1+\tau)}\, \eta_n(d(\lambda,\theta,\gamma)) < \infty.$$

Remark 2. *In fact, one can extend the distribution F_η to mixture measures η taking infinite mass at negative infinity. The only condition which needs to be satisfied is*

$$\int_{[0,\infty]\times\mathbb{R}^2} e^{\theta} + e^{\gamma}\, \eta(d(\lambda,\theta,\gamma)) < \infty.$$

Note that the one-dimensional marginals of F_η are Gumbel distributed with different location parameters, for instance

$$-\log F_\eta(x,\infty) = \exp\big[-x + \log \int\limits_{[0,\infty]\times\mathbb{R}^2} e^{\theta}\, \eta(d(\lambda,\theta,\gamma))\big].$$

Moreover, F_η is a max-stable distribution since

$$F_\eta^n(x + \log n, y + \log n) = F_\eta(x,y),$$

for all $n \in \mathbb{N}$. This is a remarkable fact, since in general row-wise maxima of triangular arrays are not max-stable, not even if the random variables in each row are identically distributed.

In order to obtain a necessary condition and to simplify the sufficient condition, we need to impose stronger assumptions on the univariate margins. We denote by $\mathcal{M}_1([0,\infty])$ the space of all probability measures on $[0,\infty]$. By Helly's theorem this space is sequentially compact.

Theorem 2. *Consider a triangular array of independent bivariate Gaussian random vectors* $\mathbf{X}_{i,n} = (X^{(1)}_{i,n}, X^{(2)}_{i,n})$*,* $n \in \mathbb{N}$ *and* $1 \le i \le n$*, where* $X^{(1)}_{i,n}$ *and* $X^{(2)}_{i,n}$ *are standard normal random variables. Denote by* $\rho_{i,n}$ *the correlation of* $\mathbf{X}_{i,n}$*. Let*

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\sqrt{b_n^2(1-\rho_{i,n})/2}} \tag{7}$$

be a probability measure on $[0,\infty]$*. For* $n \to \infty$*,*

$$\max_{i=1,\dots,n} b_n(\mathbf{X}_{i,n} - b_n) \tag{8}$$

converges in distribution if and only if ν_n *converges weakly to some probability measure* ν *on* $[0,\infty]$*, i.e.* $\nu_n \Rightarrow \nu$*. In this case, the limit of* (8) *has distribution function* F_ν *given by*

$$-\log F_\nu(x,y) = \int_0^\infty \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x} + \Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y}\,\nu(d\lambda), \tag{9}$$

$x, y \in \mathbb{R}$*. Furthermore,* F_ν *depends continuously on* ν*, in the sense that if* $\nu_n \Rightarrow \nu$*, as* $n \to \infty$*, and* $\nu_n, \nu \in \mathcal{M}_1([0,\infty])$*, then* F_{ν_n} *converges pointwise to* F_ν*.*

Remark 3. *For an arbitrary probability measure* $\nu \in \mathcal{M}_1([0,\infty])$*, let* $(R_i)_{i\in\mathbb{N}}$ *be a sequence of i.i.d. samples of* ν*. Putting* $\rho_{i,n} = \max(1 - 2R_i^2/b_n^2, -1)$ *in Theorem 2 yields*

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{R_i} \Rightarrow \nu, \quad a.s.,$$

by the law of large numbers. Hence, (8) *converges a.s. in distribution to* F_ν*. On the other hand, for two probability measures* $\nu, \tilde\nu \in \mathcal{M}_1([0,\infty])$ *with* $\nu \ne \tilde\nu$*, it follows from the proof of Theorem 2 below that* $F_\nu \ne F_{\tilde\nu}$*.*

Remark 4. *If* ν *is a probability measure on* $[0,\infty)$*, an alternative construction of the distribution* F_ν *is the following [17, Section 3]: Let* $\sum_{i=1}^\infty \delta_{U_i}$ *be a Poisson point process on* $\mathbb{R}$ *with intensity* $e^{-u}du$ *and suppose that* B *has the normal distribution* $N(-2S^2, 4S^2)$ *with random mean and variance, where* S *is* ν*-distributed. Then, for a sequence* $(B_i)_{i\in\mathbb{N}}$ *of i.i.d. copies of* B*, the bivariate random vector* $\max_{i\in\mathbb{N}}(U_i, U_i + B_i)$ *has distribution* F_ν*.*

The above theorem can be applied to completely characterize the maxima of a sequence of independent bivariate Gaussian random vectors with unit variance.

Corollary 1. *Suppose that* $\mathbf{X}_i = (X^{(1)}_i, X^{(2)}_i)$*,* $n \in \mathbb{N}$ *and* $1 \le i \le n$*, is a sequence of independent bivariate Gaussian random vectors where* $X^{(1)}_i$ *and* $X^{(2)}_i$ *are standard* *normal random variables. Denote by* ρ_i *the correlation of* $\mathbf{X}_i$ *and let*

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\sqrt{b_n^2(1-\rho_i)/2}}$$

be a probability measure on $[0,\infty]$. *For* $n \to \infty$,

$$\max_{i=1,\dots,n} b_n(\mathbf{X}_i - b_n) \tag{10}$$

converges in distribution if and only if ν_n *converges weakly to some probability measure* ν *on* $[0,\infty]$. *In this case, the limit of* (10) *has distribution function* F_ν *as in* (9). *Furthermore, for all* $\nu \in \mathcal{M}_1([0,\infty])$, F_ν *is attained as a limit of* (10) *for a suitable sequence* $(\mathbf{X}_i)_{i\in\mathbb{N}}$.

Remark 5. *It is worthwhile to note that in general, the class of max-selfdecomposable distributions, i.e. the max-limits of* sequences *of independent (not necessarily identically distributed) random variables, is a proper subclass of max-infinitely-divisible distributions, i.e. the max-limits of* triangular arrays *with i.i.d. random variables in each row. The latter coincides with the class of max-limits of* triangular arrays, *where the rows are merely independent but not identically distributed [1, 9]. In the (bivariate) Gaussian case the above shows that the max-limits of i.i.d.* triangular arrays, *namely the Hüsler-Reiss distributions in* (2), *are a proper subclass of max-limits of independent* triangular arrays, *namely the distributions in* (9), *which, on the other hand, coincide with the max-limits of independent* sequences.

The max-stable distributions F_ν in Theorem 2 for $\nu \in \mathcal{M}_1([0,\infty])$ are max-mixtures of Hüsler-Reiss distributions with different dependency parameters. They constitute a large class of new bivariate max-stable distributions. We derive two of them explicitly by evaluating the integral in (9).

Rayleigh distributed ν

The Rayleigh distribution has density

$$f_\sigma(\lambda) = \frac{\lambda}{\sigma^2} e^{-\frac{\lambda^2}{2\sigma^2}}, \quad \lambda \ge 0, \tag{11}$$

for $\sigma > 0$. Choosing the dependence parameter λ according to the Rayleigh distribution ν_σ, we obtain the bivariate distribution function

$$-\log F_{\nu_\sigma}(x,y) = \int_0^\infty \left[\Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x} + \Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y}\right]\frac{\lambda}{\sigma^2}e^{-\frac{\lambda^2}{2\sigma^2}}\,d\lambda, \tag{12}$$

for $x, y \in \mathbb{R}$. In order to evaluate this integral, we apply partial integration and use formulae 3.471.9 and 3.472.3 in Gradshteyn and Ryzhik [11]. Equation (12) then simplifies to

$$F_{\nu_\sigma}(x,y) = \exp\left[-e^{-\min(x,y)} - \frac{1}{\eta} e^{-\frac{y+x}{2}} e^{-\frac{|y-x|\eta}{2}}\right], \quad x, y \in \mathbb{R}, \tag{13}$$

where $\eta = \sqrt{1 + 1/\sigma^2} \in (1, \infty)$. Note that σ parameterizes the dependence of F_{ν_σ}. As σ goes to 0 (i.e., η goes to ∞), then the margins become equal. On the other hand, as σ goes to ∞ (i.e., η goes to 1), then the margins become completely independent.

Type-2 Gumbel distributed ν

The Type-2 Gumbel distribution has density

$$f_b(\lambda) = 2b\lambda^{-3} e^{-\frac{b}{\lambda^2}}, \quad \lambda \geq 0, \tag{14}$$

for $b > 0$. With similar arguments as for the Rayleigh distribution the distribution function F_{ν_b}, where ν_b has density f_b, is given by

$$F_{\nu_b}(x,y) = \exp\left[-e^{-x} - e^{-y} + e^{-\frac{y+x}{2}} e^{-\sqrt{\left(\frac{y-x}{2}\right)^2 + 2b}}\right], \quad x, y \in \mathbb{R}.$$

Also in this case, the parameter $b \in (0, \infty)$ interpolates between complete independence and complete dependence of the bivariate distribution. In particular, if $b \to 0$, then the margins are equal and, on the other hand, if $b \to \infty$ then the margins are independent.

3. The multivariate case

Similarly as in Hüsler and Reiss [15], the results for standard bivariate Gaussian random vectors can be generalized to d-dimensional random vectors. To this end, define a triangular array of independent d-dimensional, non-degenerate (i.e. with positive definite covariance matrix) Gaussian random vectors $\mathbf{X}_{i,n} = (X_{i,n}^{(1)}, \dots, X_{i,n}^{(d)})$, $n, d \in \mathbb{N}$ and $1 \leq i \leq n$, where $X_{i,n}^{(j)}$, $j \in \{1, \dots, d\}$, are standard normal random variables. Denote by $\Sigma_{i,n} = (\rho_{j,k}(i,n))_{1 \leq j,k \leq d}$ the correlation matrix of $\mathbf{X}_{i,n}$. Let $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^d$ and

$$\eta_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\sqrt{b_n^2(\mathbf{1}\mathbf{1}^\top - \Sigma_{i,n})/2}} \tag{15}$$

be a probability measure on the metric space $[0, \infty)^{d \times d}$, equipped with the product metric. Throughout this paper, square roots of matrices are to be understood componentwise. For a measure τ on $[0, \infty)^{d \times d}$ we will denote by τ^2 the measure given by $\tau^2(d\Lambda) =$

$\tau(d\sqrt{\Lambda})$. Further, let $D \subset [0,\infty)^{d\times d}$ be the subspace of strictly (conditionally) negative definite matrices which are symmetric and have zeros on the diagonal, i.e.

$$D := \Bigg\{ (a_{j,k})_{1\le j,k\le d} = A \in [0,\infty)^{d\times d} : \mathbf{x}^\top A\mathbf{x} < 0 \text{ for all } \mathbf{x}\in\mathbb{R}^d \setminus \{0\} \text{ s.t. } \sum_{i=1}^d x_i = 0,\ a_{j,k} = a_{k,j}, a_{j,j} = 0 \text{ for all } 1 \le j,k \le d \Bigg\}.$$

In particular, note that D is a suitable subspace for the measures η_n^2 since $\eta_n^2(D) = 1$ for all $n \in \mathbb{N}$. For $\Lambda = (\lambda_{j,k})_{1\le j,k\le d} \in [0,\infty)^{d\times d}$, define a family of transformed matrices by

$$\Gamma_{l,m}(\Lambda) = 2\left(\lambda^2_{m_j,m_l} + \lambda^2_{m_k,m_l} - \lambda^2_{m_j,m_k}\right)_{1\le j,k\le l-1},$$

where $2 \le l \le d$ and $m = (m_1,\dots,m_l)$ with $1 \le m_1 < \cdots < m_l \le d$. It follows from the proof of Lemma 2.1 in Berg et al. [3] that if $\Lambda \in D$, then $\Gamma_{l,m}(\sqrt{\Lambda})$ is a (strictly) positive definite matrix.

Denote by $S(\,\cdot\,|\Psi)$ the so-called survivor function of an l-dimensional normal random vector with mean vector $\mathbf{0}$ and covariance matrix Ψ. That is, if $\mathbf{X} \sim N(\mathbf{0},\Psi)$ and $\mathbf{x} \in \mathbb{R}^l$, then $S(\mathbf{x}|\Psi) = \mathbb{P}(X_1 > x_1,\dots,X_l > x_l)$. If Ψ is not a covariance matrix we put $S(\mathbf{x}|\Psi) = 0$.

For a fixed $\Lambda = (\lambda_{j,k})_{1\le j,k\le d} \in [0,\infty)^{d\times d}$, let

$$H_\Lambda(\mathbf{x}) = \exp\left(\sum_{l=1}^d (-1)^l \sum_{m:1\le m_1<\cdots<m_l\le d} h_{l,m,\Lambda}(x_{m_1},\dots,x_{m_l})\right),$$

where

$$h_{l,m,\Lambda}(y_1,\dots,y_l) = \int_{y_l}^\infty S\left(\left(y_i - z + 2\lambda^2_{m_i,m_l}\right)_{i=1,\dots,l-1} |\Gamma_{l,m}(\Lambda)\right) e^{-z}\,dz,$$

for $2 \le l \le d$ and $h_{1,m,\Lambda}(y) = e^{-y}$ for $m = 1,\dots,d$. For an alternative representation of the multivariate Hüsler-Reiss distribution H_Λ see Joe [16]. With this notation we are now in a position to state the following theorem.

Theorem 3. *Consider a triangular array of independent d-dimensional Gaussian random vectors as above. If for $n \to \infty$ the measure η_n in (15) converges weakly to some probability measure η on $[0,\infty)^{d\times d}$, i.e. $\eta_n \Rightarrow \eta$, s.t. $\eta^2(D) = 1$, then*

$$\max_{i=1,\dots,n} b_n(\mathbf{X}_{i,n} - b_n)$$

converges in distribution to a random vector with distribution function H_η given by

$$H_\eta(x_1,\dots,x_d) = \exp\left(\int_{[0,\infty)^{d\times d}} \log H_\Lambda(x)\,\eta(d\Lambda)\right), \quad x \in \mathbb{R}^d. \tag{16}$$

Remark 6. *We believe that the above theorem also holds in the case when η has positive measure on* non-strictly *conditionally negative definite matrices, i.e. $\eta^2(D) < 1$. Our proof of this theorem however breaks down in this situation such that another technique might be necessary.*

4. Application to Brown-Resnick processes

The d-dimensional Hüsler-Reiss distributions arise in the theory of maxima of Gaussian random fields as the finite dimensional distributions of Brown-Resnick processes [6] and its generalizations [18]. In this section we introduce a new class of max-stable processes with finite dimensional distributions given by (16) for suitable measures η. In fact, they are the max-limits of Gaussian random fields with random scaling in space.

Let us briefly recall the definition of the processes introduced in Kabluchko et al. [18]. For a zero-mean Gaussian process $\{W(t), t \in \mathbb{R}^d\}$ with stationary increments, variance $\sigma^2(t)$ and variogram $\gamma(t) = \mathbb{E}(W(t) - W(0))^2$, consider i.i.d. copies $\{W_i, i \in \mathbb{N}\}$ of W and a Poisson point process $\sum_{i=1}^{\infty} \delta_{U_i}$ on $\mathbb{R}$ with intensity $e^{-u}du$, independent of the family $W_i, i \in \mathbb{N}$. Kabluchko et al. [18] showed that the stochastic process

$$\xi(t) = \max_{i \in \mathbb{N}} (U_i + W_i(t) - \sigma(t)^2/2), \quad t \in \mathbb{R}^d, \tag{17}$$

is max-stable and stationary with standard Gumbel margins and that its law depends only on the variogram γ. Furthermore, their results imply that the subclass belonging to variograms of fractional Brownian motions on $\mathbb{R}^d$, i.e. $\gamma(t) = \|t\|^\alpha, \alpha \in (0, 2)$, are the max-limits of suitably rescaled Gaussian processes whose covariance functions satisfy the following regular variation condition [18, Theorem 17]. Let $\{X(t), t \in \mathbb{R}^d\}$ be a zero-mean, unit variance Gaussian process with covariance function $C(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)]$. Assume that

$$\lim_{\epsilon \to 0} \frac{1 - C(\epsilon t_1, \epsilon t_2)}{L(\epsilon)\epsilon^\alpha} = 2\gamma(t_1 - t_2) \tag{18}$$

holds uniformly for bounded $t_1, t_2 \in \mathbb{R}^d$, where L is continuous and slowly varying at 0 and $\gamma(t) = \|t\|^\alpha$, $\alpha \in (0, 2)$, is a continuous variogram. Further, define normalizing sequences b_n as above and

$$s_n = \inf\{s > 0 : L(s)s^\alpha = b_n^{-2}\}. \tag{19}$$

In the next theorem we prove, that by introducing a random space scaling in Theorem 17 in Kabluchko et al. [18], we obtain new stochastic processes which have finite dimensional distributions given by (16) for a certain measure η.

Theorem 4. *Let X_i, $i \in \mathbb{N}$, be independent copies of the process X above, satisfying the regular variation assumption (18). Further, let S_i, $i \in \mathbb{N}$, be independent random variables* *distributed according to a probability measure* ν *on* $(0,\infty)$*. Then the finite dimensional distributions of the process*

$$Y_n(t) = \max_{i=1,\dots,n} b_n(X_i(S_i^{2/\alpha} s_n t) - b_n), \quad t \in \mathbb{R}^d,$$

converge to the distribution in (16)*. More precisely, for* $t_1,\dots,t_m \in \mathbb{R}^d$*, the respective measure* η *is concentrated on the subspace* $\{\lambda\Lambda_0, \lambda > 0\}$ *with* $\Lambda_0 = (\sqrt{\gamma(t_j - t_k)})_{1\le j,k\le m}$ *and is given by*

$$\eta(d\lambda\Lambda_0) = \nu(d\lambda), \tag{20}$$

and by 0 *everywhere else.*

Remark 7. *Note that, by Kolmogorov's extension theorem, the family of finite dimensional limit distributions in the above theorem gives rise to a new stationary, max-stable stochastic process* $\{Y(t), t \in \mathbb{R}^d\}$ *on* $\mathbb{R}^d$*.*
In fact, it is possible to define this max-stable processes via another construction which allows for a broader class of variograms than those in (18)*. Let*

$$V_d = \big\{\gamma : \mathbb{R}^d \to [0,\infty) : \gamma(0) = 0, \gamma \text{ conditionally negative definite}\big\}$$

denote the space of all variograms on $\mathbb{R}^d$*, equipped with the product* σ*-algebra. Further, let* $\mathbb{Q}$ *be an arbitrary probability measure on this space and* γ_i*,* $i \in \mathbb{N}$*, be an i.i.d. sequence of random variables with distribution* $\mathbb{Q}$*. For each* $i \in \mathbb{N}$*, let* ξ_i *be a Brown-Resnick process as in* (17) *with variogram* $4\gamma_i$*. Consider the max-mixtures* κ_n *of the processes* ξ_i*, given by*

$$\kappa_n(t) = \max_{i=1,\dots,n} \xi_i(t) - \log n, \quad t \in \mathbb{R}^d,$$

for $n \in \mathbb{N}$*. It can be shown that, as* $n \to \infty$*, the finite dimensional distributions of the processes* κ_n *converge to those of a max-stable, stationary process* κ*. The latter are given by the distribution in* (16) *with* η *induced by* $\mathbb{Q}$*.*

5. Spectral measure and Extremal Correlation function

In multivariate and spatial extreme value theory it is important to have flexible and tractable models for spatial dependence of extremal events. On this account, in section we show how the mixtures of Hüsler-Reiss distributions give rise to new models for bivariate spectral densities and, in the spatial domain, to new classes of extremal correlation functions.

5.1. Spectral measure

Every multivariate max-stable distribution admits a spectral representation [20, Chapter 5], where the spectral measure contains all information about the extremal dependence. Recently, Cooley et al. [7] and Ballani and Schlather [2] constructed new parametric models for spectral measures. For the bivariate Hüsler-Reiss distribution, de Haan and Pereira [8] give an explicit form of its spectral density on the positive sphere $S^1_+ = \{(x_1, x_2) \in [0,\infty)^2, x_1^2 + x_2^2 = 1\}$. More precisely, they show that for $\lambda \in (0,\infty)$

$$-\log F_\lambda(x,y) = \int_0^{\pi/2} \max\left\{e^{-x}\sin\theta, e^{-y}\cos\theta\right\} s_\lambda(\theta)\,d\theta, \quad x, y \in \mathbb{R},$$

and give a rather complicated expression for s_λ. Using the equation

$$\phi\left(\lambda - \frac{\log\tan\theta}{2\lambda}\right) = \frac{\sin\theta}{\cos\theta}\phi\left(\lambda + \frac{\log\tan\theta}{2\lambda}\right), \quad \lambda \in (0,\infty),\ \theta \in [0,\pi/2],$$

their expression can be considerably simplified and the spectral density becomes

$$s_\lambda(\theta) = \frac{1}{2\lambda\sin\theta\cos^2\theta}\,\phi\left(\lambda + \frac{\log(\tan\theta)}{2\lambda}\right), \quad \theta \in [0,\pi/2].$$

For the spectral density s_ν of the Hüsler-Reiss mixture distribution F_ν as in (9), where ν does neither have an atom at 0 nor at ∞, we have the relation

$$s_\nu(\theta) = \int_0^\infty s_\lambda(\theta)\,\nu(d\lambda), \quad \theta \in [0,\pi/2].$$

For the two examples at the end of Section 2 we can compute the corresponding spectral densities.

Proposition 1. *For the Rayleigh distribution with parameter $\sigma > 0$, s_{ν_σ} is given by*

$$s_{\nu_\sigma}(\theta) = \frac{e^{-\frac{1}{\sqrt{2}}|\log\tan\theta|\sqrt{1+1/\sigma^2}}}{4\sqrt{\sigma^4+\sigma^2}\,(\sin\theta\cos\theta)^{3/2}}, \quad \theta \in [0,\pi/2].$$

Similarly, for the Type-2 Gumbel distribution with parameter $b > 0$, the spectral density has the form

$$s_{\nu_b}(\theta) = \frac{e^{-u_b(\theta)}}{4\,(\sin\theta\cos\theta)^{3/2}}\left(1 - \frac{(\log\tan\theta)^2}{4u_b(\theta)^2}\right)\left(1 + \frac{1}{u_b(\theta)}\right), \quad \theta \in [0,\pi/2],$$

with $u_b(\theta) = \sqrt{(\log\tan\theta)^2/4 + 2b}$.

Figure 1 illustrates how these spectral measures interpolate between complete independence and complete dependence for different parameters.

5.2. Extremal Correlation function

For a stationary, max-stable random field $\{X(t), t \in \mathbb{R}^d\}$ with Gumbel margins, a natural approach to measure bivariate extremal dependencies is the extremal correlation function ρ [21]. For $h \in \mathbb{R}^d$ it is determined by

$$\mathbb{P}(X(0) \leq x, X(h) \leq x) = \mathbb{P}(X(0) \leq x)^{2-\rho(h)},$$

for some (and hence all) $x \in \mathbb{R}$. For instance, for the process in (17) ρ is given by

$$\rho_\gamma(h) = 2\left(1 - \Phi\left(\sqrt{\gamma(h)}/2\right)\right), \quad h \in \mathbb{R}^d.$$

Remark 7 defines new max-stable and stationary processes and thus also extremal correlation functions. Moreover, from the construction it is obvious that processes with this dependence structure can be simulated easily as max-mixtures of Brown-Resnick processes. In fact, for an arbitrary variogram γ and mixture measure ν on $(0, \infty)$, we let the measure $\mathbb{Q}$ in this remark be the law of $S^2\gamma$, where S is ν-distributed. The corresponding process κ possesses the extremal correlation function

$$\rho_{\gamma,\nu}(h) = \int_0^\infty 2\left(1 - \Phi\left(s\sqrt{\gamma(h)}\right)\right) \nu(ds), \quad h \in \mathbb{R}^d. \tag{21}$$

Gneiting [10] analyzes this kind of scale mixtures of the complementary error function in a more general framework. The following proposition is a consequence of Theorem 3.7 and 3.8 therein.

Proposition 2. *For a fixed variogram γ the class of extremal correlation functions in (21) is given by all functions $\varphi(\sqrt{\gamma(h)})$, $h \in \mathbb{R}^d$, where $\varphi : [0, \infty) \to \mathbb{R}$ is a continuous function with $\varphi(0) = 1$, $\lim_{h\to\infty} \varphi(h) = 0$, and*

$$(-1)^k \frac{d^k}{dh^k}[-\varphi'(\sqrt{h})] \tag{22}$$

is nonnegative for infinitely many positive integers k.

For instance, if ν_1 is the Rayleigh distribution (11) with density f_1, we obtain

$$\rho_{\gamma,\nu_1}(h) = 2\left(1 - \int_0^\infty \Phi(\lambda) f_{\sqrt{\gamma(h)}}(\lambda)\, d\lambda\right) = 1 - \left(\frac{\gamma(h)}{\gamma(h)+1}\right)^{1/2}, \quad h \in \mathbb{R}^d,$$

immediately from equation (13). In fact, $\rho_{\gamma,\nu_1}(h) = \psi(\gamma(h))$, where $\psi(x) = 1-(x/(x+1))^{1/2}$ is a completely monotone member of the Dagum family [4]. However, it is interesting to note, that when writing $\rho_{\gamma,\nu_1}(h) = \varphi(\sqrt{\gamma(h)})$ with $\varphi(x) = 1 - \left(x^2/(x^2+1)\right)^{1/2}$ as in Proposition 2, the function φ merely satisfies (22) but is not completely monotone.

Similarly, for the Type-2 Gumbel distribution with $b = 1$, the extremal correlation function is given by $\rho(h) = \exp(-\sqrt{2\gamma(h)})$. In particular, it follows that for any variogram γ and any $r > 0$ the function

$$\rho(h) = \exp\left(-r\sqrt{\gamma(h)}\right), \quad h \in \mathbb{R}^d,$$

is an extremal correlation function. Since this class of extremal correlation functions is closed under the operation of mixing with respect to probability measures, this implies that for any measure $\mu \in \mathcal{M}_1((0,\infty))$ the Laplace transform $\mathcal{L}\mu$ yields an extremal correlation function

$$\rho_\mu(h) = \mathcal{L}\mu(\sqrt{\gamma(h)}) = \int_0^\infty e^{-r\sqrt{\gamma(h)}}\,\mu(dr), \quad h \in \mathbb{R}^d.$$

Equivalently, for any completely monotone function ψ with $\psi(0) = 1$, the function $\psi(\sqrt{\gamma(h)})$ is an extremal correlation function. A corresponding max-stable, stationary random field is given by a max-mixture of Brown-Resnick processes with suitable $\nu \in \mathcal{M}_1((0,\infty))$.

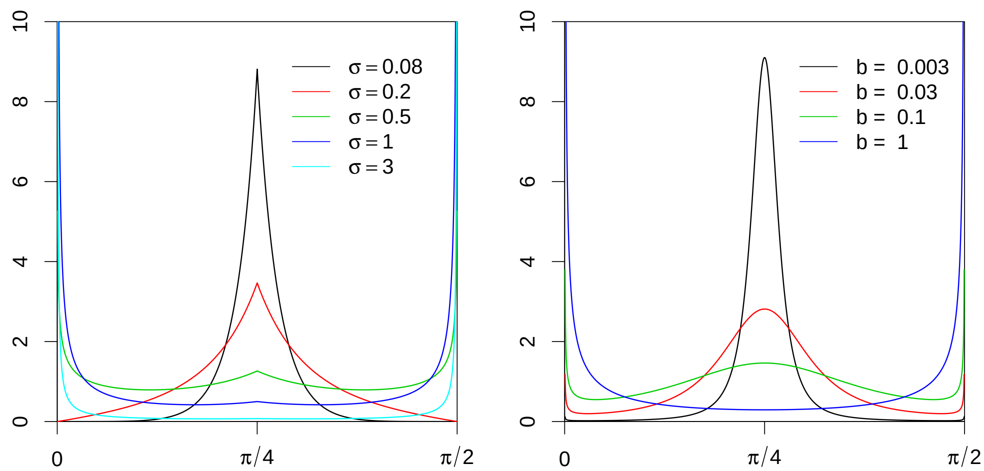

Figure 1. Spectral densities of the Rayleigh (left) and Type-2 Gumbel (right) mixture distribution for different parameters σ and b, respectively

6. Proofs

Proof of Theorem 1. Let $x, y \in \mathbb{R}$ and put $u_n(z) = b_n + z/b_n$, for $z \in \mathbb{R}$.

$$\begin{aligned}
&\log \mathbb{P}\left(\max_{i=1,\dots,n} X_{i,n}^{(1)} \le u_n(x), \max_{i=1,\dots,n} X_{i,n}^{(2)} \le u_n(y)\right) \\
&= \sum_{i=1}^n \log\left(1 - \left[\mathbb{P}(X_{i,n}^{(1)} > u_n(x)) + \mathbb{P}(X_{i,n}^{(2)} > u_n(y)) - \mathbb{P}\left(X_{i,n}^{(1)} > u_n(x), X_{i,n}^{(2)} > u_n(y)\right)\right]\right) \\
&= -\sum_{i=1}^n \mathbb{P}(X_{i,n}^{(1)} > u_n(x)) - \sum_{i=1}^n \mathbb{P}(X_{i,n}^{(2)} > u_n(y)) \\
&\qquad\qquad + \sum_{i=1}^n \mathbb{P}\left(X_{i,n}^{(1)} > u_n(x), X_{i,n}^{(2)} > u_n(y)\right) + R_n
\end{aligned} \tag{23}$$

where R_n is a remainder term from the Taylor expansion of log which can be bounded by

$$\begin{aligned}
|R_n| \le \frac{1}{2} \max_{i=1,\dots n} &\left[\mathbb{P}(X_{i,n}^{(1)} > u_n(x)) + \mathbb{P}(X_{i,n}^{(2)} > u_n(y))\right] \\
&\sum_{i=1}^n \left[\mathbb{P}(X_{i,n}^{(1)} > u_n(x)) + \mathbb{P}(X_{i,n}^{(2)} > u_n(y))\right].
\end{aligned} \tag{24}$$

For the one-dimensional margins we observe

$$\begin{aligned}
-\sum_{i=1}^n \mathbb{P}\left(X_{i,n}^{(1)} > u_n(x)\right) &= -\sum_{i=1}^n \int_{u_n(x)/\sigma_{i,n,1}}^{\infty} \phi(z)\, dz \\
&= -\sum_{i=1}^n \int_{x/\sigma_{i,n,1} - b_n^2(1-1/\sigma_{i,n,1})}^{\infty} \frac{1}{b_n} \phi(u_n(z))\, dz \\
&= -\int_{[0,\infty]\times\mathbb{R}^2} \int_{(1-\theta/b_n^2)x-\theta}^{\infty} e^{-z-z^2/(2b_n^2)}\, dz\, \eta_n(d(\lambda,\theta,\gamma)),
\end{aligned}$$

where we used $b_n = n\phi(b_n)$ for the last equation. For $n \in \mathbb{N}$, let

$$h_n(\theta) = \int_{(1-\theta/b_n^2)x-\theta}^{\infty} e^{-z-z^2/(2b_n^2)}\, dz, \quad \theta \in \mathbb{R}.$$

Clearly, as $n \to \infty$, h_n converges uniformly on compact sets to the function $h(\theta) = \exp(\theta - x)$. Note that h and h_n are continuous functions on $\mathbb{R}$. Put $\omega = (\lambda, \theta, \gamma)$ and observe for $K > 0$ that

$$
\begin{aligned}
&\left| \int_{[0,\infty]\times\mathbb{R}^2} h_n(\theta)\eta_n(d\omega) - \int_{[0,\infty]\times\mathbb{R}^2} h(\theta)\eta(d\omega) \right| \\
&\quad \le \left| \int_{[0,\infty]\times\mathbb{R}^2} h_n(\theta)\mathbf{1}_{h_n>K}\eta_n(d\omega) - \int_{[0,\infty]\times\mathbb{R}^2} h(\theta)\mathbf{1}_{h>K}\eta(d\omega) \right| \\
&\qquad + \left| \int_{[0,\infty]\times\mathbb{R}^2} h_n(\theta)\mathbf{1}_{h_n<K}\eta_n(d\omega) - \int_{[0,\infty]\times\mathbb{R}^2} h(\theta)\mathbf{1}_{h<K}\eta(d\omega) \right|. \qquad (25)
\end{aligned}
$$

By Theorem 5.5 in Billingsley [5] (see also the remark after the theorem), $\eta_n h_n^{-1}$ converges weakly to ηh^{-1}. Moreover, since $h\mathbf{1}_{h<K}$ and the $h_n\mathbf{1}_{h_n<K}$ are uniformly bounded in n, the second summand in (25) converges to 0 as $n \to \infty$, for arbitrary $K > 0$. By the uniform integrability condition (5) and Fatou's Lemma we have $\int_{[0,\infty]\times\mathbb{R}^2} h(\theta)\eta(d\omega) < \infty$ and hence, also the first summand in (25) tends to zero as $K, n \to \infty$. Consequently,

$$
-\sum_{i=1}^{n} \mathbb{P}\left(X_{i,n}^{(1)} > u_n(x)\right) \to -\int_{[0,\infty]\times\mathbb{R}^2} \exp\left[-(x-\theta)\right]\eta(d\omega), \qquad (26)
$$

Similarly, we get

$$
-\sum_{i=1}^{n} \mathbb{P}\left(X_{i,n}^{(2)} > u_n(y)\right) \to -\int_{[0,\infty]\times\mathbb{R}^2} \exp\left[-(y-\gamma)\right]\eta(d\omega). \qquad (27)
$$

Let us now show that the remainder term R_n converges to zero for $n \to \infty$. To this end, note that

$$
\begin{aligned}
\max_{i=1,\dots,n} \mathbb{P}\left(X_{i,n}^{(1)} > u_n(x)\right) &= \max_{i=1,\dots,n} \frac{1}{n} h_n(\theta_{i,n}) \\
&\le \max_{i=1,\dots,n} \frac{1}{n} e^{-x} e^{\theta_{i,n}(1+x/b_n^2)} \to 0, \quad n \to \infty, \qquad (28)
\end{aligned}
$$

where $\theta_{i,n} = b_n^2(1 - 1/\sigma_{i,n,1})$. In fact, by the uniform integrability (5), there is an $\epsilon > 0$ such that for any $\delta > 0$ we can find an $M > 0$, s.t.

$$
\sup_{n\in\mathbb{N}} \int_{[0,\infty]\times((\log M+x)/(1+\epsilon),\infty)\times\mathbb{R}} e^{\theta(1+\epsilon)}\,\eta_n(d\omega) < \delta.
$$

Choose $n_0 \in \mathbb{N}$ s.t. $M/n_0 < \delta$ and $|x|/b_n^2 < \epsilon$ to obtain for all $n \ge n_0$

$$
\begin{aligned}
&\max_{i=1,\dots,n} \frac{1}{n} e^{-x} e^{\theta_{i,n}(1+x/b_n^2)} \\
&\qquad \le \max\left\{ M/n, \int_{[0,\infty]\times((\log M+x)/(1+\epsilon),\infty)\times\mathbb{R}} e^{-x} e^{\theta(1+\epsilon)}\,\eta_n(d\omega) \right\} < \delta,
\end{aligned}
$$

where we separated the $\theta_{i,n}$ into those smaller or larger than $(\log M + x)/(1+\epsilon)$, respectively, and for the latter we bounded the maximum by the sum, i.e. the integral. It follows from (24), (26), (27) and (28) that R_n converges to zero as $n \to \infty$. We now turn to the third term in (23).

$$
\begin{aligned}
&\sum_{i=1}^{n} \mathbb{P}(X_{i,n}^{(1)}/\sigma_{i,n,1} > u_n(x)/\sigma_{i,n,1}, X_{i,n}^{(2)}/\sigma_{i,n,2} > u_n(y)/\sigma_{i,n,2}) \\
&\quad = \sum_{i=1}^{n} \int_{u_n(y)/\sigma_{i,n,2}}^{\infty} \left[1 - \Phi\left(\frac{u_n(x)/\sigma_{i,n,1} - \rho_{i,n} z}{(1-\rho_{i,n}^2)^{1/2}}\right)\right] \phi(z)\, dz \\
&\quad = \frac{1}{n} \sum_{i=1}^{n} \int_{y/\sigma_{i,n,2} - b_n^2(1-1/\sigma_{i,n,2})}^{\infty} \left[1 - \Phi\left(\frac{u_n(x)/\sigma_{i,n,1} - \rho_{i,n} u_n(z)}{(1-\rho_{i,n}^2)^{1/2}}\right)\right] e^{-z - z^2/(2b_n^2)}\, dz \\
&\quad = \int_{[0,\infty]\times\mathbb{R}^2} \int_{(1-\gamma/b_n^2)y-\gamma}^{\infty} \left[1 - \Phi\left(s_n(\lambda,\theta,z,x)\right)\right] e^{-z-z^2/(2b_n^2)}\, dz\, \eta_n(d\omega),
\end{aligned}
$$

where we used $b_n = n\phi(b_n)$ for the second last equation and

$$
s_n(\lambda,\theta,z,x) := \frac{\lambda}{(1-\lambda^2/b_n^2)^{1/2}} + \frac{(1-\theta/b_n^2)x - z - \theta}{(1-\lambda^2/b_n^2)^{1/2} 2\lambda} + \frac{\lambda z}{(1-\lambda^2/b_n^2)^{1/2} b_n^2}.
$$

For $n \in \mathbb{N}$, let

$$
g_n(\lambda,\theta,\gamma) = \mathbf{1}_{\lambda \le b_n} \int_{(1-\gamma/b_n^2)y-\gamma}^{\infty} \left[1 - \Phi\left(s_n(\lambda,\theta,z,x)\right)\right] e^{-z-z^2/(2b_n^2)}\, dz
$$

be a measurable function on $[0,\infty]\times\mathbb{R}^2$. It is easy to see, that as $n \to \infty$, g_n converges pointwise to the function

$$
g(\lambda,\theta,\gamma) = \int_{y-\gamma}^{\infty} \left[1 - \Phi\left(s(\lambda,\theta,z,x)\right)\right] e^{-z}\, dz,
$$

with

$$
s(\lambda,\theta,z,x) := \lambda + \frac{x - z - \theta}{2\lambda}.
$$

Note that g is a continuous function on $[0,\infty]\times\mathbb{R}^2$ and $g(0,\theta,\gamma) = g_n(0,\theta,\gamma) = e^{-\max(x-\theta,y-\gamma)}$ and $g(\infty,\theta,\gamma) = g_n(\infty,\theta,\gamma) = 0$, for any $(\theta,\gamma) \in \mathbb{R}^2$ and n sufficiently large. In order to establish the weak convergence $\eta_n g_n^{-1} \Rightarrow \eta g^{-1}$, we show that g_n converges uniformly on compact sets to g as $n \to \infty$. To this end, let $C =$

$[0,\infty]\times[\theta_0,\theta_1]\times[\gamma_0,\gamma_1]$ be an arbitrary compact set in $[0,\infty]\times\mathbb{R}^2$ and let $\epsilon>0$ be given. First, note that instead of g_n it suffices to consider the function $\tilde{g}_n$, defined as

$$\tilde{g}_n(\lambda,\theta,\gamma)=\mathbf{1}_{\lambda\le b_n}\int\limits_{(1-\gamma/b_n^2)y-\gamma}^{\infty}\left[1-\Phi\left(s_n(\lambda,\theta,z,x)\right)\right]e^{-z}\,dz,$$

since for n large enough

$$\sup_{(\lambda,\theta,\gamma)\in C}|g_n(\lambda,\theta,\gamma)-\tilde{g}_n(\lambda,\theta,\gamma)|\le\mathbf{1}_{\lambda\le b_n}\int\limits_{-2|y|-\gamma_1}^{\infty}e^{-z}(1-e^{-z^2/(2b_n^2)})dz\to0,$$

as $n\to\infty$, by dominated convergence. Further, for any $\epsilon>0$, let $z_1>-\log\epsilon$ which implies $\int_{z_1}^{\infty}e^{-z}dz<\epsilon$. We note that for n large enough

$$\begin{aligned}s_n(\lambda,\theta,z,x)&\ge\left(1-\lambda^2/b_n^2\right)^{-1/2}\left(\lambda\left(1+\frac{-2|y|-\gamma_1}{b_n^2}\right)+\frac{-2|x|-z_1-\theta_1}{2\lambda}\right)\\&\ge\left(\frac{\lambda}{2}+\frac{-2|x|-z_1-\theta_1}{2\lambda}\right),\end{aligned}$$

for all $\lambda\le b_n$, $-2|y|-\gamma_1\le z\le z_1$ and $(\lambda,\theta,\gamma)\in C$, independently of $n\in\mathbb{N}$. Hence, there is a $\lambda_1>0$ s.t. for all $\lambda_1\le\lambda\le b_n$

$$1-\Phi\left(s_n(\lambda,\theta,z,x)\right)<\epsilon\,e^{-2|y|-\gamma_1}.$$

Thus, for all $n\in\mathbb{N}$ large enough,

$$\sup_{(\lambda,\theta,\gamma)\in C,\lambda\ge\lambda_1}\tilde{g}_n(\lambda,\theta,\gamma)\le\mathbf{1}_{\lambda\le b_n}\left(\int_{-2|y|-\gamma_1}^{z_1}\epsilon\,e^{-2|y|-\gamma_1}e^{-z}\,dz+\int_{z_1}^{\infty}e^{-z}\,dz\right)\le2\epsilon,$$

and in the same manner, $\sup_{(\lambda,\theta,\gamma)\in C,\lambda\ge\lambda_1}g(\lambda,\theta,\gamma)\le2\epsilon$. Furthermore, we observe

$$\lim_{\lambda\to0}\Phi\left(s_n(\lambda,\theta,z,x)\right)=\mathbf{1}_{z<(1-\theta/b_n^2)x-\theta}\quad\text{and}\quad\lim_{\lambda\to0}\Phi\left(s(\lambda,\theta,z,x)\right)=\mathbf{1}_{z<x-\theta}.$$

Choose $n_0\in\mathbb{N}$ such that for all $n>n_0$ and all $\theta\in[\theta_0,\theta_1]$ we find an open interval (a_θ,b_θ) of size $\epsilon/2$ that contains $\{(1-\theta/b_n^2)x-\theta,x-\theta\}$. Put $I_\theta=(a_\theta-\epsilon/4,b_\theta+\epsilon/4)$, then we find a $\lambda_0>0$, s.t. for all $(\lambda,\theta,\gamma)\in C,\lambda\le\lambda_0$, $z\in I_\theta$ and $n>n_0$, we have $|\Phi\left(s_n(\lambda,\theta,z,x)\right)-\Phi\left(s(\lambda,\theta,z,x)\right)|\le\epsilon$. Consequently,

$$\begin{aligned}&\sup_{(\lambda,\theta,\gamma)\in C,\lambda\le\lambda_0}|\tilde{g}_n(\lambda,\theta,\gamma)-g(\lambda,\theta,\gamma)|\\&\qquad\le\sup_{(\lambda,\theta,\gamma)\in C,\lambda\le\lambda_0}\int_{-2|y|-\gamma_1}^{\infty}\left(\mathbf{1}_{z\in I_\theta}+\epsilon\,\mathbf{1}_{z\in\mathbb{R}\setminus I_\theta}\right)e^{-z}dz\le2\epsilon e^{2|y|+\gamma_1}.\end{aligned}$$

Choose $n_1 \in \mathbb{N}$, s.t. $b_{n_1} > \lambda_1$. For $\lambda_0 \le \lambda \le \lambda_1$ and $n > n_1$,

$$\begin{aligned} &|s_n(\lambda,\theta,z,x) - s(\lambda,\theta,z,x)| \\ &\quad = \left| \left(\lambda + \frac{x-z-\theta}{2\lambda}\right)\left(1 - \frac{1}{(1-\lambda_1^2/b_n^2)^{1/2}}\right) - \frac{\lambda^2 z - \theta}{(1-\lambda_1^2/b_n^2)^{1/2} b_n^2 2\lambda} \right| \\ &\quad \le M_1 \left|1 - \frac{1}{(1-\lambda_0^2/b_n^2)^{1/2}}\right| + \frac{M_2}{(1-\lambda_1^2/b_n^2)^{1/2} b_n^2} \to 0 \end{aligned} \tag{29}$$

for $n \to \infty$, uniformly in $z \in [-2|y| - \gamma_1, z_1]$ and $(\lambda,\theta,\gamma) \in C$ with $\lambda_0 \le \lambda \le \lambda_1$. Here, M_1 and M_2 are positive constants that only depend on $x, y, \lambda_0, \lambda_1, \theta_0, \theta_1, \gamma_1$. Let $n_2 \in \mathbb{N}$, s.t. for all $n > \max(n_1, n_2)$ the difference in (29) is less than or equal to $\epsilon\, e^{-2|y|-\gamma_1}$. By the Lipschitz continuity of Φ, we obtain for all $\lambda_0 \le \lambda \le \lambda_1$ and $(\lambda,\theta,\gamma) \in C$,

$$\begin{aligned} &\int_{-2|y|-\gamma_1}^{\infty} |\Phi(s_n(\lambda,\theta,z,x)) - \Phi(s(\lambda,\theta,z,x))|\, e^{-z}\, dz \\ &\quad \le \int_{-2|y|-\gamma_1}^{z_1} |s_n(\lambda,\theta,z,x) - s(\lambda,\theta,z,x)|\, e^{-z}\, dz + \int_{z_1}^{\infty} e^{-z}\, dz \\ &\quad \le \int_{-2|y|-\gamma_1}^{z_1} \epsilon\, e^{-2|y|-\gamma_1}\, e^{-z}\, dz + \int_{z_1}^{\infty} e^{-z}\, dz \le 2\epsilon. \end{aligned}$$

Putting the parts together yields

$$\lim_{n\to\infty} \sup_{(\lambda,\theta,\gamma)\in C} |\tilde{g}_n(\lambda,\theta,\gamma) - g(\lambda,\theta,\gamma)| = 0.$$

The assumptions of Theorem 5.5 in Billingsley [5] are satisfied and therefore $\eta_n g_n^{-1}$ converges weakly to ηg^{-1}. By a similar argument as in (25) together with the uniform integrability condition (5) we obtain for $n \to \infty$

$$\sum_{i=1}^{n} \mathbb{P}(X_{i,n}^{(1)} > u_n(x), X_{i,n}^{(2)} > u_n(y)) \to \int_{[0,\infty]\times\mathbb{R}^2} g(\lambda,\theta,\gamma)\eta(d(\lambda,\theta,\gamma)).$$

Finally, partial integration gives

$$\begin{aligned} g(\lambda,\theta,\gamma) = e^{-(y-\gamma)} + e^{-(x-\theta)} &- \Phi\left(\lambda + \frac{y-x+\theta-\gamma}{2\lambda}\right) e^{-(x-\theta)} \\ &- \Phi\left(\lambda - \frac{y-x+\theta-\gamma}{2\lambda}\right) e^{-(y-\gamma)}. \end{aligned}$$

Together with (23), (26), (27) and the fact that R_n converges to zero, this implies the desired result. □

Proof of Theorem 2. The sufficient part is a simple consequence of Theorem 1, where the covariance matrix of $\mathbf{X}_{i,n}$ is given by

$$\begin{pmatrix} 1 & \rho_{i,n} \\ \rho_{i,n} & 1 \end{pmatrix}.$$

For the necessary part, suppose that the sequence $(\max_{i=1,\dots,n} b_n(\mathbf{X}_{i,n}-b_n))_{n\in\mathbb{N}}$ of bivariate random vectors converges in distribution to some random vector Y. Let the ν_n, $n \in \mathbb{N}$, be defined as in (7) and assume that the sequence $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_1([0,\infty])$ does not converge. Then, by sequential compactness, it has at least two different accumulation points $\nu, \tilde{\nu} \in \mathcal{M}_1([0,\infty])$. By the first part of this theorem, $(\max_{i=1,\dots,n} b_n(\mathbf{X}_{i,n} - b_n))_{n\in\mathbb{N}}$ converges in distribution to $F_\nu \equiv F_{\tilde{\nu}}$. It now suffices to show that $F_\nu \equiv F_{\tilde{\nu}}$ implies $\nu \equiv \tilde{\nu}$ to conclude that $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_1([0,\infty])$ converges to some measure ν and that Y has distribution F_ν.
The fact that there is a one-to-one correspondence between Hüsler-Reiss distributions F_λ and the dependence parameter $\lambda \in [0,\infty]$ is straightforward [18]. Showing a similar result in our case, however, requires more effort.
To this end, for two measures $\nu_1, \nu_2 \in \mathcal{M}_1([0,\infty])$ define random variables Y_1 and Y_2 with distribution F_{ν_1} and F_{ν_2}, respectively. First, suppose that $\nu_1(\{\infty\}) = \nu_2(\{\infty\}) = 0$. For $j = 1,2$, by Remark 4 we have the stochastic representation $Y_j = \max_{i\in\mathbb{N}}(U_{i,j}, U_{i,j}+B_{i,j})$, where $\sum_{i=1}^\infty \delta_{U_{i,j}}$ are Poisson point process on $\mathbb{R}$ with intensity $e^{-u}du$ and the $(B_{i,j})_{i\in\mathbb{N}}$ are i.i.d. copies of the random variable B_j with normal distribution $N(-2S_j^2, 4S_j^2)$, where S_j is ν_j-distributed. Assume that

$$F_{\nu_1}(x,y) = F_{\nu_2}(x,y), \quad \text{for all } x,y \in \mathbb{R}, \tag{30}$$

i.e. the max-stable distributions of Y_1 and Y_2 are equal. Since a Poisson point process is determined by its intensity on a generating system of the σ-algebra, it follows that the point processes $\Pi_1 = \sum_{i=1}^\infty \delta_{(U_{i,1},U_{i,1}+B_{i,1})}$ and $\Pi_2 = \sum_{i=1}^\infty \delta_{(U_{i,2},U_{i,2}+B_{i,2})}$ are equal in distribution. Therefore, the measurable mapping

$$h : \mathbb{R}^2 \to \mathbb{R}^2, (x_1,x_2) \mapsto (x_1, x_2 - x_1)$$

induces two Poisson point processes $h(\Pi_1)$ and $h(\Pi_2)$ on $\mathbb{R}^2$ with coinciding intensity measures $e^{-u}du\,\mathbb{P}_{B_1}(dx)$ and $e^{-u}du\,\mathbb{P}_{B_2}(dx)$, respectively. Hence, B_1 and B_2 have the same distribution. Denote by ψ_j the Laplace transform of the Gaussian mixture B_j, $j = 1,2$. A straightforward calculation yields for $u \in (0,1)$

$$\psi_j(u) = \mathbb{E}\exp(uB_j) = \int_{[0,\infty)} \exp(-2\lambda^2(u-u^2))\nu_j(d\lambda), \quad j = 1,2.$$

By Lemma 7 in Kabluchko et al. [18] this implies the equality of measures $\nu_1^2(d\lambda) = \nu_2^2(d\lambda)$, where $\nu_j^2(d\lambda) = \nu_j(d\sqrt{\lambda})$ for $j = 1,2$. Hence, it also holds that $\nu_1 \equiv \nu_2$.
For arbitrary $\nu_1, \nu_2 \in \mathcal{M}_1([0,\infty])$, we first need to show that $\nu_1(\{\infty\}) = \nu_2(\{\infty\})$. For $j = 1,2$, observe that for $n \in \mathbb{N}$

$$\begin{aligned}
&-\log F_{\nu_j}(-n,0) + \log F_{\nu_j}(-n,n) \\
&= \int_{[0,\infty)} \Phi\left(\lambda + \frac{n}{2\lambda}\right)e^n + \Phi\left(\lambda - \frac{n}{2\lambda}\right) - \Phi\left(\lambda + \frac{n}{\lambda}\right)e^n - \Phi\left(\lambda - \frac{n}{\lambda}\right)e^{-n}\,\nu_j(d\lambda) \\
&\qquad + (1-e^{-n})\nu_j(\{\infty\}).
\end{aligned}$$

Since the curvature of Φ is negative on the positive real line, we have the estimate

$$e^n \left| \Phi\left(\lambda + \frac{n}{2\lambda}\right) - \Phi\left(\lambda + \frac{n}{\lambda}\right) \right| \leq \frac{n}{2\lambda\sqrt{2\pi}} e^n e^{-(\lambda + n/(2\lambda))^2/2},$$

where the latter term converges pointwise to zero as $n \to \infty$. Moreover, it is uniformly bounded in $n \in \mathbb{N}$ and $\lambda \in [0, \infty)$ by a constant and hence, by dominated convergence

$$\lim_{n\to\infty} -\log F_{\nu_j}(-n, 0) + \log F_{\nu_j}(-n, n) = \nu_j(\{\infty\}), \quad j = 1, 2.$$

It therefore follows from (30) that $\nu_1(\{\infty\}) = \nu_2(\{\infty\})$. If $\nu_1(\{\infty\}) < 1$ we apply the above to the restricted probability measures $\nu_j(\cdot \cap [0,\infty))/(1 - \nu_j(\{\infty\}))$ on $[0, \infty)$, $j = 1, 2$, to obtain $\nu_1 \equiv \nu_2$.
The last claim of the theorem follows from the fact, that the integrand in (9) is bounded and continuous in λ for fixed $x, y \in \mathbb{R}$, and hence, for $\nu, \nu_n \in \mathcal{M}_1([0, \infty])$, $n \in \mathbb{N}$, weak convergence of ν_n to ν ensures the pointwise convergence of the distribution functions.

□

Proof of Corollary 1. The first statement is a consequence of Theorem 2, because every sequence of random vectors can be understood as a triangular array where the columns contain equal random vectors.
For the second claim, let $\nu \in \mathcal{M}_1([0, \infty])$ be an arbitrary probability measure. Similarly as in Remark 3, define an i.i.d. sequence $(R_i)_{i\in\mathbb{N}}$ of samples of ν. Choosing $\rho_i = \max(1 - 2R_i^2/b_i^2, -1)$ as correlation of $\mathbf{X}_i$ yields

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{R_i b_n/b_i}.$$

For $y \in [0, \infty]$ with $\nu(\{y\}) = 0$ we observe

$$\nu_n([0, y]) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{[0,y]}(R_i b_n/b_i). \tag{31}$$

Fix $\epsilon > 0$ and recall from (1) that $b_n/\sqrt{2\log n} \to 1$ as $n \to \infty$. Hence, choose n large enough such that $i > n^{1/(1+\epsilon)^2}$ implies $b_n/b_i < 1 + \epsilon$. Let n_ϵ denote the smallest integer which is strictly larger than $n^{1/(1+\epsilon)^2}$, then (31) yields

$$\begin{aligned}\left|\nu_n([0, y]) - \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{[0,y]}(R_i)\right| &\leq \frac{n_\epsilon}{n} + \frac{1}{n}\left|\sum_{i=n_\epsilon}^{n} \mathbf{1}_{[0,y]}(R_i b_n/b_i) - \sum_{i=n_\epsilon}^{n} \mathbf{1}_{[0,y]}(R_i)\right| \\ &\leq \frac{n_\epsilon}{n} + \frac{1}{n}\sum_{i=n_\epsilon}^{n} \mathbf{1}_{(y/(1+\epsilon),y]}(R_i).\end{aligned}$$

Letting $n \to \infty$ gives

$$\lim_{n\to\infty}\left|\nu_n([0, y]) - \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{[0,y]}(R_i)\right| \leq \nu((y/(1+\epsilon), y]), \quad \text{a.s.}$$

Since ϵ was arbitrary and $\nu(\{y\}) = 0$, it follows from the law of large numbers that ν_n converges a.s. weakly to ν, as $n \to \infty$. $\square$

Proof of Theorem 3. Let $u_n(z) = b_n + z/b_n$ for $z \in \mathbb{R}$, $u_n(\mathbf{x}) = (u_n(x_1), \dots, u_n(x_d))^\top$ for $\mathbf{x} \in \mathbb{R}^d$ and for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ write $\mathbf{x} > \mathbf{y}$ if $x_i > y_i$ for all $1 \le i \le d$.
Let $\mathbf{x} = (x_1, \dots, x_d)^\top \in \mathbb{R}^d$ be a fixed vector and $A^l_{i,n} = \left\{ X^{(l)}_{i,n} \le u_n(x_l) \right\}$ for $n \in \mathbb{N}, 1 \le i \le n$ and $1 \le l \le d$.

$$\log \mathbb{P}\left(\max_{i=1,\dots,n} X^{(1)}_{i,n} \le u_n(x_1), \dots, \max_{i=1,\dots,n} X^{(d)}_{i,n} \le u_n(x_d) \right) \\ = \sum_{i=1}^n \log \mathbb{P}\left[\bigcap_{l=1}^d A^l_{i,n} \right] = -\sum_{i=1}^n \mathbb{P}\left[\bigcup_{l=1}^d \left(A^l_{i,n}\right)^C \right] + R_n \tag{32}$$

where R_n is a remainder term from the Taylor expansion of log. Using the same arguments as for the remainder term in (24), we conclude that R_n converges to zero as $n \to \infty$. By the additivity formula we have

$$-\mathbb{P}\left[\bigcup_{l=1}^d \left(A^l_{i,n}\right)^C \right] = \sum_{l=1}^d (-1)^l \sum_{m: 1 \le m_1 < \dots < m_l \le d} \mathbb{P}\left[\bigcap_{k=1}^l \left(A^{m_k}_{i,n}\right)^C \right]. \tag{33}$$

Consequently, by (32) and (33) it suffices to show that

$$\lim_{n\to\infty} \sum_{i=1}^n \mathbb{P}\left(\mathbf{X}_{i,n} > u_n(\mathbf{x})\right) = \int_{[0,\infty)^{d\times d}} h_{d,(1,\dots,d),\Lambda}(x_1, \dots, x_d)\, \eta(d\Lambda). \tag{34}$$

Let $\mathbf{Z} = (Z_1, \dots, Z_d)$ be a standard normal random vector with independent margins and let $K = \{1, \dots, d-1\}$. For a vector $\mathbf{x} \in \mathbb{R}^d$ let $\mathbf{x}_K = (x_1, \dots, x_{d-1})$. If $A = (a_{j,k})_{1 \le j,k \le d} \in \mathbb{R}^{d\times d}$ is a matrix, let $A_{d,K} = (a_{d,1}, \dots, a_{d,d-1})$, $A_{K,d} = (a_{1,d}, \dots, a_{d-1,d})$ and $A_{K,K} = (a_{j,k})_{j,k\in K}$. Similarly as in the proof of Theorem 1.1 in Hashorva et al. [14], we define a new matrix $B_{i,n} \in \mathbb{R}^{(d-1)\times(d-1)}$ by

$$B_{i,n} B_{i,n}^\top = (\Sigma_{i,n})_{K,K} - \sigma_{i,n}\sigma_{i,n}^\top, \quad \sigma_{i,n} = (\Sigma_{i,n})_{K,d}, \tag{35}$$

which is well-defined since $(\Sigma_{i,n})_{K,K} - \sigma_{i,n}\sigma_{i,n}^\top$ is positive definite as the Schur complement of $(\Sigma_{i,n})_{d,d}$ in the positive definite matrix $\Sigma_{i,n}$. This enables us to write the vector $\mathbf{X}_{i,n}$ as the joint stochastic representation

$$\left(X^{(1)}_{i,n}, \dots, X^{(d-1)}_{i,n} \right) \stackrel{d}{=} B_{i,n}\mathbf{Z}_K + Z_d \sigma_{i,n}, \quad X^{(d)}_{i,n} \stackrel{d}{=} Z_d.$$

Therefore, since Z_d is independent of $\mathbf{Z}_K$,

$$\begin{aligned}
&\mathbb{P}\left(\mathbf{X}_{i,n} > u_n(\mathbf{x})\right) = \mathbb{P}\left(B_{i,n}\mathbf{Z}_K + Z_d\sigma_{i,n} > u_n(\mathbf{x}_K), Z_d > u_n(x_d)\right) \\
&\quad = \int_{x_d}^{\infty} \mathbb{P}\left(B_{i,n}\mathbf{Z}_K + u_n(s)\sigma_{i,n} > u_n(\mathbf{x}_K)\right) b_n^{-1}\phi(b_n) e^{-s-s^2/(2b_n^2)}\, ds \\
&\quad = \frac{1}{n}\int_{x_d}^{\infty} S\left(\left(b_n^2(\mathbf{1}\mathbf{1}^\top - \Sigma_{i,n})\right)_{K,d} + x_K - s\mathbf{1} + sb_n^{-2}\left(b_n^2(\mathbf{1}\mathbf{1}^\top - \Sigma_{i,n})\right)_{K,d} \,\middle|\, b_n^2 B_{i,n}B_{i,n}^\top\right) \\
&\qquad\qquad e^{-s-s^2/(2b_n^2)}\, ds.
\end{aligned} \tag{36}$$

It follows from the definition of $B_{i,n}$ in equation (35) that

$$\begin{aligned}
B_{i,n}B_{i,n}^\top = (\mathbf{1}\mathbf{1}^\top - \Sigma_{i,n})_{K,d}\mathbf{1}^\top + \mathbf{1}(\mathbf{1}\mathbf{1}^\top - \Sigma_{i,n})_{d,K} - (\mathbf{1}\mathbf{1}^\top - \Sigma_{i,n})_{K,K} \\
- (\mathbf{1}\mathbf{1}^\top - \Sigma_{i,n})_{K,d}(\mathbf{1}\mathbf{1}^\top - \Sigma_{i,n})_{d,K}.
\end{aligned}$$

Together with (36) and the definition of η_n this yields

$$\sum_{i=1}^{n} \mathbb{P}\left(\mathbf{X}_{i,n} > u_n(\mathbf{x})\right) = \int_D p_n(A)\, \eta_n^2(dA),$$

where p_n is a measurable function from D to $[0,\infty)$ given by

$$\begin{aligned}
p_n(A) = \int_{x_d}^{\infty} S\left(2A_{K,d} + x_K - s\mathbf{1} + 2b_n^{-2}sA_{K,d} \,\middle|\, \Gamma_{d,(1,\dots,d)}(\sqrt{A}) - 4b_n^{-2}A_{K,d}A_{d,K}\right) \\
e^{-s-s^2/(2b_n^2)}\, ds.
\end{aligned}$$

Further, let p be the measurable function from D to $[0,\infty)$

$$p(A) = \int_{x_d}^{\infty} S\left(2A_{K,d} + x_K - s\mathbf{1} \,\middle|\, \Gamma_{d,(1,\dots,d)}(\sqrt{A})\right) e^{-s}\, ds.$$

Note that $\eta_n \Rightarrow \eta$ if and only if $\eta_n^2 \Rightarrow \eta^2$. In view of (34) it suffices to show that

$$\lim_{n\to\infty} \int_D p_n(A)\, \eta_n^2(dA) = \int_D p(A)\, \eta^2(dA). \tag{37}$$

To this end, let $A_0 \in D$ and $\{A_n, n \in \mathbb{N}\}$ be a sequence in D that converges to A_0. We will show that $p_n(A_n) \to p(A_0)$ as $n \to \infty$. By dominated convergence it is sufficient to show the convergence of the survivor functions. Since A_0 is in D, recall that $\Gamma_{d,(1,\dots,d)}(\sqrt{A_0})$ is in the space $\mathcal{M}_{(d-1)}$ of $(d-1)$-dimensional, non-degenerate covariance matrices. Moreover, since $\mathcal{M}_{(d-1)} \subset \mathbb{R}^{(d-1)\times(d-1)}$ is open and $\Gamma_{d,(1,\dots,d)}(\sqrt{A_n}) - b_n^{-2}4(A_n)_{K,d}(A_n)_{d,K}$ converges to $\Gamma_{d,(1,\dots,d)}(\sqrt{A_0})$, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\Gamma_{d,(1,\dots,d)}(\sqrt{A_n}) - b_n^{-2}4(A_n)_{K,d}(A_n)_{d,K} \in \mathcal{M}_{(d-1)}$. Since also $2(A_n)_{K,d} + x_K - s\mathbf{1} + b_n^{-2}s2(A_n)_{K,d}$ converges to $2(A_0)_{K,d} + x_K - s\mathbf{1}$ as $n \to \infty$, we conclude that the survivor functions converge and consequently $p_n(A_n) \to p(A_0)$. Applying Theorem 5.5 in Billingsley [5] yields (37) and therefore concludes the proof.

□

Proof of Theorem 4. First, note that s_n defined in (19) goes to 0 for $n \to \infty$. For $m \in \mathbb{N}$ and pairwise different $t_1, \dots, t_m \in \mathbb{R}^d$ we interpret the random vector $(Y(t_1), \dots, Y(t_m))$, conditionally on $\{S_i\}_{i\in\mathbb{N}}$, as the normalized maximum of n independent, yet differently distributed, m-variate normal random vectors

$$\mathbf{Z}_{i,n} = \left(X_i(S_i^{2/\alpha} s_n t_1), \dots, X_i(S_i^{2/\alpha} s_n t_m)\right), \quad n \in \mathbb{N}, i \in \{1, \dots, n\}.$$

In view of (18), their covariance matrices are specified by

$$\begin{aligned}(1/2)b_n^2(\mathbf{1}\mathbf{1}^\top - \mathrm{Cov}(\mathbf{Z}_{i,n})) &= \left(b_n^2 \gamma(S_i^{2/\alpha}(t_j - t_k))L(s_n)s_n^\alpha + b_n^2 o(L(s_n)s_n^\alpha)\right)_{1\le j,k\le m} \\ &= \left(\gamma(S_i^{2/\alpha}(t_j - t_k)) + o(1)\right)_{1\le j,k\le m},\end{aligned}$$

because of (19) and the continuity of L. The remainder term goes to 0 for $n \to \infty$ uniformly in $S_i^{2/\alpha} t_j, S_i^{2/\alpha} t_k \in \mathbb{R}^d$, as long as they stay bounded. In order to apply Theorem 3 we define η_n as in (15) by

$$\eta_n = \frac{1}{n}\sum_{i=1}^n \delta_{\left(\sqrt{\gamma(S_i^{2/\alpha}(t_j - t_k)) + o(1)}\right)_{1\le j,k\le m}}.$$

The proof of the following lemma will be given below.

Lemma 1. *For $q \in \mathbb{N}$, let Z be random vector in $\mathbb{R}^q$ and $V_i, i \in \mathbb{N}$, a sequence of i.i.d. copies of V. Further, suppose $\epsilon_{i,n}, n \in N$ and $1 \le i \le n$, is a triangular array of errors such that $\|\epsilon_{i,n}\|$ goes to 0 a.s. uniformly for bounded V_i, as $n \to \infty$. For $f \in C_b(\mathbb{R}^q)$ the following convergence holds almost surely,*

$$\frac{1}{n}\sum_{i=1}^n f(V_i + \epsilon_{i,n}) \to \mathbb{E}f(V), \quad \text{as } n \to \infty.$$

For a continuous, bounded function $f \in C_b([0,\infty)^{m\times m})$, this lemma yields with $V = (\sqrt{\gamma(S_i^{2/\alpha}(t_j - t_k))})_{1\le j,k\le m}$ that

$$\int_{[0,\infty)^{m\times m}} f(\Lambda)\,\eta_n(d\Lambda) \to \int_{[0,\infty)^{m\times m}} f(\Lambda)\,\eta(d\Lambda), \quad \text{as } n \to \infty,$$

where η is as in (20). Thus, η_n converges weakly to η. It remains to check the condition $\eta^2(D) = 1$, which is fulfilled if $\{\lambda\Lambda_0^2, \lambda > 0\} \subset D$. Since γ is a variogram, Λ_0^2 is conditionally negative definite for $\lambda > 0$. Suppose that there is an $\mathbf{x} \in \mathbb{R}^m \setminus \{0\}$ s.t. $\sum_{i=1}^m x_i = 0$ and $\mathbf{x}^\top \Lambda_0^2 \mathbf{x} = 0$. For some $t_0 \notin \{t_1, \dots, t_m\}$ this gives

$$0 = -\sum_{i,j=1}^m x_i x_j \gamma(t_i - t_j) = \sum_{i,j=1}^m x_i x_j[\gamma(t_i - t_0) + \gamma(t_j - t_0) - \gamma(t_i - t_j)] = \mathbf{x}^\top \mathbf{C}\mathbf{x}, \tag{38}$$

where $C = (C(t_i, t_j))_{1\le i,j\le m}$ is a covariance matrix of a fractional Brownian motion $\{X(t) : t \in \mathbb{R}^d\}$ on $\mathbb{R}^d$ with $X(t_0) = 0$ and covariance function $C(s,t) = \|s - t_0\|^\alpha + \|t - t_0\|^\alpha - \|s - t\|^\alpha, s, t \in \mathbb{R}^d$. Thus, (38) implies that the Gaussian random vector $(X(t_1), \dots, X(t_m))$ is degenerate. However, this contradicts the properties of fractional Brownian motion stated in Lemma 7.1 in Pitt [19] and therefore $\lambda\Lambda_0^2 \in D$ for all $\lambda > 0$. Hence, conditionally on $\{S_i\}_{i\in\mathbb{N}}$, the distribution function of $(Y(t_1), \dots, Y(t_m))$ converges by Theorem 3 pointwise to the distribution function H_η. Finally, it suffices to note that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\mathbb{P}\left(Y(t_1) \le y_1, \dots, Y(t_m) \le y_m | \{S_i\}_{i\in\mathbb{N}}\right)\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} \mathbb{P}\left(Y(t_1) \le y_1, \dots, Y(t_m) \le y_m | \{S_i\}_{i\in\mathbb{N}}\right)\right] \\
&= \mathbb{E}H_\eta(y_1, \dots, y_m) = H_\eta(y_1, \dots, y_m),
\end{aligned}$$

where the first equation follows by the dominated convergence theorem. □

Proof of Lemma 1. For arbitrary $N > 0$ note that

$$\frac{1}{n}\sum_{i=1}^n f(V_i + \epsilon_{i,n}) = \frac{1}{n}\sum_{i=1}^n f(V_i + \epsilon_{i,n})\mathbf{1}_{\|V_i\|\le N} + \frac{1}{n}\sum_{i=1}^n f(V_i + \epsilon_{i,n})\mathbf{1}_{\|V_i\|> N}. \tag{39}$$

Since for arbitrary $\delta > 0$ we find $n_0 \in \mathbb{N}$ such that $\epsilon_{i,n} < \delta$ a.s. for all $n > n_0$ and $1 \le i \le n$ with $\|V_i\| \le N$, it follows from the uniform continuity of f on compact sets that almost surely

$$\frac{1}{n}\sum_{i=1}^n \left|f(V_i + \epsilon_{i,n}) - f(V_i)\right| \mathbf{1}_{\|V_i\|\le N} \to 0, \quad \text{as } n \to \infty. \tag{40}$$

Further, observe that as $n \to \infty$, the absolute value of the second sum in (39) is a.s. bounded by $\mathbb{P}(\|V\| > N) \sup_{x\in\mathbb{R}^q} |f(x)|$, which converges to zero as $N \to \infty$. Thus, the assertion follows from the law of large numbers, the triangle inequality, (39) and (40). □

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