

# ON SOME REASONS FOR DOUBTING THE RIEMANN HYPOTHESIS

ALEKSANDAR IVIĆ

ABSTRACT. *Several arguments against the truth of the Riemann hypothesis are extensively discussed. These include the Lehmer phenomenon, the Davenport–Heilbronn zeta-function, large and mean values of  $|\zeta(\frac{1}{2} + it)|$  on the critical line, and zeros of a class of convolution functions closely related to  $\zeta(\frac{1}{2} + it)$ . The first two topics are classical, and the remaining ones are connected with the author’s recent research. By means of convolution functions a conditional disproof of the Riemann hypothesis is given.*

## 0. Foreword (“Audiatur et altera pars”)

This is the unabridged version of the work that was presented at the Bordeaux Conference in honour of the Prime Number Theorem Centenary, Bordeaux, January 26, 1996 and later during the 39th Taniguchi International Symposium on Mathematics “*Analytic Number Theory*”, May 13-17, 1996 in Kyoto and its forum, May 20-24, 1996. The abridged printed version, with a somewhat different title, is [62]. The multiplicities of zeros are treated in [64]. A plausible conjecture for the coefficients of the main term in the asymptotic formula for the  $2k$ -th moment of  $|\zeta(\frac{1}{2} + it)|$  (see (4.1)–(4.2)) is given in [67].

In the years that have passed after the writing of the first version of this paper, it appears that the subject of the Riemann Hypothesis has only gained in interest and importance. This seem particularly true in view of the Clay Mathematical Institute prize of one million dollars for the proof of the Riemann Hypothesis, which is called as one of the mathematical “Problems of the Millenium”. A comprehensive account is to be found in E. Bombieri’s paper [65]. It is the author’s belief that the present work can still be of interest, especially since the Riemann Hypothesis may be still very far from being settled. Inasmuch the Riemann Hypothesis is commonly believed to be true, and for several valid reasons, I feel that the arguments that disfavour it should also be pointed out.

One of the reasons that the original work had to be shortened and revised before being published is the remark that “The Riemann hypothesis is in the process of being proved” by powerful methods from Random matrix theory (see e.g., B. Conrey’s survey article [66]). Random matrix theory has undisputably found its place in the theory of  $\zeta(s)$  and allied functions (op. cit. [66], [67]). However, almost ten years have passed since its advent, but the Riemann hypothesis seems as distant now as it was then.

## 1. Introduction

A central place in Analytic number theory is occupied by the Riemann zeta-function  $\zeta(s)$ , defined for  $\Re s > 1$  by

$$(1.1) \quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1},$$

and otherwise by analytic continuation. It admits meromorphic continuation to the whole complex plane, its only singularity being the simple pole  $s = 1$  with residue 1. For general information on  $\zeta(s)$  the reader is referred to the monographs [7], [16], and [61]. From the functional equation

$$(1.2) \quad \zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s),$$

which is valid for any complex  $s$ , it follows that  $\zeta(s)$  has zeros at  $s = -2, -4, \dots$ . These zeros are traditionally called the “trivial” zeros of  $\zeta(s)$ , to distinguish them from the complex zeros of  $\zeta(s)$ , of which the smallest ones (in absolute value) are  $\frac{1}{2} \pm 14.134725\dots i$ . It is well-known that all complex zeros of  $\zeta(s)$  lie in the so-called “critical strip”  $0 < \sigma = \Re s < 1$ , and if  $N(T)$  denotes the number of zeros  $\rho = \beta + i\gamma$  ( $\beta, \gamma$  real) of  $\zeta(s)$  for which  $0 < \gamma \leq T$ , then

$$(1.3) \quad N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right)$$

with

$$(1.4) \quad S(T) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + iT\right) = O(\log T).$$

This is the so-called Riemann–von Mangoldt formula. *The Riemann hypothesis* (henceforth RH for short) is the conjecture, stated by B. Riemann in his epoch-making memoir [52], that *very likely all complex zeros of  $\zeta(s)$  have real parts equal to  $1/2$* . For this reason the line  $\sigma = 1/2$  is called the “critical line” in the theory of  $\zeta(s)$ . Notice that Riemann was rather cautious in formulating the RH, and that he used the wording “very likely” (“sehr wahrscheinlich” in the German original) in connection with it. Riemann goes on to say in his paper: “One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation”. The RH is undoubtedly one of the most celebrated and difficult open problems in whole Mathematics. Its proof (or disproof) would have very important consequences in multiplicative number theory, especially in problems involving the distribution of primes. It would also very likely lead to generalizations to many other zeta-functions (Dirichlet series) having similar properties as  $\zeta(s)$ .

The RH can be put into many equivalent forms. One of the classical is

$$(1.5) \quad \pi(x) = \operatorname{li} x + O(\sqrt{x} \log x),$$

where  $\pi(x)$  is the number of primes not exceeding  $x$  ( $\geq 2$ ) and

$$(1.6) \quad \operatorname{li} x = \int_0^x \frac{dt}{\log t} = \lim_{\varepsilon \rightarrow 0^+} \left( \int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \right) \frac{dt}{\log t} = \sum_{n=1}^N \frac{(n-1)!x}{\log^n x} + O\left(\frac{x}{\log^{N+1} x}\right)$$

for any fixed integer  $N \geq 1$ . One can give a purely arithmetic equivalent of the RH without mentioning primes. Namely we can define recursively the Möbius function  $\mu(n)$  as

$$\mu(1) = 1, \quad \mu(n) = - \sum_{d|n, d < n} \mu(d) \quad (n > 1).$$

Then the RH is equivalent to the following assertion: For any given integer  $k \geq 1$  there exists an integer  $N_0 = N_0(k)$  such that, for integers  $N \geq N_0$ , one has

$$(1.7) \quad \left( \sum_{n=1}^N \mu(n) \right)^{2k} \leq N^{k+1}.$$

The above definition of  $\mu(n)$  is elementary and avoids primes. A non-elementary definition of  $\mu(n)$  is through the series representation

$$(1.8) \quad \sum_{n=1}^{\infty} \mu(n) n^{-s} = \frac{1}{\zeta(s)} \quad (\Re s > 1),$$

and an equivalent form of the RH is that (1.8) holds for  $\sigma > 1/2$ . The inequality (1.7) is in fact the bound

$$(1.9) \quad \sum_{n \leq x} \mu(n) \ll_{\varepsilon} x^{\frac{1}{2} + \varepsilon}$$

in disguise, where  $\varepsilon$  corresponds to  $1/(2k)$ ,  $x$  to  $N$ , and the  $2k$ -th power avoids absolute values. The bound (1.9) (see [16] and [61]) is one of the classical equivalents of the RH. The sharper bound

$$\left| \sum_{n \leq x} \mu(n) \right| < \sqrt{x} \quad (x > 1)$$

was proposed in 1897 by Mertens on the basis of numerical evidence, and later became known in the literature as *the Mertens conjecture*. It was disproved in 1985 by A.M. Odlyzko and H.J.J. te Riele [47].

Instead of working with the complex zeros of  $\zeta(s)$  on the critical line it is convenient to introduce the function

$$(1.10) \quad Z(t) = \chi^{-1/2}(\tfrac{1}{2} + it)\zeta(\tfrac{1}{2} + it),$$

where  $\chi(s)$  is given by (1.2). Since  $\chi(s)\chi(1-s) = 1$  and  $\overline{\Gamma(s)} = \Gamma(\bar{s})$ , it follows that  $|Z(t)| = |\zeta(\frac{1}{2} + it)|$ ,  $Z(t)$  is even and

$$\overline{Z(t)} = \chi^{-1/2}(\tfrac{1}{2} - it)\zeta(\tfrac{1}{2} - it) = \chi^{1/2}(\tfrac{1}{2} - it)\zeta(\tfrac{1}{2} + it) = Z(t).$$

Hence  $Z(t)$  is real if  $t$  is real, and the zeros of  $Z(t)$  correspond to the zeros of  $\zeta(s)$  on the critical line. Let us denote by  $0 < \gamma_1 \leq \gamma_2 \leq \dots$  the positive zeros of  $Z(t)$  with multiplicities counted (all known zeros are simple). If the RH is true, then it is known (see [61]) that

$$(1.11) \quad S(T) = O\left(\frac{\log T}{\log \log T}\right),$$

and this seemingly small improvement over (1.4) is significant. If (1.11) holds, then from (1.3) one infers that  $N(T+H) - N(T) > 0$  for  $H = C/\log \log T$ , suitable  $C > 0$  and  $T \geq T_0$ . Consequently we have the bound, on the RH,

$$(1.12) \quad \gamma_{n+1} - \gamma_n \ll \frac{1}{\log \log \gamma_n}$$

for the gap between consecutive zeros on the critical line. For some unconditional results on  $\gamma_{n+1} - \gamma_n$ , see [17], [18] and [25].

We do not know exactly what motivated Riemann to conjecture the RH. Some mathematicians, like Felix Klein, thought that he was inspired by a sense of general beauty and symmetry in Mathematics. Although doubtlessly the truth of the RH would provide such harmonious symmetry, we also know now that Riemann undertook rather extensive numerical calculations concerning  $\zeta(s)$  and its zeros. C.L. Siegel [57] studied Riemann's unpublished notes, kept in the Göttingen library. It turned out that Riemann had computed several zeros of the zeta-function and had a deep understanding of its analytic behaviour. Siegel provided rigorous proof of a formula that had its genesis in Riemann's work. It came to be known later as *the Riemann-Siegel formula* (see [16], [57] and [61]) and, in a weakened form, it says that

$$(1.13) \quad Z(t) = 2 \sum_{n \leq (t/2\pi)^{1/2}} n^{-1/2} \cos\left(t \log \frac{\sqrt{t/2\pi}}{n} - \frac{t}{2} - \frac{\pi}{8}\right) + O(t^{-1/4}),$$

where the O-term in (1.13) is actually best possible, namely it is  $\Omega_{\pm}(t^{-1/4})$ . As usual  $f(x) = \Omega_{\pm}(g(x))$  (for  $g(x) > 0$  when  $x \geq x_0$ ) means that we have  $f(x) = \Omega_{+}(x)$  and  $f(x) = \Omega_{-}(x)$ , namely that both

$$\limsup_{x \rightarrow \infty} \frac{f(x)}{g(x)} > 0, \quad \liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)} < 0$$

are true. The Riemann–Siegel formula is an indispensable tool in the theory of  $\zeta(s)$ , both for theoretical investigations and for the numerical calculations of the zeros.

Perhaps the most important concrete reason for believing the RH is the impressive numerical evidence in its favour. There exists a large and rich literature on numerical calculations involving  $\zeta(s)$  and its zeros (see [38], [44], [45], [46], [51], which contain references to further work). This literature reflects the development of Mathematics in general, and of Numerical analysis and Analytic number theory in particular. Suffice to say that it is known that the first 1.5 billion complex zeros of  $\zeta(s)$  in the upper half-plane are simple and do have real parts equal to  $1/2$ , as predicted by the RH. Moreover, many large blocks of zeros of much greater height have been thoroughly investigated, and all known zeros satisfy the RH. However, one should be very careful in relying on numerical evidence in Analytic number theory. A classical example for this is the inequality  $\pi(x) < \text{li } x$  (see (1.5) and (1.6)), noticed already by Gauss, which is known to be true for all  $x$  for which the functions in question have been actually computed. But the inequality  $\pi(x) < \text{li } x$  is false; not only does  $\pi(x) - \text{li } x$  assume positive values for some arbitrarily large values of  $x$ , but J.E. Littlewood [37] proved that

$$\pi(x) = \text{li } x + \Omega_{\pm} \left( \sqrt{x} \frac{\log \log \log x}{\log x} \right).$$

By extending the methods of R. Sherman Lehman [56], H.J.J. te Riele [50] showed that  $\pi(x) < \text{li } x$  fails for some (unspecified)  $x < 6.69 \times 10^{370}$ . For values of  $t$  which are this large we may hope that  $Z(t)$  will also show its true asymptotic behaviour. Nevertheless, we cannot compute by today’s methods the values of  $Z(t)$  for  $t$  this large, actually even  $t = 10^{100}$  seems out of reach at present. To assess why the values of  $t$  where  $Z(t)$  will “really” exhibit its true behaviour must be “very large”, it suffices to compare (1.4) and (1.11) and note that the corresponding bounds differ by a factor of  $\log \log T$ , which is a very slowly varying function.

Just as there are deep reasons for believing the RH, there are also serious grounds for doubting its truth, although the author certainly makes no claims to possess a disproof of the RH. It is in the folklore that several famous mathematicians, which include P. Turán and J.E. Littlewood, believed that the RH is not true. The aim of this paper is to state and analyze some of the arguments which cast doubt on the truth of the RH. In subsequent sections we shall deal with the Lehmer phenomenon, the Davenport-Heilbronn zeta-function, mean value formulas on the critical line, large values on the critical line and the distribution of zeros of a class of convolution functions. These independent topics appear to me to be among the most salient ones which point against the truth of the RH. The first two of them, the Lehmer phenomenon and the Davenport-Heilbronn zeta-function, are classical and fairly well known. The remaining ones are rather new and are connected with the author’s research, and for these reasons the emphasis will be on them. A sharp asymptotic formula for the convolution function  $M_{Z,f}(t)$ , related to  $Z(t)$ , is given in Section 8. Finally a conditional disproof of the RH, based on the use of the functions  $M_{Z,f}(t)$ , is given at the end of the paper in Section 9. Of course, nothing short of *rigorous* proof or disproof will settle the truth of the RH.

**Acknowledgement.** I want to thank Professors M. Jutila, K. Matsumoto, Y. Motohashi and A.M. Odlyzko for valuable remarks.

## 2. Lehmer’s phenomenon

The function  $Z(t)$ , defined by (1.10), has a negative local maximum  $-0.52625\dots$  at  $t = 2.47575\dots$ . This is the only known occurrence of a negative local maximum, while no positive local minimum is known. *Lehmer’s phenomenon* (named after D.H. Lehmer, who in his works [35], [36] made significant contributions to the subject) is the fact (see [46] for a thorough discussion) that the graph of  $Z(t)$  sometimes barely crosses the  $t$ -axis. This means that the absolute value of the maximum or minimum of  $Z(t)$  between its two consecutive zeros is small. For instance, A.M. Odlyzko found (in the version of [46] available to the author, but Odlyzko kindly informed me that many more examples occur in the computations that are going on now) 1976 values of  $n$  such that  $|Z(\frac{1}{2}\gamma_n + \frac{1}{2}\gamma_{n+1})| < 0.0005$  in the block that he investigated. Several

extreme examples are also given by van de Lune et al. in [38]. The Lehmer phenomenon shows the delicacy of the RH, and the possibility that a counterexample to the RH may be found numerically. For should it happen that, for  $t \geq t_0$ ,  $Z(t)$  attains a negative local maximum or a positive local minimum, then the RH would be disproved. This assertion follows (see [7]) from the following

**Proposition 1.** If the RH is true, then the graph of  $Z'(t)/Z(t)$  is monotonically decreasing between the zeros of  $Z(t)$  for  $t \geq t_0$ .

Namely suppose that  $Z(t)$  has a negative local maximum or a positive local minimum between its two consecutive zeros  $\gamma_n$  and  $\gamma_{n+1}$ . Then  $Z'(t)$  would have at least two distinct zeros  $x_1$  and  $x_2$  ( $x_1 < x_2$ ) in  $(\gamma_n, \gamma_{n+1})$ , and hence so would  $Z'(t)/Z(t)$ . But we have

$$\frac{Z'(x_1)}{Z(x_1)} < \frac{Z'(x_2)}{Z(x_2)},$$

which is a contradiction, since  $Z'(x_1) = Z'(x_2) = 0$ .

To prove Proposition 1 consider the function

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

so that  $\xi(s)$  is an entire function of order one (see Ch. 1 of [16]), and one has unconditionally

$$(2.1) \quad \frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

with

$$B = \log 2 + \frac{1}{2} \log \pi - 1 - \frac{1}{2}C_0,$$

where  $\rho$  denotes complex zeros of  $\zeta(s)$  and  $C_0 = \Gamma'(1)$  is Euler's constant. By (1.2) it follows that

$$Z(t) = \chi^{-1/2}\left(\frac{1}{2} + it\right)\zeta\left(\frac{1}{2} + it\right) = \frac{\pi^{-it/2}\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)\zeta\left(\frac{1}{2} + it\right)}{|\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)|},$$

so that we may write

$$\xi\left(\frac{1}{2} + it\right) = -f(t)Z(t), \quad f(t) := \frac{1}{2}\pi^{-1/4}(t^2 + \frac{1}{4})|\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)|.$$

Consequently logarithmic differentiation gives

$$(2.2) \quad \frac{Z'(t)}{Z(t)} = -\frac{f'(t)}{f(t)} + i \frac{\xi'\left(\frac{1}{2} + it\right)}{\xi\left(\frac{1}{2} + it\right)}.$$

Assume now that the RH is true. Then by using (2.1) with  $\rho = \frac{1}{2} + i\gamma$ ,  $s = \frac{1}{2} + it$  we obtain, if  $t \neq \gamma$ ,

$$\left( \frac{i\xi'\left(\frac{1}{2} + it\right)}{\xi\left(\frac{1}{2} + it\right)} \right)' = -\sum_{\gamma} \frac{1}{(t-\gamma)^2} < -C(\log \log t)^2 \quad (C > 0)$$

for  $t \geq t_0$ , since (1.12) holds. On the other hand, by using Stirling's formula for the gamma-function and  $\log |z| = \Re \log z$ , it is readily found that

$$\frac{d}{dt} \left( \frac{f'(t)}{f(t)} \right) \ll \frac{1}{t},$$

so that from (2.2) it follows that  $(Z'(t)/Z(t))' < 0$  if  $t \geq t_0$ , which implies Proposition 1. Actually the value of  $t_0$  may be easily effectively determined and seen not to exceed 1000. Since  $Z(t)$  has no positive local minimum or negative local maximum for  $3 \leq t \leq 1000$ , it follows that the RH is false if we find (numerically) the occurrence of a single negative local maximum (besides the one at  $t = 2.47575\dots$ ) or a positive local minimum of  $Z(t)$ . It seems appropriate to quote in concluding Edwards [7], who says that Lehmer's phenomenon "must give pause to even the most convinced believer of the Riemann hypothesis".

### 3. The Davenport-Heilbronn zeta-function

This is a zeta-function (Dirichlet series) which satisfies a functional equation similar to the classical functional equation (1.2) for  $\zeta(s)$ . It has other analogies with  $\zeta(s)$ , like having infinitely many zeros on the critical line  $\sigma = 1/2$ , but for this zeta-function the analogue of the RH does not hold. This function was introduced by H. Davenport and H. Heilbronn [6] as

$$(3.1) \quad f(s) = 5^{-s} \left( \zeta\left(s, \frac{1}{5}\right) + \tan \theta \zeta\left(s, \frac{2}{5}\right) - \tan \theta \zeta\left(s, \frac{3}{5}\right) - \zeta\left(s, \frac{4}{5}\right) \right),$$

where  $\theta = \arctan(\sqrt{10 - 2\sqrt{5}} - 2)/(\sqrt{5} - 1)$  and, for  $\Re s > 1$ ,

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s} \quad (0 < a \leq 1)$$

is the familiar *Hurwitz zeta-function*, defined for  $\Re s \leq 1$  by analytic continuation. With the above choice of  $\theta$  (see [6], [32] or [61]) it can be shown that  $f(s)$  satisfies the functional equation

$$(3.2) \quad f(s) = X(s)f(1-s), \quad X(s) = \frac{2\Gamma(1-s)\cos(\frac{\pi s}{2})}{5^{s-\frac{1}{2}}(2\pi)^{1-s}},$$

whose analogy with the functional equation (1.2) for  $\zeta(s)$  is evident. Let  $1/2 < \sigma_1 < \sigma_2 < 1$ . Then it can be shown (see Ch. 6 of [32]) that  $f(s)$  has infinitely many zeros in the strip  $\sigma_1 < \sigma = \Re s < \sigma_2$ , and it also has (see Ch. 10 of [61]) an infinity of zeros in the half-plane  $\sigma > 1$ , while from the product representation in (1.1) it follows that  $\zeta(s) \neq 0$  for  $\sigma > 1$ , so that in the half-plane  $\sigma > 1$  the behaviour of zeros of  $\zeta(s)$  and  $f(s)$  is different. Actually the number of zeros of  $f(s)$  for which  $\sigma > 1$  and  $0 < t = \Im s \leq T$  is  $\gg T$ , and similarly each rectangle  $0 < t \leq T, 1/2 < \sigma_1 < \sigma \leq \sigma_2 \leq 1$  contains at least  $c(\sigma_1, \sigma_2)T$  zeros of  $f(s)$ . R. Spira [58] found that  $0.808517 + 85.699348i$  (the values are approximate) is a zero of  $f(s)$  lying in the critical strip  $0 < \sigma < 1$ , but not on the critical line  $\sigma = 1/2$ . On the other hand, A.A. Karatsuba [31] proved that the number of zeros  $\frac{1}{2} + i\gamma$  of  $f(s)$  for which  $0 < \gamma \leq T$  is at least  $T(\log T)^{1/2-\varepsilon}$  for any given  $\varepsilon > 0$  and  $T \geq T_0(\varepsilon)$ . This bound is weaker than A. Selberg's classical result [53] that there are  $\gg T \log T$  zeros  $\frac{1}{2} + i\gamma$  of  $\zeta(s)$  for which  $0 < \gamma \leq T$ . From the Riemann-von Mangoldt formula (1.3) it follows that, up to the value of the  $\ll$ -constant, Selberg's result on  $\zeta(s)$  is best possible. There are certainly  $\ll T \log T$  zeros  $\frac{1}{2} + i\gamma$  of  $f(s)$  for which  $0 < \gamma \leq T$  and it may be that almost all of them lie on the critical line  $\sigma = 1/2$ , although this has not been proved yet. The Davenport-Heilbronn zeta-function is not the only example of a zeta-function that exhibits the phenomena described above, and many so-called *Epstein zeta-functions* also have complex zeros off their respective critical lines (see the paper of E. Bombieri and D. Hejhal [5] for some interesting results).

What is the most important difference between  $\zeta(s)$  and  $f(s)$  which is accountable for the difference of distribution of zeros of the two functions, which occurs at least in the region  $\sigma > 1$ ? It is most likely that the answer is the lack of the Euler product for  $f(s)$ , similar to the one in (1.1) for  $\zeta(s)$ . But  $f(s)$  can be written as a linear combination of two  $L$ -functions which have Euler products (with a common factor) and this fact plays the crucial rôle in Karatsuba's proof of the lower bound result for the number of zeros of  $f(s)$ .

In any case one can argue that it may likely happen that the influence of the Euler product for  $\zeta(s)$  will not extend all the way to the line  $\sigma = 1/2$ . In other words, the existence of zeta-functions such as  $f(s)$ , which share many common properties with  $\zeta(s)$ , but which have infinitely many zeros off the critical line, certainly disfavours the RH.

Perhaps one should at this point mention the *Selberg zeta-function*  $\mathcal{Z}(s)$  (see [55]). This is an entire function which enjoys several common properties with  $\zeta(s)$ , like the functional equation and the Euler product. For  $\mathcal{Z}(s)$  the corresponding analogue of the RH is true, but it should be stressed that  $\mathcal{Z}(s)$  is *not* a classical Dirichlet series. Its Euler product

$$\mathcal{Z}(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}) \quad (\Re s > 1)$$

is not a product over the rational primes, but over norms of certain conjugacy classes of groups. Also  $\mathcal{Z}(s)$  is an entire function of order 2, while  $(s-1)\zeta(s)$  is an entire function of order 1. For these reasons  $\mathcal{Z}(s)$  cannot be compared too closely to  $\zeta(s)$ .

#### 4. Mean value formulas on the critical line

For  $k \geq 1$  a fixed integer, let us write the  $2k$ -th moment of  $|\zeta(\frac{1}{2} + it)|$  as

$$(4.1) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt = T P_{k^2}(\log T) + E_k(T),$$

where for some suitable coefficients  $a_{j,k}$  one has

$$(4.2) \quad P_{k^2}(y) = \sum_{j=0}^{k^2} a_{j,k} y^j.$$

An extensive literature exists on  $E_k(T)$ , especially on  $E_1(T) \equiv E(T)$  (see F.V. Atkinson's classical paper [2]), and the reader is referred to [20] for a comprehensive account. It is known that

$$P_1(y) = y + 2C_0 - 1 - \log(2\pi),$$

and  $P_4(y)$  is a quartic polynomial whose leading coefficient equals  $1/(2\pi^2)$  (see [22] for an explicit evaluation of its coefficients). One hopes that

$$(4.3) \quad E_k(T) = o(T) \quad (T \rightarrow \infty)$$

will hold for each fixed integer  $k \geq 1$ , but so far this is known to be true only in the cases  $k = 1$  and  $k = 2$ , when  $E_k(T)$  is a true error term in the asymptotic formula (4.1). In fact heretofore it has not been clear how to define properly (even on heuristic grounds) the values of  $a_{j,k}$  in (4.2) for  $k \geq 3$  (see [24] for an extensive discussion concerning the case  $k = 3$ ). The connection between  $E_k(T)$  and the RH is indirect, namely there is a connection with the *Lindelöf hypothesis* (LH for short). The LH is also a famous unsettled problem, and it states that

$$(4.4) \quad \zeta(\frac{1}{2} + it) \ll_{\varepsilon} t^{\varepsilon}$$

for any given  $\varepsilon > 0$  and  $t \geq t_0 > 0$  (since  $\overline{\zeta(\frac{1}{2} + it)} = \zeta(\frac{1}{2} - it)$ ,  $t$  may be assumed to be positive). It is well-known (see [61] for a proof) that the RH implies

$$(4.5) \quad \zeta(\frac{1}{2} + it) \ll \exp\left(\frac{A \log t}{\log \log t}\right) \quad (A > 0, t \geq t_0),$$

so that obviously the RH implies the LH. In the other direction it is unknown whether the LH (or (4.5)) implies the RH. However, it is known that the LH has considerable influence on the distribution of zeros of  $\zeta(s)$ . If  $N(\sigma, T)$  denotes the number of zeros  $\rho = \beta + i\gamma$  of  $\zeta(s)$  for which  $\sigma \leq \beta$  and  $|\gamma| \leq T$ , then it is known (see Ch. 11 of [16]), that the LH implies that  $N(\sigma, T) \ll T^{2-2\sigma+\varepsilon}$  for  $1/2 \leq \sigma \leq 1$  (this is a form of *the density hypothesis*) and  $N(\frac{3}{4} + \delta, T) \ll T^\varepsilon$ , where  $\varepsilon = \varepsilon(\delta)$  may be arbitrarily small for any  $0 < \delta < \frac{1}{4}$ .

The best unconditional bound for the order of  $\zeta(s)$  on the critical line, known at the time of the writing of this text is

$$(4.6) \quad \zeta\left(\frac{1}{2} + it\right) \ll_\varepsilon t^{c+\varepsilon}$$

with  $c = 89/570 = 0.15614\dots$ . This is due to M.N. Huxley [13], and represents the last in a long series of improvements over the past 80 years. The result is obtained by intricate estimates of exponential sums of the type  $\sum_{N < n \leq 2N} n^{it}$  ( $N \ll \sqrt{t}$ ), and the value  $c = 0.15$  appears to be the limit of the method.

Estimates for  $E_k(T)$  in (4.1) (both pointwise and in the mean sense) have many applications. From the knowledge about the order of  $E_k(T)$  one can deduce a bound for  $\zeta(\frac{1}{2} + iT)$  via the estimate

$$(4.7) \quad \zeta\left(\frac{1}{2} + iT\right) \ll (\log T)^{(k^2+1)/(2k)} + \left(\log T \max_{t \in [T-1, T+1]} |E_k(t)|\right)^{1/(2k)},$$

which is Lemma 4.2 of [20]. Thus the best known upper bound

$$(4.8) \quad E(T) \equiv E_1(T) \ll T^{72/227} (\log T)^{679/227}$$

of M.N. Huxley [14] yields (4.6) with  $c = 36/227 = 0.15859\dots$ . Similarly the sharpest known bound

$$(4.9) \quad E_2(T) \ll T^{2/3} \log^C T \quad (C > 0)$$

of Y. Motohashi and the author (see [20], [26], [28]) yields (4.6) with the classical value  $c = 1/6$  of Hardy and Littlewood. Since the difficulties in evaluating the left-hand side of (4.1) greatly increase as  $k$  increases, it is reasonable to expect that the best estimate for  $\zeta(\frac{1}{2} + iT)$  that one can get from (4.7) will be when  $k = 1$ .

The LH is equivalent to the bound

$$(4.10) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt \ll_{k,\varepsilon} T^{1+\varepsilon}$$

for any  $k \geq 1$  and any  $\varepsilon > 0$ , which in turn is the same as

$$(4.11) \quad E_k(T) \ll_{k,\varepsilon} T^{1+\varepsilon}.$$

The enormous difficulty in settling the truth of the LH, and so *a fortiori* of the RH, is best reflected in the relatively modest upper bounds for the integrals in (4.10) (see Ch. 8 of [16] for sharpest known results). On the other hand, we have  $\Omega$ -results in the case  $k = 1, 2$ , which show that  $E_1(T)$  and  $E_2(T)$  cannot be always small. Thus J.L. Hafner and the author [11], [12] proved that

$$(4.12) \quad E_1(T) = \Omega_+ \left( (T \log T)^{\frac{1}{4}} (\log \log T)^{\frac{3+\log 4}{4}} e^{-C\sqrt{\log \log \log T}} \right)$$

and

$$(4.13) \quad E_1(T) = \Omega_- \left( T^{\frac{1}{3}} \exp \left( \frac{D(\log \log T)^{\frac{1}{4}}}{(\log \log \log T)^{\frac{3}{4}}} \right) \right)$$



for some absolute constants  $C, D > 0$ . Moreover the author [19] proved that there exist constants  $A, B > 0$  such that, for  $T \geq T_0$ , every interval  $[T, T + B\sqrt{T}]$  contains points  $t_1, t_2$  for which

$$E_1(t_1) > At_1^{1/4}, \quad E_1(t_2) < -At_2^{1/4}.$$

Numerical investigations concerning  $E_1(T)$  were carried out by H.J.J. te Riele and the author [29].

The  $\Omega$ -result

$$(4.14) \quad E_2(T) = \Omega(\sqrt{T})$$

(meaning  $\lim_{T \rightarrow \infty} E_2(T)T^{-1/2} \neq 0$ ) was proved by Y. Motohashi and the author (see [26], [28] and Ch. 5 of [20]). The method of proof involved differences of values of the functions  $E_2(T)$ , so that (4.14) was the limit of the method. The basis of this, as well of other recent investigations involving  $E_2(T)$ , is Y. Motohashi's fundamental explicit formula for

$$(4.15) \quad (\Delta\sqrt{\pi})^{-1} \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + it + iT)|^4 e^{-(t/\Delta)^2} dt \quad (\Delta > 0),$$

obtained by deep methods involving spectral theory of the non-Euclidean Laplacian (see [40], [41], [43], [63] and Ch. 5 of [16]). On p. 310 of [20] it was pointed out that a stronger result than (4.14), namely

$$\limsup_{T \rightarrow \infty} |E_2(T)|T^{-1/2} = +\infty$$

follows if certain quantities connected with the discrete spectrum of the non-Euclidean Laplacian are linearly independent over the integers. Y. Motohashi [42] recently unconditionally improved (4.14) by showing that

$$(4.16) \quad E_2(T) = \Omega_{\pm}(\sqrt{T})$$

holds. Namely he proved that the function

$$Z_2(\xi) := \int_1^{\infty} |\zeta(\frac{1}{2} + it)|^4 t^{-\xi} dt,$$

defined initially as a function of the complex variable  $\xi$  for  $\Re \xi > 1$ , is meromorphic over the whole complex plane. In the half-plane  $\Re \xi > 0$  it has a pole of order five at  $\xi = 1$ , infinitely many simple poles of the form  $\frac{1}{2} \pm \kappa i$ , while the remaining poles for  $\Re \xi > 0$  are of the form  $\rho/2, \zeta(\rho) = 0$ . Here  $\kappa^2 + \frac{1}{4}$  is in the discrete spectrum of the non-Euclidean Laplacian with respect to the full modular group. By using (4.1) and integration by parts it follows that

$$(4.17) \quad Z_2(\xi) = C + \xi \int_1^{\infty} P_4(\log t) t^{-\xi} dt + \xi \int_1^{\infty} E_2(t) t^{-\xi-1} dt$$

with a suitable constant  $C$ , where the integrals are certainly absolutely convergent for  $\Re \xi > 1$  (actually the second for  $\Re \xi > 1/2$  in view of (4.20)). Now (4.16) is an immediate consequence of (4.17) and the following version of a classical result of E. Landau (see [1] for a proof).

**Proposition 2.** Let  $g(x)$  be a continuous function such that

$$G(\xi) := \int_1^{\infty} g(x) x^{-\xi-1} dx$$

converges absolutely for some  $\xi$ . Let us suppose that  $G(\xi)$  admits analytic continuation to a domain including the half-line  $[\sigma, \infty)$ , while it has a simple pole at  $\xi = \sigma + i\delta$  ( $\delta \neq 0$ ), with residue  $\gamma$ . Then

$$\limsup_{x \rightarrow \infty} g(x) x^{-\sigma} \geq |\gamma|, \quad \liminf_{x \rightarrow \infty} g(x) x^{-\sigma} \leq -|\gamma|.$$

It should be pointed out that (4.14) shows that the well-known analogy between  $E_1(T)$  and  $\Delta_2(x)$  ( $= \Delta(x)$ , the error term in the formula for  $\sum_{n \leq x} d(n)$ ), which is discussed e.g., in Ch. 15 of [16], cannot be extended to general  $E_k(T)$  and  $\Delta_k(x)$ . The latter function denotes the error term in the asymptotic formula for  $\sum_{n \leq x} d_k(n)$ , where  $d_k(n)$  is the general divisor function generated by  $\zeta^k(s)$ . The LH is equivalent to either  $\alpha_k \leq 1/2$  ( $k \geq 2$ ) or  $\beta_k = (k-1)/(2k)$  ( $k \geq 2$ ), where  $\alpha_k$  and  $\beta_k$  are the infima of the numbers  $a_k$  and  $b_k$  for which

$$\Delta_k(x) \ll x^{a_k}, \quad \int_1^x \Delta_k^2(y) dy \ll x^{b_k}$$

hold, respectively. We know that  $\beta_k = (k-1)/(2k)$  for  $k = 2, 3, 4$ , and it is generally conjectured that  $\alpha_k = \beta_k = (k-1)/(2k)$  for any  $k$ . At first I thought that, analogously to the conjecture for  $\alpha_k$  and  $\beta_k$ , the upper bound for general  $E_k(T)$  should be of such a form as to yield the LH when  $k \rightarrow \infty$ , but in view of (4.14) I am certain that this cannot be the case.

It may be asked then how do the  $\Omega$ -results for  $E_1(T)$  and  $E_2(T)$  affect the LH, and thus indirectly the RH? A reasonable conjecture is that these  $\Omega$ -results lie fairly close to the truth, in other words that

$$(4.18) \quad E_k(T) = O_{k,\varepsilon}(T^{\frac{k}{4} + \varepsilon})$$

holds for  $k = 1, 2$ . This view is suggested by estimates in the mean for the functions in question. Namely the author [15] proved that

$$(4.19) \quad \int_1^T |E_1(t)|^A dt \ll_\varepsilon T^{1 + \frac{A}{4} + \varepsilon} \quad (0 \leq A \leq \frac{35}{4}),$$

and the range for  $A$  for which (4.19) holds can be slightly increased by using the best known estimate (4.6) in the course of the proof. Also Y. Motohashi and the author [27], [28] proved that

$$(4.20) \quad \int_0^T E_2(t) dt \ll T^{3/2}, \quad \int_0^T E_2^2(t) dt \ll T^2 \log^C T \quad (C > 0).$$

The bounds (4.19) and (4.20) show indeed that, in the mean sense, the bound (4.18) does hold when  $k = 1, 2$ . Curiously enough, it does not seem possible to show that the RH implies (4.18) for  $k \leq 3$ . If (4.18) holds for any  $k$ , then in view of (4.7) we would obtain (4.6) with the hitherto sharpest bound  $c \leq 1/8$ , or equivalently  $\mu(1/2) \leq 1/8$ , where for any real  $\sigma$  one defines

$$\mu(\sigma) = \limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{\log t},$$

and it will be clear from the context that no confusion can arise with the Möbius function. What can one expect about the order of magnitude of  $E_k(T)$  for  $k \geq 3$ ? It was already mentioned that the structure of  $E_k(T)$  becomes increasingly complex as  $k$  increases. Thus we should not expect a smaller exponent than  $k/4$  in (4.18) for  $k \geq 3$ , as it would by (4.7) yield a result of the type  $\mu(1/2) < 1/8$ , which in view of the  $\Omega$ -results is not obtainable from (4.18) when  $k = 1, 2$ . Hence by analogy with the cases  $k = 1, 2$  one would be led to conjecture that

$$(4.21) \quad E_k(T) = \Omega(T^{k/4})$$

holds for any fixed  $k \geq 1$ . But already for  $k = 5$  (4.21) yields, in view of (4.1),

$$(4.22) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^{10} dt = \Omega_+(T^{5/4}),$$

which contradicts (4.10), thereby disproving *both the LH and the RH*. It would be of great interest to obtain more detailed information on  $E_k(T)$  in the cases when  $k = 3$  and especially when  $k = 4$ , as the latter probably represents a turning point in the asymptotic behaviour of mean values of  $|\zeta(\frac{1}{2} + it)|$ . Namely the above phenomenon strongly suggests that either the LH fails, or the shape of the asymptotic formula for the left-hand side of (4.1) changes (in a yet completely unknown way) when  $k = 4$ . In [24] the author proved that  $E_3(T) \ll_\varepsilon T^{1+\varepsilon}$  conditionally, that is, provided that a certain conjecture involving the ternary additive divisor problem holds. Y. Motohashi ([40] p. 339, and [42]) proposes, on heuristic grounds based on analogy with explicit formulas known in the cases  $k = 1, 2$ , a formula for the analogue of (4.15) for the sixth moment, and also conjectures (4.21) for  $k = 3$ . Concerning the eighth moment, it should be mentioned that N.V. Kuznetsov [33] had an interesting approach based on applications of spectral theory, but unfortunately his proof of

$$(4.23) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^8 dt \ll T \log^C T$$

had several gaps (see the author's review in Zbl. 745.11040 and the Addendum of Y. Motohashi [40]), so that (4.23) is still a conjecture. If (4.23) is true, then one must have  $C \geq 16$  in (4.23), since by a result of K. Ramachandra (see [16] and [48]) one has, for any rational number  $k \geq 0$ ,

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt \gg_k T(\log T)^{k^2}.$$

The LH (see [61]) is equivalent to the statement that  $\mu(\sigma) = 1/2 - \sigma$  for  $\sigma < 1/2$ , and  $\mu(\sigma) = 0$  for  $\sigma \geq 1/2$ . If the LH is not true, what would then the graph of  $\mu(\sigma)$  look like? If the LH fails, it is most likely that  $\mu(1/2) = 1/8$  is true. Since  $\mu(\sigma)$  is (unconditionally) a non-increasing, convex function of  $\sigma$ ,

$$\mu(\sigma) = \frac{1}{2} - \sigma \quad (\sigma \leq 0), \quad \mu(\sigma) = 0 \quad (\sigma \geq 1),$$

and by the functional equation one has  $\mu(\sigma) = \frac{1}{2} - \sigma + \mu(1 - \sigma)$  ( $0 < \sigma < 1$ ), perhaps one would have

$$(4.24) \quad \mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \sigma \leq \frac{1}{4}, \\ \frac{3}{8} - \frac{\sigma}{2} & \frac{1}{4} \leq \sigma < \frac{3}{4}, \\ 0 & \sigma \geq \frac{3}{4}, \end{cases}$$

or the slightly weaker

$$(4.25) \quad \mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \sigma \leq 0, \\ \frac{2-3\sigma}{4} & 0 < \sigma < \frac{1}{2}, \\ \frac{(1-\sigma)}{4} & \frac{1}{2} \leq \sigma \leq 1, \\ 0 & \sigma > 1. \end{cases}$$

A third candidate is

$$(4.26) \quad \mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \sigma \leq 0, \\ \frac{1}{2}(1 - \sigma)^2 & 0 < \sigma < 1, \\ 0 & \sigma \geq 1, \end{cases}$$

which is a quadratic function of  $\sigma$  in the critical strip. Note that (4.26) sharpens (4.25) for  $0 < \sigma < 1$ , except when  $\sigma = 1/2$ , when (4.24)–(4.26) all yield  $\mu(\sigma) = 1/8$ . So far no exact value of  $\mu(\sigma)$  is known when  $\sigma$  lies in the critical strip  $0 < \sigma < 1$ .

## 5. Large values on the critical line

One thing that has constantly made the author skeptical about the truth of the RH is: How to draw the graph of  $Z(t)$  when  $t$  is large? By this the following is meant. R. Balasubramanian and K. Ramachandra (see [3], [4], [48], [49]) proved unconditionally that

$$(5.1) \quad \max_{T \leq t \leq T+H} |\zeta(\tfrac{1}{2} + it)| > \exp\left(\frac{3}{4} \left(\frac{\log H}{\log \log H}\right)^{1/2}\right)$$

for  $T \geq T_0$  and  $\log \log T \ll H \leq T$ , and probably on the RH this can be further improved (but no results seem to exist yet). Anyway (5.1) shows that  $|Z(t)|$  assumes large values relatively often. On the other hand, on the RH one expects that the bound in (1.11) can be also further reduced, very likely (see [46]) to

$$(5.2) \quad S(T) \ll_{\varepsilon} (\log T)^{\frac{1}{2} + \varepsilon}.$$

Namely, on the RH, H.L. Montgomery [39] proved that

$$S(T) = \Omega_{\pm} \left( \left( \frac{\log T}{\log \log T} \right)^{1/2} \right),$$

which is in accord with (5.2). K.-M. Tsang [59], improving a classical result of A. Selberg [54], has shown that one has unconditionally

$$S(T) = \Omega_{\pm} \left( \left( \frac{\log T}{\log \log T} \right)^{1/3} \right).$$

Also K.-M. Tsang [60] proved that (unconditionally;  $\pm$  means that the result holds both with the  $+$  and the  $-$  sign)

$$\left( \sup_{T \leq t \leq 2T} \log |\zeta(\tfrac{1}{2} + it)| \right) \left( \sup_{T \leq t \leq 2T} \pm S(t) \right) \gg \frac{\log T}{\log \log T},$$

which shows that either  $|\zeta(\frac{1}{2} + it)|$  or  $|S(t)|$  must assume large values in  $[T, 2T]$ . It may be pointed out that the calculations relating to the values of  $S(T)$  (see e.g., [45], [46]) show that all known values of  $S(T)$  are relatively small. In other words they are not anywhere near the values predicted by the above  $\Omega$ -results, which is one more reason that supports the view that the values for which  $\zeta(s)$  will exhibit its true asymptotic behaviour must be *really very large*.

If on the RH (5.2) is true, then clearly (1.12) can be improved to

$$(5.3) \quad \gamma_{n+1} - \gamma_n \ll_{\varepsilon} (\log \gamma_n)^{\varepsilon - 1/2}.$$

This means that, as  $n \rightarrow \infty$ , the gap between the consecutive zeros of  $Z(t)$  tends to zero not so slowly. Now take  $H = T$  in (5.1), and let  $t_0$  be the point in  $[T, 2T]$  where the maximum in (5.1) is attained. This point falls into an interval of length  $\ll (\log T)^{\varepsilon - 1/2}$  between two consecutive zeros, so that in the vicinity of  $t_0$  the function  $Z(t)$  must have very large oscillations, which will be carried over to  $Z'(t), Z''(t), \dots$  etc. For example, for  $T = 10^{5000}$  we shall have

$$(5.4) \quad |Z(t_0)| > 2.68 \times 10^{11},$$

while  $(\log T)^{-1/2} = 0.00932\dots$ , which shows how large the oscillations of  $Z(t)$  near  $t_0$  will be. Moreover, M. Jutila [30] unconditionally proved the following

**Proposition 3.** There exist positive constants  $a_1, a_2$  and  $a_3$  such that, for  $T \geq 10$ , we have

$$\exp(a_1(\log \log T)^{1/2}) \leq |Z(t)| \leq \exp(a_2(\log \log T)^{1/2})$$

in a subset of measure at least  $a_3 T$  of the interval  $[0, T]$ .

For  $T = 10^{5000}$  one has  $e^{(\log \log T)^{1/2}} = 21.28446 \dots$ , and Proposition 3 shows that relatively large values of  $|Z(t)|$  are plentiful, and in the vicinity of the respective  $t$ 's again  $Z(t)$  (and its derivatives) must oscillate a lot. The RH and (5.3) imply that, as  $t \rightarrow \infty$ , the graph of  $Z(t)$  will consist of tightly packed spikes, which will be more and more condensed as  $t$  increases, with larger and large oscillations. This I find hardly conceivable. Of course, it could happen that the RH is true and that (5.3) is not.

## 6. A class of convolution functions

It does not appear easy to put the discussion of Section 5 into a quantitative form. We shall follow now the method developed by the author in [21] and [23] and try to make a self-contained presentation, resulting in the proof of Theorem 1 (Sec. 8) and Theorem 2 (Sec. 9). The basic idea is to connect the order of  $Z(t)$  with the distribution of its zeros and the order of its derivatives (see (7.5)). However it turned out that if one works directly with  $Z(t)$ , then one encounters several difficulties. One is that we do not know yet whether the zeros of  $Z(t)$  are all distinct (simple), even on the RH (which implies by (1.11) only the fairly weak bound that the multiplicities of zeros up to height  $T$  are  $\ll \log T / \log \log T$ ). This difficulty is technical, and we may bypass it by using a suitable form of divided differences from Numerical analysis, as will be shown a little later in Section 7. A.A. Lavrik [34] proved the useful result that, uniformly for  $0 \leq k \leq \frac{1}{2} \log t$ , one has

$$(6.1) \quad Z^{(k)}(t) = 2 \sum_{n \leq (t/2\pi)^{1/2}} n^{-1/2} \left( \log \frac{(t/2\pi)^{1/2}}{n} \right)^k \cos \left( t \log \frac{(t/2\pi)^{1/2}}{n} - \frac{t}{2} - \frac{\pi}{8} + \frac{\pi k}{2} \right) + O \left( t^{-1/4} \left( \frac{3}{2} \log t \right)^{k+1} \right).$$

The range for which (6.1) holds is large, but it is difficult to obtain good uniform bounds for  $Z^{(k)}(t)$  from (6.1). To overcome this obstacle the author introduced in [21] the class of convolution functions

$$(6.2) \quad M_{Z,f}(t) := \int_{-\infty}^{\infty} Z(t-x) f_G(x) dx = \int_{-\infty}^{\infty} Z(t+x) f\left(\frac{x}{G}\right) dx,$$

where  $G > 0$ ,  $f_G(x) = f(x/G)$ , and  $f(x) (\geq 0)$  is an even function belonging to the class of smooth ( $C^\infty$ ) functions  $f(x)$  called  $S_\alpha^\beta$  by Gel'fand and Shilov [9]. The functions  $f(x)$  satisfy for any real  $x$  the inequalities

$$(6.3) \quad |x^k f^{(q)}(x)| \leq CA^k B^q k^{k\alpha} q^{q\beta} \quad (k, q = 0, 1, 2, \dots)$$

with suitable constants  $A, B, C > 0$  depending on  $f$  alone. For  $\alpha = 0$  it follows that  $f(x)$  is of bounded support, namely it vanishes for  $|x| \geq A$ . For  $\alpha > 0$  the condition (6.3) is equivalent (see [9]) to the condition

$$(6.4) \quad |f^{(q)}(x)| \leq CB^q q^{q\beta} \exp(-a|x|^{1/\alpha}) \quad (a = \alpha/(eA^{1/\alpha}))$$

for all  $x$  and  $q \geq 0$ . We shall denote by  $E_\alpha^\beta$  the subclass of  $S_\alpha^\beta$  with  $\alpha > 0$  consisting of even functions  $f(x)$  such that  $f(x)$  is not the zero-function. It is shown in [9] that  $S_\alpha^\beta$  is non-empty if  $\beta \geq 0$  and  $\alpha + \beta \geq 1$ . If these conditions hold then  $E_\alpha^\beta$  is also non-empty, since  $f(-x) \in S_\alpha^\beta$  if  $f(x) \in S_\alpha^\beta$ , and  $f(x) + f(-x)$  is always even.

One of the main properties of the convolution function  $M_{Z,f}(t)$ , which follows by  $k$ -fold integration by parts from (6.2), is that for any integer  $k \geq 0$

$$(6.5) \quad M_{Z,f}^{(k)}(t) = M_{Z^{(k)},f}(t) = \int_{-\infty}^{\infty} Z^{(k)}(t+x) f\left(\frac{x}{G}\right) dx = \left(\frac{-1}{G}\right)^k \int_{-\infty}^{\infty} Z(t+x) f^{(k)}\left(\frac{x}{G}\right) dx.$$

This relation shows that the order of  $M^{(k)}$  depends only on the orders of  $Z$  and  $f^{(k)}$ , and the latter is by (6.4) of exponential decay, which is very useful in dealing with convergence problems etc. The salient point of our approach is that the difficulties inherent in the distribution of zeros of  $Z(t)$  are transposed to the distribution of zeros of  $M_{Z,f}(t)$ , and for the latter function (6.5) provides good uniform control of its derivatives.

Several analogies between  $Z(t)$  and  $M_{Z,f}(t)$  are established in [21], especially in connection with mean values and the distribution of their respective zeros. We shall retain here the notation introduced in [21], so that  $N_M(T)$  denotes the number of zeros of  $M_{Z,f}(t)$  in  $(0, T]$ , with multiplicities counted. If  $f(x) \in E_\alpha^\beta$ ,  $f(x) \geq 0$  and  $G = \delta / \log(T/(2\pi))$  with suitable  $\delta > 0$ , then Theorem 4 of [21] says that

$$(6.6) \quad N_M(T+V) - N_M(T-V) \gg \frac{V}{\log T}, \quad V = T^{c+\varepsilon}, \quad c = 0.329021\dots,$$

for any given  $\varepsilon > 0$ . The nonnegativity of  $f(x)$  was needed in the proof of this result. For the function  $Z(t)$  the analogous result is that

$$(6.7) \quad N_0(T+V) - N_0(T-V) \gg V \log T, \quad V = T^{c+\varepsilon}, \quad c = 0.329021\dots,$$

where as usual  $N_0(T)$  denotes the number of zeros of  $Z(t)$  (or of  $\zeta(\frac{1}{2} + it)$ ) in  $(0, T]$ , with multiplicities counted. Thus the fundamental problem in the theory of  $\zeta(s)$  is to estimate  $N(T) - N_0(T)$ , and the RH may be reformulated as  $N(T) = N_0(T)$  for  $T > 0$ . The bound (6.7) was proved by A.A. Karatsuba (see [32] for a detailed account). As explained in [21], the bound (6.6) probably falls short (by a factor of  $\log^2 T$ ) from the expected (true) order of magnitude for the number of zeros of  $M_{Z,f}(t)$  in  $[T-V, T+V]$ . This is due to the method of proof of (6.6), which is not as strong as the classical method of A. Selberg [54] (see also Ch. 10 of [61]). The function  $N_M(T)$  seems much more difficult to handle than  $N_0(T)$  or  $N(T)$ . The latter can be conveniently expressed (see [16] or [61]) by means of a complex integral from which one infers then (1.3) with the bound (1.4). I was unable to find an analogue of the integral representation for  $N_M(T)$ . Note that the bound on the right-hand side of (6.7) is actually of the best possible order of magnitude.

In the sequel we shall need the following technical result, which we state as

**Lemma 1.** *If  $L = (\log T)^{\frac{1}{2}+\varepsilon}$ ,  $P = \sqrt{\frac{T}{2\pi}}$ ,  $0 < G < 1$ ,  $L \ll V \leq T^{\frac{1}{3}}$ ,  $f(x) \in E_\alpha^\beta$ , then*

$$(6.8) \quad \int_{T-VL}^{T+VL} |M_{Z,f}(t)| e^{-(T-t)^2 V^{-2}} dt \geq GV \{ |\widehat{f}(\frac{G}{2\pi} \log P)| + O(T^{-1/4} + V^2 T^{-3/4} L^2) \}.$$

**Proof.** In (6.8)  $\widehat{f}(x)$  denotes the Fourier transform of  $f(x)$ , namely

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(u) e^{2\pi i x u} du = \int_{-\infty}^{\infty} f(u) \cos(2\pi x u) du + i \int_{-\infty}^{\infty} f(u) \sin(2\pi x u) du = \int_{-\infty}^{\infty} f(u) \cos(2\pi x u) du$$

since  $f(x)$  is even. From the Riemann–Siegel formula (1.13) we have, if  $|T-t| \leq VL$  and  $|x| \leq \log^C T$  ( $C > 0$ ),

$$Z(t+x) = 2 \sum_{n \leq P} n^{-1/2} \cos\left( (t+x) \log \frac{((t+x)/(2\pi))^{1/2}}{n} - \frac{t+x}{2} - \frac{\pi}{8} \right) + O(T^{-1/4}).$$

Simplifying the argument of the cosine by Taylor's formula it follows that

$$(6.9) \quad Z(t+x) = 2 \sum_{n \leq P} n^{-1/2} \cos\left( (t+x) \log \frac{P}{n} - \frac{T}{2} - \frac{\pi}{8} \right) + O(T^{-1/4} + V^2 T^{-3/4} L^2).$$

Hence from (6.2) and (6.9) we have, since

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

and  $f(x)$  is even,

$$\begin{aligned} M_{Z,f}(t) &= 2G \sum_{n \leq P} n^{-1/2} \cos\left(t \log \frac{P}{n} - \frac{T}{2} - \frac{\pi}{8}\right) \int_{-\infty}^{\infty} f(x) \cos\left(Gx \log \frac{P}{n}\right) dx + O(GT^{-1/4} + GV^2T^{-3/4}L^2) \\ (6.10) \quad &= 2G \sum_{n \leq P} n^{-1/2} \cos\left(t \log \frac{P}{n} - \frac{T}{2} - \frac{\pi}{8}\right) \widehat{f}\left(\frac{G}{2\pi} \log \frac{P}{n}\right) + O(GT^{-1/4} + GV^2T^{-3/4}L^2). \end{aligned}$$

Therefore we obtain from (6.10)

$$\int_{T-VL}^{T+VL} |M_{Z,f}(t)| e^{-(T-t)^2V^{-2}} dt \geq GI + O(GVT^{-1/4} + GV^3T^{-3/4}L^2),$$

say, where

$$I := \int_{T-VL}^{T+VL} \left| \sum_{n \leq P} n^{-1/2} \widehat{f}\left(\frac{G}{2\pi} \log \frac{P}{n}\right) \left( \exp\left(it \log \frac{P}{n} - \frac{iT}{2} - \frac{i\pi}{8}\right) + \exp\left(-it \log \frac{P}{n} + \frac{iT}{2} + \frac{i\pi}{8}\right) \right) \right| e^{-(T-t)^2V^{-2}} dt.$$

By using the fact that  $|\exp(it \log P - \frac{iT}{2} - \frac{i\pi}{8})| = 1$  and the classical integral

$$\int_{-\infty}^{\infty} \exp(Ax - Bx^2) dx = \sqrt{\frac{\pi}{B}} \exp\left(\frac{A^2}{4B}\right) \quad (\Re B > 0)$$

we shall obtain

$$I \geq |I_1 + I_2|,$$

where

$$\begin{aligned} I_1 &= \int_{T-VL}^{T+VL} \sum_{n \leq P} n^{-1/2} \widehat{f}\left(\frac{G}{2\pi} \log \frac{P}{n}\right) \exp(-it \log n - (T-t)^2V^{-2}) dt \\ &= \sum_{n \leq P} n^{-1/2} \widehat{f}\left(\frac{G}{2\pi} \log \frac{P}{n}\right) \exp(-iT \log n) \int_{-VL}^{VL} \exp(-ix \log n - x^2V^{-2}) dx \\ &= \sum_{n \leq P} n^{-1/2} \widehat{f}\left(\frac{G}{2\pi} \log \frac{P}{n}\right) \left\{ \sqrt{\pi}V \exp(-iT \log n - \frac{1}{4}V^2 \log^2 n) + O(\exp(-\log^{1+2\varepsilon} T)) \right\} \\ &= \sqrt{\pi}V \widehat{f}\left(\frac{G}{2\pi} \log P\right) + O(T^{-C}) \end{aligned}$$

for any fixed  $C > 0$ . Similarly we find that

$$I_2 = \int_{T-VL}^{T+VL} \sum_{n \leq P} n^{-1/2} \widehat{f}\left(\frac{G}{2\pi} \log \frac{P}{n}\right) \exp\left(-it \log\left(\frac{T}{2\pi n}\right) + iT + \frac{\pi i}{4} - (T-t)^2V^{-2}\right) dt =$$

$$\begin{aligned} \sum_{n \leq P} n^{-1/2} \widehat{f}\left(\frac{G}{2\pi} \log \frac{P}{n}\right) \exp\left(-it \log\left(\frac{T}{2\pi n}\right) + iT + \frac{\pi i}{4}\right) \left(\sqrt{\pi} V \exp\left\{-\frac{1}{4}\left(V \log\left(\frac{T}{2\pi n}\right)\right)^2\right\} + O\{\exp(-\log^{1+2\varepsilon} T)\}\right) \\ = O(T^{-C}) \end{aligned}$$

again for any fixed  $C > 0$ , since  $\log\left(\frac{T}{2\pi n}\right) \geq \log \frac{T}{2\pi P} = \frac{1}{2} \log\left(\frac{T}{2\pi}\right)$ . From the above estimates (6.8) follows.

## 7. Technical preparation

In this section we shall lay the groundwork for the investigation of the distribution of zeros of  $Z(t)$  via the convolution functions  $M_{Z,f}(t)$ . To do this we shall first briefly outline a method based on a generalized form of the mean value theorem from the differential calculus. This can be conveniently obtained from the expression for the  $n$ -th divided difference associated to the function  $F(x)$ , namely

$$\begin{aligned} [x, x_1, x_2, \dots, x_n] &:= \\ &= \frac{F(x)}{(x-x_1)(x-x_2)\cdots(x-x_n)} + \frac{F(x_1)}{(x_1-x)(x_1-x_2)\cdots(x_1-x_n)} + \cdots + \frac{F(x_n)}{(x_n-x)(x_n-x_1)\cdots(x_n-x_{n-1})} \end{aligned}$$

where  $x_i \neq x_j$  if  $i \neq j$ , and  $F(t)$  is a real-valued function of the real variable  $t$ . We have the representation

$$(7.1) \quad \begin{aligned} [x, x_1, x_2, \dots, x_n] &= \\ &= \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} F^{(n)}(x_1 + (x_2 - x_1)t_1 + \cdots + (x_n - x_{n-1})t_{n-1} + (x - x_n)t_n) dt_n \cdots dt_1 = \frac{F^{(n)}(\xi)}{n!} \end{aligned}$$

if  $F(t) \in C^n[I]$ ,  $\xi = \xi(x, x_1, \dots, x_n)$  and  $I$  is the smallest interval containing all the points  $x, x_1, \dots, x_n$ . If we suppose additionally that  $F(x_j) = 0$  for  $j = 1, \dots, n$ , then on comparing the two expressions for  $[x, x_1, x_2, \dots, x_n]$ , it follows that

$$(7.2) \quad F(x) = (x-x_1)(x-x_2)\cdots(x-x_n) \frac{F^{(n)}(\xi)}{n!},$$

where  $\xi = \xi(x)$  if we consider  $x_1, \dots, x_n$  as fixed and  $x$  as a variable. The underlying idea is that, if the (distinct) zeros  $x_j$  of  $F(x)$  are sufficiently close to one another, then (7.2) may lead to a contradiction if  $F(x)$  is assumed to be large and one has good bounds for its derivatives.

To obtain the analogue of (7.2) when the points  $x_j$  are not necessarily distinct, note that if  $F(z)$  is a regular function of the complex variable  $z$  in a region which contains the distinct points  $x, x_1, \dots, x_n$ , then for a suitable closed contour  $\mathcal{C}$  containing these points one obtains by the residue theorem

$$[x, x_1, x_2, \dots, x_n] = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(z)}{(z-x)(z-x_1)\cdots(z-x_n)} dz.$$

A comparison with (7.1) yields then

$$(7.3) \quad \begin{aligned} &\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(z)}{(z-x)(z-x_1)\cdots(z-x_n)} dz \\ &= \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} F^{(n)}(x_1 + (x_2 - x_1)t_1 + \cdots + (x_n - x_{n-1})t_{n-1} + (x - x_n)t_n) dt_n \cdots dt_1. \end{aligned}$$



Now (7.3) was derived on the assumption that the points  $x, x_1, \dots, x_n$  are distinct. But as both sides of (7.3) are regular functions of  $x, x_1, \dots, x_n$  in some region, this assumption may be dropped by analytic continuation. Thus let the points  $x, x_1, \dots, x_n$  coincide with the (distinct) points  $z_k$ , where the multiplicity of  $z_k$  is denoted by  $p_k; k = 0, 1, \dots, \nu; \sum_{k=0}^{\nu} p_k = n + 1$ . If we set

$$z_0 = x, p_0 = 1, Q(z) = \prod_{k=1}^{\nu} (z - z_k)^{p_k},$$

then the complex integral in (7.3) may be evaluated by the residue theorem (see Ch.1 of A.O. Gel'fond [10]). It equals

$$(7.4) \quad \frac{F(x)}{Q(x)} - \sum_{k=1}^{\nu} \sum_{m=0}^{p_k-1} \frac{F^{(p_k-m-1)}(z_k)}{(p_k-m-1)!} \sum_{s=0}^m \frac{1}{(m-s)!} \frac{d^{m-s}}{dz^{m-s}} \left( \frac{(z-z_k)^{p_k}}{Q(z)} \right) \Big|_{z=z_k} \cdot (x-x_k)^{-s-1}.$$

If  $x_1, x_2, \dots, x_n$  are the zeros of  $F(z)$ , then  $F^{(p_k-m-1)}(z_k) = 0$  for  $k = 1, \dots, \nu$  and  $m = 0, \dots, p_k-1$ , since  $p_k$  is the multiplicity of (the zero)  $z_k$ . Hence if  $F(x) \neq 0$ , then on comparing (7.1), (7.3) and (7.4) one obtains

$$(7.5) \quad |F(x)| \leq \prod_{k=1}^n |x - x_k| \frac{|F^{(n)}(\xi)|}{n!} \quad (\xi = \xi(x)),$$

and of course (7.5) is trivial if  $F(x) = 0$ .

Now we shall apply (7.5) to  $F(t) = M_{Z,f}(t), f(x) \in E_{\alpha}^{\beta}$ , with  $n$  replaced by  $k$ , to obtain

$$(7.6) \quad |M_{Z,f}(t)| \leq \prod_{t-H \leq \gamma \leq t+H} |\gamma - t| \frac{|M_{Z,f}^{(k)}(\tau)|}{k!},$$

where  $\gamma$  denotes the zeros of  $M_{Z,f}(t)$  in  $[t-H, t+H], \tau = \tau(t, H) \in [t-H, t+H], |t-T| \leq T^{1/2+\varepsilon}$  and  $k = k(t, H)$  is the number of zeros of  $M_{Z,f}(t)$  in  $[t-H, t+H]$ . We shall choose

$$(7.7) \quad H = \frac{A \log_3 T}{\log_2 T} \quad (\log_r T = \log(\log_{r-1} T), \log_1 T \equiv \log T)$$

for a sufficiently large  $A > 0$ . One intuitively feels that, with a suitable choice (see (8.1) and (8.2)) of  $G$  and  $f$ , the functions  $N(T)$  and  $N_M(T)$  will not differ by much. Thus we shall suppose that the analogues of (1.3) and (1.11) hold for  $N_M(T)$ , namely that

$$(7.8) \quad N_M(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + S_M(T) + O(1)$$

with a continuous function  $S_M(T)$  satisfying

$$(7.9) \quad S_M(T) = O\left(\frac{\log T}{\log \log T}\right),$$

although it is hard to imagine what should be the appropriate analogue for  $S_M(T)$  of the defining relation  $S(T) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + iT\right)$  in (1.4). We also suppose that

$$(7.10) \quad \int_T^{T+U} (S_M(t+H) - S_M(t-H))^{2m} dt \ll U(\log(2+H \log T))^m$$

holds for any fixed integer  $m \geq 1, T^a < U \leq T, 1/2 < a \leq 1, 0 < H < 1$ . Such a result holds unconditionally (even in the form of an asymptotic formula) if  $S_M(T)$  is replaced by  $S(T)$ , as shown in the works of A. Fujii

[8] and K.-M. Tsang [59]. Thus it seems plausible that (7.10) will also hold. It was already mentioned that it is reasonable to expect that  $S(T)$  and  $S_M(T)$  will be close to one another. One feels that this “closeness” should hold also in the mean sense, and that instead of (7.10) one could impose a condition which links directly  $S_M(T)$  and  $S(T)$ , such as that for any fixed integer  $m \geq 1$  one has

$$(7.11) \quad \int_T^{T+U} (S_M(t) - S(t))^{2m} dt \ll U(\log \log T)^m \quad (T^a < U \leq T, \frac{1}{2} < a \leq 1).$$

If (7.8) holds, then

$$(7.12) \quad \begin{aligned} k &= N_M(t+H) - N_M(t-H) + O(1) \\ &= \frac{(t+H)}{2\pi} \log\left(\frac{t+H}{2\pi}\right) - \frac{t+H}{2\pi} - \frac{t-H}{2\pi} \log\left(\frac{t-H}{2\pi}\right) + \frac{t-H}{2\pi} + S_M(t+H) - S_M(t-H) + O(1) \\ &= \frac{H}{\pi} \log\left(\frac{T}{2\pi}\right) + S_M(t+H) - S_M(t-H) + O(1). \end{aligned}$$

To bound from above the product in (7.6) we proceed as follows. First we have trivially

$$\prod_{|\gamma-t| \leq 1/\log_2 T} |\gamma-t| \leq 1.$$

The remaining portions of the product with  $t-H \leq \gamma < t-1/\log_2 T$  and  $t+1/\log_2 T < \gamma \leq t+H$  are treated analogously, so we shall consider in detail only the latter. We have

$$\begin{aligned} &\log\left(\prod_{t+1/\log_2 T < \gamma \leq t+H} |\gamma-t|\right) \\ &= \sum_{t+1/\log_2 T < \gamma \leq t+H} \log(\gamma-t) = \int_{t+1/\log_2 T+0}^{t+H} \log(u-t) dN_M(u) \\ &= \frac{1}{2\pi} \int_{t+1/\log_2 T+0}^{t+H} \log(u-t) \log\left(\frac{u}{2\pi}\right) du + \int_{t+1/\log_2 T+0}^{t+H} \log(u-t) d(S_M(u) + O(1)). \end{aligned}$$

By using integration by parts and (7.9) it follows that

$$\int_{t+1/\log_2 T+0}^{t+H} \log(u-t) d(S_M(u) + O(1)) = O\left(\frac{\log T \log_3 T}{\log_2 T}\right) - \int_{t+1/\log_2 T}^{t+H} \frac{S_M(u) + O(1)}{u-t} du \ll \frac{\log T \log_3 T}{\log_2 T},$$

and we have

$$\begin{aligned} &\frac{1}{2\pi} \int_{t+1/\log_2 T+0}^{t+H} \log(u-t) \log\left(\frac{u}{2\pi}\right) du = \frac{1}{2\pi} \log\left(\frac{T}{2\pi}\right) \cdot (1 + O(T^{\varepsilon-1/2})) \int_{t+1/\log_2 T}^{t+H} \log(u-t) du \\ &= \frac{1}{2\pi} \log\left(\frac{T}{2\pi}\right) \cdot \left(H \log H - H + O\left(\frac{\log_3 T}{\log_2 T}\right)\right). \end{aligned}$$

By combining the above estimates we obtain

**Lemma 2.** *Suppose that (7.8) and (7.9) hold. If  $\gamma$  denotes zeros of  $M_{Z,f}(t)$ ,  $H$  is given by (7.7) and  $|T-t| \leq T^{1/2+\varepsilon}$ , then*

$$(7.13) \quad \prod_{t-H \leq \gamma \leq t+H} |\gamma-t| \leq \exp\left\{\frac{1}{\pi} \log\left(\frac{T}{2\pi}\right) \cdot \left(H \log H - H + O\left(\frac{\log_3 T}{\log_2 T}\right)\right)\right\}.$$

## 8. The asymptotic formula for the convolution function

In this section we shall prove a sharp asymptotic formula for  $M_{Z,f}(t)$ , which is given by Theorem 1. This will hold if  $f(x)$  belongs to a specific subclass of functions from  $E_\alpha^0$  ( $\alpha > 1$  is fixed), and for such  $M_{Z,f}(t)$  we may hope that (7.8)–(7.11) will hold. To construct this subclass of functions first of all let  $\varphi(x) \geq 0$  (but  $\varphi(x) \not\equiv 0$ ) belong to  $E_\alpha^0$ . Such a choice is possible, since it is readily checked that  $f^2(x) \in S_\alpha^\beta$  if  $f(x) \in S_\alpha^\beta$ , and trivially  $f^2(x) \geq 0$ . Thus  $\varphi(x)$  is of bounded support, so that  $\varphi(x) = 0$  for  $|x| \geq a$  for some  $a > 0$ . We normalize  $\varphi(x)$  so that  $\int_{-\infty}^{\infty} \varphi(x) dx = 1$ , and for an arbitrary constant  $b > \max(1, a)$  we put

$$\Phi(x) := \int_{x-b}^{x+b} \varphi(t) dt.$$

Then  $0 \leq \Phi(x) \leq 1$ ,  $\Phi(x)$  is even (because  $\varphi(x)$  is even) and nonincreasing for  $x \geq 0$ , and

$$\Phi(x) = \begin{cases} 0 & \text{if } |x| \geq b+a, \\ 1 & \text{if } |x| \leq b-a. \end{cases}$$

One can also check that  $\varphi(x) \in S_\alpha^0$  implies that  $\Phi(x) \in S_\alpha^0$ . Namely  $|x^k \Phi(x)| \leq (b+a)^k$ , and for  $q \geq 1$  one uses (6.3) (with  $k = 0$ ,  $f^{(q)}$  replaced by  $\varphi^{(q-1)}$ ,  $(\alpha, \beta) = (0, \alpha)$ ) to obtain

$$\begin{aligned} |x^k \Phi^{(q)}(x)| &\leq (b+a)^k |\Phi^{(q)}(x)| \leq (b+a)^k \left( |\varphi^{(q-1)}(x+b)| + |\varphi^{(q-1)}(x-b)| \right) \\ &\leq (b+a)^k 2CB^{q-1} (q-1)^{(q-1)\alpha} \leq \frac{2C}{B} (b+a)^k B^q q^{q\alpha}, \end{aligned}$$

hence (6.3) will hold for  $\Phi$  in place of  $f$ , with  $A = b+a$  and suitable  $C$ . Let

$$f(x) := \int_{-\infty}^{\infty} \Phi(u) e^{-2\pi i x u} du = \int_{-\infty}^{\infty} \Phi(u) \cos(2\pi x u) du.$$

A fundamental property of the class  $S_\alpha^\beta$  (see [9]) is that  $\widehat{S}_\alpha^\beta = S_\beta^\alpha$ , where in general  $\widehat{U} = \{\widehat{f}(x) : f(x) \in U\}$ . Thus  $f(x) \in S_\alpha^0$ ,  $f(x)$  is even (because  $\Phi(x)$  is even), and by the inverse Fourier transform we have  $\widehat{f}(x) = \Phi(x)$ . The function  $f(x)$  is not necessarily nonnegative, but this property is not needed in the sequel.

Henceforth let

$$(8.1) \quad G = \frac{\delta}{\log(\frac{T}{2\pi})} \quad (\delta > 0).$$

In view of (1.3) it is seen that, on the RH,  $G$  is of the order of the average spacing between the zeros of  $Z(t)$ . If  $f(x)$  is as above, then we have

**THEOREM 1.** For  $|t - T| \leq VL$ ,  $L = \log^{\frac{1}{2}+\varepsilon} T$ ,  $\log^\varepsilon T \leq V \leq \frac{T^{\frac{1}{4}}}{\log T}$ ,  $0 < \delta < 2\pi(b-a)$  and any fixed  $N \geq 1$  we have

$$(8.2) \quad M_{Z,f}(t) = G(Z(t) + O(T^{-N})).$$

**Proof.** Observe that the weak error term  $O(T^{-1/4})$  in (8.2) follows from (6.10) (with  $x = 0$ ) and (6.10) when we note that

$$\widehat{f}\left(\frac{G}{2\pi} \log \frac{P}{n}\right) = \widehat{f}\left(\frac{\delta}{2\pi} \left(\frac{1}{2} - \frac{\log n}{\log(\frac{T}{2\pi})}\right)\right) = 1$$

since by construction  $\widehat{f}(x) = 1$  for  $|x| < b - a$ , and

$$\left| \frac{\delta}{2\pi} \left( \frac{1}{2} - \frac{\log n}{\log(\frac{T}{2\pi})} \right) \right| \leq \frac{\delta}{4\pi} < b - a \quad \left( 1 \leq n \leq P = \sqrt{\frac{T}{2\pi}} \right).$$

Also the hypotheses on  $t$  in the formulation of Theorem 1 can be relaxed.

In order to prove (8.2) it will be convenient to work with the real-valued function  $\theta(t)$ , defined by

$$(8.3) \quad Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) = \chi^{-1/2}\left(\frac{1}{2} + it\right) \zeta\left(\frac{1}{2} + it\right).$$

From the functional equation (1.2) in the symmetric form

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

one obtains

$$\chi^{-1/2}\left(\frac{1}{2} + it\right) = \frac{\pi^{-it/2} \Gamma^{1/2}\left(\frac{1}{4} + \frac{it}{2}\right)}{\Gamma^{1/2}\left(\frac{1}{4} - \frac{it}{2}\right)},$$

and consequently

$$(8.4) \quad \theta(t) = \Im \log \Gamma\left(\frac{1}{4} + \frac{1}{2}it\right) - \frac{1}{2}t \log \pi.$$

We have the explicit representation (see Ch. 3 of [32])

$$(8.5) \quad \theta(t) = \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \Delta(t)$$

with  $(\psi(x) = x - [x] - \frac{1}{2})$

$$(8.6) \quad \Delta(t) := \frac{t}{4} \log\left(1 + \frac{1}{4t^2}\right) + \frac{1}{4} \arctan \frac{1}{2t} + \frac{t}{2} \int_0^\infty \frac{\psi(u) du}{(u + \frac{1}{4})^2 + t^2}.$$

This formula is very useful, since it allows one to evaluate explicitly all the derivatives of  $\theta(t)$ . For  $t \rightarrow \infty$  it is seen that  $\Delta(t)$  admits an asymptotic expansion in terms of negative powers of  $t$ , and from (8.4) and Stirling's formula for the gamma-function it is found that ( $B_k$  is the  $k$ -th Bernoulli number)

$$(8.7) \quad \Delta(t) \sim \sum_{n=1}^{\infty} \frac{(2^{2n} - 1) |B_{2n}|}{2^{2n} (2n - 1) 2n t^{2n-1}},$$

and the meaning of (8.7) is that, for an arbitrary integer  $N \geq 1$ ,  $\Delta(t)$  equals the sum of the first  $N$  terms of the series in (8.7), plus the error term which is  $O_N(t^{-2N-1})$ . In general we shall have, for  $k \geq 0$  and suitable constants  $c_{k,n}$ ,

$$(8.8) \quad \Delta^{(k)}(t) \sim \sum_{n=1}^{\infty} c_{k,n} t^{1-2n-k}.$$

For complex  $s$  not equal to the poles of the gamma-factors we have the Riemann-Siegel formula (this is equation (56) of C.L. Siegel [57])

$$(8.9) \quad \begin{aligned} & \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \\ &= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \int_{0 \swarrow 1} \frac{e^{i\pi x^2} x^{-s}}{e^{i\pi x} - e^{-i\pi x}} dx + \pi^{(s-1)/2} \Gamma\left(\frac{1-s}{2}\right) \int_{0 \searrow 1} \frac{e^{-i\pi x^2} x^{s-1}}{e^{i\pi x} - e^{-i\pi x}} dx. \end{aligned}$$

Here  $0 \swarrow 1$  (resp.  $0 \searrow 1$ ) denotes a straight line which starts from infinity in the upper complex half-plane, has slope equal to 1 (resp. to -1), and cuts the real axis between 0 and 1. Setting in (8.9)  $s = \frac{1}{2} + it$  and using the property (8.4) it follows that

$$Z(t) = 2 \Re \left( e^{-i\theta(t)} \int_{0 \searrow 1} \frac{e^{-i\pi z^2} z^{-1/2+it}}{e^{i\pi z} - e^{-i\pi z}} dz \right).$$

As  $\Re(iw) = -\Im w$ , this can be conveniently written as

$$(8.10) \quad Z(t) = \Im \left( e^{-i\theta(t)} \int_{0 \searrow 1} e^{-i\pi z^2} z^{-1/2+it} \frac{dz}{\sin(\pi z)} \right).$$

Since

$$|\sin z|^2 = \sin^2(\Re z) + \sinh^2(\Im z), \quad z^{-1/2+it} = |z|^{-1/2+it} e^{-\frac{i}{2} \arg z - t \arg z},$$

and for  $z = \eta + u e^{3\pi i/4}$ ,  $u$  real,  $0 < \eta < 1$  we have

$$|e^{-i\pi z^2}| = e^{-\pi u^2 + \eta \sqrt{2} \pi u},$$

it follows that the contribution of the portion of the integral in (8.10) for which  $|z| \geq \log t$  is  $\ll e^{-\log^2 t}$ . Hence

$$(8.11) \quad Z(t) = \Im \left( e^{-i\theta(t)} \int_{0 \searrow 1, |z| < \log t} e^{-i\pi z^2} z^{-1/2+it} \frac{dz}{\sin(\pi z)} \right) + O(e^{-\log^2 t}).$$

From the decay property (6.4) it follows that

$$(8.12) \quad M_{Z,f}(t) = \int_{-\infty}^{\infty} Z(t+x) f\left(\frac{x}{G}\right) dx = \int_{-\log^{2\alpha-1} t}^{\log^{2\alpha-1} t} Z(t+x) f\left(\frac{x}{G}\right) dx + O(e^{-c \log^2 t}),$$

where  $c$  denotes positive, absolute constants which may not be the same ones at each occurrence. Thus from (8.10) and (8.12) we obtain that

$$(8.13) \quad M_{Z,f}(t) = \Im \left( \int_{0 \searrow 1, |z| < \log t} \frac{e^{-i\pi z^2} z^{-1/2+it}}{\sin(\pi z)} \int_{-\log^{2\alpha-1} t}^{\log^{2\alpha-1} t} e^{-i\theta(t+x)} z^{ix} f\left(\frac{x}{G}\right) dx dz \right) + O(e^{-c \log^2 t}).$$

By using Taylor's formula we have

$$(8.14) \quad \theta(t+x) = \theta(t) + \frac{x}{2} \log \frac{t}{2\pi} + x \Delta'(t) + R(t, x)$$

with  $\Delta'(t) \ll t^{-2}$  and

$$R(t, x) = \sum_{n=2}^{\infty} \left( \frac{(-1)^n}{2n(n-1)t^{n-1}} + \frac{\Delta^{(n)}(t)}{n!} \right) x^n.$$

Now we put

$$(8.15) \quad e^{-iR(t,x)} = 1 + S(t, x),$$

say, and use (8.5), (8.6), (8.8) and (8.14). We obtain

$$(8.16) \quad S(t, x) = \sum_{k=1}^{\infty} \frac{(-i)^k R^k(t, x)}{k!} = \sum_{n=2}^{\infty} g_n(t) x^n,$$

where each  $g_n(t)$  ( $\in C^\infty(0, \infty)$ ) has an asymptotic expansion of the form

$$(8.17) \quad g_n(t) \sim \sum_{k=0}^{\infty} d_{n,k} t^{-k - [(n+1)/2]} \quad (t \rightarrow \infty)$$

with suitable constants  $d_{n,k}$ . From (8.13)-(8.15) we have

$$(8.18) \quad M_{Z,f}(t) = \Im(I_1 + I_2) + O(e^{-c \log^2 t}),$$

where

$$(8.19) \quad I_1 := \int_{0 \setminus 1, |z| < \log t} \frac{e^{-i\pi z^2} z^{-1/2+it}}{\sin(\pi z)} \int_{-\log^{2\alpha-1} t}^{\log^{2\alpha-1} t} e^{-i\theta(t) - \frac{i\pi}{2} \log \frac{t}{2\pi} - ix\Delta'(t)} z^{ix} f\left(\frac{x}{G}\right) dx dz,$$

$$(8.20) \quad I_2 := \int_{0 \setminus 1, |z| < \log t} \frac{e^{-i\pi z^2} z^{-1/2+it}}{\sin(\pi z)} \int_{-\log^{2\alpha-1} t}^{\log^{2\alpha-1} t} e^{-i\theta(t) - \frac{i\pi}{2} \log \frac{t}{2\pi} - ix\Delta'(t)} z^{ix} f\left(\frac{x}{G}\right) S(t, x) dx dz.$$

In  $I_1$  we write  $z = e^{i \arg z} |z|$ , which gives

$$I_1 := \int_{0 \setminus 1, |z| < \log t} e^{-\theta(t)} \frac{e^{-i\pi z^2} z^{-1/2+it}}{\sin(\pi z)} h(z) dz + O(e^{-c \log^2 t}),$$

where

$$\begin{aligned} h(z) &:= \int_{-\infty}^{\infty} e^{-x \arg z} f\left(\frac{x}{G}\right) \exp\left(ix \log \frac{|z| e^{-\Delta'(t)}}{\sqrt{t/2\pi}}\right) dx \\ &= G \int_{-\infty}^{\infty} e^{-Gy \arg z} f(y) \exp\left(iyG \log \frac{|z| e^{-\Delta'(t)}}{\sqrt{t/2\pi}}\right) dy \\ &= G \sum_{n=0}^{\infty} \frac{(-G \arg z)^n}{n!} \int_{-\infty}^{\infty} y^n f(y) \exp\left(2i\pi y \cdot \frac{G}{2\pi} \log \frac{|z| e^{-\Delta'(t)}}{\sqrt{t/2\pi}}\right) dy. \end{aligned}$$

Change of summation and integration is justified by absolute convergence, since  $|z| < \log t$ ,  $G = \delta / \log(T/2\pi)$ ,  $\Delta'(t) \ll t^{-2}$ ,  $t \sim T$ , and  $f(x)$  satisfies (6.4). But

$$\int_{-\infty}^{\infty} f(y) \exp\left(2i\pi y \cdot \frac{G}{2\pi} \log \frac{|z| e^{-\Delta'(t)}}{\sqrt{t/2\pi}}\right) dy = \hat{f}\left(\frac{G}{2\pi} \log \frac{|z| e^{-\Delta'(t)}}{\sqrt{t/2\pi}}\right) = 1$$

for  $\delta < 2\pi(b-a)$ , since

$$\left| \frac{G}{2\pi} \log \frac{|z| e^{-\Delta'(t)}}{\sqrt{t/2\pi}} \right| = \frac{\delta}{2\pi \log\left(\frac{T}{2\pi}\right)} \left( \log \sqrt{\frac{t}{2\pi}} - \log |z| + \Delta'(t) \right) = \left( \frac{\delta}{4\pi} + o(1) \right) < \frac{\delta}{2\pi} < b-a,$$

and  $\widehat{f}(x) = 1$  for  $|x| < b - a$ . Moreover for  $n \geq 1$  and  $|x| < b - a$  we have

$$\widehat{f}^{(n)}(x) = (2\pi i)^n \int_{-\infty}^{\infty} y^n e^{2\pi i x y} f(y) dy = 0,$$

hence it follows that

$$\int_{-\infty}^{\infty} y^n f(y) \exp\left(2i\pi y \cdot \frac{G}{2\pi} \log \frac{|z|e^{-\Delta'(t)}}{\sqrt{t/2\pi}}\right) dy = 0 \quad (n \geq 1, 1 < 2\pi(b-a)).$$

Thus we obtain

$$(8.21) \quad I_1 = G \int_{0 \searrow 1} \frac{e^{-\theta(t)} e^{-i\pi z^2} z^{-1/2+it}}{\sin(\pi z)} dz + O(e^{-c \log^2 t}).$$

Similarly from (8.16) and (8.20) we have

$$\begin{aligned} I_2 &= \int_{0 \searrow 1, |z| < \log t} \frac{e^{-i\pi z^2} z^{-1/2+it}}{\sin(\pi z)} \int_{-\log^{2\alpha-1} t}^{\log^{2\alpha-1} t} e^{-i\theta(t) - \frac{ix}{2} \log \frac{t}{2\pi} - ix\Delta'(t)} z^{ix} f\left(\frac{x}{G}\right) \sum_{n=2}^{\infty} x^n g_n(t) dx dz \\ &= \int_{0 \searrow 1, |z| < \log t} \frac{e^{-i\pi z^2} z^{-1/2+it}}{\sin(\pi z)} \int_{-\log^{2\alpha-1} t}^{\log^{2\alpha-1} t} \left( \sum_{n=1}^N \frac{P_n(x)}{t^n} + O\left(\frac{1+x^{N+2}}{t^{N+1}}\right) \right) e^{-i\theta(t) - \frac{ix}{2} \log \frac{t}{2\pi} - ix\Delta'(t)} z^{ix} f\left(\frac{x}{G}\right) dx dz \\ &= \sum_{n=1}^N e^{-i\theta(t)} t^{-n} \int_{0 \searrow 1, |z| < \log t} \frac{e^{-i\pi z^2} z^{-1/2+it}}{\sin(\pi z)} \int_{-\log^{2\alpha-1} t}^{\log^{2\alpha-1} t} P_n(x) e^{-\frac{ix}{2} \log \frac{t}{2\pi} - ix\Delta'(t)} z^{ix} f\left(\frac{x}{G}\right) dx dz \\ &\quad + O\left( \frac{1}{t^{N+1}} \int_{-\log^{2\alpha-1} t}^{\log^{2\alpha-1} t} (1+x^{N+2}) |f\left(\frac{x}{G}\right)| \left| \int_{0 \searrow 1} \frac{e^{-i\pi z^2} z^{-1/2+it+ix}}{\sin(\pi z)} dz \right| dx \right) + O(e^{-c \log^2 t}), \end{aligned}$$

where each  $P_n(x)$  is a polynomial in  $x$  of degree  $n(\geq 2)$ . The integral over  $z$  in the error term is similar to the one in (8.10). Hence by the residue theorem we have ( $Q = \lfloor \sqrt{t/2\pi} \rfloor$ )

$$\int_{0 \searrow 1} \frac{e^{-i\pi z^2} z^{-1/2+it+ix}}{\sin(\pi z)} dz = 2\pi i \sum_{n=1}^Q \operatorname{Res}_{z=n} \frac{e^{-i\pi z^2} z^{-1/2+it+ix}}{\sin(\pi z)} + \int_{Q \searrow Q+1} \frac{e^{-i\pi z^2} z^{-1/2+it+ix}}{\sin(\pi z)} dz,$$

similarly as in the derivation of the Riemann-Siegel formula. It follows that

$$\left| \int_{0 \searrow 1} \frac{e^{-i\pi z^2} z^{-1/2+it+ix}}{\sin(\pi z)} dz \right| \ll t^{1/4}.$$

Thus analogously as in the case of  $I_1$  we find that, for  $n \geq 1$ ,

$$\int_{-\log^{2\alpha-1} t}^{\log^{2\alpha-1} t} P_n(x) e^{-i\theta(t) - \frac{ix}{2} \log \frac{t}{2\pi} - ix\Delta'(t)} z^{ix} f\left(\frac{x}{G}\right) dx = \int_{-\infty}^{\infty} P_n(x) \cdots dx + O(e^{-c \log^2 t}) = O(e^{-c \log^2 t}).$$

Hence it follows that, for any fixed integer  $N \geq 1$ ,

$$(8.22) \quad I_2 \ll_N T^{-N}.$$

Theorem 1 now follows from (8.10) and (8.18)–(8.22), since clearly it suffices to assume that  $N$  is an integer. One can generalize Theorem 1 to derivatives of  $M_{Z,f}(t)$ .

Theorem 1 shows that  $Z(t)$  and  $M_{Z,f}(t)/G$  differ only by  $O(T^{-N})$ , for any fixed  $N \geq 0$ , which is a very small quantity. This certainly supports the belief that, for this particular subclass of functions  $f(x)$ , (7.8)–(7.11) will be true, but *proving* it may be very hard. On the other hand, nothing precludes the possibility that the error term in Theorem 1, although it is quite small, represents a function possessing many small “spikes” (like  $t^{-N} \sin(t^{N+2})$ , say). These spikes could introduce many new zeros, thus violating (7.8)–(7.11). Therefore it remains an open question to investigate the distribution of zeros of  $M_{Z,f}(t)$  of Theorem 1, and to see to whether there is a possibility that Theorem 1 can be used in settling the truth of the RH.

## 9. Convolution functions and the RH

In this section we shall discuss the possibility to use convolution functions to disprove the RH, of course assuming that it is false. Let us denote by  $T_\alpha^\beta$  the subclass of  $S_\alpha^\beta$  with  $\alpha > 1$  consisting of functions  $f(x)$ , which are not identically equal to zero, and which satisfy  $\int_{-\infty}^{\infty} f(x) dx > 0$ . It is clear that  $T_\alpha^\beta$  is non-empty. Our choice for  $G$  will be the same one as in (8.1), so that for suitable  $\delta$  we shall have

$$(9.1) \quad \widehat{f}\left(\frac{G}{4\pi} \log\left(\frac{T}{2\pi}\right)\right) = \widehat{f}\left(\frac{\delta}{4\pi}\right) \gg 1.$$

In fact by continuity (9.1) will hold for  $|\delta| \leq C_1$ , where  $C_1 > 0$  is a suitable constant depending only on  $f$ , since if  $f(x) \in T_\alpha^\beta$ , then we have

$$(9.2) \quad \widehat{f}(0) = \int_{-\infty}^{\infty} f(x) dx > 0.$$

Moreover if  $f(x) \in S_\alpha^0$ , then  $\widehat{f}(x) \in S_0^\alpha$  and thus it is of bounded support, and consequently  $G \ll 1/\log T$  must hold if the bound in (9.1) is to be satisfied. This choice of  $f(x)$  turns out to be better suited for our purposes than the choice made in Section 8, which perhaps would seem more natural in view of Theorem 1. The reason for this is that, if  $f(x) \in S_\alpha^\beta$  with parameters  $A$  and  $B$ , then  $\widehat{f}(x) \in S_\beta^\alpha$  with parameters  $B + \varepsilon$  and  $A + \varepsilon$ , respectively (see [9]). But for  $f(x)$  as in Section 8 we have  $A = a + b$ , thus for  $\widehat{f}(x)$  we would have (in [9]  $\widehat{f}$  is defined without the factor  $2\pi$ , which would only change the scaling factors)  $B = a + b + \varepsilon$ , and this value of  $B$  would eventually turn out to be too large for our applications. In the present approach we have more flexibility, since only (9.2) is needed. Note that  $f(x)$  is not necessarily nonnegative.

Now observe that if we replace  $f(x)$  by  $f_1(x) := f(Dx)$  for a given  $D > 0$ , then obviously  $f_1(x) \in S_\alpha^0$ , and moreover uniformly for  $q \geq 0$  we have

$$f_1^{(q)}(x) = D^q f^{(q)}(Dx) \ll (BD)^q \exp(-aD^{1/\alpha}|x|^{1/\alpha}).$$

In other words the constant  $B$  in (6.3) or (6.4) is replaced by  $BD$ . Take now  $D = \eta/B$ , where  $\eta > 0$  is an arbitrary, but fixed number, and write  $f$  for  $Df_1$ . If the RH holds, then from (4.5), (6.4) and (6.5) and we have, for  $k$  given by (7.12),

$$(9.3) \quad M_{Z,f}^{(k)}(t) \ll \left(\frac{\eta}{G}\right)^k \exp\left(\frac{B_1 \log t}{\log \log t}\right)$$

with a suitable constant  $B_1 > 0$ .



We shall assume now that the RH holds and that (7.8), (7.11) hold for some  $f(x) \in T_\alpha^0$  (for which (9.3) holds, which is implied by the RH), and we shall obtain a contradiction. To this end let  $U := T^{1/2+\varepsilon}$ , so that we may apply (7.10) or (7.11),  $V = T^{1/4}/\log T$ ,  $L = \log^{1/2+\varepsilon} T$ . We shall consider the mean value of  $|M_{Z,f}(t)|$  over  $[T-U, T+U]$  in order to show that, on the average,  $|M_{Z,f}(t)|$  is not too small. We have

$$(9.4) \quad \begin{aligned} I &:= \int_{T-U}^{T+U} |M_{Z,f}(t)| dt \geq \sum_{n=1}^N \int_{T_n-VL}^{T_n+VL} |M_{Z,f}(t)| dt \\ &\geq \sum_{n=1}^N \int_{T_n-VL}^{T_n+VL} |M_{Z,f}(t)| \exp\left(-(T_n-t)^2 V^{-2}\right) dt, \end{aligned}$$

where  $T_n = T - U + (2n - 1)VL$ , and  $N$  is the largest integer for which  $T_N + VL \leq T + U$ , hence  $N \gg UV^{-1}L^{-1}$ . We use Lemma 1 to bound from below each integral over  $[T_n - VL, T_n + VL]$ . It follows that

$$(9.5) \quad I \gg GV \sum_{n=1}^N \left( \left| \widehat{f}\left(\frac{G}{4\pi} \log\left(\frac{T_n}{2\pi}\right)\right) \right| + O(T^{-1/4}) \right) \gg GV(N + O(NT^{-1/4})) \gg GUL^{-1}$$

for sufficiently small  $\delta$ , since (9.1) holds and

$$\widehat{f}\left(\frac{G}{4\pi} \log\left(\frac{T_n}{2\pi}\right)\right) = \widehat{f}\left(\frac{\delta}{4\pi} \cdot \frac{\log\left(\frac{T_n}{2\pi}\right)}{\log\left(\frac{T}{2\pi}\right)}\right) = \widehat{f}\left(\frac{\delta}{4\pi} + O\left(\frac{U}{T}\right)\right).$$

We have assumed that (7.11) holds, but this implies that (7.10) holds also. Namely it holds unconditionally with  $S(t)$  in place of  $S_M(t)$ . Thus for any fixed integer  $m \geq 1$  we have

$$\begin{aligned} &\int_T^{T+U} (S_M(t+H) - S_M(t-H))^{2m} dt \\ &\ll \int_{T+H}^{T+H+U} (S_M(t) - S(t))^{2m} dt + \int_T^{T+U} (S(t+H) - S(t-H))^{2m} dt + \int_{T-H}^{T-H+U} (S(t) - S_M(t))^{2m} dt \\ &\ll U(\log \log T)^m \quad (T^a < U \leq T, \frac{1}{2} < a \leq 1). \end{aligned}$$

Let  $\mathcal{D}$  be the subset of  $[T-U, T+U]$  where

$$(9.6) \quad |S_M(t+H) - S_M(t-H)| \leq \log^{1/2} T$$

fails. The bound (7.10) implies that

$$(9.7) \quad m(\mathcal{D}) \ll U \log^{-C} T$$

for any fixed  $C > 0$ . If we take  $C = 10$  in (9.7) and use the Cauchy-Schwarz inequality for integrals we shall have

$$(9.8) \quad \int_{\mathcal{D}} |M_{Z,f}(t)| dt \leq (m(\mathcal{D}))^{1/2} \left( \int_{T-U}^{T+U} M_{Z,f}^2(t) dt \right)^{1/2} \ll GU \log^{-4} T,$$

since

$$\begin{aligned}
\int_{T-U}^{T+U} M_{Z,f}^2(t) dt &\leq \int_{T-U}^{T+U} \int_{-\infty}^{\infty} Z^2(t+x) |f(\frac{x}{G})| dx \int_{-\infty}^{\infty} |f(\frac{y}{G})| dy dt \\
&\ll G \int_{T-U}^{T+U} \int_{-G \log^{2\alpha-1} T}^{G \log^{2\alpha-1} T} |\zeta(\frac{1}{2} + it + ix)|^2 |f(\frac{x}{G})| dx dt + G \\
&\ll G \int_{-G \log^{2\alpha-1} T}^{G \log^{2\alpha-1} T} \left( \int_{T-2U}^{T+2U} |\zeta(\frac{1}{2} + iu)|^2 du \right) |f(\frac{x}{G})| dx + G \ll G^2 U \log T.
\end{aligned}$$

The last bound easily follows from mean square results on  $|\zeta(\frac{1}{2} + it)|$  (see [16]) with the choice  $U = T^{\frac{1}{2} + \varepsilon}$ . Therefore (9.4) and (9.8) yield

$$(9.9) \quad GUL^{-1} \ll \int_{\mathcal{D}'} |M_{Z,f}(t)| dt,$$

where  $\mathcal{D}' = [T-U, T+U] \setminus \mathcal{D}$ , hence in (9.9) integration is over  $t$  for which (9.6) holds. If  $t \in \mathcal{D}'$ ,  $\gamma$  denotes the zeros of  $M_{Z,f}(t)$ , then from (7.7) and (7.12) we obtain (recall that  $\log_r t = \log(\log_{r-1} t)$ )

$$(9.10) \quad k = k(t, T) = \frac{H}{\pi} \log\left(\frac{T}{2\pi}\right) \cdot \{1 + O((\log T)^{\varepsilon - \frac{1}{2}})\}, \quad \log k = \log H - \log \pi + \log_2\left(\frac{T}{2\pi}\right) + O((\log T)^{\varepsilon - 1/2})$$

for any given  $\varepsilon > 0$ . To bound  $M_{Z,f}(t)$  we use (7.6), with  $k$  given by (9.10),  $\tau = \tau(t, k)$ , (9.3) and

$$k! = \exp(k \log k - k + O(\log k)).$$

We obtain, denoting by  $B_j$  positive absolute constants,

$$\begin{aligned}
(9.11) \quad GUL^{-1} &\ll \int_{\mathcal{D}'} \prod_{|\gamma-t| \leq H} |\gamma - t| \frac{|M_{Z,f}^{(k)}(\tau)|}{k!} dt \\
&\ll \exp\left(\frac{B_2 \log T}{\log_2 T}\right) \int_{\mathcal{D}'} \exp\{k(\log \frac{\eta}{G} - \log k + 1)\} \prod_{|\gamma-t| \leq H} |\gamma - t| dt \\
&\ll \exp\left(\frac{B_3 \log T}{\log_2 T} + \frac{H}{\pi} \log\left(\frac{T}{2\pi}\right) \cdot \left(\log \frac{\eta}{\delta} + \log_2\left(\frac{T}{2\pi}\right) - \log H + \log \pi - \log_2\left(\frac{T}{2\pi}\right) + 1\right)\right) \int_{\mathcal{D}'} \prod_{|\gamma-t| \leq H} |\gamma - t| dt.
\end{aligned}$$

It was in evaluating  $k \log k$  that we needed (9.10), since only the bound (7.9) would not suffice (one would actually need the bound  $S_M(T) \ll \log T / (\log_2 T)^2$ ). If the product under the last integral is bounded by (7.13), we obtain

$$GUL^{-1} \ll U \exp\left(\frac{H}{\pi} \log\left(\frac{T}{2\pi}\right) \cdot \left(\log \frac{\eta}{\delta} + B_4\right)\right),$$

and thus for  $T \geq T_0$

$$(9.12) \quad 1 \leq \exp\left(\frac{H}{\pi} \log\left(\frac{T}{2\pi}\right) \cdot \left(\log \frac{\eta}{\delta} + B_5\right)\right).$$

Now we choose e.g.,

$$\eta = \delta^2, \quad \delta = \min(C_1, e^{-2B_5}),$$

where  $C_1$  is the constant for which (9.1) holds if  $|\delta| \leq C_1$ , so that (9.12) gives

$$1 \leq \exp\left(\frac{-B_5 H}{\pi} \log\left(\frac{T}{2\pi}\right)\right),$$

which is a contradiction for  $T \geq T_1$ . Thus we have proved the following

**THEOREM 2.** *If (7.8) and (7.11) hold for suitable  $f(x) \in T_\alpha^0$  with  $G$  given by (8.1), then the Riemann hypothesis is false.*

Theorem 2 is similar to the result proved also in [23]. Actually the method of proof of Theorem 2 gives more than the assertion of the theorem. Namely it shows that, under the above hypotheses, (4.5) cannot hold for any fixed  $A > 0$ . Perhaps it should be mentioned that (7.11) is not the only condition which would lead to the disproof of the RH. It would be enough to assume, under the RH, that one had (7.8)–(7.10) for a suitable  $f(x)$ , or

$$(9.18) \quad N_M(t) = N(t) + O\left(\frac{\log T}{(\log \log T)^2}\right)$$

for  $t \in [T - U, T + U]$  with a suitable  $U (= T^{1/2+\varepsilon}$ , but smaller values are possible), to derive a contradiction. The main drawback of this approach is the necessity to impose conditions like (7.8)–(7.10) which can be, for all we know, equally difficult to settle as the assertions which we originally set out to prove (or disprove). For this reason our results can only be conditional. Even if the RH is false it appears plausible that, as  $T \rightarrow \infty$ ,  $N_0(T) = (1 + o(1))N(T)$ . In other words, regardless of the truth of the RH, almost all complex zeros of  $\zeta(s)$  should lie on the critical line. This is the conjecture that the author certainly believes in. No plausible conjectures seem to exist (if the RH is false) regarding the order of  $N(T) - N_0(T)$ .

## REFERENCES

- [1] R.J. Anderson and H.M. Stark, *Oscillation theorems*, in LNM's **899**, Springer-Verlag, Berlin-Heidelberg-New York, 1981, 79-106.
- [2] F.V. Atkinson, *The mean value of the Riemann zeta-function*, Acta Math. **81**(1949), 353-376.
- [3] R. Balasubramanian, *On the frequency of Titchmarsh's phenomenon for  $\zeta(s)$  IV*, Hardy-Ramanujan J. **9**(1986), 1-10.
- [4] R. Balasubramanian and K. Ramachandra, *On the frequency of Titchmarsh's phenomenon for  $\zeta(s)$  III*, Proc. Indian Acad. Sci. Section A **86**(1977), 341-351.
- [5] E. Bombieri and D. Hejhal, *Sur les zéros des fonctions zeta d'Epstein*, Comptes Rendus Acad. Sci. Paris **304**(1987), 213-217.
- [6] H. Davenport and H. Heilbronn, *On the zeros of certain Dirichlet series I,II*, J. London Math. Soc. **11**(1936), 181-185 and ibid. 307-312.
- [7] H.M. Edwards, *Riemann's zeta-function*, Academic Press, New York-London, 1974.
- [8] A. Fujii, *On the distribution of the zeros of the Riemann zeta-function in short intervals*, Bull. Amer. Math. Soc. **81**(1975), 139-142.
- [9] I.M. Gel'fand and G.E. Shilov, *Generalized functions (vol. 2)*, Academic Press, New York-London, 1968.
- [10] A.O. Gel'fond, *The calculus of finite differences (Russian)*, Nauka, Moscow, 1967.
- [11] J.L. Hafner and A. Ivić, *On the mean square of the Riemann zeta-function on the critical line*, J. Number Theory **32**(1989), 151-191.
- [12] J.L. Hafner and A. Ivić, *On some mean value results for the Riemann zeta-function*, in Proc. International Number Theory Conf. Québec 1987, Walter de Gruyter and Co., Berlin-New York, 348-358.
- [13] M.N. Huxley, *Exponential sums and the Riemann zeta-function IV*, Proceedings London Math. Soc. (3)**66**(1993), 1-40.
- [14] M.N. Huxley, *A note on exponential sums with a difference*, Bulletin London Math. Soc. **29**(1994), 325-327.
- [15] A. Ivić, *Large values of the error term in the divisor problem*, Invent. Math. **71**(1983), 513-520.
- [16] A. Ivić, *The Riemann zeta-function*, John Wiley and Sons, New York, 1985 (2nd ed. Dover, 2003).
- [17] A. Ivić, *On consecutive zeros of the Riemann zeta-function on the critical line*, Séminaire de Théorie des Nombres, Université de Bordeaux 1986/87, Exposé no. **29**, 14 pp.
- [18] A. Ivić, *On a problem connected with zeros of  $\zeta(s)$  on the critical line*, Monatshefte Math. **104**(1987), 17-27.
- [19] A. Ivić, *Large values of certain number-theoretic error terms*, Acta Arithmetica **56**(1990), 135-159.
- [20] A. Ivić, *Mean values of the Riemann zeta-function*, LN's **82**, Tata Institute of Fundamental Research, Bombay, 1991 (distr. by Springer Verlag, Berlin etc.).
- [21] A. Ivić, *On a class of convolution functions connected with  $\zeta(s)$* , Bulletin CIX Acad. Serbe des Sciences et des Arts, Classe des Sciences mathématiques et naturelles, Math. No **20**(1995), 29-50.
- [22] A. Ivić, *On the fourth moment of the Riemann zeta-function*, Publications Inst. Math. (Belgrade) **57(71)**(1995), 101-110.
- [23] A. Ivić, *On the distribution of zeros of a class of convolution functions*, Bulletin CXI Acad. Serbe des Sciences et des Arts, Classe des Sciences mathématiques et naturelles, Sciences mathématiques No. **21**(1996), 61-71.
- [24] A. Ivić, *On the ternary additive divisor problem and the sixth moment of the zeta-function*, in "Sieve Methods, Exponential Sums, and their Applications in Number Theory" (eds. G.R.H. Greaves, G. Harman, M.N. Huxley), Cambridge University Press (Cambridge, UK), 1996, 205-243.

- [25] A. Ivić and M. Jutila, *Gaps between consecutive zeros of the Riemann zeta-function*, Monatshefte Math. **105**(1988), 59-73.
- [26] A. Ivić and Y. Motohashi, *A note on the mean value of the zeta and L-functions VII*, Proc. Japan Acad. Ser. A **66**(1990), 150-152.
- [27] A. Ivić and Y. Motohashi, *The mean square of the error term for the fourth moment of the zeta-function*, Proc. London Math. Soc. (3)**66**(1994), 309-329.
- [28] A. Ivić and Y. Motohashi, *The fourth moment of the Riemann zeta-function*, Journal Number Theory **51**(1995), 16-45.
- [29] A. Ivić and H.J.J. te Riele, *On the zeros of the error term for the mean square of  $|\zeta(\frac{1}{2} + it)|$* , Math. Comp. **56** No **193**(1991), 303-328.
- [30] M. Jutila, *On the value distribution of the zeta-function on the critical line*, Bull. London Math. Soc. **15**(1983), 513-518.
- [31] A.A. Karatsuba, *On the zeros of the Davenport-Heilbronn function lying on the critical line (Russian)*, Izv. Akad. Nauk SSSR ser. mat. **54** no. 2 (1990), 303-315.
- [32] A.A. Karatsuba and S.M. Voronin, *The Riemann zeta-function*, Walter de Gruyter, Berlin-New York, 1992.
- [33] N.V. Kuznetsov, *Sums of Kloosterman sums and the eighth moment of the Riemann zeta-function*, Papers presented at the Ramanujan Colloquium, Bombay 1989, publ. for Tata Institute (Bombay) by Oxford University Press, Oxford, 1989, pp. 57-117.
- [34] A.A. Lavrik, *Uniform approximations and zeros of derivatives of Hardy's Z-function in short intervals (Russian)*, Analysis Mathem. **17**(1991), 257-259.
- [35] D.H. Lehmer, *On the roots of the Riemann zeta function*, Acta Math. **95**(1956), 291-298.
- [36] D.H. Lehmer, *Extended computation of the Riemann zeta-function*, Mathematika **3**(1956), 102-108.
- [37] J.E. Littlewood, *Sur la distribution des nombres premiers*, Comptes rendus Académie Sci. (Paris) **158**(1914), 1869-1872.
- [38] J. van de Lune, H.J.J. te Riele and D.T. Winter, *On the zeros of the Riemann zeta-function in the critical strip IV*, Math. Comp. **46**(1987), 273-308.
- [39] H.L. Montgomery, *Extreme values of the Riemann zeta-function*, Comment. Math. Helv. **52**(1977), 511-518.
- [40] Y. Motohashi, *The fourth power mean of the Riemann zeta-function*, in "Proceedings of the Amalfi Conference on Analytic Number Theory 1989", eds. E. Bombieri et al., Università di Salerno, Salerno, 1992, 325-344.
- [41] Y. Motohashi, *An explicit formula for the fourth power mean of the Riemann zeta-function*, Acta Math. **170**(1993), 181-220.
- [42] Y. Motohashi, *A relation between the Riemann zeta-function and the hyperbolic Laplacian*, Ann. Sc. Norm. Sup. Pisa, Cl. Sci. IV ser. **22**(1995), 299-313.
- [43] Y. Motohashi, *The Riemann zeta-function and the non-Euclidean Laplacian*, Sugaku Expositions, AMS **8**(1995), 59-87.
- [44] A.M. Odlyzko, *On the distribution of spacings between the zeros of the zeta-function*, Math. Comp. **48**(1987), 273-308.
- [45] A.M. Odlyzko, *Analytic computations in number theory*, Proc. Symposia in Applied Math. **48**(1994), 451-463.
- [46] A.M. Odlyzko, *The  $10^{20}$ -th zero of the Riemann zeta-function and 175 million of its neighbors*, to appear.
- [47] A.M. Odlyzko and H.J.J. te Riele, *Disproof of the Mertens conjecture*, J. reine angew. Math. **357**(1985), 138-160.
- [48] K. Ramachandra, *Progress towards a conjecture on the mean value of Titchmarsh series*, in "Recent Progress in Analytic Number Theory", symposium Durham 1979 (Vol. 1), Academic Press, London, 1981, 303-318.

- [49] K. Ramachandra, *On the mean-value and omega-theorems for the Riemann zeta-function*, LNs 85, Tata Institute of Fundamental Research, Bombay, 1995 (distr. by Springer Verlag, Berlin etc.).
- [50] H.J.J. te Riele, *On the sign of the difference  $\pi(x) - \text{li } x$* , Math. Comp. **48**(1987), 323-328.
- [51] H.J.J. te Riele and J. van de Lune, *Computational number theory at CWI in 1970-1994*, CWI Quarterly **7**(4) (1994), 285-335.
- [52] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebener Grösse*, Monats. Preuss. Akad. Wiss. (1859-1860), 671-680.
- [53] A. Selberg, *On the zeros of Riemann's zeta-function*, Skr. Norske Vid. Akad. Oslo **10**(1942), 1-59.
- [54] A. Selberg, *Contributions to the theory of the Riemann zeta-function*, Arch. Math. Naturvid. **48**(1946) No. 5, 89-155.
- [55] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20**(1956), 47-87.
- [56] R. Sherman Lehman, *On the difference  $\pi(x) - \text{li } x$* , Acta Arith, **11**(1966), 397-410.
- [57] C.L. Siegel, *Über Riemanns Nachlaß zur analytischen Zahlentheorie*, Quell. Stud. Gesch. Mat. Astr. Physik **2**(1932), 45-80 (also in *Gesammelte Abhandlungen, Band I*, Springer Verlag, Berlin etc., 1966, 275-310).
- [58] R. Spira, *Some zeros of the Titchmarsh counterexample*, Math. Comp. **63**(1994), 747-748.
- [59] K.-M. Tsang, *Some  $\Omega$ -theorems for the Riemann zeta-function*, Acta Arith. **46**(1986), 369-395.
- [60] K.-M. Tsang, *The large values of the Riemann zeta-function*, Mathematika **40**(1993), 203-214.
- [61] E.C. Titchmarsh, *The theory of the Riemann zeta-function* (2nd ed.), Clarendon Press, Oxford, 1986.

#### References added in November 2003:

- [62] A. Ivić, *On some results concerning the Riemann Hypothesis*, in "Analytic Number Theory" (Kyoto, 1996) ed. Y. Motohashi, LMS LNS **247**, Cambridge University Press, Cambridge, 1997, 139-167.
- [63] Y. Motohashi, *Spectral theory of the Riemann zeta-function*, Cambridge University Press, 1997.
- [64] A. Ivić, *On the multiplicity of zeros of the zeta-function*, Bulletin CXVIII de l'Académie Serbe des Sciences et des Arts - 1999, Classe des Sciences mathématiques et naturelles, Sciences mathématiques No. **24**, 119-131.
- [65] E. Bombieri, *Problems of the Millenium: the Riemann Hypothesis*, [http://www.ams.org/claymath/prize\\_problems/riemann.pdf](http://www.ams.org/claymath/prize_problems/riemann.pdf), Amer. Math. Soc., Providence, R.I., 2000, 12pp.
- [66] J.B. Conrey, *L-functions and random matrices*, in "Mathematics Unlimited" (Part I), B. Engquist and W. Schmid eds., Springer, 2001, 331-352.
- [67] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein and N.C. Snaith, *Integral moments of L-functions*, 2003, 58pp, arXiv:math.NT/0206018, <http://front.math.ucdavis.edu/mat.NT/0206018>.

ALEKSANDAR IVIĆ  
 KATEDRA MATEMATIKE RGF-A  
 UNIVERSITET U BEOGRADU  
 ĐUŠINA 7, 11000 BEOGRAD  
 SERBIA (YUGOSLAVIA)  
 aivic@matf.bg.ac.yu, aivic@rgf.bg.ac.yu