2-Factors in Edge Chromatic Critical Graphs with Large Maximum Degree

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Abstract. In 1968, Vizing conjectured that, if \( G \) is an \( n \)-vertex edge chromatic critical graph with \( \chi'(G) = \Delta(G) + 1 \), then \( G \) contains a 2-factor. In this paper, we verify this conjecture for \( n \leq 2\Delta(G) \).

Keywords. Edge chromatic index; Critical graphs; Tutte’s 2-factor Theorem

1 Introduction

In this paper, we only consider simple graphs. Let \( G \) be a graph. We fix the notation \( \Delta \) for the maximum degree of \( G \) throughout the paper. A \( k \)-vertex of \( G \) is a vertex of degree exactly \( k \) in \( G \). We use \( V_\Delta \) to denote the set of \( \Delta \)-vertices in \( G \). A connected graph is critical if the chromatic index \( \chi'(G - e) < \chi'(G) \) for any edge \( e \in E(G) \). This definition indicates that if \( G \) is not critical, then we can delete some edges of \( G \) until the remaining graph becomes critical. For this reason, when study edge chromatic index of a graph, the graph is usually assumed to be critical. A critical graph is called \( k \)-critical, if \( \chi'(G) = k + 1 \) and \( \chi'(G - e) = k \) for every edge \( e \in E(G) \). In particular, if \( G \) is \( \Delta \)-critical, then \( \chi'(G) = \Delta + 1 \).

In 1965, Vizing\[7\] showed that a graph of maximum degree \( \Delta \) has edge chromatic index either \( \Delta \) or \( \Delta + 1 \). If \( \chi'(G) = \Delta \), then \( G \) is said to be of class 1; otherwise, it is said to be of class 2. Motivated by this classification problem, Vizing studied critical class 2 graphs, or \( \Delta \)-critical graphs. One of the conjectures that Vizing made on \( \Delta \)-critical graphs is the following.

Conjecture 1 (Vizing’s 2-factor Conjecture). Let \( G \) be a \( \Delta \)-critical graph. Then \( G \) contains a 2-factor; that is, a 2-regular subgraph \( H \) of \( G \) with \( V(H) = V(G) \).

There had been no progress towards this conjecture until in 2004, Grünwald and Steffen\[3\] established Vizing’s 2-factor conjecture for graphs with many edges; in particular, for overfull graphs,
i.e., graphs with $|E(G)| > \Delta(G)|V(G)|/2$. In 2012, Luo and Zhao[5] proved that if $G$ is an $n$-vertex $\Delta$-critical graph with $\Delta \geq \frac{6n}{7}$, then $G$ contains a Hamiltonian cycle, and thus a 2-factor with exactly one cycle. Still consider $\Delta$-critical graphs with large maximum degree, we obtain the following result:

**Theorem 1.1.** Let $G$ be an $n$-vertex $\Delta$-critical graph. Then $G$ has a 2-factor if $\Delta \geq n/2$.

To prove Theorem 1.1, we will use several known results listed below.

**Lemma 1.1 (Vizing’s Adjacency Lemma).** Let $G$ be a $\Delta$-critical graph. Then for any edge $xy \in E(G)$, $x$ is adjacent to at least $\Delta - d_G(y) + 1 \Delta$-vertices.

The following result due to Luo and Zhao[4] verifies Vizing’s independence number conjecture for $\Delta \geq n/2$.

**Lemma 1.2.** Let $G$ be an $n$-vertex $\Delta$-critical graph with $\Delta \geq n/2$. Then $\alpha(G) \leq n/2$, where $\alpha(G)$ is the independence number of $G$.

Additionally, we need the result below which guarantees a matching in a bipartite graph. A simple proof of this result can be found in [2].

**Lemma 1.3.** Let $H$ be a bipartite graph with partite sets $X$ and $Y$. If there is no isolated vertex in $Y$ and $d_H(y) \geq d_H(x)$ holds for every edge $xy$ with $x \in X$ and $y \in Y$, then $H$ has a matching which saturates $Y$.

The following result is a generalization of Lemma 1.3.

**Lemma 1.4.** Let $H$ be a bipartite graph with partite sets $S$ and $T$. Suppose that $T$ has no isolated vertices, and that for each edge $xy \in E(H)$ with $x \in S$ and $y \in T$ either $d_H(y) \geq d_H(x)$ holds or if $d_H(y) < d_H(x)$, then one neighbor of $y$ has degree 1 in $H$. Then $H$ has a matching which saturates $T$.

**Proof.** We may assume that there exists $xy \in E(H)$ with $x \in S$ and $y \in T$ such that $d_H(x) > d_H(y)$. For otherwise, we are done by Lemma 1.3. Let $x^*$ be a neighbor of $y$ which is of degree 1 in $H$. We remove the pair of vertices $x^*, y$ from $H$. It is clear that the resulted bipartite graph still satisfy the condition in the claim. Hence, we can remove all such pairs one by one, until no such pair exists. Let the finally resulted bipartite graph be $H^*[S^*, T^*]$, where $S^*$ is the set of unremoved vertices in $S$ and $T^*$ is the set of unremoved vertices in $T$. As each removed pair of vertices are adjacent, all the removed pairs of vertices induce a matching, say $M_m$ in $H$. The matching $M_m$ saturates all the removed vertices in $T$ (also all the removed vertices in $S$). As $T$ has no isolated vertices, and the removing of vertices will not affect the degrees of other vertices in $T$, we have in $H^*$, either $T^* = \emptyset$, and
or $T^*$ contains vertices, it has no isolated vertices in $H^*$, and for each edge $xy \in E(H^*)$ with $x \in S^*$ and $y \in T^*$, $d_{H^*}(y) \geq d_{H^*}(x)$. In the first case, $M_m$ gives a matching which saturates $T$ in $H$; in the second case, by Lemma 1.3, $H^*$ has a matching $M^*$ which saturates $T^*$. Then $M_m \cup M^*$ gives a matching which saturates $T$. ■

The following lemma is a generalization of a result in [4].

**Lemma 1.5.** Let $G$ be a $\Delta$-critical graph with $V(G) = T \cup S$, where $T$ is an independent set; and let $H = G - E(G[S])$. Assume that there are $\delta_0$ $\Delta$-vertices in $T$. Then for each edge $xy \in E(H)$ with $x \in S$ and $y \in T$, $d_H(y) \geq d_H(x) + 1 - \delta_0 + \sigma_x$, where $\sigma_x$ is the number of non $\Delta$-degree neighbors of $x$ in $S$. In particular, if $\delta_0 \leq 1$, then there is a matching which saturates $T$ in $H$.

**Proof.** As the graph is $\Delta$-critical, it is connected. Together with the fact that $T$ is an independent set in $G$, we see that $T$ has no isolated vertex in $H$. Let $xy \in E(H)$ with $x \in S$ and $y \in T$. By Vizing’s Adjacency Lemma, $x$ is adjacent to at least $\Delta - d_G(y) + 1$ $\Delta$-vertices in $G$. As $T$ has $\delta_0$ $\Delta$-vertices, we know $x$ is adjacent to at least $\Delta - d_G(y) + 1 - \delta_0$ $\Delta$-vertices in $S$ (notice that the quantity is meaningful only if $\Delta - d_G(y) + 1 - \delta_0 > 0$). Let $\sigma_x$ be the number of all non $\Delta$-degree neighbors of $x$ in $S$. Then, $d_H(x) + \Delta - d_G(y) + 1 - \delta_0 + \sigma_x \leq d_G(x) \leq \Delta$. By noting that $d_G(y) = d_H(y)$, the inequality implies that $d_H(y) \geq d_H(x) + 1 - \delta_0 + \sigma_x$.

When $\delta_0 \leq 1$, for every edge $xy \in E(H)$ with $x \in S$ and $y \in T$, the relation $d_H(y) \geq d_H(x) + \sigma_x \geq d_H(x)$ holds. By applying Lemma 1.3, we see there is a matching which saturates $T$ in $H$. ■

In the end of this section, we introduce some notations. For a vertex $x$ of a graph $G$, we denote by $d_G(x)$ the degree of $x$ in $G$. For a set of vertices $S$ in $G$, we define $N_G(S)$ by $N_G(S) = \bigcup_{x \in S} N_G(x)$. For disjoint sets of vertices $S$ and $T$ in $G$, we denote by $e_G(S, T) = |E_G(S, T)|$, the number of edges that has one end vertex in $S$ and the other in $T$. If $S$ is a singleton set $S = \{s\}$, we write $e_G(s, T)$ instead of $e_G(\{s\}, T)$.

A matching of a graph $G$ is a set of independent edges in $G$. If $M$ is a matching of $G$, then let $V(M)$ denote the set of end vertices of the edges in $M$. For $X \subset V(G)$, $M$ is said to saturate $X$ if $X \subset V(M)$.

2 A Detour to Tutte’s 2-Factor Theorem

One of the main proof ingredients of Theorem 1.1 is to apply Tutte’s 2-factor Theorem. Hence, we take a detour here to introduce it.

Let $S$ and $T$ be disjoint sets of vertices of a graph $G$. Let $C$ be a component of $G - (S \cup T)$. Then $C$
is said to be an odd component (resp. even component) if \( e_G(C, T) \equiv 1 \pmod{2} \) (resp. \( e_G(C, T) \equiv 0 \pmod{2} \)). Let \( \mathcal{H}_G(S, T) \) be the set of odd components of \( G - (S \cup T) \) and let \( h_G(S, T) = |\mathcal{H}_G(S, T)| \).

Let \( \delta_G(S, T) = 2|S| + \sum_{x \in T} d_{G - S}(x) - 2|T| - h_G(S, T) \). It is easy to see \( \delta_G(S, T) \equiv 0 \pmod{2} \) for every \( S, T \subseteq V(G) \) with \( S \cap T = \emptyset \). We use the following criterion for the existence of a 2-factor, which is a restricted form of Tutte’s f-Factor Theorem.

**Lemma 2.1** (Tutte [6]). A graph \( G \) has a 2-factor if and only if \( \delta_G(S, T) \geq 0 \) for every \( S, T \subseteq V(G) \) with \( S \cap T = \emptyset \).

An ordered pair \((S, T)\) consists of disjoint sets of vertices \( S \) and \( T \) in a graph \( G \) is called a barrier if \( \delta_G(S, T) \leq -2 \). By Theorem 2.1, if \( G \) does not have a 2-factor, then \( G \) has a barrier. A barrier \((S, T)\) is called a minimum barrier if \( |S \cup T| \) is smallest among all the barriers of \( G \).

A minimum barrier of a graph without a 2-factor has some nice properties, see[1, 2] for examples. We will use the listed properties in the following lemma in our proof.

**Lemma 2.2.** Let \( G \) be a graph which does not have a 2-factor, and let \((S, T)\) be a minimum barrier of \( G \). Then

1. \( T \) is independent,
2. if \( C \) is an even component with respect to \((S, T)\), then \( e_G(T, C) = 0 \),
3. if \( C \) is an odd component with respect to \((S, T)\), then \( e_G(v, C) \leq 1 \) for every \( v \in T \).

### 3 Proof of Theorem 1.1

We may assume that \( \Delta \geq 3 \). For otherwise, the assertion in Theorem 1.1 is trivial. This assumption, together with the fact that \( G \) is critical, implies that \( \delta(G) \geq 2 \). And for every edge \( xy \in E(G), \ d_G(x) + d_G(y) \geq 5 \).

Assume, to the contrary, that \( G \) does not have a 2-factor. Then by Tutte’s 2-factor Theorem (Theorem 2.1), \( G \) has a barrier. Let \((S, T)\) be a minimum barrier such that the number of odd components of \( G - (S \cup T) \) is as small as possible. Let \( U = V(G) - (S \cup T) \) and let \( C_k \) be the set of components \( C \) of \( G - (S \cup T) \) with \( e_G(C, T) = k \). Then we have \( \mathcal{H}_G(S, T) = \bigcup_{k \geq 0} C_{2k+1} \). For each \( v \in T \) which is adjacent to a vertex in \( C_1 \), we define \( C_{1v} = \{ C \in C_1 | e(v, C) = 1 \} \). For the choice of a minimum barrier \((S, T)\) with \( \bigcup_{k \geq 0} C_{2k+1} \) smallest, we have the following claim.

**Claim 3.1.** Let \( v \in T \) with \( |C_{1v}| \geq 2 \). Then each \( C \in C_{1v} \) satisfies \( |C| \geq 2 \).
Proof. Suppose on the contrary that there exists \( v \in T \) such that \(|C_{1v}| \geq 2\) and \( C^* \in C_{1v} \) is a single vertex. Let \( v^* \) be the vertex of \( C^* \). Let \( T^* = T - \{v\} + \{v^*\} \). We claim that \((S, T^*)\) is a minimum barrier with \( h_G(S, T^*) < h_G(S, T) \). This will give a contradiction to the choice of \((S, T)\).

To see that \((S, T^*)\) is a minimum barrier we calculate \( \delta_G(S, T^*) \). Let \( C_v \) be the component of \( G - (S \cup T^*) \) combined by all the \( d_{G-S}(v) - 1 \) odd components of \( G - (S \cup T) \) (except \( C^* \)) through the vertex \( v \). As \( e(T^*, C_v) = e(v^*, v) = d_{G-S}(v^*) = 1 \), \( C_v \) is an odd component of \( G - (S \cup T^*) \). Furthermore, we have \( h_G(S, T^*) = h_G(S, T) - d_{G-S}(v) + 1 \). So,

\[
\delta_G(S, T^*) = 2|S| - 2|T^*| + \sum_{v \in T^*} d_{G-S}(v) - h_G(S, T^*) \\
= 2|S| - 2|T| + \sum_{v \in T} d_{G-S}(v) - d_{G-S}(v) + d_{G-S}(v^*) - (h_G(S, T) - d_{G-S}(v) + 1) \\
= 2|S| - 2|T| + \sum_{v \in T} d_{G-S}(v) - h_G(S, T) \\
\leq -2;
\]

by noticing that \( d_{G-S}(v^*) = 1 \).

As \(|S \cup T^*| = |S \cup T|\), \((S, T^*)\) is a minimum barrier. However, as \( d_{G-S}(v) \geq |C_{1v}| \geq 2\), \( h_G(S, T^*) = h_G(S, T) - d_{G-S}(v) + 1 < h_G(S, T) \). This gives a contradiction to the choice of \((S, T)\).

Furthermore, we have

**Claim 3.2.** Let \( v \in T \) with \(|C_{1v}| \geq 2\). Then each \( C \in C_{1v} \) satisfies \(|C| \geq 4\).

Proof. By Claim 3.1, we have \(|C| \geq 2\). We claim that in \( C \), every vertex in \( V(C) - N_G(v) \) has degree at least 2.

Suppose \( C \) has a vertex \( v^* \in V(C) - N_G(v) \) which has degree 1 in \( G[V(C)] \). Then by letting \( T^* = T - \{v\} + \{v^*\} \), similarly as shown in the proof of Claim 3.1, we can show that \((S, T^*)\) is a minimum barrier with smaller number of odd components in \( G - (S \cup T^*) \) than that in \( G - (S \cup T) \), gives a contradiction to the choice of \((S, T)\).

By the above argument and Claim 3.1, we see that \(|C| \geq 3\); and if \(|C| = 3\), then \( C = K_3 \).

Then we claim that \( C \) cannot be \( K_3 \). Otherwise, let \( v^* \) be the vertex adjacent to \( v \) on \( C \), and let \( T^* = T - \{v\} + \{v^*\} \). Again, similar calculation as in the proof of Claim 3.1 indicates that \((S, T^*)\) is a minimum barrier. However, in \( H_G(S, T^*) \), except \( C \), all the other \( d_{G-S}(v) - 1 \) odd components connected to \( v \) in \( G - (S \cup T) \) are combined into a single odd component, and \( C - \{v^*\} \) is an even component of \( G - (S \cup T^*) \). This indicates that \( h_G(S, T^*) < h_G(S, T) \), as \(|C_{1v}| \geq 2\), showing a contradiction to the choice of \((S, T)\).
A similar proof as in Claim 3.1 also gives that

**Claim 3.3.** Suppose that $v \in T$ with $|C_{1v}| = 1$ and $C \in C_{1v}$ is a single vertex. Then $v$ is not adjacent to any other odd components in $G - (S \cup T)$.

**Claim 3.4.** $|T| > |S| + \sum_{k \geq 1} k \cdot |C_{2k+1}|$.

**Proof.** Since $(S, T)$ is a barrier,

$$
\delta_G(S, T) = 2|S| - 2|T| + \sum_{v \in T} d_{G-S}(v) - h_G(S, T)
$$

$$
= 2|S| - 2|T| + \sum_{v \in T} d_{G-S}(v) - \sum_{k \geq 0} |C_{2k+1}| < 0.
$$

By Lemma 2.2 (1) and (2),

$$
\sum_{v \in T} d_{G-S}(v) = \sum_{v \in T} e_G(v, U) = e_G(T, U) = \sum_{k \geq 0} (2k + 1)|C_{2k+1}|.
$$

Therefore, we have

$$
0 > 2|S| - 2|T| + \sum_{k \geq 0} (2k + 1)|C_{2k+1}| - \sum_{k \geq 0} |C_{2k+1}|,
$$

which yields $|T| > |S| + \sum_{k \geq 1} k |C_{2k+1}|$. \qed

We perform the following operations to $G$.

(1) Remove all even components and all odd components in $C_1$.

(2) Remove all edges in $G[S]$.

(3) For a component $C \in C_{2k+1}$ with $k \geq 1$ if $C \notin C_3$ or $C$ is not a single vertex component, introduce a set of $k$ independent vertices $U^C = \{u^C_1, u^C_2, \ldots, u^C_k\}$ and replace $C$ with $U^C$. By Lemma 2.2 (3), $|N_G(C) \cap T| = e_G(T, C) = 2k + 1$. Let $N_G(C) \cap T = \{v_0, v_1, \ldots, v_{2k}\}$. Add two new edges $u^C_i v_{2i-1}$ and $u^C_i v_{2i}$ for each $i$ with $1 \leq i \leq k$. Moreover, add one extra edge $u^C_1 v_0$. In particular, if $C \in C_3$ is a single vertex component, this operation (3) does nothing to $C$.

Let $H$ be the resulting graph. Note that the construction of $H$ here is a modification of the bipartite graph constructed in [2].

Let $U^C = \bigcup_{k \geq 1} \left( \bigcup_{C \in C_{2k+1}} U^C \right)$ and $X = S \cup U^C$. We may also use notations such as $U^C_1, U^C_{2k+3}$ later on, they are defined similarly as $U^C$. By the construction, the graph $H$ satisfies the following properties.
1. $H$ is a bipartite graph with partite sets $X$ and $T$,

2. $|X| = |S| + \sum_{k \geq 1} k|C_{2k+1}|$, and

3. For each $k \geq 1$ and each $C \in C_{2k+1}$, $d_H(u_i^C) = 3$ and $d_H(u_i^C) = 2$ for each $i$ with $2 \leq i \leq k$.

Our proof strategy from here is to show that there is a matching in $H$ which saturates $T$, which gives that $|S| \geq |T|$, and thus gives a contradiction to Claim 3.4.

For notation simplicity, we denote $\bigcup_{C \in C_{2k+1}} V(C)$ by $V(C_{2k+1})$ for $k \geq 0$. In particular, $C_{\geq (2t+1)} = \bigcup_{k \geq t} C_{2k+1}$, and $V(C_{\geq (2t+1)}) = \bigcup_{k \geq t} V(C_{2k+1})$.

We may assume that $|X| < n/2$. For otherwise, as $|T| \leq n/2$ by Lemma 1.2, we see that $|X| \geq |T|$, gives a contradiction to Claim 3.4. Furthermore, we may assume that $|X| \leq n/2 - 3/2$ when $n$ is odd. Otherwise, assume $|X| = n/2 - 1/2$. By Lemma 1.2, we have $|T| \leq n/2 - 1/2$, which in turn gives that $|X| - |T| \geq 0$, a contradiction. Furthermore, we have the following claim.

**Claim 3.5.** $T$ has no $\Delta$-vertex.

**Proof.** We divide the vertices in $T$ into three groups to consider its degree in $G$. Let $v \in T$ be a vertex. If $v$ is not adjacent to any components in $C_1$, we then have $d_G(v) = d_H(v) \leq |X| < n/2$. If $v$ is adjacent to at least two components in $C_1$, that is, $|C_{1v}| \geq 2$. Then by Claim 3.1, each $C \in C_{1v}$ has order at least 2. So, $|C_{1v}| \leq |V(C_{1v})|/2 \leq |V(C_1)|/2$. Notice that $d_H(v) \leq |S| + |C_{\geq 3}|$, and $|S| + |C_{\geq 3}| < |T|$ by Claim 3.4, we have

$$d_G(v) \leq |S| + |C_{\geq 3}| + |C_{1v}| < \frac{|S| + |C_{\geq 3}| + |T|}{2} + \frac{|V(C_{1v})|}{2} \leq \frac{|S| + |T| + |V(C_{\geq 3})| + |V(C_{1v})|}{2} \leq n/2.$$  

We consider now $v \in T$ which is adjacent to exactly one component in $C_1$. For such a vertex $v$, it holds that $d_G(v) = d_H(v) + 1 \leq |X| + 1$. If $v$ is adjacent to at most $|X| - 1$ vertices in $X$, then $d_G(v) = d_H(v) + 1 \leq |X| < n/2 \leq \Delta$. So we assume that there is $v \in T$ with $|C_{1v}| = 1$, and $v$ is adjacent to all vertices in $X$. If $|X| \leq n/2 - 3/2$, then $d_G(v) < n/2 \leq \Delta$. So we assume $n/2 - 3/2 < |X| < n/2$. Recall from previous argument that if $n$ is odd, then $|X| \leq n/2 - 3/2$. Hence we have $n$ is even and $|X| = n/2 - 1$ and $d_G(v) = n/2$. As $|X| < |T| \leq n/2$ by Claim 3.4 and Lemma 1.2, $|T| = n/2$. This in turn indicates that $|S \cup U| = n/2$. By (3) of Claim 3.4, $e(w, C) \leq 1$ for every $w \in T$ and every odd component $C$. As $v$ is adjacent to every vertex in $S \cup U$, we know each odd component of $G - (S \cup T)$ consists of a single vertex, and $|C_1| = 1$. This, together with the facts
that \( |S \cup U| = n/2 \) and \( |X| = n/2 - 1 \), we know \( C_{\geq 5} = \emptyset \) and \( G - (S \cup T) \) has no even component. Since \( |C_1| = 1 \), \( v \) is the only vertex in \( T \) which is adjacent to the vertex of the component in \( C_1 \), and thus it is the only \( \Delta \)-vertex in \( T \). In fact, we will show below that under the above assumption such a \( \Delta \)-vertex can not exist.

**Claim.** If \( w \) is a \( \Delta \)-vertex in \( T \), then \( |X| \geq |T| \).

**Proof.** Let \( S^* = S \cup U \), and let \( H^* = G - E(G[S^*]) \) be the bipartite graph with bipartitions \( S^* \) and \( T \). Recall that \( w \) is the \( \Delta \)-vertex in \( T \), we let \( w^* \) be the vertex to which \( w \) is adjacent in \( C_1 \). Applying Lemma 1.5 with \( \delta_0 = 1 \) on \( H^* \), we see that for every edge \( xy \in H^* \) with \( x \in S^* \) and \( y \in T \), the relation \( d_{H^*}(y) \geq d_{H^*}(x) \) holds. Hence, there is matching \( M \) in \( H^* \) which saturates \( T \). If \( V(M) \cap S^* \) does not contain \( w^* \), we have \( |T| < |S^*| \). Otherwise, \( M \) contains \( w^* \). Notice that \( w^* \) is adjacent to only \( w \) in \( T \), hence \( d_{H^*}(w^*) = 1 \) and \( ww^* \in M \). Also, notice that for any \( v \in T \), \( d_{H^*}(v) = d_G(v) \geq 2 \). Then

\[
e(V(M) - T, T) = \sum_{xy \in M - \{ww^*\} \atop x \in S^*, y \in T} d_{H^*}(y) + d_{H^*}(w) > \sum_{xy \in M - \{ww^*\} \atop x \in S^*, y \in T} d_{H^*}(x) + d_{H^*}(w^*) \geq e(V(M) - T, T),
\]
gives a contradiction. Thus, \( V(M) \cap S^* \) does not contain \( w^* \) and \( |T| < |S^*| \).

We now explain that \( X = S^* - \{w^*\} \), which implies that \( |X| \geq |T| \). The assertion follows by noticing that \( G - (S \cup T) \) has no even component, \( C_{\geq 5} = \emptyset \), \( |C_1| = 1 \) and the component in \( C_1 \) consists of a single vertex, and each component in \( C_3 \) consists of a single vertex.

By all the above argument, we see that \( T \) has no \( \Delta \)-vertex.

Let \( S^* = V(G) - T \), and let \( H^* = G - E(G[S^*]) \) be the bipartite graph with partite sets \( S^* \) and \( T \) similar as in the above proof. We will induce the degree relation between \( d_H(x) \) and \( d_H(y) \) for an edge \( xy \in E(H) \) with \( x \in X \) and \( y \in T \) by using the corresponding relations between \( d_{H^*}(x) \) and \( d_{H^*}(y) \). Thereby to get a sufficient condition for implying a matching which saturates \( T \) in \( H \), and thus get a contradiction to Claim 3.4.

Let \( C_3^1 \) be the set of components in \( C_3 \) which consists of a single vertex. Notice that all the vertices in \( C_3^1 \) are kept in \( H \). We divide the vertices in the subset \( S' := S \cup V(C_3^1) \) of \( S^* \) into two parts. Let

\[
S_1 = \{ x \in S' \mid x \text{ has a non } \Delta \text{-degree neighbor in } S^* \}, \quad \text{and} \quad S_0 = S' - S_1.
\]

For a vertex \( x \in S^* \), define \( \sigma_x \) as the number of non \( \Delta \)-degree neighbors of \( x \) in \( S^* \). Then in \( H^* \),

\[
\sigma_x \begin{cases} 
\geq 1, & \text{if } x \in S_1; \\
= 0, & \text{if } x \in S_0.
\end{cases}
\]
As vertices in $C_1$ are deleted in the bipartite graph $H$, we do not consider $\sigma_x$ for $x \in V(C_1)$ here.

As $T$ has no $\Delta$-vertex, applying Lemma 1.5, for each edge $xy \in E(H^*)$ with $x \in S^*$ and $y \in T$, we have

**Claim 3.6.**

$$d_{H^*}(y) \geq \begin{cases} d_{H^*}(x) + 2, & \text{if } x \in S_1; \\ d_{H^*}(x) + 1, & \text{otherwise}. \end{cases}$$

Now let us look at the degree relations of the two ends of an edge in $H$. Recall that $H = H[X,T]$, where $X = S \cup U^C$, and $E(H) = E(S \cup V(C_{\geq 3}), T)$; and $H^* = H^*[S \cup U, T]$ with $E(H^*) = E(S \cup U, T)$.

In particular,

**Fact 1.**

- $d_H(y) = d_{H^*}(y) - |C_{1y}|$, by (3) of Claim 3.4 that for each $y \in T$, $v$ is adjacent to at most one vertex of an odd component of $G - (S \cup T)$, and by the construction of $H$, vertices in $V(C_1)$ are deleted, and the degrees of a vertex $v \in T$ in $V(G) - V(C_1)$ are kept;
- $d_H(x) = d_{H^*}(x)$ for each $x \in S$;
- For each $k \geq 1$ and each $C \in C_{2k+1}$, $d_H(u_1^C) = 3$ and $d_H(u_i^C) = 2$ for each $i$ with $2 \leq i \leq k$.

The following claim gives some information on the degree relation of the two ends of an edge $xy \in E(H)$ with $x \in U^C$ and $y \in T$.

**Claim 3.7.** For each edge $xy \in E(H^*)$ with $x \in V(C_{\geq 3})$ and $y \in T$, either $d_{H^*}(y) \geq 3$ holds or $d_{H^*}(y) = 2$ and in $S$ $y$ has a neighbor of degree 1.

**Proof.** As $d_{H^*}(v) = d_G(v) \geq 2$ for each $v \in T$. We assume that for some edge $xy \in E(H^*)$ with $x \in V(C_{\geq 3})$ and $y \in T$, $d_{H^*}(y) = 2$. By Claim 3.6, we see $x \notin S_0$ and $d_{H^*}(x) = 1$. Similarly, the other neighbor of $y$, say $x_1$ is also not contained in $S_1$. By Vizing’s Adjacency Lemma, both $x_1$ and $x$ are $\Delta$-vertices. By (3) of Claim 3.4, if none of $x_1$ and $x$ is contained in $S$, then they, respectively, belongs to two odd components. By Claim 3.8, one of $x_1$ and $x$ must be adjacent to at least two vertices in $T$, showing a contradiction to the fact that $d_{H^*}(x) = d_{H^*}(x_1) = 1$. Hence one of $x_1$ and $x$ is contained in $S$. \hfill $\Box$

As given in Fact 1, all vertices in $T$ keeps the same degree as in $G$, except those which is adjacent to some vertices in $C_1$. For each $v \in T$ we associate a value

$$p_v = |C_{1v}| = e(v, V(C_1))$$

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with $v$. In the bipartite graph $H$, each such vertex $v$ losses $p_v$ in degrees in comparison with that in $H^*$. In the next several paragraph, we will show that for each $v \in T$ with $p_v > 0$, we can find a way for compensation of the degree losses, and thereby find a sufficient condition for a matching which saturates $T$ in $H$.

**Claim 3.8.** Let $C_1, C_2 \in C_{\geq 1}$ be two components. If $C_1$ contains a $\Delta$-vertex $x_1$ and $C_2$ contains a $\Delta$-vertex $x_2$, then $\max\{e(x_1, T), e(x_2, T)\} \geq 2$.

**Proof.** Suppose on the contrary that $\max\{e(x_1, T), e(x_2, T)\} \leq 1$, we get

$$|S| + |C_{11}| - 1 + 1 \geq n/2 \quad \text{and} \quad |S| + |C_{12}| - 1 + 1 \geq n/2,$$

As $|T| > |S|$, we get

$$n \leq 2|S| + |C_{11}| + |C_{12}| < |S| + |T| + |C_{11}| + |C_{12}| \leq n,$$

showing a contradiction. □

The above claim indicates that there is at most one component in $C_1$ containing a $\Delta$-vertex.

A similar proof as above gives that

**Claim 3.9.** If $C \in C_{\geq 1}$ is a component which contains a $\Delta$-vertex, then $|C| > |V(C_{\geq 1})| - |C|$.

**Claim 3.10.** We may assume that $C_1 \neq \emptyset$.

**Proof.** Suppose on the contrary, that $C_1 = \emptyset$. Then for each vertex in $T$, the degree of the vertex in $H$ is the same as that in $H^*$. We will find a matching which saturates $T$ in $H$.

As in the case when $C_1 = \emptyset$, the degree of each vertex in $T$ keeps the same in both $H$ and $H^*$, by Claim 3.7, we have for each edge $xy \in E(H)$ with $x \in U^C$ and $y \in T$, either $d_H(y) \geq d_H(x)$ holds or $d_H(y) < d_H(x)$ and $y$ has a neighbor of degree 1 in $S$. For all other edge $xy \in E(H)$ with $x \in S$ and $y \in T$, we have $d_H(x) = d_{H^*}(x) \leq d_{H^*}(y) - 1 = d_H(y) - 1$. By Lemma 1.4, there is a matching which saturates $T$ in $H$, gives a contradiction to Claim 3.4. □

For each $C \in C_1$, by (3) of Claim 3.4, there is a unique vertex $t_c \in T$ which is adjacent to a unique vertex $v_c$ on it. We call $v_c$ and $t_c$ the partners of each other. We divide the components in $C_1$ into two subgroups in order to consider the degrees of the partner vertices in $T$.

Let

$$C_{11} = \{C \in C_1 \mid e(t_c, C_1) = 1\} \quad \text{and} \quad C_{12} = \{C \in C_1 \mid e(t_c, C_1) \geq 2\}.$$ 

By the definition of $C_{12}$, it is clear that if $C_{12}$ is not empty, then it contains at least two components. Also, by Claim 3.2, each component in $C_{12}$ consists of at least 4 vertices.
Furthermore, we divide the components in $C_{11}$ into two groups as follows.

$$C_{11}^1 = \{C \in C_{11} \mid |C| = 1\} \quad \text{and} \quad C_{11}^2 = \{C \in C_{11} \mid |C| \geq 2\}.$$  

Let

$$|C_{11}^1| = m_{11}, \quad |C_{11}^2| = m_{12}, \quad \text{and} \quad |C_{12}| = m_2.$$  

**Claim 3.11.** We may assume that none vertices in $V(C_{11}^1)$ is a $\Delta$-vertex.

**Proof.** Suppose on the country that $v_c$ is the single vertex in a component of $C_{11}^1$, and it is of degree $\Delta$ in $G$. This indicates that $|S| = \Delta - 1 \geq n/2 - 1$. As $|S| < |T| \leq n/2$ by Lemma 1.2, $|T| = n/2$. We consider the bipartite graph $H^*$ with bipartitions $S^* = S \cup U = S \cup \{v_c\}$ and $T$. By Claim 3.6, $H^*$ contains a matching $M$ which saturates $T$. If $M$ does not contain $v_c$, we obtain $|S| \geq |T|$. If $M$ contains $v_c$, as $v_c$ has degree 1 in $H^*$, and every vertex in $T$ has degree at least 2 in $H^*$, we will get a contradiction by counting the edges in $E(V(M) - T, T)$ in two ways as in the end of the proof of Claim 3.5. \hfill $\square$

Corresponding to the partition of $C_1$, we partition vertices in $T$ into subgroups, as follows.

$$T_1^1 = \{v \in T \mid e(v, V(C_{11}^1)) = 1\}, \quad T_1^2 = \{v \in T \mid e(v, V(C_{11}^2)) = 1\};$$

$$T_0 = \{v \in T \mid e(v, V(C_1)) = 0\}, \quad \text{and} \quad T_2 = \{v \in T \mid e(v, V(C_{12})) \geq 2\}.$$  

Recall that $C_{11}^3$ is the set of components in $C_3$ which consists of a single vertex, let $C_{11}^2 = C_3 - C_{11}^3$. Then each component in $C_{11}^2$ contains at least two vertices. Similarly, we define $C_{12}^1$ and $C_{12}^2$. Let $|C_{12}^3| = m_3$. Notice that

$$n \geq |S'| + |T| + |V(C_{12}^2)| + |V(C_{11}^1)| + |V(C_{11}^2)| + |V(C_{12})|$$

$$\geq |S'| + |T| + 2m_3 + m_{11} + 2m_{12} + 4m_2.$$  

**Claim 3.12.** Let $xy \in E(H^*)$ be an edge with $x \in V(C) \subseteq V(C_{>5})$ and $y \in T$, and let $y_c$ be the vertex in $U^C$ which is adjacent to $y$ in $H$. Then either $d_{H^*}(y) = d_H(y) + p_y \geq d_H(y_c) + 3$ or $C$ contains at least two vertices.

**Proof.** We may assume that $C$ is a single vertex. Then the vertex in $C$ has degree at least 5 in $H^*$. By Claim 3.6, we know $y$ has degree at least 6 in $H^*$. Since the largest degree of vertices in $U^C$ in $H$ is 3, $d_{H^*}(y) = d_H(y) + p_y \geq d_H(y_c) + 3$. \hfill $\square$

We consider now the degrees of vertices in $T_1^1 \cup T_1^2 \cup T_2$.

**Claim 3.13.** Each $v \in T_1^1$ (if exists) satisfies

$$|N_G(v) \cap V_{\Delta} \cap S| \geq |S_0| + (m_{11} + 1)/2 + m_{12} + 2m_2 + m_3.$$
Proof. Let \( t_c \in T_1^1 \) and \( v_c \) the partner of \( t_c \) in \( C_{11}^1 \). Instead, we show \( d_G(v_c) \leq \Delta - (|S| + (m_{11} + 1)/2 + m_{12} + 2m_2 + m_3 + 1) \). By Vizing’s Adjacency Lemma, \( N_G(t_c) \) has at least \( \Delta - d_G(v_c) + 1 \geq (|S| + (m_{11} + 1)/2 + m_{12} + 2m_2 + m_3 - 1) + 1 \Delta \)-vertices in \( G \).

Notice that \( v_c \) is adjacent to only one vertex \( t_c \) in \( T \), and \( v_c \) is not adjacent to any vertices in the odd components not containing \( v_c \). By Claim 3.11, \( v_c \) is not a \( \Delta \)-vertex. Recall that \( S_1 = \{ x \in S' \mid x \text{ has a non } \Delta \text{-degree neighbor in } S^* \} \), and \( S_0 = S' - S_1 \). Thus, \( v_c \) is not adjacent to any vertices in \( S_0 \), as vertices in \( S_0 \) (if any) only adjacent to \( \Delta \)-vertices in \( S^* = V(G) - T \). Let \( C \) be the component containing \( v_c \). Thus, as \( |C| = 1 \),
\[
d_G(v_c) \leq |S_1| + |C| - 1 + 1 \]
\[
= |S_1| + 1.
\]
Hence, by Inequality 1, the relation that \(|S'| < |T|\) (as \(|S'| \leq |X| < |T| \) by Claim 3.4) and \(|S'| = |S_1| + |S_0|\),
\[
\Delta - d_G(v_c) + 1 \geq n/2 - |S_1| - 1 + 1
\]
\[
\geq |S'| + |T| + 2m_3 + m_{11} + m_{12} + 2m_2 + 4m_2 - |S_1|
\]
\[
\geq |S| + (m_{11} + 1)/2 + m_{12} + 2m_2 + m_3.
\]
By Claim 3.3, \( t_c \) is not adjacent to any vertex in other odd components of \( G - (S \cup T) \) except \( C \), all those \( \Delta \)-vertices are contained in \( S \). \( \square \)

Let \( C_{\text{max}}^1 \) be the component of largest order among all components in \( C_{11}^2 \). By the remark after the Claim 3.8 that there is at most one component in \( C_1 \) which contains a \( \Delta \)-vertex, and by the Claim 3.9 that a component of \( G - (S \cup T) \) containing a \( \Delta \)-vertex has largest order among all the components, we see if there is a component in \( C_1 \) contains a \( \Delta \)-vertex, then it must be the component \( C_{\text{max}}^1 \) (by Claim 3.11, no component in \( C_{11}^1 \) has a \( \Delta \)-vertex).

Claim 3.14. Each \( v \in T_2^2 \) (if exists) satisfies
\[
|N_G(v) \cap V_{\Delta} \cap (S \cup V(C_{\geq 3}))| \geq \begin{cases} 
|S| + (m_{11} + 1)/2 + (m_{12} - 2) + 2m_2 + m_3, & \text{if } m_{12} \geq 2 \text{ and } v_c \notin C_{\text{max}}^1; \\
1, & \text{if } v_c \in C_{\text{max}}^1 \text{ or } m_{12} = 1; 
\end{cases}
\]
where \( v_c \) is the partner of \( t_c \) in \( C_{11}^2 \). Additionally, in \( H \), there is at most one vertex \( v \) which is adjacent to a non single-vertex component \( C_{\text{max}}^1 \) in \( C_1 \).

Proof. Let \( t_c \in T_1^2 \) and \( v_c \) the partner of \( t_c \) in \( C_{11}^2 \). We first assume that the component, say \( C \) in \( C_{11}^2 \) containing \( v_c \) is not \( C_{\text{max}}^1 \) (this indicates that \( m_{12} \geq 2 \)). Then \( C \) has no \( \Delta \)-vertex by the argument prior to Claim 3.14.
Then \( t_c \) is adjacent to at least \( \Delta - d_G(v_c) + 1 \) \( \Delta \)-vertices in \( G \). Since \( t_c \) is adjacent to exactly one component in \( C_{12}^2 \), and the component has no \( \Delta \)-vertex, all those \( \Delta - d_G(v_c) + 1 \) \( \Delta \)-vertices are contained in \( S \cup V(C_{\geq 3}) \).

Notice that \( |S'| < |T| \) and \( |S'| = |S_1| + |S_0| \), we get

\[
\Delta - d_G(v_c) + 1 \geq \frac{|S'| + |T| + m_{11} + |C| + |C_{\max}^1| + 2(m_{12} - 2) + 4m_2 + 2m_3}{2} - |S_1| - |C|
\]

\[
\geq |S_0| + (m_{11} + 1)/2 + (m_{12} - 2) + 2m_2 + m_3.
\]

Suppose now that \( C = C_{\max}^1 \). As \( d_G(t_c) \geq 2 \), and \( t_c \) is adjacent to exactly one vertex \( v_c \) in \( C_1 \), the other neighbor of \( t_c \) is contained in \( S \cup V(C_{\geq 3}) \).

There is at most one vertex \( v \) which is adjacent to a non single-vertex component \( C_{\max}^1 \) in \( C_1 \). If such a vertex exists, we label it as \( v_{\text{special}} \).

**Claim 3.15.** Each \( v \in T_2 \) (if exists) satisfies

\[
|N_G(v) \cap V_\Delta \cap (S \cup V(C_{\geq 3}))| \geq |S_0| + m_{11}/2 + m_{12} + 2(m_{2} - 2) + m_3 + 1.
\]

**Proof.** By the definition of \( C_{12} \), if \( m_2 > 0 \), then \( m_2 \geq 2 \). Let \( C_{\max}^2 \) be a component with largest cardinality in \( C_{12} \). Let \( v \in T_2 \), and let \( C_1 \neq C_{\max}^2 \) be a component in \( C_{12} \cap C_{1v} \). By Claim 3.9, \( C_1 \) has no \( \Delta \)-vertex, and thus no vertex in \( C_1 \) is adjacent to any vertex in \( S_0 \). Let \( v^* \) be the neighbor of \( v \) in \( C_1 \). Then by Vizing’s Adjacency Lemma, \( v \) is adjacent to at least \( \Delta - d_G(v^*) + 1 \) \( \Delta \)-vertices in \( G \). As there is at most one component in \( C_{12} \) contains \( \Delta \)-vertices, \( t_c \) has at least \( \Delta - d_G(v^*) \) \( \Delta \)-degree neighbors in \( S \cup V(C_{\geq 3}) \).

If \( C_{\max}^2 \) has no \( \Delta \)-vertex, we get

\[
\Delta - d_G(v^*) + 1 \geq \frac{|S'| + |T| + m_{11} + 2m_{12} + |C_1| + |C_{\max}^1| + 4(m_{2} - 2) + 2m_3}{2} - |S_1| - |C_1| + 1
\]

\[
\geq |S_0| + (m_{11} + 1)/2 + m_{12} + 2(m_{2} - 2) + m_3 + 1.
\]

If \( C_{\max}^2 \) has a \( \Delta \)-vertex, then \( |C_{\max}^2| > |C_1| \), we get

\[
\Delta - d_G(v^*) \geq \frac{|S'| + |T| + m_{11} + 2m_{12} + |C_1| + |C_{\max}^1| + 4(m_{2} - 2) + 2m_3}{2} - |S_1| - |C_1|
\]

\[
\geq |S_0| + m_{11}/2 + m_{12} + 2(m_{2} - 2) + m_3 + 1.
\]
We are now ready to find a matching which saturates $T$ in $H$. Recall that $H$ is the bipartite graph with partite sets $X = S \cup U^C$ and $T$. Let $U^3$ be the vertices in $U^C_{\geq 3}$ such that each of them has degree 3 in $H$. As each subset $U^C$ of $C \in C_{\geq 3}^2$ contains exactly one vertex of degree 3, we have $|U^3| = m_3$. Recall that each $v \in T$ $p_v = |C_1| = e(v, V(C))$ is defined previously.

**Claim 3.16.** Let $xy \in E(H)$ be an edge with $x \in X$ and $y \in T$, then each of the following holds.

1. $d_H(y) + p_y \geq d_H(x) + 2$, if $x \in S_1$ and $y \in T$;
2. $d_H(y) + p_y \geq d_H(x) + 1$, if $x \in S_0$ and $y \in T$;
3. $d_H(y) + p_y \geq d_H(x) + 3$, if $x \in U^C_{\geq 5}$ and $y \in T$;
4. $d_H(y) + p_y \geq d_H(x) = 3$, if $x \in U^3$ and $y \in T$;
5. $d_H(y) + p_y = 2$, $d_H(x) = 3$, if $x \in U^C_{\geq 3}$, $y \in T$ and in $S$, $y$ has a neighbor which has degree 1 in $H$;
6. $d_H(y) + p_y \geq d_H(x) = 2$, if $x \in U^C_{\geq 3}$, $y \in T$.

**Proof.** As $d_H(y) + p_y = d_{H^*}(y)$, and $d_H(x) = d_{H^*}(x)$, (1)-(2) follows from Claim 3.6. By Claim 3.12, we get (3). The remaining (4)-(6) follows from Claim 3.7. \hfill \Box

Let $y \in T$ be a vertex of degree 2 in $H$. By Claim 3.7, $y$ has a neighbor $x$ in $S$ which has degree 1 in $H$. As the deletion of the pair of vertices $x$ and $y$ does not affect the degrees of any other vertices in $T$, and the deletion deduces each of the sizes of $X$ and $T$ by 1 at the same time, we may assume that $T$ has no vertex of degree 2.

**Claim 3.17.** $H$ has a matching which saturates $T$.

**Proof.** Suppose on the contrary, then by Hall’s Theorem, there is an nonempty subset $A \subseteq T$ such that $|N_H(A)| < |A|$, among all such subsets with this property, we take $A$ with smallest cardinality.

Let $N_H(A) = B$. Then there is a matching in the induced bipartite graph $H' = H[A \cup B]$ which saturates $B$. For otherwise, if not, there exists $\emptyset \neq B' \subseteq B$ such that $|N_{H'}(B')| < |B'|$. Since $B' \subseteq B = N_H(A)$, $|N_{H'}(B')| \geq 1$. Then $N_H(A - N_{H'}(B')) = B - B'$, and $|A - N_{H'}(B')| = |A| - |N_{H'}(B')| > |B| - |N_{H'}(B')| > |B| - |B'|$, gives a contradiction to the choice of $A$.

Let $M$ be a matching of $H' = H[A \cup B]$ which saturates $B$. We consider three cases below.

**Case 1.** $A \subseteq T_0$. 

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In this case, all vertices in $y \in T_0$ has $p_y = 0$, by Claim 3.16 and Lemma 1.3, we find a matching of $H'$ which saturates $A$. This implies that $|A| \geq |B|$, showing a contradiction to the assumption that $|A| < |B|$.

**Case 2.** $A \cap (T^1_1 \cup T^2_1 \cup T_2) = A \cap T^2_1 = \{v_{\text{special}}\}$.

We may assume $V(M) \cap A$ contains $v_{\text{special}}$, otherwise it gets back to Case 1. Note that $p_{v_{\text{special}}} = 1$. Then

$$e(A, B) \leq \sum_{x,y \in M} d_H(x) \leq \sum_{x,y \in M, x \in B, y \in A} d_H(y) + (d_H(v_{\text{special}}) + 1) < \sum_{x,y \in M, x \in B, y \in A} d_H(y) + 1 < e(A, B),$$

where the last inequality follows from the fact that $B - (V(M) \cap B) \neq \emptyset$, and the vertices in $B - (V(M) \cap B) \subseteq T_0$, and have degree at least 2 in $H$. The inequalities above give a contradiction.

**Case 3.** $A$ contains a vertex from $T^1_1 \cup T^2_1 \cup T_2$ other than $v_{\text{special}}$.

We take $B_1 \subset B$ such that $B_1$ only contains the neighbors of a vertex from $A \cap (T^1_1 \cup T^2_1 \cup T_2)$ such that it has most neighbors in $B$ among all vertices in $A \cap (T^1_1 \cup T^2_1 \cup T_2)$ in the original graph $G$. By (3) of Claim 3.4, we have $|B_1 \cap U^{C_{2,3}}| \leq m_3$. Let $\overline{B_1} = B - B_1$.

By Claims 3.13, 3.14, and 3.15, in notching that if $m_2 > 0$ then $m_2 \geq 2$, we have

$$|B_1| \geq \begin{cases} |S_0| + (m_{11} + 1)/2 + m_3, & \text{if } m_{12} = m_2 = 0 \text{ or } m_{12} = 1, m_2 = 0; \\ |S_0| + m_{11}/2 + 2(m_2 - 2) + m_3 + 1, & \text{if } m_{12} = 0, m_2 \geq 2; \\ |S_0| + (m_{11} + 1)/2 + (m_{12} - 2) + m_3, & \text{if } m_{12} \geq 2 \text{ and } m_2 = 0; \\ |S_0| + m_{11}/2 + m_{12} + 2(m_2 - 2) + m_3 + 1, & \text{if } m_{12} \geq 2 \text{ and } m_2 \geq 2. \end{cases}$$

By Claim 3.16, we know that there are at most $m_3$ edges $xy \in M$ with $x \in V(M) \cap B_1$ and $y \in V(M) \cap A$ satisfying $d_H(y) \geq d_H(x) = 3$ or $d_H(y) \geq d_H(x) + 1 = 3$, and at most $|S_0|$ edges $xy \in M$ with $x \in V(M) \cap B_1$ and $y \in V(M) \cap A$ satisfying $d_H(y) \geq d_H(x) + 1$, all other edges $xy \in M$ with $x \in V(M) \cap B_1$ and $y \in V(M) \cap A$ satisfying $d_H(y) \geq d_H(x) + 2$. 

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\[ \begin{align*}
\epsilon_H(A, B) & \leq \sum_{x \in M} d_H(x) + \sum_{x \in B, y \not\in A} d_H(x) \\
& \leq \sum_{x \in B \cap S_0, y \in A} (d_H(y) + p_y - 1) + \sum_{x \in (B \cap S_0) \cup U_{\geq 3}, y \in A} (d_H(y) + p_y) + \\
& \quad \sum_{x \in (B \cap S_0) \cup U_{\geq 3}, y \in A} d_H(y) \\
& \leq \sum_{x \in B, y \in A} d_H(y) - |B \cap S_0| - 2|B \cap (S_0 \cup U_{\geq 3})| + \sum_{y \in A \cap V(M)} p_y
\end{align*} \]

As \(|B \cap S_0| \leq |S_0|, and |B \cap U_{\geq 3}| \leq m_3,

\[
2|B - (S_0 \cup U_{\geq 3})| \geq \begin{cases} 
  m_{11} + 1 \geq m_{11}, & \text{if } m_{12} = 0, 1, m_2 = 0; \\
  m_{11} + 4(m_2 - 2) + 2 \geq m_{11} + m_2, & \text{if } m_{12} = 0, m_2 \geq 2; \\
  m_{11} + 1 + 2(m_{12} - 2) \geq m_{11} + m_{12}, & \text{if } m_{12} \geq 2 \text{ and } m_2 = 0; \\
  m_{11} + 2m_{12} + 4(m_2 - 2) + 2 \geq m_{11} + m_{12} + m_2, & \text{if } m_{12} \geq 1 \text{ and } m_2 \geq 2.
\end{cases}
\]

Notice that \(\sum_{y \in A \cap V(M)} p_y\) is at most the quantity above under each of the conditions. Hence,

\[
\begin{align*}
\epsilon_H(A, B) & \leq \sum_{x \in B, y \not\in A} d_H(x) \\
& \leq \sum_{x \in B, y \not\in A} d_H(y) \\
& < \sum_{y \in A} d_H(y) \text{ (as } A - (V(M) \cap A) \neq \emptyset) \\
& = \epsilon_H(A, B),
\end{align*}
\]

gives a contradiction. 

\[\square\]

The proof of Theorem 1.1 is then completed. \[\blacksquare\]

**Acknowledgements**

The authors would like to thank Rong Luo and Yue Zhao for their discussions.
References


