ASYMPTOTICALLY CONICAL CALABI-YAU MANIFOLDS, III

RONAN J. CONLON AND HANS-JOACHIM HEIN

Abstract. In a recent preprint [20], Chi Li proved that asymptotically conical complex manifolds with regular tangent cone at infinity admit holomorphic compactifications (his result easily extends to the quasiregular case). In this short note, we show that if the open manifold is Calabi-Yau, then Chi Li's compactification is projective algebraic. This has two applications. First, every Calabi-Yau manifold of this kind can be constructed using our refined Tian-Yau type theorem in [6]. Secondly, we prove classification theorems for such manifolds via deformation to the normal cone. This includes Kronheimer's classification of ALE spaces and a uniqueness theorem for Stenzel's metric.

1. Introduction

Let us begin by recalling what we mean by an asymptotically conical (AC) Calabi-Yau manifold. See [5, Section 1.3] for details of the following definition.

Definition 1.1. Let (C, g_0, Ω_0) be a Calabi-Yau cone with metric g_0 and holomorphic volume form Ω_0. Let (M, g, Ω) be a Calabi-Yau manifold with metric g and holomorphic volume form Ω. We say that (M, g, Ω) is *asymptotically conical* (AC) *of rate* $\lambda < 0$ *with asymptotic cone* C if there exists a diffeomorphism $\Phi : C \setminus K \to M \setminus K'$ away from compact sets K, K' such that for all $j \in \mathbb{N}_0$,

$$|\nabla^j_{g_0}(\Phi^* g - g_0)|_{g_0} + |\nabla^j_{g_0}(\Phi^*\Omega - \Omega_0)|_{g_0} = O(r^{\lambda - j}).$$

Here, r denotes the radius function of the cone metric g_0.

Let us also recall the notion of a quasiregular Calabi-Yau cone. Let D be a Kähler-Einstein Fano orbifold, possibly with $\mathbb{C}$-codimension-1 singularities. Assume that the total space of the canonical orbibundle K_D is smooth. Moreover, assume that K_D is divisible by $k \in \mathbb{N}$ as a line orbibundle. By the Calabi ansatz, we can endow $C = (\frac{1}{k}K_D)^\times$ (i.e. the k-th root of K_D with its zero section blown down) with the structure of a Calabi-Yau cone (C, g_0, Ω_0). We call a Calabi-Yau cone constructed in this manner *quasiregular*, and *regular* if D is actually smooth.

Now suppose that (M, g, Ω) is AC Calabi-Yau with quasiregular asymptotic cone (C, g_0, Ω_0). By [5, Lemma 2.14], the complex structures J on M and J_0 on C satisfy $|\nabla^j_{g_0}(\Phi^* J - J_0)|_{g_0} = O(r^{\lambda - j})$ for all $j \in \mathbb{N}_0$. A very recent result of Li [20, Theorem 1.2] (see Appendix A) then tells us that M is biholomorphic to $X \setminus D$, where X is a compact complex orbifold without divisorial singularities, containing D as a complex suborbifold with positive normal orbibundle $N_D = -\frac{1}{k}K_D$.

Theorem A. *The complex orbifold* $X = M \cup D$ *satisfies the following properties.*

- (i) *We have that* $-K_X = (k+1)[D]$ *as complex line orbibundles on* X.
- (ii) *There exists a holomorphic map* $p : X \to Y$ *onto a normal projective variety* Y *such that* p *is an isomorphism onto its image in a neighbourhood of* D, *all of the singularities of* $Y \setminus p(D)$ *are isolated and canonical, the restriction* $p|_M : M \to Y \setminus p(D)$ *is a crepant resolution of the singularities of* $Y \setminus p(D)$, *and the* $\mathbb{Q}$*-Cartier divisor* $p_*[D]$ *is ample on* Y.
- (iii) X *is projective algebraic and satisfies* $h^{i,0}(X) = 0$ *for all* $i > 0$.
- (iv) *Every Kähler form on* M *is cohomologous to the restriction to* M *of a Kähler form on* X.

Corollary B. *Every* AC *Calabi-Yau manifold with quasiregular asymptotic cone can be obtained from the refined Tian-Yau construction of* [6, Theorem A]. *In particular, if the asymptotic cone is actually regular, then the optimal value of the rate* λ *is strictly less than* -1 [6, Corollary B].

Date: July 1, 2014.

Remark 1.2. We expect that the optimal value of the rate λ is always less than -1 whenever the asymptotic cone is quasiregular. Notice also that, at least in the regular case, it essentially follows from Li [20, Proposition 1.2, Theorem 1.2], together with our work in [5], that the optimal value of λ is a rational number: $-\min\{2, \frac{n}{k}\ell\}$ if the Kähler class $\mathfrak{k} \notin H^2_c(M)$ and $-\min\{2n, \frac{n}{k}\ell\}$ if $\mathfrak{k} \in H^2_c(M)$, where n denotes the complex dimension of M and $\ell \in \mathbb{N} \cup \{\infty\}$ is the maximal integer such that D can be $(\ell - 1)$-comfortably embedded in a projective manifold X satisfying the assumptions of [6, Theorem A]. This statement is rigorous if either $\frac{n}{k}\ell \leq 2n$ and $\mathfrak{k} \in H^2_c(M)$ (which is satisfied for the complete intersections considered in [5, Section 5], thereby resolving the conjecture posed there), or $\ell = \infty$ (in which case M is a crepant resolution of C; see [5, Section 4] and [6, Section 3]).

We now explain how Theorem A(ii), which already follows from the proof of [6, Proposition 2.4], can be used to classify AC Calabi-Yau manifolds in terms of their asymptotic cones.

Our first example is Kronheimer's classification of ALE spaces. The argument is close in spirit to Kronheimer's. Let (M, g, Ω) be an AC Calabi-Yau surface, so that the asymptotic cone (C, g_0, Ω_0) is isomorphic to $\mathbb{C}^2/\Gamma$ with its standard metric and holomorphic volume form for some finite group $\Gamma \subset \mathrm{SU}(2)$ acting freely on $\mathbb{S}^3$. Then the family $(M, tg, t\Omega)$ $(t \in \mathbb{R}^+)$ converges to (C, g_0, Ω_0) in the Gromov-Hausdorff sense as $t \to 0$. We would now like to upgrade this Gromov-Hausdorff family to an algebraic one, parametrised by $t \in \mathbb{C}$ rather than $t \in [0, \infty)$, in order to be able to use that the algebraic deformations of the algebraic surface singularity $\mathbb{C}^2/\Gamma$ can be classified.

Corollary C (Kronheimer [18])**.** *Every* AC *Calabi-Yau surface is a Kronheimer* ALE *space* [17]*, i.e. a crepant resolution of a deformation of a Kleinian singularity* $\mathbb{C}^2/\Gamma$ *with* Γ *as above.*

Proof. Consider the normal projective surface Y of Theorem A(ii), which contains D as an ample suborbifold divisor; see [6, Appendix A.1] for the orbifold structure of Y near D. Let Y_0 denote the orbifold normal cone to D in Y, i.e. the algebraic 1-point compactification of the total space of the normal orbibundle N_D. Then a "deformation to the normal cone" (compare [5, Proposition 5.1], [7, Section 6.1], [20, p.2], and [26, Section 7]) gives us a flat affine morphism $\mathcal{Y} \to \mathbb{C}$ with $\mathcal{Y}_1 = Y \setminus D$ and $\mathcal{Y}_0 = Y_0 \setminus D = \mathbb{C}^2/\Gamma$. By construction, $\mathcal{Y}$ carries a natural $\mathbb{C}^*$-action, covering the usual $\mathbb{C}^*$-action on $\mathbb{C}$ and restricting to the $\mathbb{C}^*$-action on $\mathcal{Y}_0 = \mathbb{C}^2/\Gamma$ induced by the scaling vector field $r\partial_r$ on $\mathbb{C}^2$. This allows us to conclude that the affine surface $\mathcal{Y}_1 = Y \setminus D$ must be a member of the well-known versal $\mathbb{C}^*$-deformation of $\mathcal{Y}_0 = \mathbb{C}^2/\Gamma$ [29, p.12, Remark 1].[1] □

Remark 1.3. All deformations of $\mathbb{C}^2/\Gamma$ have negative grading [24, Chapter 4], i.e. they converge back to $\mathbb{C}^2/\Gamma$ at infinity. This is not true for Calabi-Yau cones in general; see [5, Remark 5.3].

Our second example uses Theorem A(ii) in a similar manner to classify certain higher-dimensional AC Calabi-Yau manifolds whose asymptotic cones are sufficiently special.

Corollary D. *Let D be a Kähler-Einstein Fano manifold. For $k \in \mathbb{N}$ dividing $c_1(D)$, let M^n be an* AC *Calabi-Yau manifold with asymptotic cone $C = (\frac{1}{k}K_D)^\times$ given by the Calabi ansatz.*

- (i) *If $D = \mathbb{P}^2$ and $k = 1$, then $M = K_D$ with Calabi's metric* [3].
- (ii) *If $D = \mathbb{P}^1 \times \mathbb{P}^1$ and $k = 1$, then either $M = K_D$ with one of Goto's deformations of Calabi's metric* [8]*, or $M = \mathbb{P}^3 \setminus \mathrm{quadric} = T^*\mathbb{RP}^3$ with a $\mathbb{Z}_2$-quotient of Stenzel's metric* [30].
- (iii) *If $D = \mathbb{P}^1 \times \mathbb{P}^1$ and $k = 2$, then C is the* 3*-fold ordinary double point, and either $M = T^*\mathbb{S}^3$, the smoothing of C, with Stenzel's metric* [30]*, or else $M = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, the small resolution of C, with Candelas-de la Ossa's metric* [4].
- (iv) *If D is a quadric in $\mathbb{P}^n$ with $n > 3$ and $k = n-1$, then C is the n-fold ordinary double point, and $M = T^*\mathbb{S}^n$, the smoothing of C, with Stenzel's metric* [30].

[1] Standard results in deformation theory (see [2, Example 4.5] in the algebraic category and [13] or [10, p.198, Theorem] in the analytic category) all require shrinking the total space $\mathcal{Y}$ of the given deformation in order to be able to pull $\mathcal{Y}$ back from the versal deformation. We can undo this shrinking here by using the existence of a $\mathbb{C}^*$-equivariant map to the versal $\mathbb{C}^*$-deformation in the analytic category [29]. A very similar point appears in [18, (2.5)].

Proof. We again deform the pair (Y, D) of Theorem A(ii) to its normal cone. The affine cone in (i) is rigid, whereas the ones in (ii), (iii), (iv) have exactly one deformation, which is smooth. (For (i) this follows from [27], for (i), (ii), (iii) from [1], and for (iii), (iv) from [13].) Alternatively, we can apply the classification theory of log-Fano varieties [28, Definition 2.1.1] to Y: in (i) and (ii), (Y, D) is a del Pezzo 3-fold of degree 9 and 8 respectively—by [28, Remark 3.2.6, Theorem 3.3.1], the only possibilities are cones and $(\mathbb{P}^3, \mathcal{O}(2))$; in (iii) and (iv), Y must be a quadric by [28, Theorem 3.1.14], and it is easy to see that a singular quadric with only isolated singularities is a cone.

It therefore remains to classify all possible crepant resolutions M of C, or at least those carrying a Kähler form; in fact, we will use that M is quasiprojective by Theorem A(iii), although this should not be necessary. For (i), (ii), (iii), we begin by observing that C has an obvious crepant resolution M_0. By a result of Mori [15, Theorem 3.5.1],[2] it therefore suffices to classify all possible flops of M_0 [15, Definition 2.2.1]. In (i) and (ii), $M_0 = K_D$ cannot be flopped because D does not contain any contractible curves, as would be required by [15, Definition 2.1.1.2]. In (iii), $M_0 = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. By [15, Proposition 2.1.6], this has a unique flop, which is isomorphic to M_0. Finally, regarding (iv), it is easy to see that C is terminal, so that the blow-down morphism $M \to C$ would have to be small. But [5, p.2879, footnote] shows that this is not possible because D has Picard rank 1.

The metric uniqueness statements are now clear by [5, Theorem 3.1]. As usual, we are ignoring the fact that all of these metrics depend on one or two scaling or diffeomorphism parameters. □

Remark 1.4. If D is a Kähler-Einstein del Pezzo surface other than $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, then $k = 1$ and in general, the cone $K_D^\times$ admits many different smooth and singular deformations (see for instance [5, Example 1.4]) as well as many different crepant resolutions (see [14, Example 4.8]; flops of K_D yield crepant resolutions of $K_D^\times$ whose exceptional set is not pure-dimensional).

Acknowledgments. We are grateful to Mark Haskins for many helpful discussions over the years, to Chi Li for sending us a draft of [20], to Richard Thomas for explaining [2, Example 4.5] to us, and to Jeff Viaclovsky for pointing out the compactification theorem [19, Lemma 4.1].

2. Proof of Theorem A

It is clear that Ω extends to a meromorphic volume form on X with a $(k+1)$-st order pole along D and no poles or zeros elsewhere. Thus, $-K_X = (k+1)[D]$. Given this, Theorem A is an obvious consequence of the following result, which also answers the questions raised in [6, p.9].

Theorem 2.1. *Let X be a compact complex orbifold without $\mathbb{C}$-codimension-1 singularities. Let D be a suborbifold divisor in X such that D contains the singularities of X and such that the normal orbibundle to D in X is positive. Writing $M = X \setminus D$, the following properties hold.*

- (i) *There exists a holomorphic map $p : X \to Y$ onto a normal projective variety Y such that p is an isomorphism onto its image in a neighbourhood of D, all of the singularities of $Y \setminus p(D)$ are isolated, the restriction $p|_M : M \to Y \setminus p(D)$ is a resolution of singularities for $Y \setminus p(D)$, and the $\mathbb{Q}$-Cartier divisor $p_*[D]$ is ample on Y.*
- (ii) *If, in addition, $-K_X = q[D]$ for some $q \in \mathbb{N}$, then $h^{0,i}(X) = 0$ for all $i > 0$. Moreover, all of the singularities of $Y \setminus p(D)$ are canonical.*
- (iii) *If, in addition, $q > 1$ and M admits a Kähler form, then X is projective algebraic and every Kähler form on M is cohomologous to the restriction to M of a Kähler form on X.*

Part (i) was already proved in [6, Proposition 2.4]. Parts (ii) and (iii) follow from [6, Section 2.3] under the additional assumption that X is Kähler. Here, we will in particular establish that X is *necessarily* Kähler by going through the relevant proofs in [6] and showing that they still work if we only assume that X is complex. All of this strongly relies on the fact that N_D is positive.[3]

[2]It seems possible that these cases can also be treated using the holomorphic isometries of C instead of Mori theory.

[3]Thus, unlike the construction of X as a complex orbifold in [20] (compare Appendix A), our proof that X is Kähler is quite different in spirit from the treatment of the asymptotically *cylindrical* case in [12].

Proof of Theorem 2.1 in the smooth case. (i) This was already shown in the proof of [6, Proposition 2.4], based on Grauert's generalisation of the Kodaira embedding theorem [9, p.343, Satz 2].

(ii) It is clear from (i) that X admits $\dim X$ algebraically independent meromorphic functions, so that X is a Moishezon manifold. Using the Hermitian metric on $[D]$ constructed in [6, Lemma 2.3] and the vanishing theorems of [25, Theorem 3, Appendix], we find that

$$h^{0,i}(X) = h^i(X, \mathcal{O}_X) = h^i(X, K_X \otimes q[D]) = 0$$

for all $i > 0$. Alternatively, we can use that p is given by the linear system $|mD|$ for $m \gg 1$; pulling back the Fubini-Study metric then shows that $[D]$ is quasi-positive as defined in [11, p.266], so that the vanishing follows from [11, Satz 2.1]. (Both versions of this argument require X to be smooth as we need the sheaf of sections of $[D]$ to be locally free in order to have that $\cdot = \otimes$ in [11, 25].)

That the singularities of $Y \setminus p(D)$ are canonical is clear by definition.

(iii) By [21, Theorem 2.2.18], the Hodge theorem holds on X. Thus, $H^2(X) = H^{1,1}(X)$ by (ii). As in [6, Proposition 2.5], it follows that the restriction map $H^{1,1}(X) \to H^2(M)$ is surjective.

Let ω be any Kähler form on M. By what we have just said, we can write $\omega = \xi + d\eta$ for some closed $(1,1)$-form ξ on X and some 1-form η on M. Since $d\eta$ is of type $(1,1)$, we may further write $d\eta = i\partial\bar\partial u$ for some $u \in C^\infty(M)$ by [5, Corollary A.3(i)]. Now let $\gamma = -i\partial\bar\partial \log h_f$ denote the good curvature form provided by [6, Lemma 2.3], and let $\chi \in C^\infty_0(M)$ be a fixed cut-off function with $d\chi$ supported in the tubular neighbourhood of D where $\gamma > 0$. Then, for $C \gg 1$ sufficiently large,

$$\xi + i\partial\bar\partial(\chi u) + C\gamma$$

clearly defines a Kähler form on X whose restriction to M is cohomologous to ω.

Now recall that a Moishezon manifold is Kähler if and only if it is projective [21, Theorem 2.2.26]. Alternatively, we can use the fact that $h^{2,0}(X) = 0$ together with Kodaira embedding. □

Proof of Theorem 2.1 in the general orbifold case. Part (i) is proved for orbifolds in [6]. If (ii) holds and if the orbifold de Rham cohomology of X satisfies the Hodge decomposition theorem, then the above proof will go through without any changes (using Kodaira embedding in the last step).

Our tool for verifying (ii) and the Hodge decomposition is the existence of a proper modification $\pi : \hat{X} \to X$, given by successive blow-ups with smooth centres not intersecting D at any stage, such that $\hat{X}$ is a projective algebraic orbifold; to find π, we follow the proof of [21, Theorem 2.2.16] with $\varphi = p^{-1}$. If $\hat{D} = \pi^{-1}(D)$, then $\hat{X} \setminus \hat{D}$ is smooth, and π is an isomorphism onto its image near $\hat{D}$.

Given the map π, the proof of [21, Theorem 2.2.18] now goes through in the orbifold case if one uses orbifold differential forms. This yields the required Hodge decomposition. Moreover, using the injectivity of π^* from this proof, (ii) follows if we can show that $H^i(\hat{X}, \mathcal{O}_{\hat{X}}) = 0$ for $i > 0$.

For simplicity, let us write $f = p \circ \pi$. By GAGA, this is an algebraic map. Since Y has canonical singularities away from $f(\hat{D})$ and f is an isomorphism onto its image close to $\hat{D}$, [16, Theorem 5.22] yields that $R^q f_* \mathcal{O}_{\hat{X}} = 0$ for $q > 0$. Thus, $H^i(\hat{X}, \mathcal{O}_{\hat{X}}) = H^i(Y, \mathcal{O}_Y)$ by the Leray spectral sequence. It therefore suffices to prove that the latter group vanishes for $i > 0$.[4]

Assume for the moment that the statement of [16, Corollary 2.68][5] holds for f and for the orbifold line bundle $L = M = q[\hat{D}]$, where we have set $a_i = 0$. (This would follow from the statement of [16, Theorem 2.64] with $L = M = q[\hat{D}] + f^*H$ for H sufficiently ample on Y.) Then $R^q f_* F = 0$ for all $q > 0$, where $F = K_{\hat{X}} + L$. Consequently, from the Leray spectral sequence, $H^i(Y, f_*F) = H^i(\hat{X}, F)$ for all i. Now $F = \sum n_i E_i$ with $n_i \in \mathbb{N}_0$, where the E_i are f-exceptional divisors, so that $f_*F = \mathcal{O}_Y$. Moreover, $H^i(\hat{X}, F) = 0$ for $i > 0$ is exactly the statement of [16, Theorem 2.64] with $L = M = q[\hat{D}]$ there. In conclusion, it remains to show that $H^i(\hat{X}, K_{\hat{X}} + L) = 0$ for all $i > 0$ for L an orbifold line bundle with nonnegative curvature on $\hat{X}$ and with strictly positive curvature on a nonempty open subset of $\hat{X}$. But this immediately follows from the proof of [25, Theorem 6]. □

[4]If the orbifold singularities of Y along $p(D)$ are log-terminal, then this already follows from [16, Theorem 2.70]. Our aim here is to prove a version of this theorem that does not require resolving these singularities.

[5]Notice that $\omega_Y \otimes M$ should be $\omega_Y \otimes L$ in [16, Corollary 2.68].

Appendix A. Li's compactification theorem

Let us begin by stating the part of [20, Theorem 1.2] that we need.

Theorem A.1. *Let D be a compact complex manifold with a holomorphic line bundle L. Fix $\delta > 0$ and a Hermitian metric h of positive curvature on L. Write $\omega_0 = \frac{i}{2}\partial\bar{\partial}h^{-\delta}$ for the associated Calabi cone metric with radius function r given by $r^2 = h^{-\delta}$ on $L \setminus 0$. Let U be a tubular neighbourhood of the zero section of L, and let J be a complex structure on $U \setminus 0$ such that*

$$|\nabla^j_{g_0}(J - J_0)|_{g_0} = O(r^{\lambda - j}) \tag{A.1}$$

for some $\lambda < 0$ and all $j \in \mathbb{N}_0$, where J_0 denotes the usual complex structure of the total space of L. Then, modulo diffeomorphism, J extends to a smooth complex structure on U.

This result is analogous to [12, Theorem 3.1], but Li's proof [20] is different from the proof in [12]. The aim of this appendix is to clarify the relation between the conical and the cylindrical case.

Fix a small coordinate ball $B \subset D$ and a trivialisation $L|_B \to B \times \mathbb{C}$. Push the set-up forward under this trivialisation, so that J_0 now also denotes the product complex structure on $B \times \mathbb{C}$. Let z denote the fibre coordinate and set $\theta = \arg z$. Then

$$g_0 = dr^2 + r^2(d\theta + A)^2 + r^2 g_D,$$

where, up to rescaling, A and ω_D are the connection and curvature forms of the Chern connection of (L, h) in our chosen trivialisation; more precisely, if we write $h = e^{-\phi}|z|^2$ with $\phi : B \to \mathbb{R}$, then $A = \frac{\delta}{2} d^c\phi$ and $\omega_D = \frac{\delta}{4} dd^c\phi$ with respect to J_0. We now define a new Riemannian metric $g_\sharp$ by

$$g_\sharp = ds^2 + s^2 d\theta^2 + s^2 g_D,$$

where $s^2 = |z|^{-2\delta} = e^{-\delta\phi}r^2$. This is a Riemannian cone metric on $B \times \mathbb{C}$ which is Hermitian with respect to J_0, but not Kähler. The key point is that g_0 and $g_\sharp$ have exactly the same scaling vector field $r\partial_r = s\partial_s$. Thus, by [5, Lemma 1.6], $|\nabla^j_{g_0}(g_0 - g_\sharp)|_{g_0} = O(r^{-j})$ and $|\nabla^j_{g_\sharp}(g_0 - g_\sharp)|_{g_\sharp} = O(s^{-j})$ for all $j \in \mathbb{N}_0$. Using this, it is easy to see that

$$|\nabla^j_{g_\sharp}(J - J_0)|_{g_\sharp} = O(s^{\lambda - j}). \tag{A.2}$$

Next observe that the pullback of $s^{-2}g_\sharp$ under $s = e^t$ is the cylinder metric $g_\infty = dt^2 + d\theta^2 + g_D$ on $\mathbb{R}^+ \times \mathbb{S}^1 \times B$, and that the pullback of J_0 is the obvious product complex structure J_∞. Identifying J with its pullback, it then follows from (A.2) that $|\nabla^j_{g_\infty}(J - J_\infty)|_{g_\infty} = O(e^{\lambda t})$. We are now able to construct the desired extension of J by appealing to (an obvious localised version of) [12, Theorem 3.1], whose proof reduces the problem to the classical results of [22, 23].

Up until (A.2), this argument is a slight modification of Li's approach; Li instead rewrites (A.1) as a comparison of J and J_0 with respect to the smooth metric $\tilde{g}_0 = d\rho^2 + \rho^2 d\theta^2 + g_D$ on $\Delta \times B$, where Δ denotes the unit disk in $\mathbb{C}$ and $\rho = |z|$ (see [20, (25)]). He then constructs J-holomorphic coordinates by redoing the analysis of [22, 23] in weighted Hölder spaces with respect to $\tilde{g}_0$.

Finally, we point out that both arguments obviously extend to the orbifold setting. Alternatively, in order to prove Corollary C, we could also use a compactification theorem from [19], which relies on twistor theory. In fact, [19, Lemma 4.1] asserts that if the cone $(L \setminus 0, \omega_0)$ is flat of dimension 2 (hence is of the form $\mathbb{C}^2/\Gamma$ for some finite group $\Gamma \subset \mathrm{U}(2)$ acting freely on $\mathbb{S}^3 \subset \mathbb{C}^2$), and if J is the parallel complex structure of a *scalar-flat* ALE Kähler metric of rate $\lambda \leq -\frac{3}{2}$, then $(U \setminus 0, J)$ admits an orbifold compactification obtained by adding on the orbifold curve $D = \mathbb{P}^1/\Gamma$.

References

1. K. Altmann, *The versal deformation of an isolated toric Gorenstein singularity*, Invent. Math. **128** (1997), 443–479.
2. M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.
3. E. Calabi, *Métriques kählériennes et fibrés holomorphes*, Ann. Sci. École Norm. Sup. (4) **12** (1979), 269–294.
4. P. Candelas and X. C. de la Ossa, *Comments on conifolds*, Nuclear Phys. B **342** (1990), 246–268.
5. R. J. Conlon and H.-J. Hein, *Asymptotically conical Calabi-Yau manifolds, I*, Duke Math. J. **162** (2013), 2855–2902.

6. ———, *Asymptotically conical Calabi-Yau manifolds, II*, arXiv:1301.5312.
7. W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1998.
8. R. Goto, *Calabi-Yau structures and Einstein-Sasakian structures on crepant resolutions of isolated singularities*, J. Math. Soc. Japan **64** (2012), 1005–1052.
9. H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.
10. ———, *Über die Deformation isolierter Singularitäten analytischer Mengen*, Invent. Math. **15** (1972), 171–198.
11. H. Grauert and O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970), 263–292.
12. M. Haskins, H.-J. Hein, and J. Nordström, *Asymptotically cylindrical Calabi-Yau manifolds*, arXiv:1212.6929.
13. A. Kas and M. Schlessinger, *On the versal deformation of a complex space with an isolated singularity*, Math. Ann. **196** (1972), 23–29.
14. J. Kollár, *Flops*, Nagoya Math. J. **113** (1989), 15–36.
15. ———, *Flips, flops, minimal models, etc*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 113–199.
16. J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge University Press, Cambridge, 1998.
17. P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. **29** (1989), 665–683.
18. ———, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. **29** (1989), 685–697.
19. C. LeBrun and B. Maskit, *On optimal 4-dimensional metrics*, J. Geom. Anal. **18** (2008), 537–564.
20. C. Li, *Remarks on deformation of cones and neighborhoods of ample divisors*, arXiv:1405.2433.
21. X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Birkhäuser Verlag, Basel, 2007.
22. A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), 391–404.
23. A. Nijenhuis and W. B. Woolf, *Some integration problems in almost-complex and complex manifolds*, Ann. of Math. (2) **77** (1963), 424–489.
24. H. C. Pinkham, *Deformations of algebraic varieties with G_m action*, Astérisque, vol. 20, Société Mathématique de France, Paris, 1974.
25. O. Riemenschneider, *Characterizing Moišezon spaces by almost positive coherent analytic sheaves*, Math. Z. **123** (1971), 263–284.
26. J. Ross and R. Thomas, *Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics*, J. Differential Geom. **88** (2011), 109–159.
27. M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222.
28. I. R. Shafarevich (ed.), *Algebraic geometry, V, Fano varieties*, Encycl. Math. Sciences, vol. 47, Springer-Verlag, Berlin, 1999.
29. P. Slodowy, *Simple singularities and simple algebraic groups*, Lecture Notes in Mathematics, vol. 815, Springer, Berlin, 1980.
30. M. B. Stenzel, *Ricci-flat metrics on the complexification of a compact rank one symmetric space*, Manuscripta Math. **80** (1993), 151–163.

Département de Mathématiques, Université du Québec à Montréal, Case Postale 8888, Succursale Centre-ville, Montréal (Québec), H3C 3P8, Canada
E-mail address: rconlon@cirget.ca

Laboratoire de Mathématiques Jean Leray, Université de Nantes, 2 rue de la Houssinière, 44322 Nantes Cedex 3, France
E-mail address: hansjoachim.hein@univ-nantes.fr